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# Non-explosion criteria for relativistic diffusions

Ismaël Bailleul<sup>\*</sup>, Jacques Franchi<sup>†</sup>

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## Abstract

General Lorentz covariant operators, associated to so-called  $\Theta$ (or  $\Xi$ )-relativistic diffusions, and making sense in any Lorentz manifold, were introduced by Franchi and Le Jan in [F-LJ-1], [F-LJ-2]. Only a few examples have been studied. We provide in this work non-explosion criteria for these diffusions, which can be used in generic cases.

## 1 Introduction

It is well known that the metric completeness of a Riemannian manifold does not imply stochastic completeness for its Brownian motion. A lot of studies provide non-explosion (i.e. stochastic completeness) criteria: see in particular [Gr1],[Gr2], [Ta2], [H], [H-Q].

Except in some particular cases ([Du], [F-LJ-1], [F], [A]), no non-explosion general statement was given till now in a Lorentzian setting. The aim of this article is to provide non-explosion criteria for some generic classes of Lorentz manifolds.

In the Riemannian case, stochastic completeness refers to completeness of Brownian trajectories. The most natural Lorentzian analogue, which we consider here, refers to a general class of relativistic diffusions, introduced successively in [F-LJ-1], [F-LJ-2], and presented in Section 2.2 below. Their trajectories represent the random motion of a small massive particle in spacetime, and make sense only at the level of the unit tangent bundle or of the orthonormal frame bundle. These diffusions being defined in purely geometric terms, it is very likely that part of the geometry of the ambient spacetime might be recovered from the probabilistic behaviour of these processes, as is the case in a Riemannian setting.

Dating back to Penrose and Hawking's incompleteness theorems, the appearance of singularities in Einstein's theory of gravitation has been recognized as unavoidable under quite natural assumptions. Although there is no agreement on what should be called a singularity of spacetime, the existence of incomplete geodesics has been widely used as an indicator of singularity. In so far as the random dynamics to be introduced below in Section 2.2 have an intrinsic character, it is natural to investigate the question of their completeness, as well.

To start our investigation of the explosion question in a Lorentzian setting, we shall take advantage in Section 3 of the bundle structure of the state space, to exhibit a one-dimensional

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sub-process whose control is possible in the class of globally hyperbolic spacetimes. This structure allows indeed to define some Lyapounov functional. As Lorentzian manifolds are not metric spaces, no natural distance function exists on them out of which constructing a Lyapounov function. The existence of such a one-dimensional process then happens to be precious, and allows us to deduce a first non-explosion criterion, using a simple and well-known observation due to Khasminsky.

Also, there is no obvious way to replace the usual metric completeness assumption used crucially in the Riemannian case. Some propositions were made to fill this gap, in particular by Busemann, Hawking and Ellis, Schmidt, Beem and Ehrlich, see ([B-E], Section 5.2). Schmidt's idea is to give a Riemannian structure to the orthonormal frame bundle. We consider the Schmidt's "b-completeness" notion in Section 4, showing actually that it may appear as too restrictive, since it entails the stochastic completeness rather directly.

The Riemannian situation, for Brownian motion, is now well understood thanks in particular to the works of Grigor'yan [Gr1],[Gr2], Takeda [Ta1],[Ta2], and very recently Hsu and Qin [H-Q], to cite but a few names. Different lines of approach have been used. Grigor'yan used the analytic counterpart of the completeness problem and investigated the well-posedness of the parabolic Cauchy problem using skillful estimates on its solutions, based on the control of the growth of the volume of the metric balls. Takeda used a purely probabilistic method based on reversibility, originated by Lyons and Zheng in [L-Z]. This approach was recently improved by Hsu and Qin in [H-Q]. Hsu used stochastic analysis in ([H], Theorem 3.5.1) to control the radial process, by estimating the Laplacian of the distance function to a fixed point. All these results require the metric completeness of the manifold to hold.

We shall adapt the strategy of [H-Q] to the Lorentzian setting in Section 5. This is however far from being straightforward, since we are working in a non-symmetric, non-elliptic setting, where the main ingredients of Takeda's and Hsu-Qin's method (use of symmetry and reflected Brownian motion on the boundary of large Riemannian balls) have no obvious Lorentzian counterpart. To overcome this difficulty, we shall use some tools from control theory, presented in Section 5.2 below, and shall take advantage of a sub-Riemannian minimal time functional, which will somehow play for us the role of the non-existing Lorentzian distance.

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## 2 Relativistic diffusions

### 2.1 Basic geometrical setting

Recall Minkowski space is the product  $\mathbb{R}^{1,d} \equiv \mathbb{R} \times \mathbb{R}^d$  equipped with the metric

$$g_M(q, q) := t^2 - |x^1|^2 - \dots - |x^d|^2, \quad \text{for any } q = (t, x) \in \mathbb{R}^{1,d},$$

where  $(t, x^1, \dots, x^d)$  denote the coordinates of  $q$  in the canonical basis  $\{\epsilon_0, \epsilon_1, \dots, \epsilon_d\}$  of  $\mathbb{R}^{1,d}$ .

Let  $(\mathbb{M}, g)$  be a (smooth)  $(1 + d)$ -dimensional Lorentzian manifold (with  $d \geq 2$ ), which we shall always suppose to be oriented and time-oriented. Given any point  $m \in \mathbb{M}$ , it is usual to consider an orthonormal basis  $\{\mathbf{e}_0, \dots, \mathbf{e}_d\}$  of the tangent space  $T_m\mathbb{M}$  as an isometry  $\mathbf{e}$  from  $(\mathbb{R}^{1,d}, g_{\mathbb{M}})$  to  $(T_m\mathbb{M}, g_m)$ ; so, strictly speaking,  $\mathbf{e}_i = \mathbf{e}(\epsilon_i)$ . The orthonormal frame bundle of  $\mathbb{M}$  is just the collection

$$\mathbb{O}\mathbb{M} = \{\Phi = (m, \mathbf{e}) \mid m \in \mathbb{M}, \mathbf{e} \text{ an orthonormal basis of } (T_m\mathbb{M}, g_m)\}.$$

We shall write  $\mathbb{O}\mathcal{U} = \{\Phi = (m, \mathbf{e}) \mid m \in \mathcal{U}, \mathbf{e} \text{ an orthonormal basis of } T_m\mathbb{M}\}$  for any subset  $\mathcal{U}$  of  $\mathbb{M}$ . For a small enough  $\mathcal{U}$  and a chart  $x : \mathcal{U} \rightarrow \mathbb{R}^{1+d}$  on it, we shall write  $\mathbf{e}_j = e_j^k \partial_{x^k}$  for each vector  $\mathbf{e}_j$  of a frame  $\mathbf{e}$ ; this decomposition provides local coordinates  $(x^i, e_j^k)$  on  $\mathbb{O}\mathcal{U}$ .

Each fibre  $\mathbb{O}_m\mathbb{M}$  is modelled on the non-compact orthogonal group  $O(1, d)$ , which has four connected components. We shall be interested in dynamics leaving these components globally fixed. We choose to consider only one of them, specified by the requirement that  $\mathbf{e}_0$  should be future-oriented and that the orientation of  $\mathbf{e}$  should be direct. We shall still denote the resulting frame bundle by  $\mathbb{O}\mathbb{M}$ , as there will be no risk of confusion. The Lorentz-Möbius group  $SO_0(1, d)$ , i.e. the connected component of the unit in  $O(1, d)$ , acts properly on  $\mathbb{O}\mathbb{M}$ . This natural action induces the canonical vertical vector fields  $(V_{ij})_{0 \leq i < j \leq d}$ . The subgroup of elements in  $SO_0(1, d)$  that fix  $\epsilon_0$  identifies with the rotation group  $SO(d)$ , and generates  $(V_{ij})_{1 \leq i < j \leq d}$ . To shorten notations we shall write  $V_j$  for  $V_{0j}$ ; it generates boosts (i.e. hyperbolic rotations in the fibres), and reads in the above local coordinates:

$$V_j = e_j^k \frac{\partial}{\partial e_0^k} + e_0^k \frac{\partial}{\partial e_j^k}. \quad (2.1)$$

Throughout this work,  $T\mathbb{M}$  and  $\mathbb{O}\mathbb{M}$  will be endowed with the Levi-Civita connection, inherited from the Lorentzian pseudo-metric  $g$ . Last, we denote by  $H_0$  the vector field generating the geodesic flow on  $\mathbb{O}\mathbb{M}$ . Denoting by  $\Gamma_{kj}^\ell$  the Christoffel coefficients, in the above local chart on  $\mathbb{O}\mathbb{M}$  we have:

$$H_0 = e_0^k \partial_{x^k} - e_0^k e_i^j \Gamma_{kj}^\ell \frac{\partial}{\partial e_i^\ell}. \quad (2.2)$$

We shall denote by  $T^1\mathbb{M}$  the future-oriented unit tangent bundle over  $\mathbb{M}$ , with generic element  $(m, \dot{m})$ . We shall denote by  $\pi_1$  the projection  $(m, \mathbf{e}) \mapsto (m, \mathbf{e}_0 \equiv \dot{m})$  from  $\mathbb{O}\mathbb{M}$  to  $T^1\mathbb{M}$ , and by  $\pi_0$  the canonical projection  $\mathbb{O}\mathbb{M} \rightarrow \mathbb{M}$ .

## 2.2 Relativistic random dynamics

Relativistic diffusions model the random motion in spacetime of a small massive particle in its proper time, providing random timelike paths; so, properly speaking, their mathematical counterpart are random trajectories  $(m_s, \dot{m}_s)$  in  $T^1\mathbb{M}$  subject to the condition  $\frac{d}{ds}m_s = \dot{m}_s$ . Yet it happens to be more convenient to define random dynamics in  $\mathbb{O}\mathbb{M}$  as this bundle bears more structure than  $T^1\mathbb{M}$ ; these diffusions on  $\mathbb{O}\mathbb{M}$  are constructed so as to have a projection on  $T^1\mathbb{M}$  which is itself a diffusion. Such a construction is reminiscent of

Malliavin-Eells-Elworthy's construction of Brownian motion on a Riemannian manifold as the projection of a diffusion on the orthonormal frame bundle.

Given any smooth non-negative function  $\Theta : T^1\mathbb{M} \rightarrow \mathbb{R}_+$ , identified to a  $SO(d)$ -invariant function on  $\mathbb{OM}$  by setting  $\Theta(\Phi) := \Theta(\pi_1(\Phi))$ , consider the following Stratonovich differential equation on  $\mathbb{OM}$  (for some  $d$ -dimensional Brownian motion  $w$ ):

$$\circ d\Phi_s = H_0(\Phi_s) ds + \frac{1}{4} \sum_{1 \leq j \leq d} V_j \Theta(\Phi_s) V_j(\Phi_s) ds + \sqrt{\Theta(\Phi_s)} \sum_{1 \leq j \leq d} V_j(\Phi_s) \circ dw_s^j, \quad (2.3)$$

where  $w$  is a  $d$ -dimensional Brownian motion and where we understand a vector field as a first order differential operator. This equation has a unique maximal strong solution, defined up to its explosion time  $\zeta$ .

It is clear on this equation that the  $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ -part of  $\Phi_s$  is irrelevant in defining the dynamics of  $(m_s, \mathbf{e}_0(s))$  since  $\Theta(\Phi)$  depends only on  $\pi_1(\Phi)$ ; this is the reason why the diffusion on  $\mathbb{OM}$  projects down in  $T^1\mathbb{M}$  onto a diffusion. Consult ([F-LJ-1], Theorem 1) and ([F-LJ-2], Theorem 3.2.1) for the details. The diffusion in  $\mathbb{OM}$  has generator

$$\mathcal{G}_\Theta = H_0 + \frac{1}{2} \sum_{1 \leq j \leq d} V_j(\Theta V_j). \quad (2.4)$$

We shall generically call these relativistic dynamics  $\Theta$ -*diffusions* ( $\equiv$  the  $\Xi$ -diffusions of [F-LJ-2]). These diffusions are *covariant*, in the sense that any isometry of  $(\mathbb{M}, g)$  maps a  $\Theta$ -diffusion to a  $\Theta$ -diffusion (with the same  $\Theta$ : the law is preserved, up to the starting point), and admit the Liouville measure as an invariant measure. The  $\pi_0$ -projections (on the base manifold  $\mathbb{M}$ ) of their trajectories are almost-surely  $C^1$  paths. A  $\Theta$ -diffusion  $(\Phi_s)_{0 \leq s < \zeta}$  solving Equation (2.3) is parametrized by *proper time*  $s \geq 0$ . The particular case  $\Theta = 0$  gives back the deterministic geodesic flow, and the case of a non-null constant  $\Theta$  gives back the relativistic diffusion as defined first in [F-LJ-1], i.e. the *basic relativistic diffusion*.

On a manifold with non-positive scalar curvature  $R$ , taking  $\Theta(\Phi) = -\varrho^2 R$  (for a constant  $\varrho \neq 0$ ), one gets dynamics which can be truly random only in non-empty parts of spacetime, and were called *R-diffusions* in [F-LJ-2]. Denote by  $\mathbf{T}$  the energy-momentum tensor of the spacetime. Taking  $\Theta(\Phi) = \varrho^2 \mathbf{T}(\mathbf{e}_0, \mathbf{e}_0)$ , we get what was named the *energy diffusion* in [F-LJ-2]. See [B2] for more general models of diffusions.

Denote by  $\nabla^v$  the gradient on  $T_m^1\mathbb{M}$ , identified with the hyperbolic space  $\mathbb{H}^d$  by means of the metric  $g_m$ , and by  $\mathcal{L}_0$  the vector field generating the geodesic flow on  $T^1\mathbb{M}$ . Note that  $T\pi_1(H_0) = \mathcal{L}_0$  and  $T\pi_1(V_j) = \nabla_j^v = e_j^k \partial_{\dot{m}^k}$  (with Einstein's summation convention). The projection on  $T^1\mathbb{M}$  of the  $\mathbb{OM}$ -valued diffusion has the following  $SO(d)$ -invariant generator:

$$\mathcal{L}_\Theta = \mathcal{L}_0 + \frac{1}{2} \nabla^v(\Theta \nabla^v).$$

For a constant  $\Theta$  the operator  $\mathcal{L}_\Theta$  has the following expression in the above local coordinates.

$$\mathcal{L}_0 + \frac{\Theta}{2} \Delta^v = \dot{m}^k \frac{\partial}{\partial m^k} + \left( \frac{d}{2} \Theta \dot{m}^k - \dot{m}^i \dot{m}^j \Gamma_{ij}^k(m) \right) \frac{\partial}{\partial \dot{m}^k} + \frac{\Theta}{2} (\dot{m}^k \dot{m}^\ell - g^{k\ell}(m)) \frac{\partial^2}{\partial \dot{m}^k \partial \dot{m}^\ell},$$

where  $\Delta^v$  denotes the vertical Laplacian. We have for a generic  $\Theta$ :

$$\mathcal{L}_\Theta = \mathcal{L}_0 + \frac{\Theta}{2} \Delta^v + \frac{1}{2} (\dot{m}^k \dot{m}^\ell - g^{k\ell}(m)) \frac{\partial \Theta}{\partial \dot{m}^k} \frac{\partial}{\partial \dot{m}^\ell}. \quad (2.5)$$

The purpose of this work is to provide some conditions under which the  $\Theta$ -diffusions have almost-surely an infinite lifetime (i.e. explosion time). In so far as we are mainly interested in the  $T^1\mathbb{M}$ -valued  $\Theta$ -diffusions as models of physical phenomena, while we shall mainly work with  $\mathbb{OM}$ -valued diffusions, it is reassuring to have the following fact, which essentially means that the possible explosion of  $(\Phi_s)_{0 \leq s < \zeta}$  is never due to its  $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ -part.

**PROPOSITION 1.** *A  $\Theta$ -diffusion on  $\mathbb{OM}$  and its  $T^1\mathbb{M}$ -projection have the same lifetime.*

**PROOF** – Write  $\Phi_s = (m_s; \dot{m}_s, e_1(s), \dots, e_d(s))$  and  $\phi_s := \pi_1(\Phi_s) = (m_s, \dot{m}_s)$ . Using local coordinates  $(x^k, e_j^\ell)_{0 \leq k, \ell \leq d; 1 \leq j \leq d}$ , Equation (2.3) defining the  $\Theta$ -diffusion reads (see Section 3.2 of [F-LJ-2] for the computation of the Itô correction) :

$$\begin{aligned} d\dot{m}_s^k &= dM_s^k - \Gamma_{i\ell}^k(m_s) \dot{m}_s^i \dot{m}_s^\ell ds + \frac{d}{2} \Theta(\phi_s) \dot{m}_s^k ds + \frac{1}{2} (\dot{m}_s^k \dot{m}_s^\ell - g^{k\ell}(m_s)) \frac{\partial \Theta}{\partial \dot{m}^\ell}(\phi_s) ds ; \\ de_j^k(s) &= \sqrt{\Theta(\phi_s)} \dot{m}_s^k dw_s^j - \Gamma_{i\ell}^k(m_s) e_j^\ell(s) \dot{m}_s^i ds + \frac{1}{2} \Theta(\phi_s) e_j^k(s) ds + \frac{1}{2} V_j \Theta(\phi_s) \dot{m}_s^k ds , \end{aligned}$$

with  $dM_s^k := \sqrt{\Theta(\phi_s)} e_j^k(s) dw_s^j$ . Setting  $e_0 = \dot{m}$  and  $\eta^{in} := \eta_i^n := \mathbf{1}_{i=n=0} - \mathbf{1}_{1 \leq i=n \leq d}$ , and noticing that the matrix  $(\eta^{in} e_n^k g_{k\ell})_{0 \leq i, \ell \leq d}$  is the inverse of the matrix  $(e_\ell^i)_{0 \leq i, \ell \leq d}$ , it follows from the above system that we have for all  $0 \leq k \leq d, 1 \leq j \leq d$  (with  $\eta_j^n \equiv \eta^{jn}$ ) :

$$\begin{aligned} de_j^k(s) &= \dot{m}_s^k \eta_j^n e_n^q(s) g_{q\ell}(m_s) dM_s^\ell - \Gamma_{i\ell}^k(m_s) e_j^\ell(s) \dot{m}_s^i ds + \frac{1}{2} \Theta(\phi_s) e_j^k(s) ds + \frac{1}{2} V_j \Theta(\phi_s) \dot{m}_s^k ds \\ &= -e_j^\ell(s) \Gamma_{i\ell}^k(m_s) \dot{m}_s^i ds + \frac{1}{2} e_j^k(s) \Theta(\phi_s) ds + \frac{1}{2} V_j \Theta(\phi_s) \dot{m}_s^k ds \\ &\quad - e_j^q(s) \dot{m}_s^k g_{q\ell}(m_s) \left[ d\dot{m}_s^\ell + \Gamma_{ip}^\ell(m_s) \dot{m}_s^i \dot{m}_s^p ds - \frac{d}{2} \Theta(\phi_s) \dot{m}_s^\ell ds - \frac{1}{2} [\dot{m}_s^p \dot{m}_s^\ell - g^{p\ell}(m_s)] \frac{\partial \Theta}{\partial \dot{m}^p}(\phi_s) ds \right]. \end{aligned}$$

So, the matrix  $(e_j^k)_{0 \leq k \leq d, 1 \leq j \leq d}$ , and then the frame-valued diffusion  $(\Phi_s)_{0 \leq s < \zeta}$ , satisfies a linear stochastic differential equation, conditionally on  $(\phi_s)_{0 \leq s < \zeta}$ . It is thus well defined up to the explosion time  $\zeta$  of the  $T^1\mathbb{M}$ -valued  $\Theta$ -diffusion.  $\triangleright$

This point being clarified, we shall work freely in the sequel with  $\Theta$ -diffusions on  $\mathbb{OM}$ .

### 3 A first non-explosion criterion

We give in this section a simple non-explosion criterion, well-suited to investigate the behaviour of the  $\Theta$ -diffusions in globally hyperbolic spacetimes. We use a Lyapounov function. This non-explosion criterion is thus of a different nature than the typical Riemannian criteria of Grigor'yan [Gr1], Takeda [Ta2], or Hsu [H].

The idea is roughly the following : if we can find a function  $f = f(\Phi)$  which has compact level sets  $\{f \leq \lambda\}$ , and does not increase along the trajectories, then the dynamics cannot explode. This was noted first by Khasminsky in a stochastic context ; we state his observation here for the relativistic diffusions.

**LEMMA 2** (Khasminsky). *If there exists a non-negative function  $f$  on  $\mathbb{OM}$  and a constant  $C$  such that*

- $\mathcal{G}_\Theta f \leq C f$  and

- $f$  goes to infinity along any timelike path leaving any compact in a finite time,

*then the  $\Theta$ -diffusion has almost-surely an infinite lifetime.*

PROOF – Denote by  $\tau_n$  the (possibly infinite) exit time from the level set  $\{f \leq n\}$ . The condition  $\mathcal{G}_\Theta f \leq C f$  implies that the real-valued process  $(e^{-Cs} f(\Phi_s))_{s < \zeta}$  is a non-negative supermartingale. Thus we have, by optional stopping:  $f(\Phi_0) \geq \mathbb{E}[e^{-C\tau_n} f(\Phi_{\tau_n})] = n \mathbb{E}[e^{-C\tau_n}]$ . This implies that  $\tau_n$  goes to infinity as  $n$  goes to infinity; as  $\zeta = \lim_{n \rightarrow \infty} \tau_n$ , this proves the Khasminsky's statement.  $\triangleright$

As a  $\Theta$ -diffusion may well explode, such a Lyapounov function will generally not exist. Yet it is possible to construct such a function in some classes of spacetimes of interest for cosmology and theoretical physics. We give below two such examples. The construction of the function  $f$  uses the same recipe in both cases: if there exists an intrinsic distinguished timelike  $C^1$  vector field  $U$  on  $\mathbb{M}$ , one can suppose  $U \in T^1\mathbb{M}$  and take

$$f(\Phi) := g(U, \dot{m}) \quad (\text{recall that } \pi_1(\Phi) = (m, \dot{m}) \in T^1\mathbb{M}). \quad (3.1)$$

For this choice of  $f(\Phi)$ , which is the hyperbolic angle between  $U$  and  $\dot{m}$ , we have  $f \geq 1$ , and

$$H_0 f(\Phi) = \nabla_{\dot{m}}(g(U, \dot{m})) = g(\nabla_{\dot{m}} U, \dot{m}). \quad (3.2)$$

The following lemma shows that such an  $f$  can be a good choice to apply Khasminsky criterion.

LEMMA 3. We have on  $\mathbb{OM}$ :  $\frac{1}{2} \sum_{j=1}^d V_j(\Theta V_j f) = \frac{d}{2} \Theta f + \frac{1}{2}(f \dot{m}^k - U^k) \frac{\partial \Theta}{\partial \dot{m}^k}$ .

PROOF – We can take locally the coordinates such that  $g^{0\cdot} = g^{ij} \partial_{x^j} x^0 = \nabla x^0 = U$ , so that  $f(\Phi) = \dot{m}^0 = e_0^0$ . Using (2.1), we have thus locally:

$$V_j f = \left( e_j^k \frac{\partial}{\partial e_0^k} + e_0^k \frac{\partial}{\partial e_j^k} \right) e_0^0 = e_j^0, \quad V_j^2 f = e_0^0 = f,$$

$$\text{and } \sum_{j=1}^d (V_j \Theta)(V_j f) = \sum_{j=1}^d e_j^0 e_j^k \frac{\partial \Theta}{\partial \dot{m}^k} = (\dot{m}^0 \dot{m}^k - g^{0k}) \frac{\partial \Theta}{\partial \dot{m}^k} = (f \dot{m}^k - U^k) \frac{\partial \Theta}{\partial \dot{m}^k}.$$

$\triangleright$

So by (2.4) and (3.2),  $\mathcal{G}_\Theta f = g(\nabla_{\dot{m}} U, \dot{m}) + \frac{d}{2} \Theta f + \frac{1}{2}(f \dot{m}^k - U^k) \frac{\partial \Theta}{\partial \dot{m}^k}$ .

Khasminsky's Criterion will thus guarantee the non-explosion of the relativistic diffusion provided  $f$  explodes along exploding trajectories and

$$g(\nabla_{\dot{m}} U, \dot{m}) + \frac{1}{2}(f \dot{m}^k - U^k) \frac{\partial \Theta}{\partial \dot{m}^k} \leq (C - \frac{d}{2} \Theta) f \quad \text{for some constant } C. \quad (3.3)$$

In order to turn this criterion into an effective tool, we restrict ourselves first to the following general class of spacetimes.

### 3.1 Globally hyperbolic spacetimes

This class of cosmological models is characterized by the existence of a global time function, i.e. a function  $\tau : \mathbb{M} \rightarrow \mathbb{R}$ , with timelike gradient, such that it has connected spacelike level sets  $S_t := \{\tau = t\}$ , and each integral curve of the vector field  $\nabla\tau$  meets each  $S_t$  in exactly one point. Thus  $\mathbb{M}$  is diffeomorphic to the product  $I \times S$  of an interval  $I$  (we shall suppose to be unbounded above) and a  $d$ -dimensional manifold  $S$ .

There is no hope to prove Inequality (3.3) without specifying further the model, as the time function is not uniquely defined and not intrinsically associated with the geometry. To proceed further, we shall look at the sub-class of **generalized warped product spacetimes**, in which the time function is supplied by the model and can be seen as an absolute time. These universes are globally hyperbolic spacetimes  $\mathbb{M} = I \times S$  whose metric tensor has the form

$$g_m(\dot{m}, \dot{m}) = a_m^2 |\dot{m}^0|^2 - h_m(\dot{m}^S, \dot{m}^S), \quad (3.4)$$

where  $\dot{m}^0$  is the image of  $\dot{m} \in T_m^1\mathbb{M}$  by the differential of the first projection  $I \times S \rightarrow I$  and  $\dot{m}^S$  the image of  $\dot{m}$  by the differential of the second projection  $I \times S \rightarrow S$ . Write  $m = (t, x) \in I \times S$ . The function  $a$  is a positive  $\mathcal{C}^1$  function on  $\mathbb{M}$ , which is assumed to be bounded on bounded above slices  $I' \times S$  (i.e. for bounded above  $I' \subset I$ ), and  $h_m$  is a positive-definite scalar product on  $T_x S$ , depending on  $m$  in a  $\mathcal{C}^1$  way. The unit pseudo-norm relation reads :

$$a_m^2 \dot{t}^2 = 1 + h_m(\dot{x}, \dot{x}) \geq 1. \quad (3.5)$$

This class of spacetimes contains all Robertson-Walker spacetimes (hence in particular de Sitter and Einstein-de Sitter spacetimes, and the universal covering of the anti-de Sitter spacetime). Let us take as vector field  $U$  in Formula (3.1) (up to a positive scalar) the gradient of the time function  $\tau$  (we have thus  $g_m(U, U) = a_m^{-2}$  a priori not equal to 1) :

$$I \times S \ni (t, x) = m \mapsto t, \quad \text{and we have } f(\Phi) = g(U, \dot{m}) = \nabla_{\dot{m}} \tau = \dot{m}^0 = \dot{t}.$$

On (the future-directed component of)  $\mathbb{O}\mathbb{M}$ , we have  $f \equiv \dot{t} > 0$ , as demanded by Lemma 2.

**THEOREM 4.** *Let  $(\mathbb{M}, g)$  be a generalised warped product spacetime. If the function*

$$T^1\mathbb{M} \ni (m, \dot{m}) \mapsto \nabla_{\dot{m}} \log a - \frac{d}{4} \Theta(m, \dot{m}) - \frac{1}{4} \left( \dot{m}^k \frac{\partial \Theta}{\partial \dot{m}^k} - \frac{\partial_{\dot{m}^0} \Theta}{a^2(m) \dot{m}^0} \right)$$

*is bounded below, then the  $\Theta$ -diffusion almost-surely cannot explode.*

**PROOF** – • We first check that if the  $\Theta$ -diffusion has a finite lifetime  $\zeta$ , then  $f(\Phi_s)$  explodes at time  $\zeta^-$ . To that end, consider a timelike trajectory  $\gamma = (m_s, \dot{m}_s)_{0 \leq s < T}$  in  $T^1\mathbb{M}$ , with  $\frac{d}{ds} m_s = \dot{m}_s$  and  $\dot{t}_s = f(\gamma_s) \leq C$ . We have then  $t_0 \leq t_s \leq t_0 + CT$ , and  $h_{m_s}(\dot{x}_s, \dot{x}_s) < C^2 a_{m_s}^2 \leq K^2$  constant, by (3.5) and by the assumption made on  $a$  (bounded on bounded above slices). This entails that  $(x_s)$  cannot exit a bounded region of  $S$ , and then that  $\gamma$  must be trapped in some finite union of sets  $\overline{J^+(m_0)} \cap \overline{J^-(q_j)}$  (for some  $q_j \in \mathbb{M}$ ), which is compact (see for example [H-E], Section 6.6). Hence  $\gamma$  cannot explode: otherwise, it would have a cluster point at which the strong causality would fail (globally hyperbolic spacetimes are indeed strongly causal, see again [H-E], Section 6.6).



• The condition of the theorem is a rephrasing of Condition (3.3). To see that, as Condition (3.3) is local, let us work in a neighbourhood  $\mathcal{V} = [t_1, t_2] \times V$  of a given point  $m_0$ , and choose coordinates  $x^j$  on  $V$ ; this provides coordinates  $(t, x^i)$  on  $\mathcal{V}$ , which induce coordinates on  $T^1\mathcal{V}$ : for  $m \in \mathcal{V}$  and  $\dot{m} \in T_m^1\mathbb{M}$ , write  $\dot{m} = \dot{m}^0 \partial_t + \sum_{1 \leq j \leq d} \dot{m}^j \partial_{x^j}$ .

We have only to compute  $\nabla_{\dot{m}} U$ . Recall from the above that we have  $U = a^{-2} \partial_t$ . Hence,

$$\nabla_{\dot{m}} U = \nabla_{\dot{m}}(a^{-2}) \partial_t + a^{-2} \nabla_{\dot{m}} \partial_t.$$

Using Christoffel's symbols we have, with usual notations,

$$(\nabla_{\dot{m}} \partial_t)^\alpha = \nabla_{\dot{m}}(a^{-2}) \delta_0^\alpha + a^{-2} \dot{m}^c \Gamma_{c0}^\alpha$$

for  $\alpha \in \{0, \dots, d\}$  and a summation over  $c$  in  $\{0, \dots, d\}$ . So

$$H_0 f = g(\nabla_{\dot{m}} T, \dot{m}) = \nabla_{\dot{m}}(\log a^{-2}) \dot{m}^0 + a^{-2} \dot{m}^c \Gamma_{c0}^\alpha g_{\alpha\beta} \dot{m}^\beta.$$

The explicit formulas for the Christoffel symbols, in terms of the metric, are

$$\Gamma_{00}^0 = \partial_t(\log a), \quad \Gamma_{k0}^0 = \partial_{x^k}(\log a), \quad \Gamma_{00}^i = \frac{1}{2} h^{i\ell} \partial_{x^\ell}(a^2), \quad \Gamma_{k0}^i = \frac{1}{2} h^{i\ell} \partial_t h_{\ell k},$$

for  $i, k \in \{1, \dots, d\}$  and a summation over  $1 \leq \ell \leq d$ . We thus have, after simplifications,

$$\begin{aligned} H_0 f &= -2 \nabla_{\dot{m}}(\log a) \dot{m}^0 + |\dot{m}^0|^2 \partial_t(\log a) - \frac{a^{-2}}{2} \dot{m}^k \partial_t(h_{\ell k}) \dot{m}^\ell \\ &= -|\dot{m}^0|^2 \partial_t \log a - 2 \dot{m}^0 \dot{m}^k \partial_{x^k} \log a - \frac{a^{-2}}{2} \dot{m}^k \partial_t(h_{\ell k}) \dot{m}^\ell. \end{aligned}$$

Using the unit pseudo-norm relation  $a^2 |\dot{m}^0|^2 - h_{\ell k} \dot{m}^k \dot{m}^\ell = 1$ , the above equality becomes:

$$H_0 f = -|\dot{m}^0|^2 \partial_t \log a - 2 \dot{m}^0 \dot{m}^k \partial_{x^k} \log a - \frac{a^{-2}}{2} |\dot{m}^0|^2 \partial_t(a^2),$$

that is,  $H_0 f = -2 \dot{m}^0 \nabla_{\dot{m}} \log a$ . The statement of the theorem follows now from (3.3).

▷

This result takes a particularly simple form in the case where  $\Theta$  depends only on the base point  $m$  (i.e. is a  $SO_0(1, d)$ -invariant function on  $\mathbb{O}\mathbb{M}$ ), as for the  $R$ -diffusions.

**COROLLARY 5.** *Let  $\mathbb{M} = I \times S$  denote a generalised warped product spacetime and  $\Theta$  be a bounded non-negative function on  $\mathbb{M}$ . Then the  $\Theta$ -diffusion does not explode if  $\nabla a$  is everywhere non-spacelike, future-directed.*

**PROOF** – The condition of Theorem 4 reads in that case:  $T^1\mathbb{M} \ni (m, \dot{m}) \mapsto \nabla_{\dot{m}} \log a$  is bounded below. To rephrase this condition into the more synthetic condition of the statement, let us work in local coordinates,  $(t, x)$  and  $(\dot{t}, \dot{x})$  for  $m$  and  $\dot{m}$  respectively.

We have  $\dot{t} = a^{-1} \text{chr}$  and  $\dot{x} = (\text{shr})\sigma$ , for some  $r \in \mathbb{R}$  and  $\sigma \in T_x S$  such that  $|\sigma|_{h(m)} = 1$ .

Define  $u := \partial_t \log a$  and  $v := \partial_x \log a \in T_x S \equiv \mathbb{R}^d$ . Then the condition of Theorem 4 reads:  $u a^{-1} \text{chr} - (v_i \sigma^i) \text{shr} \geq C$ , for any  $r$  and  $\sigma$ . Letting  $r \rightarrow \pm\infty$ , this yields  $a^{-1} u \geq |v_i \sigma^i| \geq 0$ . As the constant  $C$  can be taken negative without loss of generality, the reciprocal is clear. Now, since  $\max_{|\sigma|_{h(m)}=1} |v_i \sigma^i| = |v|_{h^{-1}(m)}$ , the condition reads:

$a^{-1} u \geq |v|_{h^{-1}(m)}$ . Finally, as  $\nabla = (a^{-2} \partial_t, -h^{ij} \partial_{x^j})$ , the vector  $\nabla \log a = (a^{-2} u, -h^{ij} v_j)$  has pseudo-norm  $g(\nabla \log a, \nabla \log a) = a^{-2} u^2 - |v|_{h^{-1}(m)}^2 \geq 0$ . ▷

## 3.2 Perfect fluids

Our second class of examples where to apply Lyapounov's method to prove non-explosion will be the set of spacetimes with normal matter whose energy-momentum tensor  $\mathbf{T}$  is that of a perfect fluid. They are characterized by the datum of a timelike vector field  $U$ , the four velocity of the fluid, and two functions  $\rho$  and  $p$  on  $\mathbb{M}$ , respectively the energy density and pressure of the fluid. See [H-E], [B-E]. We have then:  $\mathbf{T} = \rho U \otimes U + p(g + U \otimes U)$ , or in local coordinates,

$$\mathbf{T}_{ij} = (\rho + p) U_i U_j + p g_{ij}.$$

Such a spacetime is said of **perfect fluid type**. Notice that contrarily to the globally hyperbolic spacetimes no topological assumption is made on a perfect fluid type spacetime.

Gödel's universe is such a spacetime. This is the manifold  $\mathbb{R}^4$  with the metric  $ds^2 = dt^2 - dx^2 + \frac{1}{2}e^{2\sqrt{2}\omega x} dy^2 - dz^2 - 2e^{\sqrt{2}\omega x} dt dy$ , where  $\omega > 0$  is a constant. It is a solution to Einstein's equation with cosmological constant  $\omega^2$  and represents a pressure free perfect fluid. It has energy-momentum tensor  $\mathbf{T} = U \otimes U$ , where  $(U_j) = (\sqrt{2}\omega, 0, \sqrt{2}\omega e^{\sqrt{2}\omega x}, 0)$  represents the four-velocity covector of the matter, and  $\omega$  is the vorticity of this field. This spacetime has constant scalar curvature  $2\omega^2$ . See Section 2.4 in [F].

As above, the function  $f$  is defined by Formula (3.1) and can be used as a Lyapounov function under some conditions. The computations made in Section 3.1 work equally well in that setting and lead to the following results.

**PROPOSITION 6.** *Let  $(\mathbb{M}, g)$  be a Lorentzian manifold of perfect fluid type, and  $f$  defined by Formula (3.1). Suppose  $f$  goes almost-surely to infinity along any exploding timelike path. If there exists a constant  $C$  such that*

$$H_0 f + \frac{d}{2} \Theta f + \frac{1}{2} (f \dot{m}^k - U^k) \frac{\partial \Theta}{\partial \dot{m}^k} \leq C f,$$

*then the  $\Theta$ -diffusion has almost-surely an infinite lifetime.*

In the particular case of Gödel's universe, the gradient  $\nabla U$  of the velocity vanishes (since  $U^i = \delta_0^i$ ), so that  $H_0 f = 0$ , by Formula (3.2); and  $f$  is the square root of the energy.

**COROLLARY 7.** *Let us work in Gödel's universe and suppose that  $3\Theta + (\dot{m}^k \frac{\partial \Theta}{\partial \dot{m}^k} - \frac{1}{f} \frac{\partial \Theta}{\partial \dot{m}^0})$  is bounded above in  $T^1\mathbb{M}$ . Then the  $\Theta$ -diffusion has almost-surely an infinite lifetime. This condition holds in particular if  $\Theta = \Theta(m)$  is  $SO_0(1, d)$ -invariant and is bounded, as this is the case for the basic relativistic diffusion (and the  $R$ -diffusion) of Gödel's universe.*

Note that this criterion does not apply to the energy diffusion in Gödel's universe. Indeed one can see in that case (see Section 2.4 of [F]) that the above quantity is equal to  $5\Theta - 4\omega^2$  and that the energy  $\Theta$  is unbounded along the trajectories of the energy diffusion.

**REMARK 8.** In Einstein-de Sitter spacetime (this Robertson-Walker universe is both a warped product and a perfect fluid type spacetime), the energy diffusion explodes with positive probability, as proved in Proposition 5.4.2 of [F-LJ-2].

## 4 b-completeness

The study of dynamics in the orthonormal frame bundle is not new at all in general relativity and essentially dates back to Cartan's moving frame method. However, B.G. Schmidt [S1] was the first to notice that the geometry of  $\mathbb{OM}$  itself may be used to provide a conceptual framework in which studying the nature of spacetime singularities. For that purpose, he introduced on the parallelizable manifold  $\mathbb{OM}$  a Riemannian metric, turning  $\{H_0, \dots, H_d, (V_{ij})_{0 \leq i < j \leq d}\}$  into a Riemannian orthonormal basis, and called it the **bundle metric**, or **b-metric**. Schmidt [S2] proved that the completeness of this metric structure on  $\mathbb{OM}$  can essentially be phrased in terms of  $\mathbb{M}$ -valued paths. To state his theorem, recall that one can associate to any  $\mathbb{M}$ -valued  $\mathcal{C}^1$  path  $\gamma : [0, T[ \rightarrow \mathbb{M}$  and  $\mathbf{e} \in T_{\gamma_0}\mathbb{M}$  a unique horizontal lift  $\gamma^\uparrow$  of  $\gamma : [0, T[ \rightarrow \mathbb{OM}$ , starting from  $(\gamma_0, \mathbf{e})$ , defined by :

$$\frac{d}{ds}\gamma_s^\uparrow \in \text{span}(H_0, \dots, H_d), \quad \text{and} \quad \pi_0(\gamma_s^\uparrow) = \gamma_s, \quad \text{for all } s \in [0, T[.$$

The **S-length** of  $\gamma$  is defined as the Riemannian length of its horizontal lift  $\gamma^\uparrow$ . In other words, given  $\mathbf{e} \in T_{\gamma_0}\mathbb{M}$ , seen as orthonormal in the Euclidian sense, the S-length of the  $\mathbb{M}$ -valued  $\mathcal{C}^1$  path  $\gamma$  is the Euclidian length of its anti-development in  $T_{\gamma_0}\mathbb{M}$ .

**THEOREM 9** (Schmidt, [S2]; see also [H-E], Section 8.3).  *$\mathbb{OM}$  is complete for the above b-metric if and only if any  $\mathcal{C}^1$  path  $\gamma : [0, T[ \rightarrow \mathbb{M}$  having a bounded S-length converges in  $\mathbb{M}$  at  $T^-$ .*

The above completeness hypothesis is usually called **b-completeness**. It might seem that this Riemannian view in a seemingly (much) more involved Lorentzian setting may provide a bridge to investigate the latter using tools of Riemannian geometry. This is indeed the case as the following proposition shows.

**PROPOSITION 10.** *Let  $\Theta$  be a bounded function on  $\mathbb{M}$  (i.e.  $SO_0(1, d)$ -invariant on  $\mathbb{OM}$ ). Then the  $\Theta$ -diffusion does not explode if  $\mathbb{OM}$  is b-complete.*

**PROOF** – • Given a  $\mathcal{C}^1$ -path  $(\gamma^\uparrow)_{0 \leq s < T}$  in  $\mathbb{OM}$ , write  $\gamma$  for its projection  $\pi_0 \circ \gamma^\uparrow$  in  $\mathbb{M}$ . For  $s < T$  denote by  $\tau_{0 \rightarrow s}^\gamma$  the parallel transport operator along the curve  $(\gamma_r)_{0 \leq r \leq s}$ , with inverse  $\tau_{0 \leftarrow s}^\gamma$ . Also, denote by  $(p_s)_{0 \leq s < T}$  the anti-development of  $\gamma$ : this  $T_{\gamma_0}\mathbb{M}$ -valued  $\mathcal{C}^1$ -path is defined for all  $s \in [0, T[$  by the formula:  $p_s = \int_0^s \tau_{0 \leftarrow r}^\gamma \dot{\gamma}_r dr$ . Last, we shall denote by  $\dot{p}_r^j$  the coordinates of  $\dot{p}_r$  in the frame  $\gamma_0^\uparrow$  and set  $\|\dot{p}_r\|_{\gamma_0^\uparrow} := \sum_{1 \leq j \leq d} (\dot{p}_r^j)^2$ . We have by construction  $d\gamma_s^\uparrow = \sum_{1 \leq j \leq d} H_j(\gamma_s^\uparrow) \dot{p}_r^j dr$  and  $\dot{\gamma}_s = \tau_{0 \rightarrow s}^\gamma \dot{p}_s$ . Denoting by  $\|\cdot\|_{\gamma_s}$  the Euclidean norm in  $(T_{\gamma_s}\mathbb{M}, \gamma_s^\uparrow)$ , we have  $\|\dot{\gamma}_s\|_{\gamma_s^\uparrow} = \|\dot{p}_s\|_{\gamma_0^\uparrow}$ . The b-completeness assumption means that if

$$\int_0^{T^-} \|\dot{p}_s\|_{\gamma_s} ds < \infty, \quad (4.1)$$

then  $\gamma$  has a limit  $\gamma_T$  in  $\mathbb{M}$  at time  $T$ .

• Now, the basic relativistic diffusion  $(m_s, \mathbf{e}_s)_{0 \leq s < \zeta}$  is by construction the development in  $\mathbb{M}$  of the relativistic Dudley diffusion in Minkowski spacetime, identified with  $T_{m_0}\mathbb{M}$ , see Theorem 3.2 in [F-LJ-1]. As the latter satisfies almost-surely the condition (4.1) for

all times, the b-completeness of  $\mathbb{O}\mathbb{M}$  is sufficient to ensure the non-explosion of the basic relativistic diffusion.

- For a generic  $\Theta$ -diffusion, Formula (2.5) implies the existence for each  $s \in [0, \zeta[$  of an orthonormal basis  $(\varphi_1(s), \dots, \varphi_d(s))$  of  $\dot{p}_s^\perp$  in  $\mathbb{R}^{1,d}$  such that one has (using that  $\Theta$  is  $SO_0(1, d)$ -invariant, i.e. depends only on  $m$ , to simplify the general expression):

$$d\dot{p}_s^k = \sum_{j=1}^d \sqrt{\Theta(m_s)} \varphi_j^k(s) dw_s^j + \frac{d}{2} \Theta(m_s) \dot{p}_s^k ds$$

for some  $d$ -dimensional Brownian motion  $w$ . The path  $(p_s, \dot{p}_s)_{0 \leq s < \zeta}$  appears then as a time change of Dudley's diffusion, by means of  $s \mapsto \inf\{u \mid \int_0^u \Theta(m_r) dr > s\}$ . The result follows directly for bounded  $\Theta$ 's.  $\triangleright$

Although this result is somehow satisfactory, one should note, however, that it is practically tricky to decide whether or not a given spacetime is b-complete. Also, while natural at first sight, the b-completeness hypothesis happens not to be satisfied by quite reasonable spacetimes. Geroch has indeed constructed in [Ge] a geodesically complete spacetime in which there exists an incomplete path with bounded acceleration, and such a pathology can even happen in a globally hyperbolic spacetime, as shown by Beem in [Be]. These examples indicate moreover that the mere geodesic completeness (in its strongest sense: all timelike, lightlike, and spacelike geodesics are complete) should generally not be a sufficient condition for stochastic completeness. Actually, it seems that no satisfactory definition of a non-singular spacetime has been brought out (even since [Ge] and [H-E]).

## 5 A volume growth non-explosion criterion

In this section we prove a non-explosion criterion which requires much weaker conditions to apply than the results of Sections 3 and 4. In particular, no hypothesis on the topology of the manifold is made. Rather, the criterion will only involve the volume growth of some boxes in  $\mathbb{O}\mathbb{M}$  and the function  $\Theta$ , as written in (5.1) below. Theorem 11 is proved in Section 5.4 following Takeda's method, as improved recently by Hsu and Qin in [H-Q]. Yet, there is a real difficulty in doing this, as we are working with a non-symmetric, hypoelliptic diffusion, furthermore on a bundle having a non-compact fibre. To overcome this difficulty we shall resort in Section 5.2 to control theory and sub-Riemannian geometry, in order to find a function which will play the role of the missing Lorentzian distance. The boxes involved in the statement of Theorem 11 will be defined in this section 5.2. To implement Takeda's approach in our setting, we shall construct in Section 5.3 a modified  $\Theta$ -diffusion on some new compact manifold. The proof of Theorem 11 will be given in Section 5.4, and a by-product in Section 5.5.

As a Lorentzian manifold is not a metric space there is no obvious way to replace the usual completeness assumption used in the Riemannian setting. Among the propositions which were made in this direction (see e.g. Section 5.2 of [B-E]), we retain the following rather mild and natural assumption due to Busemann in 1967 and adapted to the Lorentzian setting by Beem in 1976; it is well-suited for our purpose.

**Busemann's assumption.**  $\mathbb{M}$  is timelike Cauchy complete: Any sequence  $(m_n)_{n \geq 0} \subset \mathbb{M}$  such that  $m_{n+1}$  belongs to the future of  $m_n$  and  $\sum_{n \geq 0} \text{dist}(m_n, m_{n+1}) < \infty$ , converges in  $\mathbb{M}$ .

Note that the Lorentzian distance  $\text{dist}(m, m')$  is well-defined for  $m'$  in the future of  $m$  (see [H-E] or [B-E]): this is the supremum of the length  $\int_0^1 \sqrt{g(\dot{\gamma}_r, \dot{\gamma}_r)} dr$ , of  $C^1$  future-directed timelike curves  $(\gamma_r)_{0 \leq r \leq 1}$  joining  $m = \gamma_0$  to  $m' = \gamma_1$ .

It will also happen to be convenient to make the following rather mild assumption on the causal structure of spacetime.

**Hypothesis.**  $(\mathbb{M}, g)$  is strongly causal ([H-E] page 192, or [B-E]): any point of  $\mathbb{M}$  contains arbitrary small neighbourhoods which no non-spacelike curve intersects more than once).

The following notations are used in Theorem 11 below.

- The boxes  $B_r \subset \mathbb{O}\mathbb{M}$ ,  $r > 0$ , are defined in Corollary 15 of Section 5.2 below. Think of them for the moment as some kind of balls in  $\mathbb{O}\mathbb{M}$ , increasing with  $r$ .
- For any  $r > 0$ , set  $\Theta_r := \sup_{\Phi \in B_r} \Theta(\Phi)$ .

We shall establish the following non-explosion criterion, in the spirit of Grigor'yan [Gr1].

**THEOREM 11.** *Let  $(\mathbb{M}, g)$  be a strongly causal Lorentzian manifold satisfying Busemann's assumption. Suppose*

$$\int_0^\infty \frac{r dr}{\Theta_r \log(\Theta_r \text{VOL}(B_r))} = \infty. \quad (5.1)$$

*Then the  $\Theta$ -diffusion has an almost-surely infinite lifetime, from any starting point.*

Clearly Condition (5.1) takes on the form of Grigor'yan's one:  $\int_0^\infty \frac{r dr}{\log \text{VOL}(B_r)} = \infty$ , for bounded (not identically vanishing)  $\Theta$ .

## 5.1 Takeda's method

Using an idea of Lyons and Zheng [L-Z], Takeda devised in [Ta1],[Ta2] a remarkably simple and sharp non-explosion criterion for Brownian motion on a Riemannian manifold  $\mathbb{V}$ . Loosely speaking, his reasoning works as follows. Suppose we have a diffusion  $(x_s)_{s \geq 0}$  on  $\mathbb{V}$  which is symmetric (with respect to the Riemannian volume measure  $\text{VOL}$ , say) and conservative; denote by  $L$  its generator, and let  $f$  be a sufficiently smooth function. Denote by  $\mathbb{P}_{\text{VOL}}$  the measure  $\int \mathbb{P}_x \text{VOL}(dx)$ , where  $\mathbb{P}_x$  is the law of the diffusion started from  $x$ . Fix a time  $T > 0$ . As the reversed process  $(x_{T-s})_{0 \leq s \leq T}$  is an  $L$ -diffusion under  $\mathbb{P}_{\text{VOL}}$ , applying Itô's formula to both  $f(x_s)$  and  $f(x_{T-s})$  provides two martingales  $M$  and  $\widetilde{M}$  (with respect to  $\sigma(x_s; 0 \leq s \leq T)$  and  $\sigma(x_{T-s}; 0 \leq s \leq T)$  respectively) such that:

$$\begin{aligned} f(x_s) &= f(x_0) + M_s + \int_0^s Lf(x_r) dr, \\ f(x_s) &= f(x_{T-(T-s)}) = f(x_T) + \widetilde{M}_{T-s} + \int_0^{T-s} Lf(x_{T-r}) dr. \end{aligned}$$

It follows that  $f(x_s) = \frac{f(x_0)+f(x_T)}{2} + \frac{M_s+\widetilde{M}_{T-s}}{2} + \int_0^T Lf(x_s) dr$ , and consequently,

$$f(x_T) - f(x_0) = \frac{1}{2}(M_T - \widetilde{M}_T).$$

If  $d\langle M \rangle_s/ds$  and  $d\langle \widetilde{M} \rangle_s/ds$  are bounded above, by 1 say, the previous identity provides a control of  $(f(x_T) - f(x_0))$  by the supremum of the absolute value of a Brownian motion over the interval  $[0, T]$ .

Back to the non-explosion problem for Brownian motion on  $\mathbb{V}$ , fix a point  $m \in \mathbb{V}$  and a radius  $R > 1$ , and consider the Brownian motion  $(x_s)_{s \geq 0}$  reflected on the boundary of the Riemannian ball  $B(m; R)$ , started under its invariant measure  $\mathbf{1}_{B(m; R)} \text{VOL}$ . It is a symmetric conservative diffusion; denote by  $\overline{\mathbb{P}}_{B(m; R)}$  its law. Using the Dirichlet forms approach to symmetric diffusions one can apply the above reasoning to the (non-smooth, but 1-Lipschitz) Riemannian distance function  $d(m, \cdot)$ . This provides the estimate

$$\overline{\mathbb{P}}_{B(m; R)} \left( x_0 \in B(m; 1), \sup_{s \leq T} d(m, x_s) = R \right) \leq \text{VOL}(B(m; R)) \times 2 \mathbb{P} \left( \sup_{s \leq T} |B_s| > R \right).$$

But as the Brownian motion on  $\mathbb{V}$  behaves in the ball  $B(m; R)$  as the Brownian motion reflected on the boundary of  $B(m; R)$ , the inequality above also gives an upper bound for the probability that the Brownian motion on  $\mathbb{V}$ , started uniformly from  $B(m; 1)$ , exits the ball  $B(m; R)$  before time  $T$ . Combining this estimate with the Borel-Cantelli lemma, Takeda proved that the Brownian motion on  $\mathbb{V}$  is conservative provided

$$\liminf_{R \rightarrow \infty} R^{-2} \log \text{VOL}(B(m; R)) < \infty,$$

re-proving in a simple way a criterion due to Karp and Li. Takeda's method has been refined by several authors, culminating with Hsu and Qin's recent work [H-Q], who give an elegant and simple proof of a sharp non-explosion criterion, due to Grigor'yan [Gr1], for Brownian motion on a Riemannian manifold in terms of volume growth, as well as an escape rate function. We shall follow their method to deal with relativistic diffusions.

The main difficulty in implementing this approach is in finding what will play the role of the pair "Riemannian distance function – reflected Brownian motion" in our Lorentzian, hypoelliptic framework. Using control theory, we are going to introduce in Section 5.2 below a function on  $\mathbb{OM}$  which will play somehow the role of the missing distance function.

In the Riemannian setting, the choice of a Brownian motion reflected on the boundary of a ball  $B(m; R)$  is guided by the fact that it is the simplest diffusion process which coincides with Brownian motion on the ball  $B(m; R)$  and has a state space with finite volume. One cannot take a smaller state space if the former property is to be satisfied. Yet, one can make different choices if one is ready to loose the minimality property.

To explain that, let us suppose that  $(\mathbb{V}, g)$  is a Cartan-Hadamard manifold. Given a point  $m \in \mathbb{V}$  let us use the exponential map  $\exp_m$  at  $m$  as a global chart on  $\mathbb{V}$ ; this identifies the geodesic ball  $B(m; R)$  on  $M$  to the (Euclidean-shaped) ball  $B'(0; R)$  in  $T_m \mathbb{V}$ . Given  $\epsilon > 0$ , let us modify the metric on  $B'(0; R + \epsilon) \setminus B(0; R)$  so as to interpolate smoothly between  $\exp_m^* g$  on  $B'(0; R)$  and the constant metric  $g_m$  outside  $B'(0; R + \epsilon)$  (primed balls refer to the pull-back metric  $\exp_m^* g$ ). Denote by  $\tilde{g}$  the restriction to  $B'(0; R + 2\epsilon)$  of this modified

metric, and define the compact space  $\mathbb{K}$  as the closed ball  $\overline{B}'(0; R + 2\epsilon)$  where we identify  $m' \in \overline{B}'(0; R + 2\epsilon) \setminus B'(0; R + 2\epsilon)$  and  $-m'$ . Then the  $\tilde{g}$ -Brownian motion on  $\mathbb{K}$  coincides with the  $\exp_m^* g$ -Brownian motion on  $B'(0; R)$  and has a state space with finite  $\tilde{g}$ -wolume  $\text{VOL}_{\tilde{g}}(\mathbb{K}) = (1 + o(\epsilon)) \text{VOL}_g(B(m; R))$ . The construction of the modified  $\Theta$ -diffusion in Section 5.3 below is made in the same spirit.

## 5.2 Sub-Riemannian minimal time.

In this section, we present the results of control theory we need. Note that this rich theory contains a lot of subtleties and traps, like the existence of abnormal minimal curves and the discontinuity of some value functions, as can be seen for example in [C-L], [C-L-E], [L], [M], [Tr1], [Tr2], [B-C], [B-CD]. Let us thank here again Emmanuel Trélat for his very kind help and efficient advices.

Let us consider the following linear sub-Riemannian control problem on  $\mathbb{OM}$ : fix  $\Phi_0 \in \mathbb{OM}$ , and associate to any  $\Phi \in \mathbb{OM}$ :

- the **admissible** curves  $\gamma = \gamma^u \in C^0([0, T(\gamma)], \mathbb{OM})$  such that

$$\dot{\gamma}_s = u_s^0 H_0(\gamma_s) + \sum_{j=1}^d u_s^j V_j(\gamma_s), \quad \text{with } \gamma_0 = \Phi_0, \gamma_{T(\gamma)} = \Phi,$$

the measurable **controls**  $u := (u_s^0, u_s^1, \dots, u_s^d)$  taking their values in the unit cube  $[-1, 1]^{1+d}$ ;

- the **minimal time**:  $\mathcal{T}(\Phi) \equiv \mathcal{T}_{\Phi_0}(\Phi) := \inf\{T(\gamma) \mid \gamma \text{ admissible}\}$ .

**REMARK 12.** It could seem more natural, concerning relativistic diffusions, to use hypo-elliptic (or "affine") controls, rather than sub-Riemannian (or "linear") ones. This means, restricting ourselves to controls  $u$  with  $u^0 \equiv 1$ . However it is unknown to us whether or not the crucial control of the derivatives (see Corollary 15 below) holds in that setting, where a priori the analogue of Corollary 15 holds without the absolute value about  $H_0\mathcal{T}$ ; so that the necessary lower control of the derivative in the  $H_0$ -direction could fail. As it is not clear also that this would yield ultimately better a criterion for stochastic completeness, we shall not pursue this line here.

**REMARK 13.** Since the Lie algebra generated by  $\{H_0, V_1, \dots, V_d\}$  equals everywhere the whole tangent space  $T\mathbb{OM}$ , there exist by Chow's Theorem ([M], Theorem 2.2) admissible curves  $\gamma$  connecting any couple of points in  $\mathbb{OM}$ . Moreover, by the ball-box theorem ([M], Theorem 2.10), the sub-Riemannian distance  $d_{SR}(\Phi_0, \Phi) := \inf_{\gamma} \left\{ \int_0^{T(\gamma)} \|u_s\|_2 ds \right\}$  (whithout constraint on the control) is locally equivalent to the Riemannian bundle distance on  $\mathbb{OM}$ . As  $\|u_s\|_{\infty} \leq \|u_s\|_2 \leq \sqrt{d} \|u_s\|_{\infty}$  on  $\mathbb{R}^{1+d}$  and  $\mathcal{T}(\Phi) = \inf_{\gamma} \left\{ \int_0^{T(\gamma)} \|u_s\|_{\infty} ds \right\}$  (whithout constraint on the measurable control  $u$ ), it follows that we have

$$\mathcal{T}(\Phi) \leq d_{SR}(\Phi_0, \Phi) \leq \sqrt{d} \mathcal{T}(\Phi).$$

Hence, a ball-box theorem for the boxes  $B_r = \{\mathcal{T} \leq r\}$  holds as well.

**THEOREM 14** (Théorème 8.3.3 of [Tr2] ; Theorem IV.2.6 of [B-CD] ; Theorem 8.3 of [L]). *The minimal time functional  $\mathcal{T}$  is finite, locally Lipschitzian on  $\mathbb{OM}$ , and solves, in the viscosity sense, the following Hamilton-Jacobi-Bellman equation :*

$$\max \left\{ \left( v^0 H_0 + \sum_{j=1}^d v^j V_j \right) \cdot \nabla \mathcal{T} \mid v \in [-1, 1]^{1+d} \right\} = 1.$$

**Remark.** Actually, Theorem 14 is stated in the references for a function  $F = F(\Phi, u)$  which is bounded, and globally Lipschitzian with respect to  $\Phi$ , uniformity with respect to the control  $u$ , and in the conclusion the minimal time is also globally Lipschitzian. In our case, we have  $F(\Phi, u) = u^0 H_0(\Phi) + \sum_{j=1}^d u^j V_j(\Phi)$ , so that the uniformity with respect to  $u$  is clear,  $F$  is locally Lipschitzian by (2.1) and (2.2), but  $H_0$  (and then  $F$ ) has generally no reason to be globally Lipschitzian on  $\mathbb{OM}$ , nor bounded. To overcome this difficulty, we can use the sub-Riemannian distance  $d_{SR} = d_{SR}(\Phi_0, \cdot)$  of Remark 13, and for an arbitrary large  $R$ , a smooth non-increasing function  $h^R$  such that  $1_{[0,R]} \leq h^R \leq 1_{[0,R+1]}$ . Set then  $H_0^R := (h^R \circ d_{SR}) \times H_0$ , and  $V_j^R := (h^R \circ d_{SR}) \times V_j$ . Thus, the corresponding function  $F^R$  is globally Lipschitzian (with respect to  $d_{SR}$ ) and bounded, so the corresponding minimal time  $\mathcal{T}^R$  is globally Lipschitzian too, by the cited results. The awaited localization follows by the uniform estimate  $d_{SR} \leq \sqrt{d} \mathcal{T}$  of Remark 13.

**COROLLARY 15.** 1. *We have on  $\mathbb{OM}$  :  $|H_0 \mathcal{T}| + \sum_{j=1}^d |V_j \mathcal{T}| = 1$ , in the sense of viscosity.*

2. *The boxes  $B_r := \{\mathcal{T} \leq r\}$  are compact.*

The viscosity sense will be sufficient for our purpose. Indeed, regularizing the 1-Lipschitz function  $\mathcal{T}$  of Theorem 14 by means of a standard kernel, we get the following corollary.

**COROLLARY 16.** *For any  $\eta, R > 0$ , there exists a smooth function  $F$  on  $\mathbb{OM}$  such that*

$$\sup_{\mathcal{T}^{-1}([0,R])} |F - \mathcal{T}| < \eta \quad \text{and} \quad \sup_{\mathcal{T}^{-1}([0,R])} \left\{ |H_0 F| + \sum_{j=1}^d |V_j F| \right\} < 2.$$

### 5.3 A modified process

We construct in this section the process which will play the role of the reflected Brownian motion in Takeda's approach. We start by 'compactifying' the space. We fix a reference frame  $\Phi_{\text{ref}} \in \mathbb{OM}$ , origin for the minimal time  $\mathcal{T} \equiv \mathcal{T}_{\Phi_{\text{ref}}}$  and centre of the boxes  $B_r$ . We fix also  $\lambda, \varepsilon > 0$ , and consider the relatively compact open region

$$\mathcal{U} := \{\lambda < \mathcal{T} \equiv \mathcal{T}_{\Phi_{\text{ref}}} < \lambda + \varepsilon\}.$$

**LEMMA 17.** *There exists in  $\mathcal{U}$  a smooth hypersurface  $V$  separating  $\partial B_\lambda$  from  $\partial B_{\lambda+\varepsilon}$  ( $\partial B_\lambda \cup \partial B_{\lambda+\varepsilon}$  does not intersect  $V$  but any continuous path from  $\partial B_\lambda$  to  $\partial B_{\lambda+\varepsilon}$  hits  $V$ ), such that the subset  $V_0 := \{\Phi \in V \mid H_0(\Phi) \in T_\Phi V\}$  is a smooth hypersurface of the hypersurface  $V$ .*



PROOF – (with thank A. Oancea and P. Pansu for their help in proving this statement) –

Let us use the function  $F$  of Corollary 16, with  $\eta < \varepsilon/4$  and  $R > \lambda + \varepsilon$ , and fix  $\eta < \varepsilon_1 < \varepsilon_2 < \varepsilon/2 - \eta$  such that  $B_\lambda \subset \{\varepsilon_1 \leq F - \lambda \leq \varepsilon_2\} \subset B_{\lambda+\varepsilon/2}$ . By Sard's Theorem, the set of regular values of  $(F - \lambda)$  is dense in the interval  $]\varepsilon_1, \varepsilon_2[$ . Fix then a regular value  $c$ , so that  $S := \{F = c\}$  is a smooth hypersurface separating  $\partial B_\lambda$  from  $\partial B_{\lambda+\varepsilon/2}$ .

We can then suppose that we are dealing with  $\mathcal{U} \equiv S \times ]0, \varepsilon/2[$ . The nearby hypersurface having the wanted transversality property with respect to  $H_0$  will be the graph of some smooth function  $f : S \rightarrow ]0, \varepsilon/2[$ .

Denote by  $Gr(T\mathcal{U})$  the Grassmannian bundle over  $\mathcal{U}$  made up of all the hyperplanes of  $T\mathcal{U}$ , and to any function  $f : S \rightarrow ]0, \varepsilon/2[$  associate the function  $G_f : S \rightarrow Gr(T\mathcal{U})$ , defined by  $G_f(s) := \{(\sigma, df_s(\sigma)) \mid \sigma \in T_s S\}$ . Let  $\mathcal{H}$  denote the smooth hypersurface of  $Gr(T\mathcal{U})$ , made of all hyperplanes containing  $H_0$ . Then  $G_f^{-1}(\mathcal{H})$  is a smooth hypersurface of  $\text{Graph}(f)$  as soon as  $G_f$  is transverse to  $\mathcal{H}$ . Therefore the statement reduces to finding a function  $f$  such that  $G_f$  be transverse to  $\mathcal{H}$ .

Consider a smooth partition of unity:  $1_S = \sum_{j=1}^n \alpha_j$ , with  $\{\alpha_j > 0\} = \psi_j(\mathcal{B}^\nu)$  diffeomorphic (under  $\psi_j$ ) to the unit ball  $\mathcal{B}^\nu \subset \mathbb{R}^\nu$  (with  $\nu = \dim(\mathbb{O}\mathbb{M}) - 1 = (d+3)d/2$ ). Let us denote by  $\mathcal{A}$  the space of affine functions on  $\mathbb{R}^\nu$  (restricted to  $\mathcal{B}^\nu$ ), and consider the function  $G : \mathcal{A}^n \times S \rightarrow Gr(T\mathcal{U})$  defined by  $G(\varphi_1, \dots, \varphi_n, s) := G_{\sum_{j=1}^n \alpha_j \times \varphi_j \circ \psi_j^{-1}}(s)$ . This is a submersion. Therefore the transversality lemma ensures that  $G_{\sum_{j=1}^n \alpha_j \times \varphi_j \circ \psi_j^{-1}}$  is transverse to  $\mathcal{H}$  for almost every  $\varphi = (\varphi_1, \dots, \varphi_n) \in \mathcal{A}^n$ . Thus a small multiple of such a  $\varphi$  provides the wanted function  $f$ .  $\triangleright$

Define  $O$  as the set of points of the ball  $B_{\lambda+\varepsilon}$  of the form  $\gamma(1)$  for some continuous path  $\gamma : [0, 1] \rightarrow B_{\lambda+\varepsilon}$  starting from a point of  $B_\lambda$  and not hitting  $V$ ; this is an open set with boundary  $V$ . Denote also by  $\mathcal{S}$  another smooth hypersurface, separating  $V$  from  $\partial B_\lambda$  and transverse to  $H_0$  except on a relative hypersurface. Let now  $\mathbb{O}'\mathbb{M}$  denote a disjoint copy of the set of past-directed frames:

$$\{(m, \mathbf{e}) \in GL\mathbb{M} \mid \mathbf{e} = (\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d) \text{ such that } (m, (-\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d)) \in \mathbb{O}\mathbb{M}\},$$

and  $O'$ ,  $V'$ ,  $V'_0$  and  $\mathcal{S}'$  be the subsets of  $\mathbb{O}'\mathbb{M}$  corresponding to  $O$ ,  $V$ ,  $V_0$  and  $\mathcal{S}$ . The equivalence relation

$$(m, (\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d)) \in V \sim (m, (-\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d)) \in V'$$

defines a manifold structure on the quotient space  $(O \cup V) \sqcup (O' \cup V') / \sim$ , which we denote by  $\mathcal{E}$ . Note that  $\mathcal{E}$  is compact and that its volume is in between  $2 \text{Vol}(B_\lambda)$  and  $2 \text{Vol}(B_{\lambda+\varepsilon})$ . Write  $\mathcal{V}$  for the image in  $\mathcal{E}$  of  $V$ , and  $\mathcal{V}_0$  for the image in  $\mathcal{E}$  of  $V_0$ .

**REMARK 18.** The geodesic flow is naturally well defined on  $\mathcal{E} \setminus \mathcal{V}_0$ , getting instantly from  $O$  to  $O'$  or from  $O'$  to  $O$  at its crossings of  $\mathcal{V} \setminus \mathcal{V}_0$ . Indeed by the above definition, for any

$\Phi \in \mathcal{V} \setminus \mathcal{V}_0$ , either  $H_0(\Phi)$  points outwards seen from  $O$  and inwards seen from  $O'$ , or  $H_0(\Phi)$  points inwards seen from  $O$  and outwards seen from  $O'$ . On the contrary, there is a priori no convenient way to continue the geodesic flow on  $\mathcal{V}_0$ . This is the reason why we take care of this exceptional set, in the previous lemma 17 and in Lemma 19 below.

We define the *modified relativistic diffusion* on the compact manifold  $\mathcal{E}$  as follows.

Let  $a : B_{\lambda+\varepsilon} \rightarrow [0, 1]$  be a smooth function equal to 1 on  $B_\lambda$ , and whose vanishing set is exactly the closed part  $\mathfrak{C}$  of  $\mathcal{U}$  in between  $\mathcal{S}$  and  $V$  (this means that  $\mathfrak{C}$  is the union of the trajectories  $\gamma[0, 1] \subset \mathcal{U}$  of continuous paths  $\gamma$  such that  $\gamma_0 \in \mathcal{S}$ ,  $\gamma_1 \in V$ , and  $\gamma_{]0, 1[}$  does not intersect the oriented hypersurface  $\mathcal{S} \cup V$ ). We extend to  $\mathcal{E}$  the restriction of  $a$  to  $O \cup V$ , by setting  $a(\mathbf{e}') = a(\mathbf{e})$  for  $\mathbf{e}' = (m, (-\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d)) \in \mathcal{O}'\mathbb{M}$  associated with  $\mathbf{e} = (m, (\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d)) \in \mathcal{O}\mathbb{M}$ .

We define the generator of the modified diffusion to be the following variant of  $\mathcal{G}_\Theta$ :

$$\mathcal{G} := H_0 + \frac{1}{2} \sum_{j=1}^d V_j(a \Theta V_j). \quad (5.2)$$

Denote by  $\text{VOL}_\mathcal{E}$  (resp.  $\text{VOL}_V$ ,  $\text{VOL}_\mathcal{S}$ ) the natural volume element on  $\mathcal{E}$  (resp.  $V$ ,  $\mathcal{S}$ ).

**LEMMA 19.** *For  $\text{VOL}_\mathcal{E}$ -almost all starting point  $\Phi_0 \in \mathcal{E}$ , the modified relativistic diffusion is a well-defined  $\mathcal{E}$ -valued process having an almost-surely infinite lifetime.*

**PROOF** – This modified diffusion has generator  $\mathcal{G}_\Theta$  in  $B_\lambda$  and in its mirror copy  $B'_\lambda$ , and reduces to the geodesic flow in the region  $\{a = 0\}$ , i.e. in between  $\mathcal{S}$  and  $\mathcal{S}'$ . Owing to Remark 18, we must first make sure that the subset  $\mathcal{V}_0$  of “bad points” is polar.

Now by Lemma 17, the orbit in  $\{a = 0\}$  of  $\mathcal{V}_0$  under the geodesic flow and its inverse, say  $\mathcal{N}$ , has null  $\text{VOL}_\mathcal{E}$ -measure, and null  $\text{VOL}_\mathcal{S}$ -measure and  $\text{VOL}_{\mathcal{S}'}$ -measure as well. Moreover, as the modified diffusion started from any  $\Phi_0 \in \{a > 0\}$  is hypoelliptic, its hitting distribution of  $\mathcal{S} \cup \mathcal{S}'$  has a density with respect to  $\text{VOL}_{\mathcal{S} \cup \mathcal{S}'}$ . Therefore the modified diffusion, started from any  $\Phi_0 \in \mathcal{E} \setminus \mathcal{N}$ , will almost surely never hit  $\mathcal{N}$ .

This proves that the modified diffusion is a well-defined  $\mathcal{E}$ -valued process, with the following two possible behaviours as it approaches its lifetime: either it crosses infinitely many times  $\mathcal{V}$ , or it eventually remains in a compact subset of  $O$  or of  $O'$ .

In the latter case where the diffusion path stays in  $O$ , say, its projection on  $\mathbb{M}$  is a timelike path confined in a compact set. As such it has a cluster point at which the strong causality condition cannot hold. Such a type of explosion is thus forbidden in the space  $\mathcal{E}$  constructed out of a strongly causal spacetime  $\mathbb{M}$ .

It remains to exclude the first possibility of explosion: by infinitely many crossings of  $\mathcal{V}$  within a finite proper time. In this case, either the path remains eventually in the region  $\{a = 0\}$ , or it performs before some finite proper time an infinite number of crossings from  $\mathcal{S} \cup \mathcal{S}'$  to  $\mathcal{V}$ . Since the geodesic flow does not explode, only the last possibility is to take into account. Now it cannot allow any explosion either, since the geodesic flow needs a traveling time bounded away from 0 to travel from  $\mathcal{S} \cup \mathcal{S}'$  to  $\mathcal{V}$ .  $\triangleright$

Note that the volume measure  $\text{VOL}_\mathcal{E}$  of the compact manifold  $\mathcal{E}$  is an invariant finite measure for the modified diffusion.

## 5.4 Crossing times and escape rate of $\Theta$ -diffusions

We keep the reference point  $\Phi_{\text{ref}} \in \mathbb{OM}$ , and  $\mathcal{T} \equiv \mathcal{T}_{\Phi_{\text{ref}}} = \mathcal{T}(\Phi_{\text{ref}}, \cdot)$ . We emphasize that  $\mathcal{T}$  is a two point function:  $\mathcal{T}(\Phi_{\text{ref}}, \Phi)$  (recall Section 5.2), so that it is easy, by the triangle inequality, to pass from  $\mathcal{T}(\Phi_{\text{ref}}, \Phi)$  to  $\mathcal{T}(\Phi_0, \Phi)$ , or the other way round.

Let  $(R_n)_{n \geq 1}$  be an increasing sequence of positive reals; associate to each  $R_n$

$$\text{the exit time } \tau_n \text{ from the box } B^{(n)} := \{\mathcal{T} \leq R_n\},$$

and set also  $\tau_0 = 0$ . It takes the diffusion an amount of proper time  $(\tau_n - \tau_{n-1})$  to go from the box  $B^{(n-1)}$  to the box  $B^{(n)}$ . The strategy in [H-Q] is to estimate  $\mathbb{P}_{\Phi}(\tau_n - \tau_{n-1} \leq t_n)$  for some suitably chosen deterministic sequence  $\{t_n\}_{n \geq 0}$  of increments of time. Set for  $n \geq 1$ :

$$T_n := \sum_{k=1}^n t_k, \quad \text{and} \quad r_n := R_n - R_{n-1}.$$

If one can show that

$$\sum_{n \geq 1} \mathbb{P}_{\Phi}(\tau_n - \tau_{n-1} \leq t_n) < \infty \tag{5.3}$$

for a convenient choice of the sequences  $(R_n)_{n \geq 1}$  and  $(T_n)_{n \geq 1}$ , then the Borel-Cantelli lemma will tell us that the diffusion does not exit  $B^{(n)}$  before time  $T_n$ , for  $n$  large enough, preventing explosion. Following [H-Q], we are going to consider the events

$$E_n := \{\tau_n - \tau_{n-1} \leq t_n, \tau_n \leq T_n\},$$

so as to be able to use our modified process run backwards from the *fixed* time  $T_n$ , when estimating the probability that the process crosses from  $B^{(n-1)}$  to  $B^{(n)}$  not too fast. Lemma 2.1 of [H-Q] (which is an easy application of the Borel-Cantelli lemma) justifies that considering these events leads to the same non-explosion conclusion as (5.3). We recall it here for the reader's convenience.

**LEMMA 20** ([H-Q]). *Fix  $\Phi \in \mathbb{OM}$ . If  $\sum_{n \geq 1} \mathbb{P}_{\Phi}(E_n) < \infty$ , then there exists  $\mathbb{P}_{\Phi}$ -almost-surely  $\delta$  such that  $\tau_n \geq T_n - \delta$  for all  $n$ .*

We shall use the results of Sections 5.2 and 5.3 to prove the fundamental estimate of Proposition 21 below. Given any compact subset  $B$  of  $\mathbb{OM}$ , denote by  $\mathbb{P}_B$  the law of the relativistic diffusion in  $\mathbb{OM}$  started under the uniform distribution in  $B$ :

$$\mathbb{P}_B(\cdot) = \frac{1}{\text{VOL}(B)} \int_B \mathbb{P}_{\Phi}(\cdot) \text{VOL}(d\Phi).$$

Similarly, and given any compact subset  $A$  of  $\mathcal{E}$ , write  $\mathbb{Q}_A$  for the law of the modified  $\Theta$ -diffusion in  $\mathcal{E}$  started under the uniform distribution in  $A$ .

**PROPOSITION 21.** *There exists a constant  $C$  such that for any  $n \geq 1$ :*

$$\mathbb{P}_{B^{(1)}}(\tau_n - \tau_{n-1} \leq t_n, \tau_n \leq T_n) \leq C \frac{\text{VOL}(B^{(n)})}{\text{VOL}(B^{(1)})} \frac{T_n \sqrt{\widehat{\Theta}_n/t_n}}{(r_n - 1 - 4t_n)} \exp \left[ -\frac{(r_n - 1 - 4t_n)^2}{32 \widehat{\Theta}_n t_n} \right],$$

where  $\widehat{\Theta}_n$  denotes the supremum of  $\Theta$  over the box  $\{\mathcal{T} \leq R_n + 1\}$ .

The proof mimics Takeda's original proof, as adapted by Hsu and Qin in [H-Q], with the noticeable difference that we are working with a non-symmetric, non-elliptic diffusion.

PROOF – We start by embedding the box  $B^{(n)}$  into the set  $\mathcal{E}^{(n)}$  constructed in Section 5.3, with  $\lambda = R_n$  and  $\varepsilon = \frac{1}{2}$ , say. From now on we work on the path space over  $\mathcal{E}^{(n)}$  and use the coordinate process  $X$ , whose filtration is denoted by  $(\mathcal{F}_s)_{s \geq 0}$ .

We still denote by  $\tau_n$  the exit time from (the image in  $\mathcal{E}^{(n)}$  of)  $B^{(n)}$ ; the event

$$E_n := \{\tau_n - \tau_{n-1} \leq t_n, \tau_n \leq T_n\}$$

belongs to  $\mathcal{F}_{\tau_n}$ . As explained above in Section 5.1, the proof has two main ingredients, the first of which is Inequality (5.4) below.

As the  $\Theta$ -diffusion and the modified  $\Theta$ -diffusion have the same law before  $\tau_n$ , we have  $\mathbb{P}_{B^{(n)}}(E_n) = \mathbb{Q}_{B^{(n)}}(E_n) = 2 \mathbb{Q}_{\mathcal{E}^{(n)}}(E_n)$ , and so

$$\mathbb{P}_{B^{(1)}}(E_n) \leq 2 \frac{\text{VOL}(B^{(n)})}{\text{VOL}(B^{(1)})} \times \mathbb{Q}_{\mathcal{E}^{(n)}}(E_n), \quad (5.4)$$

by the obvious inequality  $\mathbb{P}_{B^{(1)}}(E_n) \leq \frac{\text{VOL}(B^{(n)})}{\text{VOL}(B^{(1)})} \mathbb{P}_{B^{(n)}}(E_n)$ . The second ingredient involves the Lyons-Zheng decomposition of  $\mathcal{T}(X_s)$  under  $\mathbb{Q}_{\mathcal{E}^{(n)}}$ . As  $\mathcal{T}$  is not a priori sufficiently regular to use the basic Itô formula, we are going to apply this decomposition to its smooth approximation  $F$  exhibited in Corollary 16 (with  $R = R_n$  and  $\eta = \frac{1}{2}$ ). The process  $X$  is under  $\mathbb{Q}_{\mathcal{E}^{(n)}}$  a diffusion with generator  $\mathcal{G}$  given by Formula (5.2), and its time-reversal  $(X_{T_n-s})_{0 \leq s \leq T_n}$  is a diffusion with generator  $\mathcal{G}^* = -H_0 + \frac{1}{2} \sum_{j=1}^d V_j(a \Theta V_j)$ .

It follows by Itô's formula that there exists a martingale  $(M_s)_{0 \leq s \leq T_n}$  with respect to the filtration  $(\mathcal{F}_s)_{s \geq 0}$  and a martingale  $(\widetilde{M}_s)_{0 \leq s \leq T_n}$  with respect to the filtration of the reversed process, such that :

$$\begin{aligned} F(X_s) &= F(X_0) + M_s + \int_0^s \mathcal{G}F(X_r) dr; \\ F(X_s) &= F(X_{T_n-(T_n-s)}) = F(X_{T_n}) + \widetilde{M}_{T_n-s} + \int_s^{T_n} \mathcal{G}^*F(X_r) dr, \end{aligned}$$

with

$$\langle M \rangle_s = \sum_{j=1}^d \int_0^s a(X_r) \Theta(X_r) |V_j F|^2(X_r) dr \leq 4 \widehat{\Theta}_n s; \quad (5.5)$$

$$\langle \widetilde{M} \rangle_s = \sum_{j=1}^d \int_0^s a(X_{T_n-r}) \Theta(X_{T_n-r}) |V_j F|^2(X_{T_n-r}) dr \leq 4 \widehat{\Theta}_n s. \quad (5.6)$$

As  $\mathcal{G} - \mathcal{G}^* = 2H_0$ , we thus have :

$$d(F(X_s)) = d\left(\frac{M_s + \widetilde{M}_s}{2}\right) + H_0 F(X_s) ds, \quad (5.7)$$

with a controlled drift term  $|H_0 F| \leq 2$ , by Corollary 15. By construction, we have

$$\sup_{0 \leq s \leq t_n} |F(X_{\tau_{n-1}+s}) - F(X_{\tau_{n-1}})| \geq r_n - 1$$

on the event  $E_n$ , where  $X$  hits the set  $\{F \geq R_n - \frac{1}{2}\}$  on the time interval  $[\tau_{n-1}, \tau_{n-1} + t_n]$ . Then, to control the  $\mathbb{Q}_{\mathcal{E}^{(n)}}$ -probability of  $E_n$ , we use Hsu and Qin's trick: cutting the interval  $[0, T_n] = \bigcup_{k=1}^{\ell_n} [(k-1)t_n, kt_n]$  into  $\ell_n := T_n/t_n$  sub-intervals of length  $t_n$  (to lighten, we shall neglect the fact that  $\ell_n$  may not be an integer; this fact causes no trouble but notational), and writing on each  $\{(k-1)t_n \leq \tau_{n-1} \leq kt_n\}$ :

$$F(X_{\tau_{n-1}+s}) - F(X_{\tau_{n-1}}) = F(X_{\tau_{n-1}+s}) - F(X_{kt_n}) + F(X_{kt_n}) - F(X_{\tau_{n-1}}),$$

we see that the event  $\left\{ \sup_{0 \leq s \leq t_n} |F(X_{\tau_{n-1}+s}) - F(X_{\tau_{n-1}})| \geq r_n - 1 \right\}$  is included in one of the  $\ell_n$  events  $\left\{ \sup_{0 \leq |s| \leq t_n} |F(X_{kt_n+s}) - F(X_{kt_n})| \geq \frac{r_n-1}{2} \right\}$  ( $1 \leq k \leq \ell_n$ ). By (5.7) and the inequality  $|H_0 F| \leq 2$ , the  $k$ -th of these  $\ell_n$  events is included in the union  $A_k \cup \tilde{A}_k$ , where:

$$A_k := \left\{ \sup_{0 \leq |s| \leq t_n} |M_{kt_n+s} - M_{kt_n}| \geq \frac{r_n-1}{2} - 2t_n \right\}, \quad \tilde{A}_k := \left\{ \sup_{0 \leq |s| \leq t_n} |\tilde{M}_{kt_n+s} - \tilde{M}_{kt_n}| \geq \frac{r_n-1}{2} - 2t_n \right\}.$$

Let  $B$  be a Brownian motion defined on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . By (5.5) we have

$$\mathbb{Q}_{\mathcal{E}^{(n)}}(A_k) \leq 2 \mathbb{P} \left[ \sup_{0 \leq s \leq t_n} |B_s| \geq \frac{r_n-1-4t_n}{4\sqrt{\hat{\Theta}_n}} \right] \leq \frac{C \sqrt{\hat{\Theta}_n}/t_n}{r_n-1-4t_n} \exp \left( -\frac{(r_n-1-4t_n)^2}{32 \hat{\Theta}_n t_n} \right)$$

for some positive constant  $C$ ; the same identity holds for  $\tilde{A}_k$ , using (5.6). Summing over  $k$  and using Inequality (5.4) yields directly the statement.  $\triangleright$

This key proposition being proved, it becomes easy to achieve the proof of Theorem 11.

PROOF OF THEOREM 11 – Taking  $R_n = 2^{n+5}$  and  $t_n \leq 2^{n+1}$  in Proposition 21, so that  $T_n \leq 2^{n+2}$ , we get for any  $n \geq 1$ :

$$\mathbb{P}_{B^{(1)}}(E_n) = \mathbb{P}_{B^{(1)}}(\tau_n - \tau_{n-1} \leq t_n, \tau_n \leq T_n) \leq C \frac{\text{VOL}(B^{(n)})}{\text{VOL}(B^{(1)})} \sqrt{\frac{\hat{\Theta}_n}{t_n}} \exp \left[ -\frac{4^n}{\hat{\Theta}_n t_n} \right]. \quad (5.8)$$

Specifying the choice of  $t_n$  by setting

$$t_n := \min \left\{ 2^{n+1}, \frac{4^{n-1}}{\left(1 + \log^+ [\hat{\Theta}_n \text{VOL}(B^{(n)})]\right) \hat{\Theta}_n} \right\},$$

the right hand side of (5.8) is seen to be bounded above by a constant multiple of  $2^{-n}$ , ensuring as a consequence the convergence of the series  $\sum_{n \geq 1} \mathbb{P}_{B^{(1)}}(E_n)$ . Indeed, we get from (5.8), with the above  $t_n$  (noticing that ignoring the trivial case  $\Theta \equiv 0$ , we can suppose  $\hat{\Theta}_n \text{VOL}(B^{(n)}) \geq 3$  for large enough  $n$ ):

$$\mathbb{P}_{B^{(1)}}(E_n) \leq C' \text{VOL}(B^{(n)}) \sqrt{\frac{\hat{\Theta}_n^2 \log [\hat{\Theta}_n \text{VOL}(B^{(n)})]}{4^n}} e^{-4 \log [\hat{\Theta}_n \text{VOL}(B^{(n)})]} \leq C''/2^n.$$

Note that the above choice of  $t_n$  is simpler than Hsu and Qin's choice in [H-Q]; there is in particular no need to introduce their auxiliary function  $h(R) \equiv \log \log R$ , to get Grigor'yan's criterion, if the second upper bound of their section 3 is not used.

To conclude that the  $\Theta$ -diffusion does not explode we need to check that  $T_n = \sum_{k=1}^n t_k$  increases to infinity. Now by our choice of  $t_n$ , we have  $\mathbb{P}_{B^{(1)}}$ -almost-surely, for  $n$  larger than some  $n_0$  and for a positive universal constant  $c$ :

$$\begin{aligned} T_n &\geq \sum_{k=n_0}^n \min \left\{ 2^{k+1}, \frac{4^{k-1}}{\Theta_{2^{k+5}+1} (\log^+ [\Theta_{2^{k+5}+1} \text{VOL}(B_{2^{k+5}})] + 1)} \right\} \\ &\geq c \int_{2^{n_0+1}}^{2^n} \min \left\{ 8, \frac{r}{\Theta_r \log [\Theta_r \text{VOL}(B_r)]} \right\} dr. \end{aligned} \quad (5.9)$$

Note that we can drop the trivial case  $\Theta \equiv 0$ , and suppose  $\Theta_r \geq 3$  (for large enough  $r$ ; recall that by its very definition, just before Theorem 11,  $r \mapsto \Theta_r$  is non-decreasing). Hence the divergence of  $(T_n)$  is equivalent to the integral criterion:

$$\int^{\infty} \min \left\{ 8, \frac{r}{\Theta_r \log [\Theta_r \text{VOL}(B_r)]} \right\} dr = \infty.$$

Now by the non-decreasing property of  $\Theta_r$  and  $\text{VOL}(B_r)$ , this condition is equivalent to

$$\sum_n \min \left\{ 8, \frac{n}{\Theta_n \log [\Theta_n \text{VOL}(B_n)]} \right\} = \infty, \text{ and then to } \sum_n \frac{n}{\Theta_n \log [\Theta_n \text{VOL}(B_n)]} = \infty,$$

since the former holds obviously if an infinite number of terms were larger than 8. Now the latter is equivalent to Condition (5.1) of Theorem 11.

Using Lemma 20 (i.e. essentially the Borel-Cantelli lemma) it follows that we have:

$$\mathbb{P}_{B^{(1)}} \left( \sup_{0 \leq s \leq T_n - \delta} \mathcal{T}(\Phi_s) \leq 2^{n+5} \text{ for any large enough } n \right) = 1, \quad (5.10)$$

which gives  $\sup_{0 \leq s \leq t} \mathcal{T}(\Phi_s) < \infty$  for all  $t > 0$ , since  $T_n$  increases to  $\infty$ . Together with the continuity of  $\mathcal{T}$  and Busemann's assumption, this prevents the explosion of the  $\Theta$ -diffusion before any time  $t > 0$ , under  $\mathbb{P}_{B^{(1)}}$ .

To prove that the same happens under any  $\mathbb{P}_{\Phi_0}$ , notice that since the non-explosion event  $\mathcal{D}$  belongs to the invariant  $\sigma$ -algebra, the function  $\mathbb{OM} \ni \Phi \mapsto \mathbb{P}_{\Phi}(\mathcal{D})$  is  $\mathcal{G}_{\Theta}$ -harmonic, hence continuous, as  $\mathcal{G}_{\Theta}$  is hypoelliptic. It follows that since

$$\mathbb{P}_{B^{(1)}}(\mathcal{D}) = \frac{1}{\text{VOL}(B^{(1)})} \int_{B^{(1)}} \mathbb{P}_{\Phi}(\mathcal{D}) \text{VOL}(d\Phi),$$

the probability  $\mathbb{P}_{\Phi}(\mathcal{D})$  must be equal to 1 for all  $\Phi \in B^{(1)}$ . But as the ball  $B^{(1)}$  was arbitrarily chosen,  $\mathbb{P}_{\Phi}(\mathcal{D})$  is identically equal to 1 everywhere.  $\triangleright$

## 5.5 Upper rate function

Using essentially the same reasoning as in Section 4 of [H-Q], the above proof yields almost for free an upper rate function for the  $\Theta$ -diffusion. We keep the preceding notations.

**COROLLARY 22.** *Let  $\mathbb{M}$  be a strongly causal Lorentzian manifold satisfying Busemann's assumption. Set  $h(\rho) \equiv \rho$  if  $\Theta \equiv 0$ , and if not, given a constant  $R_0$  such that  $\Theta_{R_0} > 0$ , set*

$$h(\rho) := \inf \left\{ R > R_0 \mid \int_{R_0}^R \frac{r \, dr}{\Theta_r \log [\Theta_r \text{VOL}(B_r)]} > \rho \right\} \quad \text{for all } \rho > 0.$$

*Then, given any  $\Phi_0 \in \mathbb{OM}$ , there exist  $R_0 > 0$  and a positive constant  $C$  such that we have  $\mathbb{P}_{\Phi_0}$ -almost-surely*

$$\mathcal{T}(\Phi_0, \Phi_s) \leq Ch(Cs).$$

**PROOF** – We follow the argument of [H-Q], Section 4, making sure that it works here as well with our choice for  $t_n$  and without their auxiliary function  $\log \log$ . Recall (5.9), in which by Proposition 23 below we can forget to take the minimum with 8, except in the trivial geodesic case  $\Theta \equiv 0$ . By (5.10), this yields  $\sup_{0 \leq s \leq ch^{-1}(2^n) - \delta} \mathcal{T}(\Phi_s) \leq 2^{n+5}$ , i.e.

$\sup_{0 \leq s \leq ch^{-1}(R) - \delta} \mathcal{T}(\Phi_s) \leq 32R$  for large enough  $R$ . Letting  $R = h((t + \delta)/c)$ , this entails

$$\sup_{0 \leq s \leq t} \mathcal{T}(\Phi_s) \leq 32h((t + \delta)/c), \quad \text{hence} \quad \sup_{0 \leq s \leq t} \mathcal{T}(\Phi_s) \leq 32h(Ct), \quad \text{for large enough } t.$$

This shows the claim under the probability  $\mathbb{P}_{B(1)}$ , and then under  $\mathbb{P}_{\Phi_0}$  as well, by the same argument already used at the end of the proof of Theorem 11.

Finally, in the geodesic case  $\Theta \equiv 0$  the same holds with  $T_n \geq c2^n = ch(2^n)$ .  $\triangleright$

## 5.6 Estimates of the volume $\text{VOL}(B_r)$ of boxes, and application

Let us begin with a lower estimate of the volume of boxes  $B_r$ , which is rather crude since it is based on the vertical expansion in the  $SO_0(1, d)$ -fibre of  $\mathbb{OM}$ , without taking into account the horizontal expansion which depends on the curvature of the base Lorentzian manifold  $\mathbb{M}$ . We used this lower bound in the proof of Corollary 22.

**PROPOSITION 23.** *We have*  $\liminf_{r \rightarrow \infty} \frac{\log \text{VOL}(B_r)}{r} \geq d - 1$ .

**PROOF** – Fix a compact neighborhood  $\mathcal{U}$  of  $m_0$  in  $\mathbb{M}$ , above which we have:  $\mathbb{OU} \equiv \mathcal{U} \times SO_0(1, d)$ . We can obviously assume that  $\Phi_0 \equiv (m_0, \mathbf{1})$ . By the ball-box Theorem (recall Remark 13),  $B_r = \{\mathcal{T} \leq r\}$  contains a neighborhood  $\mathcal{V} \times B(\mathbf{1}, \varepsilon)$  of  $\Phi_0$ , for some  $\varepsilon > 0$  and for  $r$  larger than some fixed  $r_1$ . Using this repeatedly (a finite number of times) and the triangle inequality for  $\mathcal{T}$ , we see that  $\{\mathcal{T} \leq r\}$  contains  $\mathcal{U} \times B(\mathbf{1}, \varrho)$  of  $\Phi_0$ , for any  $\varrho > 0$  and for  $r$  larger than some fixed  $r_0 = r_0(\mathcal{U}, \varrho)$ .

We take  $\varrho$  larger than the diameter of  $SO(d)$ .

Recall the very definition of  $\mathcal{T}$  (in Section 5.2), and restrict all possible controls  $u$ , after time  $r_0$ , to those such that  $u^0 = 0$ . This shows up that as  $r$  increases, the box  $\{\mathcal{T} \leq r\}$  dilates in the vertical directions  $V_1, \dots, V_d$  with speed  $r$ . We find thus that

for large enough  $r$ ,  $\{\mathcal{T} \leq r\}$  contains the product of  $\mathcal{U}$  by the ball of radius  $(r - r_0)$  in  $SO_0(1, d)$ . This yields in turn the lower estimate of  $\text{VOL}(\{\mathcal{T} \leq r\})$  by some constant multiple of the volume of the hyperbolic ball of radius  $(r - r_0)$ , whence for some positive  $c$ :  $\log \text{VOL}(B_r) \geq (d - 1)r + \log c$ , for large enough  $r$ .  $\triangleright$

The volume growth integral criterion of Theorem 11 is not very concrete, in particular because of the subriemannian minimal time  $\mathcal{T}$ . Let us look for a sufficient condition, which resembles more the Grigor'yan's condition for Brownian motion. We shall express such a condition by bounding above  $\mathcal{T}$ , and separating the vertical and horizontal expansions of the subriemannian boxes. As already mentioned, no natural distance exists on  $\mathbb{M}$ . In order to get a convenient kind of measurement on  $\mathbb{M}$ , we use the S-length introduced in Section 4.

**PROPOSITION 24.** *Fix  $\Phi_0 \in \mathbb{OM}$ , and define the **S-radius**  $\rho_{\Phi_0}^S(m)$  of any  $m \in \mathbb{M}$  as the infimum of the S-lengths of  $C^1$  curves joining  $m_0 = \pi_0(\Phi_0)$  to  $m$ . Define the **S-ball**  $B_{\Phi_0}^S(r)$  of radius  $r$  as the set  $B_{\Phi_0}^S(r) := \{m \in \mathbb{M} \mid \rho_{\Phi_0}^S(m) \leq r\}$ , and set*

$$V^S(r) := \text{VOL}_{\mathbb{M}}(B_{\Phi_0}^S(r)).$$

*Then there exists a constant  $C$  such that we have  $\log \text{VOL}(B_r) \leq C + (d - 1)r + \log V^S(Ce^r)$  for all  $r > 0$ .*

Note that the S-balls  $B_{\Phi_0}^S(r)$  and  $V^S(r)$  relate actually to the base manifold  $\mathbb{M}$  (and to the initial frame  $\Phi_0 \in \mathbb{O}_{m_0}\mathbb{M}$ ) only, not to the bundle  $\mathbb{OM}$ . We noticed indeed in Section 4 that the S-length of a curve in  $\mathbb{M}$  is the Euclidian length of its anti-development in  $T_{\gamma_0}\mathbb{M}$ .

**PROOF** – By the definitions in Sections 4 and 5.2, the b-distance from  $\Phi_0$  to any  $\Phi \in \mathbb{OM}$  is not larger than  $\mathcal{T}_{\Phi_0}(\Phi)$ :  $B_r \subset B^b(\Phi_0; r)$ , where  $B^b$  denotes the ball relating to the Schmidt b-metric. Vertically, that is to say in the frame  $\tau_{0 \rightarrow s}^\gamma(\Phi_0)$  parallelly transported along a minimizing curve  $\gamma$ , the maximal hyperbolic distance reached by the velocity component  $\dot{m}_s$  of  $\gamma_s$  is  $s$ , which is responsible for a maximal vertical volume  $\mathcal{O}(e^{(d-1)r})$ .

Having accelerated till reaching a maximal velocity  $\mathcal{O}(e^r)$ , a minimizing curve in  $B^b(\Phi_0; r)$  can perform a maximal horizontal displacement  $\mathcal{O}(e^r)$ . Hence we have

$$B_{\Phi_0}^S(r) \subset \pi_0(B^b(\Phi_0; r)) \subset B_{\Phi_0}^S(\mathcal{O}(e^r)), \text{ whence } \text{VOL}(B_r) \leq C e^{(d-1)r} \times V^S(Ce^r). \quad \triangleright$$

Applying Proposition 24 to the volume growth integral condition of Theorem 11 yields instead of (5.1), in the case of a bounded  $\Theta$ :  $\int_0^\infty \frac{r dr}{r + \log V^S(e^r)} = \infty$ . Now, owing to the increase of  $(r \mapsto V^S(e^r))$ , discretizing, and distinguishing whether there are infinitely many  $n$  such that  $\log V^S(e^n) \leq n$  or not, we see easily that this is equivalent to  $\int_0^\infty \frac{r dr}{\log V^S(e^r)} = \infty$ . Hence we get the following, somehow more similar to the Grigor'yan's criterion than Condition (5.1) of Theorem 11 (even in the case of a constant  $\Theta$ ).

**COROLLARY 25.** *Let  $(\mathbb{M}, g)$  be a strongly causal Lorentz manifold satisfying the Busemann's assumption, and the volume growth condition:  $\int_0^\infty \frac{r dr}{\log V^S(e^r)} = \infty$ .*

*Then  $\mathbb{M}$  is stochastically complete; precisely, for  $\Theta$  bounded, the  $\Theta$ -diffusion does not explode.*



This volume growth integral criterion does not depend on the chosen origin  $\Phi_0 \in \mathbb{O}\mathbb{M}$ . Contrary to Proposition 23, it relies on the horizontal expansion and not on the vertical one.

This criterion does not apply to Gödel's universe, for which  $\log V^S(e^r)$  is of order  $e^r$ . Recall that in this case, our first criterion is better, according to Corollary 7. On the contrary it applies for example to Lorentz manifolds which are topologically  $\mathbb{R}^{1+d}$  and have a pseudo-metric  $g$  such that  $g, g^{-1}$ , and the first order derivatives of  $g$  be bounded, since then  $\log V^S(e^r)$  will be of the same order (i.e.  $\mathcal{O}(r)$ ) as in the Minkowski case.

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