The # product in combinatorial Hopf algebras

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Abstract. We show that the #-product of binary trees introduced by Aval and Viennot [1] is in fact defined at the level of the free associative algebra, and can be extended to most of the classical combinatorial Hopf algebras.

1. Introduction

There is a well-known Hopf algebra structure, due to Loday and Ronco [8], on the set of planar binary trees. Using a new description of the product of this algebra, (denoted here by **PBT**) in terms of Catalan alternative tableaux, Aval and Viennot [1] introduced a new product, denoted by #, which is compatible with the original graduation shifted by 1.

Since then, Chapoton [2] has given a functorial interpretation of this operation.

Most classical combinatorial Hopf algebras, including **PBT**, admit a realization in terms of special families of noncommutative polynomials. We shall see that on the realization, the # product has a simple interpretation. It can in fact be defined at the level of words over the auxiliary alphabet. Then, it preserves in particular the algebras based on parking functions (**PQSym**), packed words (**WQSym**), permutations (**FQSym**), planar binary trees (**PBT**), plane trees (the free tridendriform algebra $\mathfrak{TD}$), segmented compositions (the free cubical algebra $\mathfrak{TC}$), Young tableaux (**FSym**), and integer compositions (**Sym**).

All definitions not recalled here can be found, *e.g.*, in [11, 12, 13].

2. A semigroup of paths

Let A be an alphabet. Words over A can be regarded as encoding paths in a complete graph with a loop on each vertex, vertices being labelled by A.

Composition of paths, denoted by #, endows the set $\Sigma(A) = A^+ \cup \{0\}$ with the structure of a semigroup:

$$ua\#bv = \begin{cases} uav & \text{if } b = a, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

For example, $baaca\#adb = baacadb$ and $ab\#cd = 0$. Thus, the # product maps $A^n \times A^m$ to A^{m+n-1}. It is graded w.r.t. the path length (*i.e.*, the number of edges in the path).

We have the following obvious compatibilities with the concatenation product:

$$(uv)\#w = u \cdot (v\#w)\,, \tag{2}$$

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$$(u\#v)\cdot w = u\#(vw)\,. \tag{3}$$

Let d_k be the linear operator on $\mathbb{K}\langle A\rangle$ (over some field $\mathbb{K}$) defined by

$$d_k(w) = \begin{cases} uav & \text{if } w = uaav \text{ for some } a, \text{ with } |u| = k-1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Then, for u of length k,

$$u\#v = d_k(uv)\,. \tag{5}$$

3. Application to combinatorial Hopf algebras

The notion of a *combinatorial Hopf algebra* is a heuristic one, referring to rich algebraic structures arising naturally on the linear spans of various families of combinatorial objects. These spaces are generally endowed with several products and coproducts, and are in particular graded connected bialgebras, hence Hopf algebras.

The most prominent combinatorial Hopf algebras can be realized in terms of ordinary noncommutative polynomials over an auxiliary alphabet A. This means that their products, which are described by combinatorial algorithms, can be interpreted as describing the ordinary product of certain bases of polynomials in an underlying totally ordered alphabet $A = \{a_1 < a_2 < \dots\}$.

We shall see that all these realizations are stable under the # product. In the case of **PBT** (planar binary trees), we recover the result of Aval and Viennot [1]. In this case, the #-product has been interpreted by Chapoton [2] in representation theoretical terms.

We shall start with the most natural algebra, **FQSym**, based on permutations. It contains as subalgebras **PBT** (planar binary trees or the Loday-Ronco algebra, the free dendriform algebra on one generator), **FSym** (free symmetric functions, based on standard Young tableaux), and **Sym** (noncommutative symmetric functions).

It is itself a subalgebra of **WQSym**, based on packed words (or set compositions), in which the role of **PBT** is played by the free dendriform trialgebra on one generator $\mathfrak{TD}$ (based on Schröder trees), the free cubical trialgebra $\mathfrak{TC}$ (segmented compositions).

Finally, all of these algebras can be embedded in **PQSym**, based on parking functions.

Note that although all our algebras are actually Hopf algebras, the Hopf structure does not play any role in this paper.

4. Free quasi-symmetric functions: **FQSym** and its subalgebras

4.1. Free quasi-symmetric functions.

4.1.1. *The operation d_k on* **FQSym**. Recall that the alphabet A is *totally ordered*. Thus, we can associate to any word over A a permutation $\sigma = \mathrm{std}(w)$, the *standardized word* $\mathrm{std}(w)$ of w, obtained by iteratively scanning w from left to right, and labelling $1, 2, \ldots$ the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. For example,

$$\mathrm{std}(365182122) = 687193245. \tag{6}$$

For a permutation σ, define

$$\mathbf{G}_\sigma = \sum_{\mathrm{std}(w)=\sigma} w\,. \tag{7}$$

We shall need the following easy property of the standardization:

Lemma 4.1. *Let $= u_1u_2\cdots u_n$ be a word over A, and $\sigma = \sigma_1\sigma_2\ldots\sigma_n = \mathrm{std}(u)$. Then, for any factor of u,*

$$\mathrm{std}(u_iu_{i+1}\cdots u_j) = \mathrm{std}(\sigma_i\sigma_{i+1}\cdots\sigma_j)\,. \tag{8}$$

This implies that **FQSym** is stable under the d_k:

$$d_k(\mathbf{G}_\sigma) = \begin{cases} \mathbf{G}_{\mathrm{std}(\sigma_1\cdots\sigma_{k-1}\sigma_{k+1}\cdots\sigma_n)} & \text{if } \sigma_{k+1} = \sigma_k + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

We shall make use of the dual basis of the $\mathbf{G}_\sigma$ when dealing with subalgebras of **FQSym**. In the dual basis $\mathbf{F}_\sigma$ defined by $\mathbf{F}_\sigma := \mathbf{G}_{\sigma^{-1}}$, the formula is

$$d_k(\mathbf{F}_\sigma) = \begin{cases} \mathbf{F}_{\mathrm{std}(\sigma_1\cdots k\widehat{k+1}\cdots\sigma_n)} & \text{if } \sigma \text{ has a factor } k\,k{+}1, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

where $\widehat{a}$ means that $a$is removed.

4.1.2. *Algebraic structure.* The $\mathbf{G}_\sigma$ span a subalgebra of the free associative algebra, denoted by **FQSym**. The product is given by

$$\mathbf{G}_\alpha\mathbf{G}_\beta = \sum_{\gamma=uv,\ \mathrm{std}(u)=\alpha,\ \mathrm{std}(v)=\beta} \mathbf{G}_\gamma\,. \tag{11}$$

The set of permutations occuring in the r.h.s. is called the convolution of α and β, and denoted by $\alpha * \beta$.

Hence,

$$\mathbf{G}_\sigma \# \mathbf{G}_\tau = d_k(\mathbf{G}_\sigma\mathbf{G}_\tau) = \sum_{\nu\in\sigma\#\tau} \mathbf{G}_\nu, \tag{12}$$

where

$$\sigma\#\tau = \{\nu|\,|\nu| = k + l - 1, \mathrm{std}(\nu_1\ldots\nu_k) = \sigma; \mathrm{std}(\nu_k\ldots\nu_{k+l-1}) = \tau\}\,. \tag{13}$$

Indeed, $\mathbf{G}_\sigma\#\mathbf{G}_\tau$ is the sum of all words of the form $w = uxv$, with $\mathrm{std}(ux) = \sigma$ and $\mathrm{std}(xv) = \tau$. For example,

$$\mathbf{G}_{132}\#\mathbf{G}_{231} = \mathbf{G}_{14352} + \mathbf{G}_{15342} + \mathbf{G}_{24351} + \mathbf{G}_{25341}. \tag{14}$$

Note that d_k induces a bijection

$$\mathfrak{S}_{n;k} := \{\sigma \in \mathfrak{S}_n | \sigma_{k+1} = \sigma_k + 1\} \longrightarrow \mathfrak{S}_{n-1}\,. \tag{15}$$

In the sequel, the notation d_k^{-1} will refer to the inverse bijection.

4.1.3. *Multiplicative bases.* The multiplicative basis $\mathbf{S}^\sigma$ of **FQSym** is defined by [4]

$$\mathbf{S}^\sigma = \sum_{\tau\le\sigma} \mathbf{G}_\tau\,, \tag{16}$$

where $\le$ is the left weak order.

Similarly, the multiplicative basis $\mathbf{E}^\sigma$ of **FQSym** is defined by [4]

$$\mathbf{E}^\sigma = \sum_{\tau\ge\sigma} \mathbf{G}_\tau\,. \tag{17}$$

For $\alpha \in \mathfrak{S}_k$ and $\beta \in \mathfrak{S}_l$, define $\alpha \vee \beta \in \mathfrak{S}_{k+l-1}$ as the output of the following algorithm:

- scan the letters of α from left to right and write

$$\begin{cases} \alpha_i + \beta_1 - 1 & \text{if } \alpha_i \le \alpha_k, \\ \alpha_i + \max(\beta) - 1 & \text{if } \alpha_i > \alpha_k, \end{cases} \tag{18}$$

- scan the letters of β starting from the second one and write

$$\begin{cases} \beta_i & \text{if } \beta_i < \beta_1, \\ \beta_i + \alpha_k - 1 & \text{if } \beta_i \ge \beta_1. \end{cases} \tag{19}$$

Similarly, define $\alpha \wedge \beta$ by:

- scan the letters of α and write

$$\begin{cases} \alpha_i & \text{if } \alpha_i < \alpha_k, \\ \alpha_i + \beta_1 - 1 & \text{if } \alpha_i \ge \alpha_k, \end{cases} \tag{20}$$

- read the letters of β starting from the second one and write

$$\begin{cases} \beta_i + \alpha_k - 1 & \text{if } \beta_i \le \beta_1, \\ \beta_i + \max(\alpha) - 1 & \text{if } \beta_i > \beta_1. \end{cases} \tag{21}$$

For example, $3412 \vee 35124 = 78346125$ and $3412 \wedge 35124 = 56148237$.

Theorem 4.2. *The permutations appearing in a #-product $\mathbf{G}_\alpha \# \mathbf{G}_\beta$ is an interval of the left weak order:*

$$\mathbf{G}_\alpha \# \mathbf{G}_\beta = \sum_{\gamma\in[\alpha\wedge\beta,\alpha\vee\beta]} \mathbf{G}_\gamma\,. \tag{22}$$

Proof – First, it is clear that

$$d_k^{-1}([\alpha\wedge\beta, \alpha\vee\beta]) \subseteq \alpha * \beta \tag{23}$$

since for any $\gamma \in [\alpha\wedge\beta, \alpha\vee\beta]$, we have $\mathrm{std}(\gamma_1\cdots\gamma_k) = \alpha$ and $\mathrm{std}(\gamma_k\cdots\gamma_{k+l-1}) = \beta$, so that $d_k^{-1}(\gamma) \in \alpha * \beta$.

Let us now show the reverse inclusion, that is,

$$(\alpha * \beta) \cap \mathfrak{S}_{n;k} \subseteq d_k^{-1}([\alpha\wedge\beta, \alpha\vee\beta]). \tag{24}$$

Let $\sigma = d_k^{-1}(\alpha\vee\beta)$, and $\tau \in (\alpha*\beta)\cap\mathfrak{S}_{n;k}$. We need to show that $\mathrm{Inv}\,(\tau) \subseteq \mathrm{Inv}\,(\sigma)$. Clearly, any inversion (i,j) of τ such that $i,j \le k$ or $i,j > k$ is also an inversion of σ, by definition of $*$.

Assume now that $i \leq k < j$ and $(i,j) \notin \mathrm{Inv}\,(\sigma)$. By definition of $\alpha \vee \beta$, the only such pairs (i,j) are those such that $\sigma_i \leq \sigma_k$ and $\sigma_j \geq \sigma_{k+1}$. Again, since $\mathrm{std}(\tau_1 \cdots \tau_k) = \mathrm{std}(\sigma_1 \cdots \sigma_k)$ and $\mathrm{std}(\tau_k \cdots \tau_{k+l-1}) = \mathrm{std}(\sigma_k \cdots \sigma_{k+l-1})$, we have $\tau_i \leq \tau_k < \tau_{k+1} \leq \tau_j$, hence $(i,j) \notin \mathrm{Inv}\,(\tau)$.

The proof of the lower bound is analogous. ■

Using only either the lower bound or the upper bound, one obtains:

Corollary 4.3. *The bases* $\mathbf{S}^\sigma$ *and* $\mathbf{E}^\sigma$ *are multiplicative for the #-product:*

$$\mathbf{S}^\alpha \# \mathbf{S}^\beta = \mathbf{S}^{\alpha\vee\beta} \qquad \textit{and} \qquad \mathbf{E}^\alpha \# \mathbf{E}^\beta = \mathbf{E}^{\alpha\wedge\beta}\,. \tag{25}$$

For example,

$$\mathbf{S}^{3412} \# \mathbf{S}^{35124} = \mathbf{S}^{78346125} \qquad \mathbf{E}^{3412} \# \mathbf{E}^{35124} = \mathbf{E}^{56148237}. \tag{26}$$

4.1.4. *Freeness.* The above description of the # product in the $\mathbf{S}$ basis implies the following result:

Theorem 4.4. *For the # product,* **FQSym** *is free on either* $\mathbf{S}^\alpha$ *or* $\mathbf{G}_\alpha$ *where* α *runs over* non-secable permutations, *that is, permutations of size* $n \geq 2$ *such that any prefix* $\alpha_1 \dots \alpha_k$ *of size* $2 \leq k < n$ *is not, up to order, the union of an interval with maximal value* σ_k *and another interval either empty or with maximal value* n*.*

Proof – Indeed, any permutation can be uniquely decomposed as a maximal $\vee$ product of non-secable permutations, so that the result holds for the $\mathbf{S}$ basis. Now, if $\sigma = \sigma_1 \vee \cdots \vee \sigma_k$ is secable, then

$$\mathbf{G}_{\sigma_1} \# \cdots \# \mathbf{G}_{\sigma_k} = \mathbf{G}_\sigma + \cdots \tag{27}$$

where the dots stand for permutations strictly smaller than σ by Theorem 4.2. Hence, **FQSym** is also free on the $\mathbf{G}_\alpha$ with α non-secable. ■

The generating series of the number of non-secable permutations (by shifted degree $d'(\sigma) = n-1$ for $\sigma \in \mathfrak{S}_n$) is Sequence A077607 of [15]

$$\mathrm{NI}(t) := 2\,t + 2\,t^2 + 8\,t^3 + 44\,t^4 + 296\,t^5 + 2312\,t^6 + \dots \tag{28}$$

or equivalently

$$1/(1 - \mathrm{NI}(t)) = \sum_{n\geq 1} n!t^{n-1}. \tag{29}$$

Although there is a canonical choice of the free generators in the $\mathbf{S}$ basis, there are other possibilities in the $\mathbf{G}$ basis, as stated by the following proposition.

Proposition 4.5. *For the # product,* **FQSym** *is free on the* $\mathbf{G}_\alpha$*, where* α *runs over* non-interval permutations*, that is permutations of size* $|\alpha| \geq 2$ *having no prefix* $\alpha_1, \dots, \alpha_i$ *of size* $2 \leq i < |\alpha|$ *which is up to order an interval* $[j, i+j-1]$*.*

The starting point is the following lemma.

Lemma 4.6. *Let* σ *and* τ *be two permutations, of respective sizes* k *and* ℓ*. There is exactly one permutation* γ *in the set* $\sigma\#\tau$ *such that the set of values* $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$ *is an interval of integers. We denote this permutation* γ *by* $\sigma \bullet \tau$*.*

Proof – Let γ be a permutation of $\sigma\#\tau$. We first observe that the entry γ_k is fixed and equal to

$$\gamma_k = \sigma_k + \tau_1 - 1. \tag{30}$$

Next, it is easy to see that there is exactly one choice for $(\gamma_1, \dots, \gamma_{k-1})$: the first $k-1$ entries of σ shifted by $\gamma_k - \sigma_k$. When this is done, the unused entries have to be ordered with respect to $(\tau_2, \dots, \tau_l)$. ∎

As an example we have:

$$25143 \bullet 45132 = 5847\,6\,9132. \tag{31}$$

Any permutation α may be decomposed in a unique way as a maximal $\bullet$ product. It is clear that permutations involved in such a maximal $\bullet$ product are non-interval permutations.

Note 4.7. The order of "breaking" α in a $\bullet$ product is irrelevant. Moreover, no new breakpoint is generated during the decomposition process. This comes from the following observation. Let us write

$$\alpha = \alpha_1 \dots \alpha_r \dots \alpha_s \dots \alpha_k. \tag{32}$$

First, if we assume that $\{\alpha_1, \dots, \alpha_r\}$ is an interval, we may write $\alpha = \sigma^{(1)} \bullet \tau^{(1)}$ with $|\sigma^{(1)}| = r$. Then $\{\tau_1^{(1)}, \dots, \tau_{s-r+1}^{(1)}\}$ is an interval if and only if $\{\alpha_1, \dots, \alpha_s\}$ is also an interval.

Now, if we suppose that $\{\alpha_1, \dots, \alpha_s\}$ is an interval, we may write $\alpha = \sigma^{(2)} \bullet \tau^{(2)}$ with $|\sigma^{(2)}| = s$. Then $\{\sigma_1^{(2)}, \dots, \sigma_r^{(2)}\}$ is an interval if and only if $\{\alpha_1, \dots, \alpha_r\}$ is also an interval.

Let us denote by $I(\alpha)$ the number of terms in such a maximal $\bullet$ decomposition of α. We observe that

$$I(\sigma \bullet \tau) = I(\sigma) + I(\tau) + 1 \tag{33}$$

and the next lemma provides a strict inequality for any other permutation appearing in the $\#$ product of σ and τ.

Lemma 4.8. *For σ and τ two permutations, let γ be a permutation in the set $\sigma\#\tau$, different from $\sigma \bullet \tau$, then $I(\gamma) < I(\sigma) + I(\tau) + 1$.*

We shall first introduce some notations, state and prove a sub-lemma, then conclude the proof of Lemma 4.8.

Let $\sigma = \sigma_1 \bullet \cdots \bullet \sigma_r$ with $r = I(\sigma)$ and $\tau = \tau_1 \bullet \cdots \bullet \tau_s$ with $s = I(\tau)$. The size of σ_i, resp. τ_i will be denoted by k_i, resp. ℓ_i. We denote by $k = 1 + \sum_{i=1}^{r}(k_i - 1)$ the size of σ.

Lemma 4.9. *With these notations, and $\gamma \neq \sigma \bullet \tau$, if the integer p is neither in the set $\{1 + \sum_{i=1}^{j}(k_i - 1),\ j = 1, \dots, r-1\}$ ("breakpoints in σ") nor in the set $\{k + \sum_{i=1}^{j}(l_i - 1),\ j = 1, \dots, s-1\}$ ("breakpoints in τ"), then $\{\gamma_1, \dots, \gamma_p\}$ is* not *an interval.*

Proof – For $p < k$, if $\{\gamma_1, \dots, \gamma_p\}$ is an interval, then so is $\{\sigma_1, \dots, \sigma_p\}$, which is absurd by Note 4.7. For $p > k$, if $\{\gamma_1, \dots, \gamma_p\}$ is an interval, then so is $\{\tau_1, \dots, \tau_{p-(k-1)}\}$, which is absurd by Note 4.7. Finally, for $p = k$, and because $\gamma \neq \sigma \bullet \tau$, this comes from Lemma 4.6. ∎

Proof of Lemma 4.8 – The proof proceeds by induction on $r + s = I(\sigma) + I(\tau)$. If $r + s = 2$, that is σ and τ are both non-interval, the result comes directly from Lemma 4.9.

If $r + s > 2$, we apply Lemma 4.9 to get that either γ is non-interval, or it may be decomposed as a $\bullet$ product, *but only* on a breakpoint relative to σ or τ. These two cases are treated in the same way, and we assume γ may be written as $\gamma = \gamma^{(1)} \bullet \gamma^{(2)}$ with $\gamma^{(2)} \in (\sigma_j \bullet \cdots \sigma_r)\#\tau$ for $j > 1$ and we conclude by induction. ∎

Proof of Proposition 4.5 – Let $\alpha_1, \dots, \alpha_r$ be non-interval permutations. By applying Lemma 4.8, we get that

$$\mathbf{G}_{\alpha_1}\#\cdots\#\mathbf{G}_{\alpha_r} = \mathbf{G}_{\alpha_1\bullet\cdots\bullet\alpha_r} + \sum \mathbf{G}_\gamma \tag{34}$$

where the γ appearing in the sum are such that $I(\gamma) < I(\alpha_1 \bullet \cdots \bullet \alpha_r) = r$. This proves the statement by triangularity. ∎

Note 4.10. *Since the inverse of a non-interval permutation is a* non-internal-interval *permutation, that is a permutation α of size at least 2 such that for any $r \geq 2$ no set of consecutive values $\{\alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_{i+r}\}$ is the set of the first r integers $\{1, \dots, r\}$,* **FQSym** *is free on $\mathbf{F}_\alpha$ where α runs over non-internal-interval permutations.*

4.2. **Young tableaux: FSym.** The algebra **FSym** of free symmetric functions [3] is the subalgebra of **FQSym** spanned by the coplactic classes

$$\mathbf{S}_t = \sum_{Q(w)=t} w = \sum_{P(\sigma)=t} \mathbf{F}_\sigma \tag{35}$$

where (P, Q) are the P-symbol and Q-symbol defined by the Robinson-Schensted correspondence. This algebra is isomorphic to the algebra of tableaux defined by Poirier and Reutenauer [14]. We shall denote by $\mathrm{STab}(n)$ the standard tableaux of size n. We introduce the conjugate $\bar{d}_k$ of the operator d_k defined by $\bar{d}_k(\sigma) = d_k(\sigma^{-1})^{-1}$ or (if we identify an injective word with its standardisation) in an equivalent manner by:

$$\bar{d}_k(\sigma) = \begin{cases} \mathrm{std}(ukv) & \text{if } \sigma = uk(k+1)v, \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

Similarly to d_k, $\bar{d}_k$ induces a bijection

$$\{\sigma \in \mathfrak{S}_n | \sigma_{k+1}^{-1} = \sigma_k^{-1} + 1\} \longrightarrow \mathfrak{S}_{n-1}\,. \tag{37}$$

We shall use the notation $\bar{d}_k^{-1}$ for the inverse bijection.

Note that

$$S_{\begin{ytableau} 3 & 4 \\ 1 & 2 \end{ytableau}} = \mathbf{G}_{2413} + \mathbf{G}_{3412}, \quad \text{so that} \quad d_1(S_{\begin{ytableau} 3 & 4 \\ 1 & 2 \end{ytableau}}) = \mathbf{G}_{312}, \tag{38}$$

which does not belong to **FSym**. Hence **FSym** is not stable under the d_k. However, we have:

Theorem 4.11. **FSym** *is stable under the #-product.*

Proof – For two standard tableaux $t' \in \mathrm{STab}(k)$ and $t'' \in \mathrm{STab}(\ell)$ we have

$$\mathbf{S}_{t'}\mathbf{S}_{t''} = \sum_{P(\sigma')=t',\ P(\sigma'')=t''} \mathbf{F}_{\sigma'}\mathbf{F}_{\sigma''} = \sum_{\tau \in \sigma' \sqcup\!\sqcup \sigma''[k]} \mathbf{F}_\tau\,, \tag{39}$$

so that $d_k(\mathbf{S}_{t'}\mathbf{S}_{t''})$ is a sum of $\mathbf{F}_\alpha$ without multiplicites. We need to show that if a given α occurs in this sum, all the α' such that $P(\alpha') = P(\alpha)$ are also present. We may assume that α' differs from α by application of a single plactic relation. We shall prove it for the relation $acb \equiv cab$, the other one being proved in the same way. Denote by σ_1 and σ_2 the two permutations (σ_1 of size k) such that $\bar{d}_k^{-1}(\alpha) \in \sigma_1 \sqcup\!\sqcup \sigma_2[k]$. Then, if $k < a$, $\bar{d}_k^{-1}(\alpha')$ belongs to $\sigma_1 \sqcup\!\sqcup \sigma_2'[k]$ where σ_2' is obtained from σ_2 by the same plactic rewriting. Then, if $k \geq c$, $\bar{d}_k^{-1}(\alpha')$ belongs to $\sigma_1' \sqcup\!\sqcup \sigma_2[k]$ where σ_1' is obtained from σ_1 by the same plactic rewriting. Finally, if $a \leq k < c$, $\bar{d}_k^{-1}(\alpha')$ also belongs to $\sigma_1 \sqcup\!\sqcup \sigma_2[k]$ since a comes from σ_1 and c from $\sigma_2[k]$. ∎

For example,

$$\mathbf{S}_{\begin{ytableau} 2 \\ 1 & 3 \end{ytableau}} \# \mathbf{S}_{\begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau}} = \mathbf{S}_{\begin{ytableau} 5 \\ 4 \\ 2 \\ 1 & 3 \end{ytableau}} + \mathbf{S}_{\begin{ytableau} 5 \\ 2 & 4 \\ 1 & 3 \end{ytableau}} \tag{40}$$

$$\mathbf{S}_{\begin{ytableau} 3 \\ 1 & 2 \end{ytableau}} \# \mathbf{S}_{\begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau}} = \mathbf{S}_{\begin{ytableau} 5 \\ 4 \\ 3 \\ 1 & 2 \end{ytableau}} \tag{41}$$

$$\mathbf{S}_{\begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau}} \# \mathbf{S}_{\begin{ytableau} 2 \\ 1 & 3 \end{ytableau}} = \mathbf{S}_{\begin{ytableau} 4 \\ 3 \\ 2 \\ 1 & 5 \end{ytableau}} \tag{42}$$

$$\mathbf{S}_{\begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau}} \# \mathbf{S}_{\begin{ytableau} 3 \\ 1 & 2 \end{ytableau}} = \mathbf{S}_{\begin{ytableau} 3 \\ 2 & 5 \\ 1 & 4 \end{ytableau}} + \mathbf{S}_{\begin{ytableau} 5 \\ 3 \\ 2 \\ 1 & 4 \end{ytableau}} \tag{43}$$

Note that those products do not have same number of terms, so that there is no natural definition of what would be the # product on the usual (commutative) symmetric functions.

For T an injective tableau and S a subset of its entries, let us denote by $T_{|S}$ the (sub-)tableau consisting of the restriction of T to its entries in S. For T, T' two skew tableaux, we denote their plactic equivalence (as for words) by $T \equiv T'$, that is we can obtain T' from T by playing *Jeu de Taquin*. The # product in **FSym** is given by the following simple combinatorial rule.

Proposition 4.12. *Let T_1 and T_2 be two standard tableaux of sizes k and ℓ. Then*

$$\mathbf{S}_{T_1} \,\#\, \mathbf{S}_{T_2} = \sum \mathbf{S}_T \tag{44}$$

where T runs over standard tableaux of size $k+\ell-1$ such that

- $T_{|\{1,\dots,k\}} = T_1$;
- $T_{|\{k,\dots,k+l-1\}} \equiv T_2$.

Proof – We shall use the same notation $\sigma_{|S}$ for the restriction of a permutation σ to a subset S of its entries, and identify a permutation with its standardisation.

Let us first consider a tableau T in the left-hand side of Equation (44) and γ a permutation of size $k+\ell-1$ such that $P(\gamma) = T$. Such permutations γ are characterized by the existence of two permutations σ of size k and τ of size ℓ satisfying

- $P(\sigma) = T_1$;
- $P(\tau) = T_2$;
- $\gamma' \in \sigma \sqcup\!\sqcup \tau[k]$;
- $\gamma = \bar{d}_k(\gamma')$.

Thus we have $P(\gamma_{|\{1,\dots,k\}}) = T_1$ and $P(\gamma_{|\{k,\dots,k+\ell-1\}}) = T_2$, which implies that T is in the right-hand side of (44).

Conversely, let us consider a tableau T in the right-hand side of Equation (44) and γ a permutation of size $k+\ell-1$ such that $P(\gamma) = T$. One has: $P(\gamma_{|\{1,\dots,k\}}) = T_1$ and $P(\gamma_{|\{k,\dots,k+\ell-1\}}) = T_2$. If we write $\gamma = ukv$, then we set $\gamma' = u'k(k+1)v' = \bar{d}_k^{-1}(\gamma) \in \mathfrak{S}_{k+\ell}$. Then we observe that:

- $\gamma' = u'k(k+1)v'$,
- $P(\gamma'_{|\{1,\dots,k\}}) = T_1$,
- $P(\gamma'_{|\{k+1,\dots,k+\ell\}}) = T_2$.

This implies that $P(\gamma) = T$ is in the left-hand side of (44). ■

With this description, it is easy to compute by hand

$$\mathbf{S}_{\begin{array}{|c|c|c|}\cline{1-1} 4 \\ \hline 1&2&3 \\ \hline\end{array}} \,\#\, \mathbf{S}_{\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1&2 \\ \hline\end{array}} = \mathbf{S}_{\begin{array}{|c|c|c|}\cline{1-1} 6 \\ \cline{1-2} 4&5 \\ \hline 1&2&3 \\ \hline\end{array}} + \mathbf{S}_{\begin{array}{|c|c|c|c|}\cline{1-1} 6 \\ \cline{1-1} 4 \\ \hline 1&2&3&5 \\ \hline\end{array}} + \mathbf{S}_{\begin{array}{|c|c|c|c|}\cline{1-2} 4&6 \\ \hline 1&2&3&5 \\ \hline\end{array}}. \tag{45}$$

4.3. **Planar binary trees: PBT.**

4.3.1. *Algebraic structure.* Recall that the natural basis of **PBT** can be defined by

$$\mathbf{P}_T = \sum_{\mathcal{T}(\sigma)=T} \mathbf{G}_\sigma \tag{46}$$

where $\mathcal{T}(\sigma)$ is the shape of the decreasing tree of σ.

Proposition 4.13. *The image of a tree by d_k is either 0 or a single tree:*

$$d_k(\mathbf{P}_T) = \begin{cases} \mathbf{P}_{T'}, \\ 0, \end{cases} \tag{47}$$

according to whether k is the left child of $k+1$ in the unique standard binary search tree of shape T (equivalently if the k-th vertex in the infix reading of T has no right child), in which case T' is obtained from T by contracting this edge, the result being 0 *otherwise.*

Proof – Recall that on the basis $\mathbf{F}_\sigma$, $\mathbf{P}_T$ is the sum of the linear extensions of the order whose Hasse diagram is the unique binary search tree T' of shape T (regarded as a poset, with the root as maximal element):

$$\mathbf{P}_T = \sum_{\sigma\in\mathcal{L}(T')} \mathbf{F}_\sigma\,. \tag{48}$$

If the right subtree of the vertex k of T' is nonempty, it must contain $k+1$ so that $k+1$ will always be before k in any linear extension of T', so in this case the result is 0. If the right subtree of k is empty, then $k+1$ must be in the left subtree of K, and is actually its left son. Then, the linear extensions of T' contain permutations having the factor $k\,k{+}1$. Erasing $k+1$ in those permutations and standardizing, we get exactly the linear extensions of the tree T'' obtained from T' by contracting the edge $(k, k+1)$ and standardizing. ∎

By the above result, any product $\mathbf{P}_{T'}\#\mathbf{P}_{T''}$ is in **PBT**. We just need to select those linear extensions which are not annihilated by d_k. Since $d_k(\mathbf{F}_\sigma)$ is nonzero iff σ has (as a word) a factor $k\,k{+}1$, the image under d_k of the surviving linear extensions are precisely those of the poset obtained by identifying the rightmost node of T' with the leftmost node of T''. Thus, # is indeed the Aval-Viennot product.

4.3.2. *Multiplicative bases.* The multiplicative basis of initial intervals [5] (corresponding to the projective elements of [2]) is a subset of the **S** basis of **FQSym**:

$$H_T = \mathbf{S}^\tau \tag{49}$$

where τ is the maximal element of the sylvester class T [5]. These maximal elements are the 132-avoiding permutations. Hence, they are preserved by the # operation, so that we recover Chapoton's result: the # product of two projective elements is a projective element. One can also apply the argument the other way round: since one easily checks that the # product of two permutations avoiding the pattern 132 also avoids this pattern, it is a simple proof that **PBT** is stable under #.

As in the case of **FQSym**, the fact that the **S** basis is still multiplicative for the # product implies that the product in the **P** basis is an interval in the Tamari order.

4.4. **Noncommutative symmetric functions: Sym.**

4.4.1. *Algebraic structure.* Recall that **Sym** is freely generated by the noncommutative complete functions

$$S_n(A) = \sum_{i_1 \le i_2 \le \ldots \le i_n} a_{i_1} a_{i_2} \cdots a_{i_n} = \mathbf{G}_{12\cdots n} \tag{50}$$

Here, we have obviously $S_n \# S_m = S_{n+m-1}$. This implies, for $l(I) = r$, $I = I' i_r$ and $J = j_1 J''$,

$$S^I \# S^J = S^{I' \cdot (i_r + j_1 - 1) \cdot J''} \tag{51}$$

and similarly

$$R_I \# R_J = R_{I' \cdot (i_r + j_1 - 1) \cdot J''} \,. \tag{52}$$

For example,

$$R_{1512} \# R_{43} = R_{15153}. \tag{53}$$

Clearly, as a #-algebra, $\mathbf{Sym}^+$ is the free graded associative algebra $\mathbb{K}\langle x, y \rangle$ over the two generators

$$x = S_2 = R_2 \qquad\qquad y = \Lambda_2 = R_{11} \tag{54}$$

of degree 1, the neutral element being $S_1 = R_1 = \Lambda_1$.

Now, define for any composition $I = (i_0, \ldots, i_r)$, the binary word

$$b(I) = 0^{i_0 - 1} 1 (0^{i_1 - 1}) 1 \ldots (0^{i_r - 1}). \tag{55}$$

On the binary coding of a composition I, one can read an expression of R_I, S^I, and Λ^I in terms of #-products of the generators x, y: replace the concatenation product by the #-product , replace 0 by respectively x, x, or y, and 1 by respectively y, $x+y$, or $x+y$, so that

$$R_I := (x^{i_0 - 1})^{\#} \# y \# (x^{i_1 - 1})^{\#} \# y \# \ldots \# (x^{i_r - 1})^{\#}, \tag{56}$$

$$S^I := (x^{i_0 - 1})^{\#} \# (x+y) \# (x^{i_1 - 1})^{\#} \# (x+y) \# \ldots \# (x^{i_r - 1})^{\#}, \tag{57}$$

and

$$\Lambda^I := (y^{i_0 - 1})^{\#} \# (x+y) \# (y^{i_1 - 1})^{\#} \# (x+y) \# \ldots \# (y^{i_r - 1})^{\#}. \tag{58}$$

Note that in particular, the maps sending S^I either to R_I or Λ^I are algebra automorphisms. This property will extend to a Hopf algebra automorphism with the natural coproduct.

4.4.2. *Coproduct.* In this case, we have a natural coproduct: the one for which x and y are primitive:

$$\nabla S_2 = S_2 \otimes S_1 + S_1 \otimes S_2, \tag{59}$$

$$\nabla \Lambda_2 = \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_2, \tag{60}$$

and, the neutral element S_1 is grouplike

$$\nabla S_1 = S_1 \otimes S_1. \tag{61}$$

Then,

$$\nabla S_n = \sum_{i=1}^{n} \binom{n-1}{i-1} S_i \otimes S_{n+1-i}. \tag{62}$$

The coproduct of generic S^I, Λ^I, and R_I all are the same: since x and y are primitive, $x+y$ is also primitive, so that, *e.g.*,

$$\nabla R_I = \sum_{w,w' | w \sqcup\!\sqcup w' = b(I)} C_{w,w'}^{b(I)} R_J \otimes R_K, \tag{63}$$

where J (resp. K) are the compositions whose binary words are w (resp. w'), and $C_{w,w'}^{b(I)}$ is the coefficient of $b(I)$ in $w \sqcup\!\sqcup w'$. Another way of presenting this coproduct is as follows: given $b(I)$, choose for each element if it appears on the left or on the right of the coproduct (hence giving $2^{|I|-1}$ terms) and compute the corresponding products of x and y.

Hence, the maps sending S^I either to R_I or Λ^I are Hopf algebra automorphisms.

4.4.3. *Duality: quasi-symmetric functions under #.* Since **Sym** is isomorphic to the Hopf algebra $\mathbb{K}\langle x, y\rangle$ on two primitive generators x and y, its dual is the shuffle algebra on two generators whose coproduct is given by deconcatenation.

Since all three bases S, R, and Λ behave in the same way for the Hopf structure, the same holds for their dual bases, so that the bases M_I, F_I, and the forgotten basis of *QSym* have the same product and coproduct formulas. In the basis F_I, this is

$$F_I \# F_J = \sum_{w \in b(I) \sqcup\!\sqcup b(J)} F_K, \tag{64}$$

where K is the composition such that $b(K) = w$.

For example,

$$F_3 \# F_{12} = 3\, F_{14} + 2\, F_{23} + F_{32}, \tag{65}$$

since

$$xx \sqcup\!\sqcup yx = 3\, yxxx + 2\, xyxx + xxyx. \tag{66}$$

Note that since the product is a shuffle on words in x and y, all elements in a product $F_I F_J$ have same length, which is $l(I) + l(J) - 1$.

5. Word quasi-symmetric functions: **WQSym** and its subalgebras

5.1. **Word quasi-symmetric functions.**

5.1.1. *Algebraic structure.* Word quasi-symmetric functions are the invariants of the quasi-symmetrizing action of the symmetric group (in the limit of an infinite alphabet), see, *e.g.*, [12].

The *packed word* $u = \mathrm{pack}\,(w)$ associated with a word $w \in A^*$ is obtained by the following process. If $b_1 < b_2 < \ldots < b_r$ are the letters occuring in w, u is the image of w by the homomorphism $b_i \mapsto a_i$. A word u is said to be *packed* if $\mathrm{pack}\,(u) = u$. Such words can be interpreted as set compositions, or as faces of the permutohedreon, and are sometimes called pseudo-permutations [6].

As in the case of permutations, we have:

Lemma 5.1. *Let $u = u_1u_2\cdots u_n$ be a word over A, and $v = v_1v_2\dots v_n = \mathrm{pack}\,(u)$. Then, for any factor of u,*

$$\mathrm{pack}\,(u_iu_{i+1}\cdots u_j) = \mathrm{pack}\,(v_iv_{i+1}\cdots v_j)\,. \tag{67}$$

The natural basis of **WQSym**, which lifts the quasi monomial basis of $QSym$, is labelled by packed words. It is defined by

$$\mathbf{M}_u = \sum_{\mathrm{pack}\,(w)=u} w\,. \tag{68}$$

Note that **WQSym** is stable under the operators d_k. We have

$$d_k(\mathbf{M}_w) = \begin{cases} \mathbf{M}_{w_1\cdots w_{k-1}w_{k+1}\cdots w_n} & \text{if } w_k = w_{k+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{69}$$

so that **WQSym** is stable under #. In this basis, the product is given by

$$\mathbf{M}_u\mathbf{M}_v = \sum_{w=u'v';\ \mathrm{pack}\,(u')=u,\,\mathrm{pack}\,(v')=v} \mathbf{M}_w\,. \tag{70}$$

Thus, **WQSym** is stable under #, and

$$\mathbf{M}_u\#\mathbf{M}_v = d_k(\mathbf{M}_u\mathbf{M}_v) = \sum_{w\in u\#v} \mathbf{M}_w, \tag{71}$$

where

$$u\#v = \{w|\,|w| = k+l-1, \mathrm{pack}\,(w_1\dots w_k) = u; \mathrm{pack}\,(w_k\dots w_{k+l-1}) = v\}. \tag{72}$$

For example,

$$\mathbf{M}_{121}\#\mathbf{M}_{12} = \mathbf{M}_{1212} + \mathbf{M}_{1213} + \mathbf{M}_{1312}. \tag{73}$$

5.1.2. *Multiplicative bases.* Recall that there exists an order on packed words generalizing the left weak order : it is the pseudo-permutohedron order. This order has a definition in terms of inversions (see [6]) similar to the definition of the left weak order. The *generalized inversion set* of a given packed word w is the union of the set of pairs (i, j) such that $i < j$ and $w_i > w_j$ with coefficient one (full inversions), and the set of pairs (i, j) such that $i < j$ and $w_i = w_j$ with coefficient one half (half-inversions).

One then says that two words u and v satisfy $u < v$ for the pseudo-permutohedron order iff the coefficient of any pair (i, j) in u is smaller than or equal to the same coefficient in v.

Note that the definition of $u \vee v$ and $u \wedge v$ (see the section about **FQSym**) does not require u and v to be permutations. One then has

Theorem 5.2. *The words appearing in the product $\mathbf{M}_u\#\mathbf{M}_v$ is an interval of the pseudo-permutohedron order:*

$$\mathbf{M}_u\#\mathbf{M}_v = \sum_{w\in[u\wedge v,u\vee v]} \mathbf{M}_w\,. \tag{74}$$

Proof – The proof of the statement is identical to that of the corresponding property of **FQSym**, simply taking into account half-inversions in the picture. ∎

The multiplicative basis $\mathbf{S}^u$ of **WQSym** is defined in [12] by

$$\mathbf{S}^u = \sum_{v \leq u} \mathbf{M}_v \,, \tag{75}$$

where $\leq$ is the pseudo-permutohedron order.

Proposition 5.3. *The* $\mathbf{S}$*-basis is multiplicative for the* $\#$*-product:*

$$\mathbf{S}^u \# \mathbf{S}^v = \mathbf{S}^{u \vee v}. \tag{76}$$

Similarly, the multiplicative basis $\mathbf{E}^u$ of **WQSym** is defined in [12] by

$$\mathbf{E}^u = \sum_{v \geq u} \mathbf{M}_v \,. \tag{77}$$

Proposition 5.4. *The* $\mathbf{E}$*-basis is multiplicative for the* $\#$*-product:*

$$\mathbf{E}^u \# \mathbf{E}^v = \mathbf{E}^{u \wedge v}. \tag{78}$$

5.1.3. *Freeness.* As in the case of **FQSym**, we can describe a set of free generators in the $\mathbf{S}$ basis for the algebra **WQSym**.

We shall say that a packed word u of size n is *secable* if there exists a prefix $u_1 \ldots u_k$ of size $2 \leq k < n$ such that:

- the set $\{u_1, \ldots, u_k\}$ is, up to order the union of an interval with maximal value u_k and another interval either empty or with maximal value the maximal entry of the whole word u;
- $\{u_1, \ldots, u_k\} \cap \{u_k, \ldots, u_n\} = \{u_k\}$.

Conversely, a packed word of size at least 2 which is not secable will be called *non-secable.*

Theorem 5.5. *For the* $\#$ *product,* **WQSym** *is free on the* $\mathbf{S}^u$ *or* $\mathbf{M}_u$ *where* u *runs over non-secable packed words.*

Proof – It is clear that any packed word can be uniquely decomposed as a maximal $\vee$ product of non-secable packed words, whence the assertion on the $\mathbf{S}^u$.

As in the case of **FQSym**, the result for the $\mathbf{M}_u$ comes by triangularity thanks to Theorem 5.2. ∎

If a packed word u is weighted by $t^{|u|-1}$, the generating series $\mathrm{PW}(t)$ of (unrestricted) packed words corresponds to Sequence A000670 of [15]:

$$\mathrm{PW}(t) = 1 + 3\,t + 13\,t^2 + 75\,t^3 + 541\,t^4 + 4683\,t^5 + 47293\,t^6 + \ldots \tag{79}$$

The generating series $NSPW$ of non-secable packed words is related to PW by

$$\mathrm{PW}(t) = 1/(1 - NSPW(t)) \tag{80}$$

which enables us to compute $NSPW$:

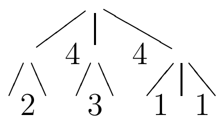
$$\mathrm{NSPW}(t) = 3\,t + 4\,t^2 + 24\,t^3 + 192\,t + 1872\,t^5 + 21168\,t^6 + \ldots \tag{81}$$

5.2. **The free tridendriform algebra $\mathfrak{TD}$.** The realization of the free dendriform trialgebra given in [11] involves the following construction. With any word w of length n, associate a plane tree $\mathcal{T}(w)$ with $n+1$ leaves, as follows: if $m = \max(w)$ and if w has exactly $k-1$ occurences of m, write

$$w = v_1\, m\, v_2 \cdots v_{k-1}\, m\, v_k\,, \tag{82}$$

where the v_i may be empty. Then, $\mathcal{T}(w)$ is the tree obtained by grafting the subtrees $\mathcal{T}(v_1), \mathcal{T}(v_2), \ldots, \mathcal{T}(v_k)$ (in this order) on a common root, with the initial condition $\mathcal{T}(\epsilon) = \emptyset$ for the empty word. For example, the tree associated with 243411 is

(83)



We shall call *sectors* the zones containing numbers and say that a sector is to the left of another sector if its number is to the left of the other one, so that the reading of all sectors from left to right of any $\mathcal{T}(w)$ gives back w.

Now define a polynomial by

$$\mathcal{M}_T := \sum_{\mathcal{T}(w)=T} \mathcal{M}_w\,. \tag{84}$$

Then, exactly as in the case of **PBT**, we have

Theorem 5.6.

$$d_k(\mathcal{M}_T) = \begin{cases} \mathcal{M}_{T'}, \\ 0, \end{cases} \tag{85}$$

depending on whether the k-th and $k+1$-th sectors are grafted on the same vertex or not. In the nonzero case, T' is obtained from T by gluing the k-th and $k+1$-th sectors.

Proof – If the k-th and $k+1$-th sectors of T are not grafted on the same vertex, they cannot have the same value, so that $d_k(\mathbf{M}_w)$ for all w satisfying $\mathcal{T}(w) = T$, implying that $d_k(\mathcal{M}_T) = 0$.

Otherwise, all w such that $\mathcal{T}(w) = T$ have equal values at places k and $k+1$, so that, applying d_k, we find all words satisfying $\mathcal{T}(w) = T'$ where T' is obtained from T by forgetting the k-th sector. ∎

Corollary 5.7. *The operation # is internal on $\mathfrak{TD}$.*

This result can be proved in a simpler way using the multiplicative basis **S** of **WQSym**. Indeed, it has been shown in [12] that $\mathfrak{TD}$ is the subalgebra of **WQSym** generated by the $\mathbf{S}^w$ where w runs over words avoiding the patterns 132 and 121. Since by construction the #-product of two such **S** also avoids both patterns 132 and 121, we get immediately that # is internal in $\mathfrak{TD}$.

The same holds for the multiplicative basis **E** of **WQSym**. It has also been shown in [12] that $\mathfrak{TD}$ is the subalgebra of **WQSym** generated by the $\mathbf{E}^w$ where w runs over words avoiding the patterns 312 and 211, and one easily checks that by construction, the #-product of two such **E** also avoids both patterns 312 and 211.

5.3. **The free cubical algebra $\mathfrak{TC}$.** Define a *segmented composition* as a finite sequence of integers, separated by vertical bars or commas, *e.g.*, $(2,1\,|\,2\,|\,1,2)$. We shall associate an ordinary composition with a segmented composition by replacing the vertical bars by commas.

There is a natural bijection between segmented compositions of sum n and sequences of length $n-1$ over three symbols $<,=,>$: start with a segmented composition $\mathbf{I}$. If i is not a descent of the underlying composition of $\mathbf{I}$, write $<$; otherwise, if i corresponds to a comma, write $=$; if i corresponds to a bar, write $>$.

Now, with each word w of length n, associate a segmented composition $S(w)$, defined as the sequence $s_1,\ldots,s_{n-1}$ where s_i is the comparison sign between w_i and w_{i+1}. For example, given $w = 1615116244543$, one gets the sequence (and the segmented composition):

$$<><>=<><=<>>\Longleftrightarrow (2|2|1,2|2,2|1|1). \tag{86}$$

Given a segmented composition $\mathbf{I}$, define

$$\mathcal{M}_{\mathbf{I}} = \sum_{S(T)=\mathbf{I}} \mathcal{M}_T\,. \tag{87}$$

It has been shown in [12] that the $\mathcal{M}_{\mathbf{I}}$ generate a Hopf subalgebra of $\mathfrak{TD}$ and that their product is given by

$$\mathcal{M}_{\mathbf{I}'}\mathcal{M}_{\mathbf{I}''} = \mathcal{M}_{\mathbf{I}'.\mathbf{I}''} + \mathcal{M}_{\mathbf{I}',\mathbf{I}''} + \mathcal{M}_{\mathbf{I}'|\mathbf{I}''}. \tag{88}$$

where $\mathbf{I}'.\mathbf{I}''$ is obtained by gluing the last part of $\mathbf{I}'$ with the first part of $\mathbf{I}''$.

As before, it is easy to see that

Theorem 5.8.

$$d_k(\mathcal{M}_{\mathbf{I}}) = \begin{cases} \mathcal{M}_{\mathbf{I}'}, \\ 0, \end{cases} \tag{89}$$

depending on whether k is not or is a descent of the underlying composition of $\mathbf{I}$. *In the nonzero case,* $\mathbf{I}'$ *is obtained from* $\mathbf{I}$ *by decreasing the entry that corresponds to the entry containing the k-th cell in the corresponding composition, that is, if* $\mathbf{I} = (i_1,\ldots,i_\ell)$ *where the i are separated by commas or vertical bars, decreasing i_n where n is the smallest integer such that $i_1+\cdots+i_n > k$.*

Proof – Trivial by definition of $\mathcal{M}_{\mathbf{I}}$. ∎

For the same reason, the following result is also true:

$$\mathcal{M}_{\mathbf{I}'}\#\mathcal{M}_{\mathbf{I}''} = \mathcal{M}_{\mathbf{I}'.'\mathbf{I}''}, \tag{90}$$

where $\mathbf{I}'.'\mathbf{I}''$ amounts to glue together the last part of $\mathbf{I}'$ with the first part of $\mathbf{I}''$ minus one, leaving the other parts unchanged.

The fact that $\#$ is internal can also be seen in a simple way, again using the multiplicative bases $\mathbf{S}$ and $\mathbf{E}$ of **WQSym**.

Indeed, it has been shown in [12] that $\mathfrak{TC}$ is the subalgebra of **WQSym** generated by the $\mathbf{S}^w$ (resp. the $\mathbf{E}^w$) where w runs over words avoiding the four patterns 132, 213, 121, and 212 (resp. 312, 231, 212, and 221). Since the #-product of two such **S** (**E**) also avoids all the given patterns, we get that # is internal in $\mathfrak{TC}$.

As a direct consequence (as in the case of **Sym**), one easily sees that $(\mathfrak{TC}, \#)$ is the free algebra on the three generators of (shifted) degree 1

$$\mathcal{M}_2 \quad \mathcal{M}_{1,1} \quad \mathcal{M}_{1|1}. \tag{91}$$

It has therefore a canonical coproduct, for which these generators are primitive.

6. Parking quasi-symmetric functions: PQSym

A *parking function* on $[n] = \{1, 2, \dots, n\}$ is a word $\mathbf{a} = a_1a_2 \cdots a_n$ of length n on $[n]$ whose non-decreasing rearrangement $\mathbf{a}^{\uparrow} = a'_1a'_2 \cdots a'_n$ satisfies $a'_i \leq i$ for all i. We shall denote by PF the set of parking functions.

For a word w over a totally ordered alphabet in which each element has a successor, one can define [13] a notion of *parkized word* $\mathrm{park}(w)$, a parking function which reduces to $\mathrm{std}(w)$ when w is a word without repeated letters.

For $w = w_1w_2 \cdots w_n$ on $\{1, 2, \dots\}$, we set

$$d(w) := \min\{i | \#\{w_j \leq i\} < i\} . \tag{92}$$

If $d(w) = n + 1$, then w is a parking function and the algorithm terminates, returning w. Otherwise, let w' be the word obtained by decrementing all the elements of w greater than $d(w)$. Then $\mathrm{park}(w) := \mathrm{park}(w')$. Since w' is smaller than w in the lexicographic order, the algorithm terminates and always returns a parking function.

For example, let $w = (3, 5, 1, 1, 11, 8, 8, 2)$. Then $d(w) = 6$ and the word $w' = (3, 5, 1, 1, 10, 7, 7, 2)$. Then $d(w') = 6$ and $w'' = (3, 5, 1, 1, 9, 6, 6, 2)$. Finally, $d(w'') = 8$ and $w''' = (3, 5, 1, 1, 8, 6, 6, 2)$, that is a parking function. Thus, $\mathrm{park}(w) = (3, 5, 1, 1, 8, 6, 6, 2)$.

Lemma 6.1. *Let $u = u_1u_2 \cdots u_n$ be a word over A, and $\mathbf{c} = c_1c_2 \dots c_n = \mathrm{park}(u)$. Then, for any factor of u,*

$$\mathrm{park}(u_iu_{i+1} \cdots u_j) = \mathrm{park}(c_ic_{i+1} \cdots c_j) . \tag{93}$$

Recall from [13], that with a parking function $\mathbf{a}$, one associates the polynomial

$$\mathbf{G_a} = \sum_{\mathrm{park}(w)=\mathbf{a}} w . \tag{94}$$

These polynomials form a basis of a subalgebra[1] **PQSym** of the free associative algebra over A. In this basis, the product is given by

$$\mathbf{G_a}\mathbf{G_b} = \sum_{\mathbf{c}=uv;\ \mathrm{park}(u)=\mathbf{a},\, \mathrm{park}(v)=\mathbf{b}} \mathbf{G_c} . \tag{95}$$

Thus,

$$\mathbf{G_a}\#\mathbf{G_b} = \sum_{\mathbf{c}\in\mathbf{a}\#\mathbf{b}} \mathbf{G_c}, \tag{96}$$

[1]Strictly speaking, this subalgebra is rather **PQSym***, the graded dual of the Hopf algebra **PQSym**, but both are actually isomorphic.

where

$$\mathbf{a}\#\mathbf{b} = \{\mathbf{c} |\, |\mathbf{c}| = k+l-1, \mathrm{park}(\mathbf{c}_1\ldots\mathbf{c}_k) = \mathbf{a}, \mathrm{park}(\mathbf{c}_k\ldots\mathbf{c}_{k+l-1}) = \mathbf{b}\}. \tag{97}$$

Indeed, $\mathbf{G_a}\#\mathbf{G_b}$ is the sum of all words of the form $w = uxv$, with $\mathrm{park}(ux) = \mathbf{a}$ and $\mathrm{park}(xv) = \mathbf{b}$.

Note that **PQSym** is not stable under the operators d_k. For example, $d_1(\mathbf{G}_{112})$ is not in **PQSym**. However, let d'_k be the linear operator defined by

$$d'_k(\mathbf{G_c}) = \begin{cases} \mathbf{G}_{\mathbf{c}_1\ldots\mathbf{c}_{k-1}\mathbf{c}_{k+1}\ldots\mathbf{c}_n} & \text{if } \mathbf{c}_k = \mathbf{c}_{k+1} \text{ and } \mathbf{c}_1\ldots\mathbf{c}_{k-1}\mathbf{c}_{k+1}\ldots\mathbf{c}_n \in \mathrm{PF}, \\ 0 & \text{otherwise.} \end{cases} \tag{98}$$

Proposition 6.2. *Then, if* $\mathbf{a}$ *is of length* k*,*

$$\mathbf{G_a}\#\mathbf{G_b} = d'_k(\mathbf{G_a}\mathbf{G_b})\,. \tag{99}$$

Proof – We already know that the l.h.s. is a sum of terms $\mathbf{G_c}$ without multiplicites. This is also true of the r.h.s., since d'_k induces a bijection between parking functions of length $n-1$, and parking functions $\mathbf{c}$ of length n such that $c_k = c_{k+1}$. Now, if a word u occurs in $\mathbf{G_a}\#\mathbf{G_b}$, its antecedent $u' = {d'}_k^{-1}(u)$ satisfies $Park(u'_1\cdots u'_k) = \mathbf{a}$ and $\mathrm{park}(u'_{k+1}\cdots u'_{k+l}) = \mathbf{b}$, so that u occurs in $d'_k(\mathbf{G_a}\mathbf{G_b})$. For the same reason, if u occurs in $d'_k(\mathbf{G_a}\mathbf{G_b})$, then u occurs in $\mathbf{G_a}\#\mathbf{G_b}$. ∎

For example,

$$\mathbf{G}_{121}\#\mathbf{G}_{1141} = \mathbf{G}_{121161} + \mathbf{G}_{121151} + \mathbf{G}_{121141}\,, \tag{100}$$

and

$$\begin{aligned} \mathbf{G}_{1411}\#\mathbf{G}_{2124} = {} & \mathbf{G}_{2722126} + \mathbf{G}_{2722125} + \mathbf{G}_{2722124} + \mathbf{G}_{2622127} \\ & + \mathbf{G}_{2622126} + \mathbf{G}_{2622125} + \mathbf{G}_{2622124} + \mathbf{G}_{2522127} \\ & + \mathbf{G}_{2522126} + \mathbf{G}_{2522125} + \mathbf{G}_{2522124}. \end{aligned} \tag{101}$$

7. Concluding remarks

7.1. **Dendriform structures.** For those algebras which are stable under the operators d_k, and which are dendriform or tridendriform, a similar structure can be defined for the # product, by taking the images of the partial products by d_k.

7.2. **Coproducts.** Since all our #-algebras are free, one may endow them with bialgebra structures by declaring primitive any complete set of free generators. However, no canonical choice has been found, apart from the trivial cases of $\mathfrak{TC}$ and **Sym**.

7.3. **Geometric interpretations.** In **WQSym**, the $\mathbf{M}_u$ can be interpreted as the characteristic functions of all faces of the hyperplane arrangement of type A. The product of **WQSym** gives then the decomposition of the characteristic function of a cartesian product of faces. Then d_k takes the intersection of this product with the hyperplane $\alpha_k = x_k - x_{k+1} = 0$.

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