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LOCALIZATION FOR THE RANDOM DISPLACEMENT MODEL

FRÉDÉRIC KLOPP[1], MICHAEL LOSS[2], SHU NAKAMURA[3], AND GÜNTER STOLZ[4]

Abstract

We prove spectral and dynamical localization for the multi-dimensional random displacement model near the bottom of its spectrum by showing that the approach through multiscale analysis is applicable. In particular, we show that a previously known Lifshitz tail bound can be extended to our setting and prove a new Wegner estimate. A key tool is given by a quantitative form of a property of a related single-site Neumann problem which can be described as "bubbles tend to the corners".

1. Introduction

We consider the *random displacement model* (RDM), a random Schrödinger operator

$$H_\omega = -\Delta + V_\omega \tag{1}$$

in $L^2(\mathbb{R}^d)$, $d \geq 1$, where the random potential has the form

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^d} q(x - i - \omega_i). \tag{2}$$

This models a random perturbation of the periodic potential $\sum_i q(x-i)$, where the single-site terms sit at exact lattices sites $i \in \mathbb{Z}^d$. The parameter $\omega = (\omega_i)_{i \in \mathbb{Z}^d}$ describes a configuration of random displacement vectors $\omega_i \in \mathbb{R}^d$. Before entering a more thorough discussion of background and assumptions, let us state the main result of our work in a simple non-trivial special case:

Theorem 1.1. *Suppose that $d \geq 2$ and that $q \in C_0^\infty(\mathbb{R}^d)$ is real-valued, sign-definite, rotation symmetric and $\operatorname{supp} q \subset \{x : |x| \leq r\}$ for some $r < 1/4$. Also assume that $(\omega_i)_{i \in \mathbb{Z}^d}$ are i.i.d. $\mathbb{R}^d$-valued random variables, uniformly distributed on $[-d_{max}, d_{max}]^d$ where $d_{max} = 1/2 - r$.*

Then H_ω is spectrally and dynamically localized at energies near the bottom of its almost sure spectrum.

The exact meaning of the latter will be recalled below.

The RDM represents a natural way to model a solid with structural disorder. It can be considered as intermediate between the *Anderson model*, which has no structural disorder and randomness instead appears in the form of coupling constants at the single site terms, and the *Poisson model*, where the structure of the medium is entirely dissolved by placing single-site scatterers at the points of a Poisson process. This point of view has recently been supported by an investigation of the integrated density of states of the RDM in [9] and [10].

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It is physically expected that multi-dimensional random Schrödinger operators such as the three models mentioned above should exhibit *localization* at energies near the bottom of the spectrum, while extended states should exist at high energy. For mathematicians the latter remains an open problem, while localization at low energy for the Anderson model (both in the continuum and in the original lattice setting) and, with more effort and more recently, for the Poisson model has been proven rigorously. The references given in Section 5.1 below may be used as a starting point into the enormous literature on localization for the Anderson model. For the multi-dimensional Poisson model localization at low energy has been proven, separately for the case of positive and negative single-site potentials, in [11] and [12].

A localization proof for the RDM provides additional challenges and does not follow from the methods alone which have led to proofs for the Anderson and Poisson models. The main reason for this is that the RDM does not have any obvious monotonicity properties with respect to the random parameters. Such properties are frequently used in an essential way in the theory of the Anderson model and, to some extend, can also be exploited for the Poisson model.

This becomes apparent most immediately when attempting to characterize the bottom of the spectrum for these models. For the Anderson model with sign-definite potential q this corresponds to choosing all couplings minimal (if q is positive) or maximal (if q is negative), respectively. For the Poisson model the bottom of the spectrum is 0 if q is positive (due to large regions devoid of any Poisson points) and $-\infty$ if q is negative (due to dense clusters of Poisson points).

Identifying a mechanism which characterizes the bottom of the spectrum, a crucial preliminary step towards the localization question, poses a non-trivial challenge for the RDM. It is not at all obvious which configurations $\omega = (\omega_i)$ of the displacements should characterize minimal energy.

Far more than a characterization of the spectral minimum is needed for a proof of localization. From the theory of the Anderson model it is well known that sufficient ingredients are smallness of the integrated density of states (IDS) at the bottom of the spectrum (e.g. in the form of *Lifshitz tails*), and sensitivity of the spectrum to the random parameters (for example in the form of *spectral averaging* or *Wegner estimates*). The usual approaches to verifying both of these ingredients make heavy use of monotonicity properties as well, providing further obstacles to a localization proof for the RDM, where such properties are not apparent.

There are two previous works in which localization properties of modified versions of the RDM (1), (2) have been shown. In [19], Klopp considered a semiclassical version $-h^2\Delta + V_\omega$ of (1) and was able to show the existence of a localized region in the spectrum for sufficiently small value of the semiclassical parameter h. In this regime neither Lifshitz tails nor an exact characterization of the spectral minimum are needed for the localization proof. More recently, Ghribi and Klopp [14] considered an RDM of the form (1) with an additional periodic background potential. For a generic non-zero choice of the latter and sufficiently small displacement vectors ω_i, they use first order perturbation arguments to recover monotonicity properties which lead to Lifshitz tails as well as a Wegner estimate, and thus localization.

Our goal here is to prove localization for the model (1), (2) without working in the semiclassical regime or modifying the background. This means that we can not hope to reveal any monotonicity properties by exclusively using perturbative arguments, at least not easily and not with first order perturbation theory.

We will make central use of symmetry properties. In particular, we will assume throughout that the real-valued single-site potential q is reflection symmetric in each coordinate, i.e.

$$q(\dots, x_{k-1}, -x_k, \dots, x_{k+1}, \dots) = q(\dots, x_{k-1}, x_k, x_{k+1}, \dots), \quad k = 1, \dots, d. \tag{3}$$

Due to the use of some higher order perturbation theory we will need some smoothness of q and for convenience assume that

$$q \in C^\infty(\mathbb{R}^d), \tag{4}$$

which could be substantially weakened. We also assume that

$$\operatorname{supp} q \subset [-r, r]^d \quad \text{for some } r < 1/4. \tag{5}$$

The latter is best understood in conjunction with the following assumption on the displacement parameters $\omega = (\omega_i)_{i\in\mathbb{Z}^d}$: They are i.i.d. $\mathbb{R}^d$-valued random variables, distributed according to a measure μ satisfying

$$\mathcal{C} \subset \operatorname{supp} \mu \subset [-d_{max}, d_{max}]^d, \tag{6}$$

where $d_{max} = \frac{1}{2} - r$. Here $\mathcal{C} := \{(\pm d_{max}, \dots, \pm d_{max})\}$ denotes the 2^d corners of the cube $[-d_{max}, d_{max}]^d$. For the proof of our main result, Theorem 1.2 below, we will need some smoothness of the distribution of μ. A convenient assumption is that μ has a density ρ, i.e. $\mu(B) = \int_B \rho(\omega_0)\, d\omega_0$ for all Borel subsets B of $[-d_{max}, d_{max}]^d$, and that

$$\rho \in C^1([-d_{max}, d_{max}]^d). \tag{7}$$

However, as will be discussed in Section 5, our proofs work for considerably more general distributions. In particular, our proof of Theorem 1.2 below only uses that μ has a C^1-density in a neighborhood of the corners $\mathcal{C}$ and can be arbitrary away from this neighborhood.

By (5) and (6), the centers of the "bubbles" $q(x - i - \omega_i)$ can move all the way into the corners $i + \mathcal{C}$ of $i + [-d_{max}, d_{max}]^d$, and the supports of the bumps stay, up to touching, mutually disjoint. See Figure 1 for a typical configuration. Note that only $r < 1/2$ would be needed to do this in a non-trivial way (give the bubble space to move), but that we will need $r < 1/4$ for technical reasons to make use of results in [2] and [21], see the discussion preceding Corollary 3.4 below.

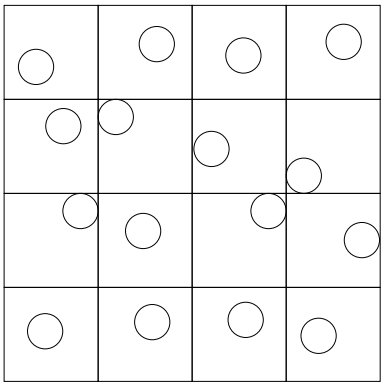

FIGURE 1. A typical configuration

The random operator H_ω is ergodic with respect to shifts in $\mathbb{Z}^d$ and thus, by the general theory of ergodic operators, its spectrum is almost surely deterministic, i.e. there is a $\Sigma \subset \mathbb{R}$ such that

$$\sigma(H_\omega) = \Sigma \quad \text{almost surely.}$$

Under the above assumptions it was shown in [1] that among all configurations ω a configuration with lowest spectral minimum is given by $\omega^* = (\omega_i^*)_{i\in\mathbb{Z}^d}$ where

$$\omega_i^* = ((-1)^{i_1} d_{max}, \dots, (-1)^{i_d} d_{max}), \quad i = (i_1, \dots, i_d) \in \mathbb{Z}^d. \tag{8}$$

As discussed in [1], under the condition (6) this also characterizes the minimum of the almost sure spectrum of the RDM, i.e.

$$E_0 := \inf \Sigma = \inf \sigma(H_{\omega^*}).$$

In the configuration ω^* the single site potentials in V_{ω^*} form densest possible clusters where 2^d neighboring bumps move into adjacent corners of their unit cells, see Figure 2.

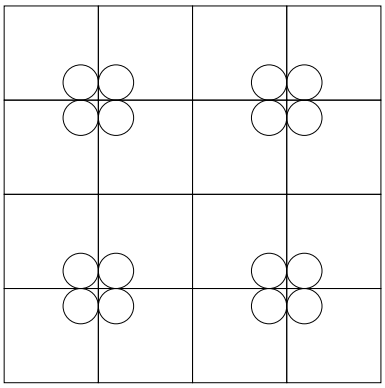

FIGURE 2. The minimizing configuration

Thus, for the set of assumptions listed above, we have an answer to the preliminary question of characterizing $E_0 = \inf \Sigma$. For a proof of localization we will need much more information. In particular, we will need quantitative bounds on the probability that other configurations have spectral minimum close to E_0. This will require one more condition on q.

To state this condition, we need to introduce Neumann operators where a single bump is placed into a unit cell at varying position. For this, and for later, set

$$\Lambda_r = \Lambda_r(0) = \left(-\frac{r}{2}, \frac{r}{2}\right)^d, \quad \Lambda_r(x) = \Lambda_r + x, \quad \chi_x = \chi_{\Lambda_1(x)},$$

the characteristic function of the unit cube centered at x.

For $a \in [-d_{max}, d_{max}]^d$ let

$$H_{\Lambda_1}^N(a) = -\Delta + q(x-a) \quad \text{on } L^2(\Lambda_1) \tag{9}$$

with Neumann boundary condition. Finally, let

$$E_0(a) = \inf \sigma(H_{\Lambda_1}^N(a))$$

be the lowest eigenvalue of $H_{\Lambda_1}^N(a)$. Note that, by symmetry of q, $E_0(a)$ is symmetric with respect to all the coordinate hyperplanes $a_k = 0$, $k = 1, \dots, d$.

In [1] the following alternative was established (only requiring boundedness of q, not smoothness):

(i) Either $E_0(a)$ is strictly maximized at $a = 0$ and strictly minimized in the corners $\mathcal{C}$ of $[-d_{max}, d_{max}]^d$,
(ii) or $E_0(a)$ is identically zero, and the ground state of $H^N_{\Lambda_1}(a)$ is constant near the boundary of Λ_1.

From this and the symmetries of q it is easy to see, e.g. [1], that ω^* given by (8) is a spectrally minimizing configuration and that the almost sure spectral minimum E_0 of H_ω is given by the minimum value of $E_0(a)$, i.e. its value at $a \in \mathcal{C}$.

While it is possible to construct non-vanishing q where (ii) holds, e.g. our remarks in Section 5.2, alternative (i) is the generic case. It holds if $q \neq 0$ is sign-definite (since then the ground state energy 0 of the Neumann Laplacian $-\Delta^N_\Lambda$ must be shifted up or down), but also for generic sign-indefinite q.

We are now able to state our main result on localization. Here, for a self-adjoint operator H and Borel function g we define $g(H)$ by the functional calculus.

Theorem 1.2. *Assume that $d \geq 2$, ω and q satisfy (3) to (7), and that $E_0(a)$ does not vanish identically in $a \in [-d_{max}, d_{max}]^d$.*

Then there exists $\delta > 0$ such that H_ω almost surely has pure point spectrum in $I = [E_0, E_0+\delta]$ with exponentially decaying eigenfunctions. Moreover, H_ω is dynamically localized in I, in the sense that for every $\zeta < 1$ there exist $C < \infty$ and $\eta > 0$ such that

$$\mathbb{E}\left(\sup_{|g|\leq 1} \|\chi_x g(H_\omega)\chi_I(H_\omega)\chi_y\|_2^2\right) \leq Ce^{-\eta|x-y|^\zeta} \tag{10}$$

for all $x, y \in \mathbb{Z}^d$. Here, the supremum is taken over all Borel functions $g : \mathbb{R} \to \mathbb{C}$ which satisfy $|g| \leq 1$ pointwise.

The proof of Theorem 1.2 proceeds via multiscale analysis and our main task will be to establish the two main ingredients into the multiscale analysis, i.e. a smallness bound on the probability that finite volume restrictions of H_ω have low lying eigenvalues (related to Lifshitz tails of the integrated density of states) and a Wegner estimate.

It is in the proof of these two ingredients where new ideas are needed. In both proofs we will use that alternative (i) can be strengthened if some smoothness is assumed for q: in this case it can be shown that the first partial derivatives of $E_0(a)$ are non-zero in all directions away from the symmetry planes. In particular, at its strict minima in the corners $\mathcal{C}$, the function $E_0(a)$ is not flat as it has non-vanishing gradient. The proof of this result, given in Section 2, starts from a second-order perturbation theory formula.

This can be considered as the crucial monotonicity property which makes the localization proof work. In the spectrally minimizing configuration ω^* of the RDM all bubbles sit in corners. Non-vanishing of the gradient of $E_0(a)$ in the corners will allow to gain quantitative control on how close the ground state energies for other configurations are to E_0.

First, in Section 3, this will lead to a Lifshitz tail bound with a proof which is based on an argument in [21]. This also uses the fact that under alternative (i) the configuration ω^* is, up to translation, the unique *periodic* configuration with spectral minimum E_0, a result established in [2], where it is also shown that this is *not* true for $d = 1$. In [21] it was shown that this uniqueness result leads to a Lifshitz tail bound for the IDS near E_0 if the distribution of the ω_i is discrete and contains all corners $\mathcal{C}$. In fact, for technical reasons the bound obtained

there is weaker than the "classical" Lifshitz tail, but it is strong enough for a localization proof (if one also has a Wegner estimate). In Section 3 we will show how non-vanishing of $\nabla E_0(a)$ in the corners can be used to get same bound on the IDS under the assumptions considered here, i.e. with distribution satisfying (6).

What allows us to push the localization argument through is that under assumption (6) we can also prove a Wegner estimate, see Section 4. This will again use that $\nabla E_0(a) \neq 0$ at the corners, which will provide us with a measure for how much the ground state energy of finite volume restrictions of H_ω is pushed upwards if the bubbles $q(x - i - \omega_i)$ move away from the corners towards the center i of $\Lambda_1(i)$. This is technically implemented in Proposition 4.3 below in form of positivity of the derivative of H_ω with respect to a suitable vector field, whose proof is close to an argument previously used in [22] for operators with random magnetic fields. Based on this result, we are able to prove a Wegner estimate by modifying an argument developed in [5] and [15] in the context of the Anderson model, which itself is a modification of the original argument due to Wegner. In particular, we obtain the correct (linear) volume dependence and can conclude Hölder continuity of the IDS as a by-product.

We finally mention that the connection between monotonicity properties of $H^N_{\Lambda_1}(a)$ and the RDM is made through surprisingly simple Neumann bracketing arguments, making crucial use of the variational characterization of the lowest eigenvalue as the minimum of the quadratic form. Thus we are able to deduce monotonicity properties of a model with infinitely many parameters from a one-parameter model. This trick, employed in Sections 3 and 4, was previously used in [20] to study the Anderson model with sign-indefinite single-site potential. It is the main reason why symmetry of the single-site potential is important for us.

Our concluding Section 5 serves two purposes. First, in Section 5.1, we briefly discuss how the Lifshitz tail bound and Wegner estimate obtained here lead to a proof of spectral and dynamical localization via multi-scale analysis. In Section 5.2 we mention some generalizations, related results and open problems.

2. Bubbles tend to the corners

In this section we will prove a property of the derivatives of $E_0(a)$, the ground state of the Neumann operators $H^N_{\Lambda_1}(a)$ defined in (9), which is crucial for all our later arguments. The main result of [1], i.e. that under the generic alternative (i) the function $E_0(a)$ is strictly minimized in the corners $\mathcal{C}$, may be dubbed as "bubbles tend to the corners". The seemingly small but important improvement to be shown here means that "bubbles tend to the corners at non-zero derivative".

In fact, in [1] two methods were developed to prove results of this type. One method, used to prove Theorem 1.3 in [1], applied to Neumann operators on rectangular domains as considered here and is mostly based on exploiting symmetries of the domain and the potential. A second, very different, method was developed in [1] to show the phenomenon that "bubbles tend to the boundary" for Neumann operators on general smooth domains and with smooth potentials, see the proof of Theorem 1.4 in [1]. What we do here is to apply the second method in the setting of rectangular domains. That "smooth methods" apply to rectangles is possible due to our use of Neumann boundary conditions, which allow to get smooth extensions by reflection. Our argument works in form of a boot-strap: Once we know from Theorem 1.3 of [1] that bubbles tend to the corners (having exploited symmetries of the problem) we can add perturbative arguments (exploiting smoothness) to get the added non-zero derivative property.

2.1. **Basic smoothness properties.** Let $D \subset \mathbb{R}^n$ be an open rectangular parallelepiped and q be a real-valued smooth function with closed support in D, i.e. $q \in C^\infty(D)$ such that $q(x) = 0$ for x in a neighborhood of ∂D. Consider the quadratic form

$$\|\nabla u\|_2^2 + \langle u, qu\rangle$$

where we use the symbol

$$\langle u, v\rangle = \int_D \overline{u(x)} v(x) dx \ .$$

This quadratic form is a closed form on $H^1(D)$ and defines a unique self–adjoint operator

$$H := -\Delta_N + q$$

where Δ_N is the Neumann Laplacian. The eigenvalues of this operator are discrete, have finite degeneracy and tend to infinity. It will be important for us that the eigenfunctions are regular up and including the boundary, i.e., $C^\infty(\overline{D})$. We state this fact as a lemma.

Lemma 2.1. *The eigenfunctions of the operator H can be extended to a neighborhood of D where they are infinitely often differentiable.*

Proof. By reflecting the potential and eigenfunctions repeatedly across the boundary one obtains a generalized eigenfunction with the same eigenvalue on the whole space $\mathbb{R}^d$. Since the potential is C^∞ it follows, by elliptic regularity, that the generalized eigenfunctions are in $C^\infty(\mathbb{R}^d)$. □

We set for $a \in \mathbb{R}^n$

$$q_a(x) = q(x - a) \ .$$

The set of points a for which the support of q_a is a subset of D is denoted by G. This is an open rectangle as well. We shall denote the eigenvalues of

$$H_a := -\Delta_N + q_a$$

in increasing order and counted with multiplicity by

$$E_n(a) \ , \ n = 0, 1, 2, \dots$$

and the corresponding real-valued normalized eigenfunctions as

$$u_n(x, a) \ .$$

In what follows we often denote the extension of the function to a larger set by the same symbol.

Lemma 2.2 (**Differentiability of eigenvalue and eigenfunction).** *The eigenvalue $E_0(a)$ as well as the eigenfunction $u_0(\cdot, a)$ are (as an $L^2(D)$-valued function) infinitely often differentiable in a neighborhood of G. In particular these two functions are in $C^\infty(\overline{G})$.*

Proof. From the formula, which holds for all z in the resolvent set of H_a,

$$(H_a - z)^{-1} - (H_{a'} - z)^{-1} = (H_a - z)^{-1} \left[q_{a'} - q_a\right] (H_{a'} - z)^{-1} \tag{11}$$

we see that the non-degenerate eigenvalue $E_0(a)$ is a continuous function. Note that this formula holds for all positions of the potential, in particular, the support of the potential does not have to be in D. Thus $E_0(a)$ is strictly separated from the rest of the spectrum locally uniformly in a. Hence there is a circle C with center $E_0(a)$ which is strictly separated from

the rest of the spectrum, locally uniformly in a. Now, repeated use of (11) shows that the projection onto the eigenfunction $u_0(x,a)$,

$$P_0(a) = \frac{1}{2\pi i} \oint_C (H_a - z)^{-1} dz \ ,$$

can be arbitrarily often differentiated with respect to a. The eigenvalue equation

$$(H_a - z)^{-1} P_0(a) = (E_0(a) - z)^{-1} P_0(a)$$

now shows that $E_0(a)$ in turn can also be differentiated as often as we please. □

Given the above smoothness properties and the non-degeneracy of $E_0(a)$ we can derive the following first and second order perturbation formulas for $E_0(a)$,

$$\partial_{a_j} E_0(a) = -\langle u_0, (\partial_{x_j} q_a) u_0 \rangle, \tag{12}$$

and

$$\partial_{a_j}^2 E_0(a) = \langle u_0, (\partial_{x_j}^2 q_a) u_0 \rangle - 2 \sum_{k \neq 0} \frac{\langle u_0, (\partial_{x_j} q_a) u_k \rangle^2}{E_k - E_0}, \tag{13}$$

for $j = 1, \dots, d$. This is done in complete analogy to the better known case of non-degenerate eigenvalues of operators of the form $A + \lambda B$, with (12) corresponding to the Feynman-Hellmann formula, see e.g. Section 2.3 of [1].

2.2. **Second order perturbation theory.** Making notations more explicit, we now write

$$D = (\alpha_1, \beta_1) \times D',$$

where

$$D' = (\alpha_2, \beta_2) \times \ldots \times (\alpha_d, \beta_d) \subset \mathbb{R}^{d-1}.$$

Then $G = I \times G'$, with an open interval $I = (-\delta_1, \delta_2)$, $\delta_1 > 0$, $\delta_2 > 0$, and an open rectangular parallelepiped $G' \subset \mathbb{R}^{d-1}$.

We will consider the dependence of $E_0(a_1, \dots, a_d)$ on the first variable $a_1 \in \overline{I}$ at fixed value of $(a_2, \dots, a_d) \in \overline{G'}$. Throughout this subsection we will abuse notation and write $q_{a_1} := q_a$, $H_{a_1} := -\Delta_N + q_{a_1}$, as well as $E_n(a_1)$ and $u_n(x, a_1)$ for its eigenvalues and eigenfunctions.

The following lemma provides us with a differential equation for the eigenvalue $E_0(a_1)$.

Lemma 2.3. *The ground state energy satisfies the equation*

$$E_0'' - 4\langle u_0, \partial_1 u_0 \rangle E_0' = -2 \sum_{k \neq 0} \frac{B(u_k, \partial_1 u_0)^2}{E_k - E_0} \ . \tag{14}$$

Here $B(u,v) := \langle u, \Delta v \rangle - \langle \Delta u, v \rangle$, E_0' and E_0'' refer to a_1-derivatives of E_0, and $\partial_1 u_0$ refers to the spatial derivative in x_1-direction.

Proof. The proof is essentially the same as in [1], and we present it with the necessary modifications. Differentiating the eigenvalue equation yields

$$(\partial_1 q_{a_1}) u_0 = E_0 \partial_1 u_0 - (-\Delta + q_{a_1}) \partial_1 u_0$$

and therefore

$$\langle u_k, (\partial_1 q_{a_1}) u_0 \rangle = -(E_k - E_0) \langle u_k, \partial_1 u_0 \rangle + B(u_k, \partial_1 u_0) \ . \tag{15}$$

Plugging this into the second order perturbation formula (13) (for $j = 1$) gives

$$\begin{aligned} E_0'' &= \langle u_0, (\partial_1^2 q_{a_1}) u_0\rangle - 2\sum_{k\neq 0}\left[(E_k - E_0)\langle \partial_1 u_0, u_k\rangle^2 - 2\langle \partial_1 u_0, u_k\rangle B(u_k, \partial_1 u_0)\right] \\ &\quad - 2\sum_{k\neq 0}\frac{B(u_k, \partial_1 u_0)^2}{E_k - E_0} \end{aligned}$$

which, using (15) once more, can be rewritten as

$$\begin{aligned} E_0'' &= \langle u_0, (\partial_1^2 q_{a_1}) u_0\rangle + 2\sum_{k\neq 0}[B(u_k, \partial_1 u_0) + \langle u_k, (\partial_1 q_{a_1}) u_0\rangle]\langle u_k, \partial_1 u_0\rangle \\ &\quad - 2\sum_{k\neq 0}\frac{B(u_k, \partial_1 u_0)^2}{E_k - E_0} \\ &= \langle u_0, (\partial_1^2 q_{a_1}) u_0\rangle + 2\sum_{k}[B(u_k, \partial_1 u_0) + \langle u_k, (\partial_1 q_{a_1}) u_0\rangle]\langle u_k, \partial_1 u_0\rangle \\ &\quad - 2[B(u_0, \partial_1 u_0) + \langle u_0, (\partial_1 q_{a_1}) u_0\rangle]\langle u_0, \partial_1 u_0\rangle \\ &\quad - 2\sum_{k\neq 0}\frac{B(u_k, \partial_1 u_0)^2}{E_k - E_0} \\ &= \langle u_0, (\partial_1^2 q_{a_1}) u_0\rangle + 2\langle \partial_1 u_0, (\partial_1 q_{a_1}) u_0\rangle + 2\sum_{k} B(u_k, \partial_1 u_0)\langle u_k, \partial_1 u_0\rangle \\ &\quad - 2[B(u_0, \partial_1 u_0) + \langle u_0, (\partial_1 q_{a_1}) u_0\rangle]\langle u_0, \partial_1 u_0\rangle \\ &\quad - 2\sum_{k\neq 0}\frac{B(u_k, \partial_1 u_0)^2}{E_k - E_0}\ , \end{aligned}$$

where finally the completeness relation of the u_k was used. Integration by parts shows that

$$\langle u_0, (\partial_1^2 q_{a_1}) u_0\rangle + 2\langle \partial_1 u_0, (\partial_1 q_{a_1}) u_0\rangle = 0$$

and, using again (15) with $k = 0$, we can simplify further and get

$$\begin{aligned} E_0'' &= 2\sum_{k} B(u_k, \partial_1 u_0)\langle u_k, \partial_1 u_0\rangle - 4\langle u_0, (\partial_1 q_{a_1}) u_0\rangle\langle u_0, \partial_1 u_0\rangle \\ &\quad - 2\sum_{k\neq 0}\frac{B(u_k, \partial_1 u_0)^2}{E_k - E_0}\ . \end{aligned} \tag{16}$$

Consider

$$\sum_{k} B(u_k, \partial_1 u_0)\langle u_k, \partial_1 u_0\rangle = \sum_{k}[\langle u_k, \Delta\partial_1 u_0\rangle - \langle \Delta u_k, \partial_1 u_0\rangle]\langle u_k, \partial_1 u_0\rangle\ . \tag{17}$$

Since $u_0 \in C^\infty(\overline{D})$ we know that $\Delta\partial_1 u_0$ is square integrable and hence

$$\sum_{k}\langle u_k, \Delta\partial_1 u_0\rangle\langle u_k, \partial_1 u_0\rangle = \langle \partial_1 u_0, \Delta\partial_1 u_0\rangle\ .$$

The second term of (17) we write as

$$\sum_{k}\langle[-\Delta + q_{a_1}]u_k, \partial_1 u_0\rangle\langle u_k, \partial_1 u_0\rangle - \sum_{k}\langle q_{a_1} u_k, \partial_1 u_0\rangle\langle u_k, \partial_1 u_0\rangle\ .$$

The second sum equals $\langle \partial_1 u_0, q_{a_1} \partial_1 u_0 \rangle$ while the first sum is

$$\begin{aligned} \sum_k E_k \langle u_k, \partial_1 u_0 \rangle^2 &= \sum_k (E_k - E_0) \langle u_k, \partial_1 u_0 \rangle^2 + E_0 \|\partial_1 u_u\|^2 \\ &= \langle (H_{a_1} - E_0)^{1/2} \partial_1 u_0, H_{a_1} - E_0)^{1/2} \partial_1 u_0 \rangle + E_0 \|\partial_1 u_0\|^2 \\ &= \int_D [|\nabla \partial_1 u_0|^2 + q_{a_1} (\partial_1 u_0)^2] \, dx \end{aligned} \tag{18}$$

by Kato's form representation theorem [17], since $\partial_1 u_0$ is in the form domain. Collecting terms we find from (17) that

$$\sum_k B(u_k, \partial_1 u_0) \langle u_k, \partial_1 u_0 \rangle = [\langle \partial_1 u_0, \Delta \partial_1 u_0 \rangle + \|\nabla \partial_1 u_0\|^2] \, .$$

Up to this moment we just have copied the proof of [1] (or, more precisely, a "fixed direction" version thereof). Applying Green's identity we find that

$$\sum_k B(u_k, \partial_1 u_0) \langle u_k, \partial_1 u_0 \rangle = \int_D \nabla \cdot (\partial_1 u_0 \nabla \partial_1 u_0) \, dx \tag{19}$$

which equals

$$\sum_j \left[\int_{S_j} (\partial_1 u_0) \partial_j \partial_1 u_0 dS - \int_{T_j} (\partial_1 u_0) \partial_j \partial_1 u_0 dS \right] \, , \tag{20}$$

where S_j and T_j are the faces of D perpendicular to the j direction. Since $\partial_j u_0 \equiv 0$ on S_j and T_j we find that the sum can be restricted to the indices $j \neq 1$. Since $\partial_j \partial_1 u_0 = \partial_1 \partial_j u_0$ and $\partial_j u_0 \equiv 0$ on S_j and T_j, the expression (20) vanishes. After substituting this and the first order perturbation formula (12) for $j = 1$ into (16) we arrive at (14).

□

Lemma 2.4. *Assume that the right hand side of (14) vanishes for some $a_{1,0} \in \overline{I}$. Then $E_0(a_{1,0}) = E_0'(a_{1,0}) = 0$ and $\partial_j \partial_1 u_0(\cdot, a_{1,0}) = 0$ for all $j = 1, \dots, d$ on all faces of D.*

Moreover, $u_0(\cdot, a_{1,0})$ is constant on the right and left faces S_1 and T_1 of D.

It will follow in Theorem 2.5 below that $u_0(\cdot, a_{1,0})$ takes the same value on S_1 and T_1.

Proof. The vanishing of the right side of (14) means that

$$\langle \Delta u_k, \partial_1 u_0 \rangle - \langle u_k, \Delta \partial_1 u_0 \rangle = 0 \, . \tag{21}$$

Using the eigenvalue equation $-\Delta u_k + q_{a_{1,0}} u_k = E_k u_k$ we can transform this into

$$(E_k - E_0) \langle u_k, \partial_1 u_0 \rangle = -\langle u_k, (\partial_1 q_{a_{1,0}}) u_0 \rangle \, . \tag{22}$$

Now pick any function $f \in C^\infty(D)$ and use the previous identity to write

$$\sum_k (E_k - E_0) \langle u_k, \partial_1 u_0 \rangle \langle f, u_k \rangle = -\sum_k (u_k, \langle \partial_1 q_{a_{1,0}}) u_0 \rangle \langle f, u_k \rangle + \langle u_0, \partial_1 q_{a_{1,0}} u_0 \rangle (f, u_0) \, . \tag{23}$$

Since the functions u_k, $k = 0, 1, 2, \dots$ form an orthonormal basis for $L^2(D)$ we can simplify these expressions and obtain

$$\sum_k \langle u_k, (\partial_1 q_{a_{1,0}}) u_0 \rangle \langle f, u_k \rangle = \langle f, (\partial_1 q_{a_{1,0}}) u_0 \rangle,$$

and, similar to (18), using that f as well as $\partial_1 u_0$ are in the form domain,

$$\sum_k (E_k - E_0)\langle u_k, \partial_1 u_0\rangle\langle f, u_k\rangle = \langle \nabla f, \nabla(\partial_1 u_0)\rangle + \langle f, q_{a_{1,0}}\partial_1 u_0\rangle - E_0\langle f, \partial_1 u_0\rangle \ .$$

Plugging the latter two identities back into (23) we arrive at

$$\langle \nabla f, \nabla(\partial_1 u_0)\rangle + \langle f, \partial_1(q_{a_{1,0}} u_0)\rangle - E_0\langle f, \partial_1 u_0\rangle = \langle u_0, (\partial_1 q_{a_{1,0}})u_0\rangle\langle f, u_0\rangle \ . \tag{24}$$

Now pick $f \in C_c^\infty(D)$ so that we can integrate by parts without boundary terms and get

$$\langle f, \partial_1(-\Delta u_0 + q_{a_{1,0}} u_0 - E_0 u_0)\rangle = \langle u_0, (\partial_1 q_{a_{1,0}})u_0\rangle\langle f, u_0\rangle \ . \tag{25}$$

Since the left side vanishes for all $f \in C_c^\infty(D)$ and the latter are dense in $L^2(D)$, we must necessarily have that

$$\langle u_0, (\partial_1 q_{a_{1,0}})u_0\rangle = 0 \ ,$$

i.e., $E_0'(a_{1,0}) = 0$ by (12). Thus we have for all $f \in C^\infty(D)$,

$$\langle \nabla f, \nabla(\partial_1 u_0)\rangle + \langle f, \partial_1(q_{a_{1,0}} u_0)\rangle - E_0\langle f, \partial_1 u_0\rangle = 0 \ . \tag{26}$$

Doing the same integration by parts as above, this time with boundary terms, yields

$$\sum_j \left[\int_{S_j} (\partial_j \partial_1 u_0) f dS - \int_{T_j} (\partial_j \partial_1 u_0) f dS\right] = 0 \tag{27}$$

for all $f \in C^\infty(D)$. This means that

$$\partial_j \partial_1 u_0 = 0 \tag{28}$$

pointwise on S_j and T_j for all $j = 1, \dots, d$. In particular, $\partial_1^2 u_0(x, a_{0,1}) \equiv 0$ on S_1 and T_1. Since the potential vanishes on the faces S_1 and T_1 we find that u_0 satisfies the equation

$$-\Delta' u_0 = -\sum_{j=2}^d \frac{\partial^2 u_0}{\partial x_j^2} = E_0(a_0) u_0 \tag{29}$$

on S_1 and T_1. The function u_0 is smooth up to and including the boundary of D, in particular it is a smooth function on the faces of D. Consider the $d-2$ dimensional 'edge' where S_1 and S_2, say, meet. The gradient of u_0 at this intersection must be of the form

$$\nabla u_0 = (0, 0, \partial_3 u_0, \cdots \partial_d u_0) \ . \tag{30}$$

Hence, u_0 is an eigenfunction of $-\Delta'$ on S_1 and T_1 with a Neumann condition on the boundary. Since u_0 has a fixed sign, $E_0(a_{1,0})$ must be the lowest eigenvalue of the Neumann Laplacian on S_1 and T_1 and hence

$$E_0(a_{1,0}) = 0 \ , \tag{31}$$

and u_0 is constant on S_1 as well as on T_1. □

Theorem 2.5. *Assume that the right side of (14) vanishes for some $a_{1,0} \in \overline{I}$. Then $E_0(a_1) = 0$ identically in I and for every $a_1 \in \overline{I}$ the eigenfunction $u_0(x, a_1)$ is constant on the two rectangles $L := [\alpha_1, \alpha_1 + \delta_1 + a_1] \times \overline{D'}$ and $R := [\beta_1 - \delta_2 + a_1, \beta_1] \times \overline{D'}$, i.e. the parts of D to the left and right of $\operatorname{supp} q_{a_1}$.*

Proof. Let us first assume that $a_{1,0} \in (-\delta_1, \delta_2)$. Then the faces S_1 and T_1 of D are in the complement of the support of $q_{a_{1,0}}$.

Pick a point on S_1. We may call this point the origin. By the reflection argument already used in the proof of Lemma 2.1 the function $u_0(x, a_{1,0})$ is harmonic in a neighborhood N of 0. Thus it has a convergent power series expansion

$$\sum_\alpha c_\alpha x^\alpha \tag{32}$$

which can also be written as

$$\sum_{k=0}^\infty x_1^k \sum_{\alpha'} c_{(k,\alpha')} x^{\alpha'} = \sum_{k=0}^\infty x_1^k P_k(x') \ , \tag{33}$$

where $x' = (x_2, \dots, x_d)$. Since u is harmonic we find the recursion

$$\Delta' P_k(x') + (k+1)(k+2) P_{k+2}(x') = 0 \ , k = 0, 1, 2, \dots \ . \tag{34}$$

By Lemma 2.4 we know that $u_0(0, x', a_{1,0})$ is constant and we find that $P_0(x')$ is constant, too, and thus $P_k(x') = 0$ for $k \geq 2$, even. Since $\partial_1 \partial_j u_0(0, x') = 0$ for $j = 2, \dots, d$ we find that $\partial_j P_1(x') = 0$ for $j = 2, \dots, d$ and hence $P_1(x')$ is constant and thus on account of (34), $P_k(x') = 0$ for all $k \geq 2$. Thus $u_0(x, a_{0,1}) = c + dx_1$ near 0, where c, d are constants. However, as $u_0(x, a_{0,1})$ satisfies Neumann boundary conditions, we must have $d = 0$. This implies that $u_0(x, a_{0,1}) = c$ on $[\alpha_1, \alpha_1 + \delta_1 + a_{1,0}] \times \overline{D'}$, where $q_{a_{1,0}} = 0$.

In the same way, now starting with a point on the face T_1 it is shows that $u_0(x, a_{0,1})$ is constant on $[\beta_1 - \delta_2 + a_{1,0}, \beta_1] \times \overline{D'}$.

If we translate the potential $q_{a_{1,0}}(x) \to q_{a_{1,0}}(x - a_1) = q(x - a_{1,0} - a_1)$ the translated function $v(x) = u_0(x - a_1, a_{1,0})$ is an eigenfunction of the Schrödinger equation $H_{a_{1,0}-a_1} v = E_0(a_{1,0}) v$ and since v has fixed positive sign, it is the ground state. Thus the eigenvalue $E_0(a_1) = 0$ identically in $a_1 \in \overline{I}$.

If $a_{1,0} = \delta_2$, then the above argument works to show constancy of $u_0(x, a_{1,0})$ on $[\alpha_1, \alpha_1 + \delta_1 + a_{1,0}] \times \overline{D'}$, while $[\beta_1 - \delta_2 + a_{1,0}, \beta_1] \times \overline{D'}$ degenerates to the face T_1, where constancy already follows from Lemma 2.4. From here the proof is completed as before, using that $u_0(x, a_{1,0})$ also satisfies Neumann conditions on T_1. The case $a_{1,0} = -\delta_1$ is done in the same way, with the role of the faces reversed.

This proves the theorem. □

So far we have had no need to require any symmetry of the potential q. But this will be required now in order to make use of a result from [1], which enters our reasoning in the form of a bootstrap argument.

Thus assume now that $(\alpha_1, \beta_1) = (-s, s)$ for some $s > 0$, $\operatorname{supp} q \subset [-r, r] \times D'$ for some $0 < r < s$, and that q is reflection-symmetric with respect to the first variable, i.e. $q(x_1, x') = q(-x_1, x')$ for all $x_1 \in (-s, s)$ and $x' \in D'$. The following result does not require smoothness of q, but merely boundedness.

Theorem 2.6. *We have the following alternative: Either*

(i) $E_0(a_1)$ is strictly decreasing for $a_1 \in [0, s - r]$ (and thus, by symmetry, strictly increasing in $[-(s - r), 0]$), or

(ii) $E_0(a_1)$ vanishes identically in $a_1 \in [-(s - r), s - r]$ and $u(x; 0)$ is constant in $D \setminus ([-r, r] \times D')$ (and, by symmetry, takes the same value in both connected components of this set).

Proof. This is, line by line, the argument provided in the proof of Theorem 3.1 in [1], where the authors saw no need to state the result in its full strength. □

If $E_0(\cdot)$ is differentiable, as true for the case of smooth q considered here, one might expect that the strict monotonicity of $E_0(a_1)$ for $a_1 \neq 0$, established under alternative (i) in Theorem 2.6, corresponds to non-vanishing derivative. Using Theorem 2.6 as a first step in a bootstrap argument, we can indeed show that this is true.

Corollary 2.7. *Let q be smooth and satisfy the assumptions of Theorem 2.6. Also assume that $E_0(a_1)$ does not vanish identically. Then $E_0'(a_1) < 0$ for all $a_1 \in (0, s-r]$ and, by symmetry, $E_0'(0) = 0$ and $E_0'(a_1) > 0$ for $a_1 \in [-(s-r), 0)$.*

Proof. It follows from Theorem 2.6 that $E_0'(a_1) \leq 0$ for $a_1 \in (0, s-r]$. Suppose that $E_0'(a_{1,0}) = 0$ for one such value $a_{1,0}$. By Theorem 2.5 the right hand side of (14) is strictly negative at $a_{1,0}$, meaning that $E_0''(a_{1,0}) < 0$.

Further, for a_1 near $a_{1,0}$,

$$E_0(a_1) = E_0(a_{1,0}) + \frac{1}{2}E_0''(a_{1,0})(a_1 - a_{1,0})^2 + o(|a_1 - a_{1,0}|^3).$$

Negativity of $E_0''(a_{1,0})$ contradicts that, by Theorem 2.6, $E_0(a_1)$ is strictly decreasing near $a_{1,0}$. This completes the proof. □

We are ready to state and complete proving the strengthened version of Theorem 1.3 of [1], which will be central to all our later considerations in the localization proof.

The domain D in $\mathbb{R}^d$ may be any open rectangular parallelepiped and q a C^∞-smooth function with closed support contained in D and having the same symmetry hyper-planes as D. We define G and $E_0(a)$ for $a \in \overline{G}$ as in Section 2.1. Then G is an open rectangular parallelepiped and we may assume that its center is in the origin and $G = (-M_1, M_1) \times \ldots \times (-M_d, M_d)$.

Corollary 2.8. *Assume that $E_0(a)$ is not identically zero on G. For all $a = (a_1, \ldots, a_d) \in \overline{G}$ and all $i = 1, \ldots, d$ we have*

$$\partial_i E_0(a) \begin{cases} < 0, & \text{if } a_i > 0, \\ = 0, & \text{if } a_i = 0, \\ > 0, & \text{if } a_i < 0. \end{cases}$$

Proof. It suffices to consider $i = 1$. Fix any values $a_j \in [-M_j, M_j]$, $j = 2, \ldots, d$. By Theorem 1.3 of [1] (or, better, its proof) the function $E_0(a_1) := E_0(a_1, a_2, \ldots, a_d)$ does not vanish identically (it decreases for $a_1 > 0$ and increases for $a_1 < 0$. Thus Corollary 2.7 completes our argument. □

Open Problem: It is an open problem to show that the phenomenon "bubbles tend to the corners", i.e. results such as Theorem 3.1 in [1] or Corollary 2.8 above, also appears if the rectangular domain D is replaced by more general polyhedra, for example regular n-gons in $\mathbb{R}^2$. Again this should need a suitable symmetry assumption on the potential q, e.g. spherical symmetry. Particularly interesting would be the case of an equilateral triangle, as all other results in this paper would be applicable to this case as well. It was shown in [1] for general smooth convex domains that the minimizing position lies at the boundary.

3. Lifshitz tails

In proving smallness of the IDS of H_ω near E_0, the almost sure spectral minimum of H_ω, we will rely on a prior result of this kind for the RDM (1), (2) established in [21]. Their result requires widely the same assumptions as made above, i.e. (3), (5), (6) and that $E_0(a)$ does not vanish identically, but needs in addition that the support of the distribution μ is *finite*. This violates our assumption (7) (and every other assumption on μ which works for our proof of a Wegner estimate in Section 4).

However, based on the results from Section 2, one might expect that configurations in which all bubbles sit in corners, as dealt with in [21], constitute the worst case scenario for smallness of the IDS near E_0. We will make this rigorous here by showing that the IDS of H_ω can be bounded from above (up to a multiplicative constant) by the IDS of a modified RDM where all bubbles have been moved to the nearest corner.

More precisely, for an $a \in \overline{G} = [-d_{max}, d_{max}]^d$, let $c(a)$ denote the element of $\mathcal{C}$ closest to a. If there is more than one such point, any of them can be chosen; for the sake of definiteness, we may order the points in $\mathcal{C}$ lexicographically and chose the first in the list. For a displacement configuration $\omega = (\omega_i)_{i\in\mathbb{Z}^d} \in \overline{G}^{\mathbb{Z}^d}$ we will write $c(\omega)$ for the *closest corner configuration* given by $(c(\omega))_i = c(\omega_i)$, $i \in \mathbb{Z}^d$.

For a non-negative integer L, let $\Lambda_{2L+1} = (-L-1/2, L+1/2)^d$ and $H_{\omega,L}$ the restriction of H_ω to Λ_L with Neumann boundary conditions. Also, let $\Lambda'_{2L+1} = \mathbb{Z}^d \cap \Lambda_{2L+1}$. We will prove

Proposition 3.1. *There exists a constant $C \in (0,\infty)$ such that, in the sense of quadratic forms,*

$$H_{\omega,L} - E_0 \geq \frac{1}{C}(H_{c(\omega),L} - E_0) \tag{35}$$

for all $\omega \in \overline{G}^{\mathbb{Z}^d}$ and all $L \geq 0$.

We start by showing this for $L = 0$, i.e. on the level of the single-site operators $H^N_{\Lambda_1}(a)$. This is achieved in the following two lemmas, which separately treat the cases where the bubble is close to a corner or not close.

Recall from Section 1 that $E_0(a)$ denotes the ground state energy of $H^N_{\Lambda_1}(a)$ for $a \in \overline{G}$ and that the almost sure spectral minimum E_0 of H_ω is given by $E_0(a)$ when a is one of the corners $\mathcal{C}$ of $\overline{G}$. As a consequence of Corollary 2.8, $E_0(a)$ grows linearly in the distance of a from the nearest corner, i.e. there exists a constant $C \in (0,\infty)$ such that,

$$E_0(a) - E_0 \geq \frac{1}{C} D(a) \tag{36}$$

where $D(a) = \min_{c\in\mathcal{C}} |a-c|$.

We will use this to prove

Lemma 3.2. *There exists $C > 0$ and $\delta > 0$ such that, if $D(a) \leq \delta$, then*

$$H^N_{\Lambda_1}(a) - E_0 \geq \frac{1}{C}\left(H^N_{\Lambda_1}(c) - E_0 + |a-c|\right). \tag{37}$$

Proof. Let $a \in \overline{G}$ and pick $c \in \mathcal{C}$ such that $D(a) = |a-c|$. As q is C^1, write

$$\begin{aligned} H^N_{\Lambda_1}(a) - E_0 &= H^N_{\Lambda_1}(c) - E_0 + q(\cdot - a) - q(\cdot - c) \\ &= H^N_{\Lambda_1}(c) - E_0 + (c-a)\cdot\nabla q(\cdot - c) + o(|a-c|). \end{aligned} \tag{38}$$

Using (36), one obtains that

$$H^N_{\Lambda_1}(c) - E_0 + (c-a)\cdot\nabla q(\cdot - c) \geq \frac{1}{C}|a-c| + o(|a-c|).$$

Hence, there exists $\rho \in (0,1)$ small such that, for $\sigma \in \mathbb{S}^{d-1}$ with $c + \rho\sigma \in \overline{G}$, one has

$$H^N_{\Lambda_1}(c) - E_0 - \rho\,\sigma\cdot\nabla q(\cdot - c) \geq \rho/2C.$$

We recall Lemma 2.1 of [20]: If we suppose $A \geq 0$ and $A + B \geq c_0 > 0$, then we have $A + tB \geq \min(1/2, c_0)\cdot(A+t)$ for $t \in [0,1/2]$, since

$$A + tB = (1-t)A + t(A+B) \geq \tfrac{1}{2}A + tc_0 \geq \min(\tfrac{1}{2}, c_0)(A+t).$$

Applying this with $A = H^N_{\Lambda_1}(c) - E_0$ and $B = \rho\sigma\cdot\nabla q(\cdot - c)$, we learn there exists $C_\rho > 0$ such that for $t \in [0,1/2]$ and for $\sigma \in \mathbb{S}^{d-1}$ with $c + \rho\sigma \in \overline{G}$, one has

$$H^N_{\Lambda_1}(c) - E_0 - t\rho\sigma\cdot\nabla q(\cdot - c) \geq \frac{1}{C_\rho}\left(H^N_{\Lambda_1}(c) - E_0 + t\right).$$

Hence, in view of (38), for $|a-c| \leq \rho/2$ and $t = |a-c|/\rho$, one has

$$H^N_{\Lambda_1}(a) - E_0 \geq \frac{1}{C_\rho}\left(H^N_{\Lambda_1}(c) - E_0 + |a-c|/\rho\right) + o(|a-c|).$$

Finally, this implies that there exists $\delta > 0$ such that for $|a-c| \leq \delta$, one has (37) and completes the proof of Lemma 3.2. □

Bubbles which are not close to a corner are easier to handle and considered in

Lemma 3.3. *Fix $\delta \in (0,1)$. There exists $C = C_\delta \in (0,\infty)$ such that, for $D(a) \geq \delta$ and all $c \in \mathcal{C}$, one has*

$$H^N_{\Lambda_1}(a) - E_0 \geq \frac{1}{C}\left(H^N_{\Lambda_1}(c) - E_0 + |a-c|\right).$$

Proof. Fix $\delta \in (0,1)$. By (36) there exists $\eta > 0$ such that, for a with $D(a) \geq \delta$, $H^N_{\Lambda_1}(a) - E_0 \geq \eta$. Hence, as $|q(x-a) - q(x-c)| \geq -2\|q\|_\infty$, there exists $C > 1$ such that, for a satisfying $D(a) \geq \delta$ and all $c \in \mathcal{C}$, one has

$$(C+1)(H^N_{\Lambda_1}(a) - E_0) - (H^N_{\Lambda_1}(c) - E_0) \geq C\eta - 2\|q\|_\infty \geq \eta \geq \frac{1}{C}|a-c|.$$

Hence,

$$\begin{aligned} H^N_{\Lambda_1}(a) - E_0 &\geq \frac{1}{C+1}\left(H^N_{\Lambda_1}(c) - E_0 + \frac{1}{C}|a-c|\right) \\ &\geq \frac{1}{C(C+1)}\left(H^N_{\Lambda_1}(c) - E_0 + |a-c|\right). \end{aligned}$$

This proves Lemma 3.3. □

Proof. (of Proposition 3.1) To complete the proof of Proposition 3.1 we need to extend the result of the previous lemmas to general boxes Λ_{2L+1}. This is done by an argument previously used in the proof of Theorem 2.1 in [20] which makes crucial use of properties of Neumann boundary conditions.

For $\psi \in H^1(\Lambda_{2L+1})$, the form domain of $H_{\omega,L}$, one has that the restriction of ψ to $\Lambda_1(i)$ is in $H^1(\Lambda_1(i))$ for each $i \in \Lambda'_{2L+1}$ and

$$\langle (H_{\omega,L} - E_0)\psi, \psi\rangle = \sum_{i\in\Lambda'_{2L+1}} \langle (-\Delta - E_0 + q(\cdot - i - \omega_i)\psi, \psi\rangle_{\Lambda_1(i)}$$

where $\langle\cdot,\cdot\rangle_A$ denotes the standard scalar product in $L^2(A)$. This may also be applied to the modified displacement model $H_{c(\omega)}$,

$$\langle (H_{c(\omega),L} - E_0)\psi, \psi\rangle = \sum_{i\in\Lambda'_{2L+1}} \langle (-\Delta - E_0 + q(\cdot - i - c(\omega_i))\psi, \psi\rangle_{\Lambda_1(i)}.$$

Hence, using Lemmas 3.2 and 3.3 on each term in the sum, we obtain that

$$\begin{aligned}\langle (H_{\omega,L} - E_0)\psi, \psi\rangle &\geq \frac{1}{C} \sum_{i\in\Lambda'_{2L+1}} \langle ((H^N_{\Lambda_1}(c(\omega)_i) - E_0 + |\omega_i - c(\omega_i)|)\psi, \psi\rangle_{\Lambda_1(i)} \\ &\geq \frac{1}{C}\langle (H_{c(\omega),L} - E_0)\psi, \psi\rangle,\end{aligned}$$

where the positive term $\sum_i \langle |\omega_i - c(\omega_i)|\psi, \psi\rangle_{\Lambda_1(i)}$ was omitted. This completes the proof of Proposition 3.1. □

The random displacement model $H_{c(\omega)} = -\Delta + \sum_{i\in\mathbb{Z}^d} q(\cdot - c(\omega_i))$ has i.i.d. displacement vectors $(c(\omega))_{i\in\mathbb{Z}^d}$, whose distribution is discrete with support given by the corners $\mathcal{C}$ of $\overline{G}$. Thus it satisfies the assumptions of Theorem 4.1 in [21]; hence, by the proof of Theorem 1.2 in [21], in particular (16) in the same work, there exist $C < \infty$ and $\mu > 1$ such that, for all L,

$$\mathbb{P}(H_{c(\omega),L} \text{ has an eigenvalue less than } E_0 + C/L^2) \leq L^d \mu^{-L}.$$

Note that this requires that $d \geq 2$. The argument leading to Theorem 4.1 of [21] uses crucially the uniqueness (up to translations) of the minimizing configuration of the potentials proved in [2], which holds only for $d \geq 2$. See Section 5.2 below for a comment on the differences for $d = 1$. The assumption $r < 1/4$ in (5) was used in [2] for a more technical reason (rather than just $r < 1/2$) and thus also enters our argument here.

Using Proposition 3.1, we immediately obtain the following finite volume bound on the probability for finding low lying eigenvalues. It is this result which enters the proof of localization via multiscale analysis.

Corollary 3.4. *There exist $C < \infty$ and $\mu > 1$ such that, for all L,*

$$\mathbb{P}(H_{\omega,L} \text{ has an eigenvalue less than } E_0 + C/L^2) \leq L^d \mu^{-L}.$$

The results of [21] also show that the integrated density of states of $H_{c(\omega)}$, say $\tilde{N}$, satisfies a Lifshitz tails estimate of the form

$$\limsup_{\substack{E\to E_0\\ E>E_0}} \frac{\log|\log \tilde{N}(E)|}{\log(E - E_0)} \leq -\frac{1}{2}. \tag{39}$$

By Proposition 3.1 we see that N, the integrated density of states of H_ω, satisfies, for $E \geq E_0$,

$$N(E) \leq \tilde{N}(E_0 + C(E - E_0)).$$

Hence, we have proven the following result which is not required in the localization proof but stated here for its independent interest.

Theorem 3.5. *The IDS N of H_ω has a Lifshitz tail of the form*

$$\limsup_{\substack{E\to E_0\\ E>E_0}} \frac{\log|\log N(E)|}{\log(E-E_0)} \le -\frac{1}{2}.$$

We expect that the Lifshitz exponent $1/2$ is not optimal and should instead be $d/2$, the standard value known from the Anderson model. We can think of two ways in which one could try to get this improvement, non of which we know how to make rigorous. One approach would be to show that (39) holds with exponent $d/2$, which would immediately give the same in Theorem 3.5. That this should hold is discussed in [21], where the exponent $1/2$ is found due to one part of the proof which uses an essentially one-dimensional argument.

Another way to argue would be to make use of the term $\sum_i |\omega_i - c(\omega_i)|\chi_{\Lambda_1(i)}$ which was dropped in the proof of Proposition 3.1. Under our assumptions (6) and (7) this means that one would have to show standard Lifshitz tails for an Anderson-type model where the unperturbed operator is the random operator $H_{c(\omega)}$. However, the known methods do not work for the irregular background potential appearing here.

4. A Wegner estimate

Throughout this section, we write $H_i(a) = H^N_{\Lambda_1(i)}(a)$ for simplicity. Our goal in this section if to prove the following Wegner estimate for energies near $E_0 = \inf_{a\in\overline{G}} E_0(a)$, where $E_0(a) = \inf\sigma(H_i(a))$:

Theorem 4.1. *There exists $\delta > 0$ such that, for any $\alpha \in (0,1)$, there exists $C_\alpha > 0$ such that, for every interval $I \subset [E_0, E_0+\delta]$ and $L \in \mathbb{N}$,*

$$\mathbb{E}(\operatorname{tr}\, \chi_I(H_{\omega,L})) \le C_\alpha |I|^\alpha L^d. \tag{40}$$

As an immediate consequence of the existence of the integrated density of states (see e.g. [25]) and Theorem 4.1, we get

Corollary 4.2. *There exists $\delta > 0$ such that, for any $\alpha \in (0,1)$, the integrated density of states of H_ω is α-Hölder continuous in $[E_0, E_0+\delta]$.*

The rest of this section will be devoted to the proof of Theorem 4.1.

For a function f on G we set

$$(\partial_c f)(a) := \frac{c(a)-a}{|c(a)-a|}\cdot\nabla f(a),$$

with $c(a)$ denoting the corner closest to a as in Section 3. Thus, ∂_c denotes the directional derivative in the direction of the closest corner, where points a with multiple closest corners will not play a role in the arguments below.

By Corollary 2.8 there exists $\delta_0 > 0$ and $r_0 > 0$ such that $\partial_c E_0(a) \le -\delta_0$ for all $a \in A_{r_0} := \{a \in G : |c(a)-a| \le r_0\}$, a neighborhood of $\mathcal{C}$.

Let $\eta \in C^\infty(\mathbb{R})$ such that $0 \le \eta \le 1$, $\eta(r) = 1$ for $r \le r_0$ and $\eta(r) = 0$ for $r \ge 2r_0$. Using this function as a cut-off, we localize the vector fields associated with ∂_c onto a neighborhood of the corners, defining

$$(\partial'_c f)(a) := \eta(|c(a)-a|)(\partial_c f)(a). \tag{41}$$

For each $i \in \mathbb{Z}^d$, we write

$$\partial'_{c,\omega_i} H_\omega = \partial'_{c,\omega_i} q(\cdot - i - \omega_i) = -\eta(|c(\omega_i)-\omega_i|)\frac{c(\omega_i)-\omega_i}{|c(\omega_i)-\omega_i|}\cdot(\nabla q)(\cdot - i - \omega_i).$$

If $\psi \in H^1(\Lambda_{2L+1})$, the form domain of $H_{\omega,L}$, then $\psi_i := \psi|_{\Lambda_1(i)} \in H^1(\Lambda_1(i))$, the form domain of $H_i(\omega_i)$, and, with the usual abuse of notation for the quadratic form,

$$\langle \psi, H_{\omega,L}\psi\rangle = \sum_{i\in\Lambda'_{2L+1}} \langle \psi_i, H_i(\omega_i)\psi_i\rangle, \tag{42}$$

as well as

$$\sum_{i\in\Lambda'_{2L+1}} \langle \psi, \partial'_{c,\omega_i} H_{\omega,L}\psi\rangle = \sum_{i\in\Lambda'_{2L+1}} \langle \psi_i, \partial'_{c,\omega_i} H_i(\omega_i)\psi_i\rangle. \tag{43}$$

Proposition 4.3. *There exist $\delta_1 > 0$ and $\delta_2 > 0$ such that*

$$-\sum_{i\in\Lambda'_{2L+1}} \langle \psi, (\partial'_{c,\omega_i} H_{\omega,L})\psi\rangle \geq \delta_1 \|\psi\|^2 \tag{44}$$

for all $L \in \mathbb{N}$, and $\psi \in H^1(\Lambda_{2L+1})$ with $\langle \psi, (H_{\omega,L} - E_0)\psi\rangle \leq \delta_2\|\psi\|^2$.

Proposition 4.3 will be established through a series of lemmas.

Let $P_i(a)$ be the eigenprojection onto the groundstate of $H_i(a)$, $\overline{P}_i(a) := I - P_i(a)$, and

$$E_1 := \inf_{a\in\overline{G}} (\sigma(H_i(a)) \setminus \{E_0(a)\}) > E_0.$$

Lemma 4.4. *If $\psi \in H^1(\Lambda_{2L+1})$ with $\langle \psi, (H_{\omega,L} - E_0)\psi\rangle \leq \delta_2\|\psi\|^2$, then*

$$\sum_{i\in\Lambda'_{2L+1}} \|\overline{P}_i(\omega_i)\psi_i\|^2 \leq \frac{\delta_2}{E_1 - E_0}\|\psi\|^2. \tag{45}$$

Proof. We have

$$\begin{aligned}\langle \psi, (H_{\omega,L} - E_0)\psi\rangle &= \sum_{i\in\Lambda'_{2L+1}} \big\{(E_0(\omega_i) - E_0)\|P_i(\omega_i)\psi_i\|^2 \\ &\qquad + \langle \overline{P}_i(\omega_i)\psi_i, (H_i(\omega_i) - E_0)\overline{P}_i(\omega_i)\psi_i\rangle\big\} \\ &\geq (E_1 - E_0) \sum_{i\in\Lambda'_{2L+1}} \|\overline{P}_i(\omega_i)\psi_i\|^2,\end{aligned}$$

which yields (45) by the assumption. □

By the results of [1] we have

$$E_{0,r_0} := \inf_{a\in\overline{G}\setminus A_{r_0}} E_0(a) > E_0.$$

Define $\Lambda''_{2L+1} := \{i \in \Lambda'_{2L+1} : \ \omega_i \notin A_{r_0}\}$.

Lemma 4.5. *If $\psi \in H^1(\Lambda_{2L+1})$ with $\langle \psi, (H_{\omega,L} - E_0)\psi\rangle \leq \delta_2\|\psi\|^2$, then*

$$\sum_{i\in\Lambda''_{2L+1}} \|\psi_i\|^2 \leq \frac{\delta_2}{E_{0,r_0} - E_0}\|\psi\|^2. \tag{46}$$

Proof. This follows from the assumption and

$$\langle \psi, (H_{\omega,L} - E_0)\psi\rangle \geq \sum_{i\in\Lambda''_{2L+1}} (E_{0,r_0} - E_0)\|\psi_i\|^2.$$

□

Lemma 4.6. *There exist $C_1 < \infty$ and $C_2 < \infty$ such that for $\varphi \in L^2(\Lambda_1(i))$,*

$$-\langle\varphi, (\partial'_{c,a}H_i(a))\varphi\rangle \geq -(\partial'_{c,a}E_0(a))\|P_i(a)\varphi\|^2 - C_1\|\varphi\|\|\overline{P}_i\varphi\| - C_2\|\overline{P}_i\varphi\|^2. \tag{47}$$

Proof. Omitting the variable a we have,

$$\partial'_c P_i = -\partial'_c\overline{P}_i = -\partial'_c(\overline{P}_i^2) = \overline{P}_i(\partial'_c P_i) + (\partial'_c P_i)\overline{P}_i$$

and

$$\begin{aligned}\partial'_c H_i &= \partial'_c(E_0P_i + \overline{P}_iH_i\overline{P}_i)\\ &= (\partial'_cE_0)P_i + E_0(\partial'_cP_i) + (\partial'_c\overline{P}_i)H_i\overline{P}_i + \overline{P}_i(\partial'_cH_i)\overline{P}_i + \overline{P}_iH_i(\partial'_c\overline{P}_i)\\ &= (\partial'_cE_0)P_i + E_0(\partial'_cP_i)\overline{P}_i + E_0\overline{P}_i(\partial'_cP_i)\\ &\quad + \overline{P}_i(\partial'_cH_i)\overline{P}_i - (\partial'_cP_i)H_i\overline{P}_i - \overline{P}_iH_i(\partial'_cP_i).\end{aligned}$$

Thus, for $\varphi \in L^2(\Lambda_1(i))$,

$$\begin{aligned}-\langle\varphi, (\partial'_cH_i)\varphi\rangle &= -(\partial'_cE_0)\langle\varphi, P_i\varphi\rangle - 2E_0\mathrm{Re}\langle\overline{P}_i\varphi, (\partial'_cP_i)\varphi\rangle\\ &\quad - \langle\overline{P}_i\varphi, (\partial'_cH_i)\overline{P}_i\varphi\rangle + 2\mathrm{Re}\langle H_i(\partial'_cP_i)\varphi, \overline{P}_i\varphi\rangle.\end{aligned}$$

This implies (47) if we can show that the operators $\partial'_{c,a}H_i(a)$, $(\partial'_{c,a}P_i)(a)$ and $H_i(a)(\partial'_{c,a}P_i)(a)$ are bounded in $L^2(\Lambda_1(i))$, uniformly in $a \in \overline{G}$. This is clear for

$$\partial'_{c,a}H_i(a) = -\eta(|c(a)-a|)\frac{c(a)-a}{|c(a)-a|}\cdot(\nabla q)(\cdot - i - a).$$

For $P_i(a)$ we write, as in Section 2,

$$P_i(a) = \frac{1}{2\pi i}\oint_C (H_i(a)-z)^{-1}\,dz,$$

where C circles around $E_0(a)$, and can be chosen locally independent of a and with distance to $E_0(a)$ which is bounded below uniformly in $a \in \overline{G}$ (as the distance of $E_0(a)$ and $E_1(a)$ is uniformly bounded below). From this we can conclude that

$$(\partial'_{c,a}P_i)(a) = \frac{1}{2\pi i}\oint_C (H_i(a)-z)^{-1}(\partial'_{c,a}H_i(a))(H_i(a)-z)^{-1}\,dz$$

is uniformly bounded in a. In this expression one can absorb an additional factor $H_i(a)$ to show uniform boundedness of $H_i(a)(\partial'_{c,a}P_i)(a)$.

□

We are now ready to complete the proof of Proposition 4.3:

Proof. By Lemmas 4.4 and 4.5 we have

$$\begin{aligned}\sum_{i\notin\Lambda''_{2L+1}}\|P_i(\omega_i)\psi_i\|^2 &= \sum_{i\notin\Lambda''_{2L+1}}(\|\psi_i\|^2 - \|\overline{P}_i(\omega_i)\psi_i\|^2)\\ &\geq \|\psi\|^2 - \sum_{i\in\Lambda''_{2L+1}}\|\psi_i\|^2 - \sum_{i\in\Lambda'_{2L+1}}\|\overline{P}_i(\omega_i)\psi_i\|^2\\ &\geq \left(1 - \frac{\delta_2}{E_1-E_0} - \frac{\delta_2}{E_{0,r_0}-E_0}\right)\|\psi\|^2.\end{aligned}$$

Let $c_3 := \sup_{a\in G} |\partial'_{c,a} E_0(a)|$. With Lemma 4.6 we find

$$\begin{aligned}
-\sum_{i\in\Lambda'_{2L+1}} \langle\psi, (\partial'_{c,\omega_i} H_{\omega,L})\psi\rangle &= -\sum_{i\in\Lambda'_{2L+1}} \langle\psi_i, (\partial'_{c,\omega_i} H_i(\omega_i))\psi_i\rangle \\
&\geq \sum_{i\in\Lambda'_{2L+1}} (\partial'_{c,\omega_i} E_0(\omega_i))\|P_i(\omega_i)\psi_i\|^2 \\
&\quad - C_1 \sum_{i\notin\Lambda'_{2L+1}} \|\psi_i\|\|\overline{P}_i(\omega_i)\psi_i\| - C_2 \sum_{i\in\Lambda'_{2L+1}} \|\overline{P}_i(\omega_i)\psi_i\|^2 \\
&\geq -\sum_{i\notin\Lambda''_{2L+1}} (\partial'_{c,\omega_i} E_0(\omega_i))\|P_i(\omega_i)\psi_i\|^2 - \sum_{i\in\Lambda''_{2L+1}} (\partial'_{c,\omega_i} E_0(\omega_i))\|P_i(\omega_i)\psi_i\|^2 \\
&\quad - \frac{C_1}{2}\|\psi\| \left(\sum_{i\in\Lambda'_{2L+1}} \|\overline{P}_i(\omega_i)\psi_i\|^2\right)^{1/2} - C_2 \sum_{i\in\Lambda'_{2L+1}} \|\overline{P}_i(\omega_i)\psi_i\|^2. \qquad (48)
\end{aligned}$$

Now we use that $-\partial'_{c,\omega_i} E_0(\omega_i)$ is bounded from below by δ_0 if $i \notin \Lambda''_{2L+1}$ (the latter means $\omega_i \in A_0$ and thus $\partial'_{c,\omega_i} E_0(\omega_i) = \partial_{c,\omega_i} E_0(\omega_i)$) and by c_3 if $i \in \Lambda''_{2L+1}$. Also using the bounds from Lemma 4.4 and 4.5 again, we arrive at

$$\begin{aligned}
-\sum_{i\in\Lambda'_{2L+1}} \langle\psi, (\partial'_{c,\omega_i} H_{\omega,L})\psi\rangle &\geq \delta_0 \left(1 - \frac{\delta_2}{E_1 - E_0} - \frac{\delta_2}{E_{0,r_0} - E_0}\right)\|\psi\|^2 \\
&\quad - c_3 \frac{\delta_2}{E_{0,r_0} - E_0}\|\psi\|^2 - \frac{C_1}{2}\sqrt{\frac{\delta_2}{E_1 - E_0}}\|\psi\|^2 \\
&\quad - C_2 \frac{\delta_2}{E_1 - E_0}\|\psi\|^2.
\end{aligned}$$

Choose $\delta_2 > 0$ such that

$$\frac{c_3\delta_2}{E_{0,r_0} - E_0} + \frac{C_1}{2}\sqrt{\frac{\delta_2}{E_1 - E_0}} + \frac{C_2\delta_2}{E_1 - E_0} < \frac{\delta_0}{4},$$

and

$$\frac{\delta_2}{E_1 - E_0} + \frac{\delta_2}{E_{0,r_0} - E_0} < \frac{\delta_0}{2}.$$

Then

$$-\sum_{i\in\Lambda'_{2L+1}} \langle\psi, (\partial'_{c,\omega_i} H_{\omega,L})\psi\rangle \geq \frac{\delta_0}{4}\|\psi\|^2.$$

This proves (44) with $\delta_1 = \delta_0/4$.

□

We now prove Theorem 4.1. We follow the approach developed in [15] based on L^p estimates of the spectral shift function (see also [5]). The method can be adapted to our model thanks to Proposition 4.3.

For δ_2 from Proposition 4.3 choose $\delta = \delta_2/2$ and let $I \subset [E_0, E_0 + \delta]$ be an interval of the form $I = [E - \varepsilon, E + \varepsilon]$.

Let $\chi \in C^\infty(\mathbb{R})$ be a real-valued function such that $\chi(x) = -1$ for $x \leq -\varepsilon$; $\chi(x) = 0$ for $x \geq \varepsilon$; $\chi' \geq 0$; and $\|\chi'\|_\infty \leq 1/\varepsilon$.

By our assumption on I and the Gohberg-Krein formula, see e.g. Proposition 2 in [23], Proposition 4.3 implies that

$$\begin{aligned}\operatorname{tr}\left(-\sum_{i\in\Lambda'_{2L+1}}\partial'_{c,\omega_i}[\chi(H_{\omega,L}-E+t)]\right) &= \operatorname{tr}\left(\chi'(H_{\omega,L}-E+t)\left(-\sum_{i\in\Lambda'_{2L+1}}(\partial'_{c,\omega_i}H_\omega)\right)\right)\\ &\geq \delta_1\operatorname{tr}\left(\chi'(H_{\omega,L}-E+t)\right).\end{aligned}$$

Then, as $\operatorname{supp}\chi'\subset[-\varepsilon,\varepsilon]$ and as $\chi'\geq 0$, one has

$$\begin{aligned}\mathbb{E}(\operatorname{tr}\,\chi_I(H_{\omega,L})) &\leq \mathbb{E}(\operatorname{tr}\int_{-2\varepsilon}^{2\varepsilon}\chi'(H_{\omega,L}-E+t)\,dt)\\ &\leq \frac{1}{\delta_1}\sum_{i\in\Lambda'_{2L+1}}\int_{-2\varepsilon}^{2\varepsilon}\mathbb{E}\left(\operatorname{tr}\,\left(-\partial'_{c,\omega_i}[\chi(H_{\omega,L}-E+t)]\right)\right)\,dt. \qquad (49)\end{aligned}$$

In the above expectation we want to write the integration with respect to ω_i over G as a sum of integrals over the intersection of G with each one of the 2^d orthants, using polar coordinates with respect to the corners $c\in\mathcal{C}$ in each orthant. For this we represent $a\in G$ by $(r,\theta,c(a))\in(0,\infty)\times\mathbb{S}^{d-1}\times\mathcal{C}$, where $c(a)$ again denotes the corner closest to a and (r,θ) polar coordinates of $a-c(a)$. For a function f supported near the corners $\mathcal{C}$ this means that

$$\int_G f(a)\rho(a)\,da=\sum_{c\in\mathcal{C}}\int_{\mathbb{S}^{d-1}_c}\int_0^\infty f(r\theta-c)\rho(r\theta-c)r^{d-1}\,dr\,d\theta,$$

with $\mathbb{S}^{d-1}_c$ denoting the intersection of $\mathbb{S}^{d-1}$ with the orthant containing $-c$.

With $a=\omega_i$ this leads to

$$\begin{aligned}&\mathbb{E}\big(\operatorname{tr}\,\big(-\partial'_{c,\omega_i}\big[\chi(H_{\omega,L}-E+t)\big]\big)\big)\\ &\qquad=\hat{\mathbb{E}}_i\Big(\operatorname{tr}\,\sum_{c\in\mathcal{C}}\int_{\mathbb{S}^{d-1}_c}\int_0^{2r_0}\partial_r[\chi(H_{\omega,L}-E+t)]\eta(r)\rho(r\theta-c)r^{d-1}\,dr\,d\theta\Big), \qquad (50)\end{aligned}$$

where $\hat{\mathbb{E}}_i$ denotes the expectation with respect to the random variables $(\omega_j)_{j\neq i}$. Here a sign-change is due to the fact that ∂_r acts in the direction opposite to ∂_{c,ω_i}. By integration by parts, we have

$$\begin{aligned}&\int_0^{2r_0}\partial_r\big[\chi(H_{\omega,L}-E+t)\big]\eta(r)\rho(r\theta-c)r^{d-1}dr\\ &\qquad=-\int_0^{2r_0}\big[\chi(H_{\omega,L}-E+t)-\chi(H_{\omega^{(i)},L}-E+t)\big]\partial_r(\eta(r)\rho(r\theta-c)r^{d-1})dr, \qquad (51)\end{aligned}$$

where $\omega^{(i)}$ is the random variable such that $\omega^{(i)}_j=\omega_j$ for $j\neq i$, and $\omega^{(i)}_i=c(\omega_i)$. Note that the second term in (51) actually integrates out to zero, as $\eta(r)r^{d-1}$ vanishes at both endpoints. But we include this term in the integral to be able to make use of bounds on $\chi(H_{\omega,L}-E+t)-\chi(H_{\omega^{(i)},L}-E+t)$.

Now (51) implies that

$$\operatorname{tr}\left(\int_0^{2r_0}\partial_r[\chi(H_{\omega,L}-E+t)]\eta(r)\rho(r\theta-c)r^{d-1}dr\right)$$
$$=-\int_0^{2r_0}\left(\int_{\mathbb{R}}\xi(\cdot\,;H_{\omega,L},H_{\omega^{(i)},L})\chi'(\lambda)d\lambda\right)\partial_r[\eta(r)\rho(r\theta-c)r^{d-1}]dr \quad (52)$$

This uses the spectral shift function $\xi(\lambda\,;H_{\omega,L},H_{\omega^{(i)},L})$ for the pair $(H_{\omega,L},H_{\omega^{(i)},L})$ which is defined so that

$$\int_{\mathbb{R}}\varphi'(\lambda)\xi(\lambda\,;H_{\omega,L},H_{\omega^{(i)},L})d\lambda=\operatorname{tr}\,(\varphi(H_{\omega,L})-\varphi(H_{\omega^{(i)},L})).$$

for all $\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$. The invariance principle for the spectral shift function (see e.g. [26] or [3]) states that the spectral shift function $\xi(\lambda\,;H_{\omega,L},H_{\omega^{(i)},L})$ can be written as

$$\xi(\lambda\,;H_{\omega,L},H_{\omega^{(i)},L})=-\xi(g_k(\lambda)\,;g_k(H_{\omega,L}),g_k(H_{\omega^{(i)},L})).$$

Here we define $g_k(\lambda)=(\lambda+M)^{-k}$ and M is picked such that ω-a.s., $\inf_x V_\omega(x)\geq -M+1$.

By definition $H_{\omega,L}-H_{\omega^{(i)},L}=q(\cdot-i-\omega_i)-q(\cdot-i-c(\omega_i))$ which is a bounded, compactly supported potential. In Section 5 of [15], it is proved that (actually for more general operators), if $k>pd/2+1$ and $p>1$, the operator $g_k(H_{\omega,L})-g_k(H_{\omega^{(i)},L})$ is super trace class of order $1/p$, i.e. its singular values to the power $1/p$ are summable; the p-th power of this sum is denoted by $\|\cdot\|_{1/p}$. Moreover, $\|g_k(H_{\omega,L})-g_k(H_{\omega^{(i)},L})\|_{1/p}\leq C_0$ for $C_0>0$ independent of ω and L.

Using a simple change of variables and the bound $\|\xi(\cdot;A,B)\|_{L^p}\leq\|A-B\|_{1/p}^{1/p}$ proven in [5] we find

$$\begin{aligned}\int_{-\varepsilon}^{\varepsilon}|\xi(\lambda;H_{\omega,L},H_{\omega^{(i)},L})|^p\,d\lambda &= \int_{-\varepsilon}^{\varepsilon}|\xi(g_k(\lambda);g_k(H_{\omega,L}),g_k(H_{\omega^{(i)},L}))|^p\,d\lambda\\ &\leq C\int_{\mathbb{R}}|\xi(s;g_k(H_{\omega,L}),g_k(H_{\omega^{(i)},L}))|^p\,ds\\ &\leq C\|g_k(H_{\omega,L})-g_k(H_{\omega^{(i)},L})\|_{1/p}\\ &\leq CC_0. \end{aligned}\quad (53)$$

As $\|\chi'\|_\infty\leq 1/\varepsilon$ and $\operatorname{supp}\chi'\subset[-\varepsilon,\varepsilon]$, (53) and the Hölder inequality imply that, for any $q\in(1,+\infty)$, there exists $C_q>0$ such that

$$\sup_\omega\left|\int_{\mathbb{R}}\xi(\cdot\,;H_{\omega,L},H_{\omega^{(i)},L})\chi'(\lambda)d\lambda\right|\leq C_q\,\varepsilon^{1/q-1}.$$

As $\partial_r[\eta(r)\rho(r\theta-c)r^{d-1}]$ is a bounded, compactly supported function uniformly in θ and c, (52) then implies that

$$\sup_\omega\left|\operatorname{tr}\left(\int_0^{2r_0}\partial_r[\chi(H_{\omega,L}-E+t)]\eta(r)\rho(r\theta-c)r^{d-1}dr\right)\right|\leq C_q\,\varepsilon^{1/q-1}.$$

Plugging this into (50) and then into (49), we get that, for any $q\in(1,+\infty)$, there exists $C_q>0$ such that

$$\mathbb{E}(\operatorname{tr}\,\chi_I(H_{\omega,L}))\leq C_q\,|\Lambda'_{2L+1}|\,\varepsilon^{1/q-1}\int_{-2\varepsilon}^{2\varepsilon}dt\leq\tilde{C}_q\,\varepsilon^{1/q}L^d.$$

This completes the proof of Theorem 4.1. □

5. Concluding remarks

5.1. **Remarks on multi-scale analysis.** It is well known to experts in the field that Lifshitz tails of the IDS, more precisely, a bound as in Corollary 3.4, combined with a Wegner estimate such as Theorem 4.1 leads to a proof of spectral and dynamical localization via multi-scale analysis (MSA). For the sake of reaching a broader audience we include some discussion of the strategy and additional tools which are behind this. Here it is very convenient to follow the presentation in [25], which we highly recommend to readers who want to understand MSA. In particular, while including applications to models other than the one considered here, [25] carefully singles out all the properties required of a model to make the various steps of MSA work. This allows us the luxury to mostly discuss the validity of these properties for our model, after which the reader can follow the details in [25] without changes. The conclusions reached in [25] are somewhat weaker than what we state in Theorem 1.2. One gets the existence of an interval $I = [E_0, E_0 + \delta']$ with pure point spectrum and exponentially decaying eigenfunctions as well as dynamical localization in the sense that

$$\mathbb{E}\left(\sup_{|g|\leq 1} \| |X|^p g(H_\omega) \chi_I(H_\omega) \chi_y \| \right) < \infty \tag{54}$$

for all $p > 0$. The bound (10), stronger than (54) both with respect to the decay in $|x - y|$ and the Hilbert-Schmidt norm used, was shown to be obtainable via MSA in [13], see also the review [18], which represents the current state of the art. This follows the same general strategy, i.e. using input much weaker than what we have obtained here to prove localization.

To link our results with the strategy employed in [25], we first note that the Wegner estimate of Theorem 4.1 is much stronger than what is required in MSA. In fact, for $E \in I_0 := [E_0, E_0 + \delta/2]$, $\theta \in (0, 1/2)$, all $q > 0$ and ℓ sufficiently large, we get from Theorem 4.1 and Chebychev's inequality that

$$\mathbb{P}\{\mathrm{dist}(\sigma(H_{\omega,\ell}), E) \leq \exp(-\ell^\theta)\} \leq \mathbb{E}(\mathrm{tr}\, \chi_{[E-\exp(-\ell^\theta), E+\exp(-\ell^\theta)]}(H_{\omega,\ell})) \leq \ell^{-q},$$

i.e. that the Wegner estimate denoted as $W(I_0, \ell, \theta, q)$ in [25] holds.

Next, we observe that the Lifshitz-tail bound from Corollary 3.4 implies the following: For any $\xi > 0$ and $\beta \in (0, 1)$ there exists $L_2 = L_2(\beta, \xi)$ such that

$$\mathbb{P}(H_{\omega,L} \text{ has an eigenvalue less than } E_0 + L^{\beta-1}) \leq L^{-\xi}$$

for all $L \geq L_2$. This follows from Corollary 3.4 with a Neumann bracketing argument exactly as in the proof of Theorem 2.2.3 in [25].

For $E \in I := [E_0, E_0 + L^{\beta-1}/2]$ one may now use a Combes-Thomas estimate (which holds for very general semi-bounded Schrödinger operators, and thus certainly in our setting) to turn this into an *initial length estimate* (ILE) for the decay of the "smeared" Green function $\|\chi_x (H_{\omega,L} - E)^{-1} \chi_y\|$ with y near the center of Λ_L and x near its boundary. In fact, for $\gamma = L^{\beta-1}$ and L sufficiently large, one obtains the ILE denoted by $G(I, L, \gamma, \xi)$ in [25].

As carried out in Chapter 3 of [25], the MSA machine can be started with the first L satisfying the above if one links the θ from the Wegner estimate and the β from the ILE by $0 < 2\theta < \beta < 1$.

[25] explicitly identifies four more "soft" ingredients which are needed along the way of running the machine to prove the exponential decay properties of Green's function necessary to conclude pure point spectrum with exponentially decaying eigenfunctions:

(i) independence (INDY) of restrictions $H_{\omega,\Lambda}$ and $H_{\omega,\Lambda'}$ of H_ω to disjoint cubes Λ and Λ',

(ii) an a-priori Weyl-type bound $\operatorname{tr} \chi_J(H_{\omega,\Lambda}) \leq C|\Lambda|$ (WEYL) for the number of eigenvalues in an interval J covering the energy regime considered,

(iii) a *geometric resolvent inequality* (GRI) which allows to carry Green function bounds from smaller cubes over to larger cubes,

(iv) an eigenfunction decay inequality (EDI) which allows to relate Green function bounds to decay of generalized eigenfunctions.

All these properties are either obvious in our setting or were proven in [25] for Schrödinger operators much more general than needed here. The same holds for the bound

(v) $\operatorname{tr}(\chi_\Lambda \chi_J(H_\omega)) \leq C|\Lambda|^\kappa$, $\kappa > 0$ fixed,

which enters the proof of dynamical localization, e.g. Theorem 3.4.1 in [25].

With this we hope to have prepared the motivated reader for consulting the details in [25]. The even more motivated reader will be rewarded by [13] with stronger results.

5.2. **Related results and problems.** (i) The Wegner estimate and the Lifshitz tails, and therefore our main result Theorem 1.2, hold under weaker assumptions on the distribution μ of the displacements. E.g. the proof as written in Section 4 only requires that μ has a C^1-density ρ near the corners $\mathcal{C}$. This is made possible through the introduction of the cut-off η supported near the corners in (41).

It is evident from (52) that the C^1 condition for ρ is only needed in the radial direction with respect to the corners. In fact, similar to [14] we could allow distributions supported on a suitable submanifold. Examples would be the uniform distribution supported on a "cross" in $d = 2$ or, in general dimension, uniform distribution supported on the boundary ∂G of $G = (-d_{max}, d_{max})^d$.

(ii) Under our assumptions and in $d \geq 2$, among all *periodic* configurations $\omega \in \overline{G}^{\mathbb{Z}^d}$, ω^* as defined in (8) is, up to translations, the unique minimizer in the sense that $\inf \sigma(H_{\omega^*}) = \inf \Sigma$. This was proven in [2] and enters the argument in [21] leading to Corollary 3.4 and the Lifshitz tail estimate Theorem 3.5 above.

As also shown in [2], in $d = 1$ there are many periodic minimizing configurations. This has strong consequences for the IDS at the bottom of the spectrum. An extreme case is given by the 1D *Bernoulli displacement model*, i.e. $\mu = \frac{1}{2}\delta_{d_{max}} + \frac{1}{2}\delta_{-d_{max}}$, whose IDS satisfies the lower bound $N(E_0 + \varepsilon) \geq C/\ln^2 \varepsilon$. This singular behavior is the extreme opposite of a Lifshitz tail.

(iii) The previous remark might mislead into expecting that the Bernoulli displacement model is not localized at low energy. Spectral localization for the 1D random displacement model (at all energies and for arbitrary non-trivial distribution of the displacements) has been proven in [4] and [24] (using methods of [7]). These methods are completely different from what is available for $d > 1$, and, in particular, do not require smallness of the IDS. One uses dynamical systems tools such as results on products of random matrices, in particular Furstenberg's theorem. In fact, one finds that the Lyapunov exponents are positive with the possible exception of a discrete set of energies. As far as dynamical localization is concerned, it might be violated at those critical energies. For examples of this see [16, 6]. Away from the critical energies, however, one also has dynamical localization.

In the case of the one-dimensional Bernoulli displacement model, the energy E_0 provides a new example of a critical energy. This is seen as follows: By the results of [2], for any $\varepsilon > 0$ almost surely there is a solution u_0 of $H_\omega u = E_0 u$ and $C > 0$ such that $\frac{1}{C}\exp(-x^{-1/2-\varepsilon}) \leq |u_0(x)| \leq C\exp(x^{1/2+\varepsilon})$. Using the lower bound and the standard reduction of order argument one shows that there is a second linearly independent solution u_1 which satisfies the same upper

bound. This shows that the transfer matrices grow sub-exponentially. Thus the Lyapunov exponent at E_0 vanishes.

(iv) An interesting open problem arises from cases of the random displacement model where, in the language used in the introduction, alternative (ii) holds, i.e. where $E_0(a)$ vanishes identically. Examples for this (non-generic) situation can be constructed as follows:

Let $0 < \varphi \in C^\infty(\Lambda_1(0))$ be constant near the boundary (but not constant throughout $\Lambda_1(0)$). In the definition (1), (2) of the random displacement H_ω model pick the single-site potential as $q = \Delta\varphi/\varphi$. By construction, this leads to alternative (ii). Moreover, for *every* displacement configuration ω a generalized ground state of H_ω to $E_0 = 0$ is given by $\phi = \sum_n \varphi(\cdot - n - \omega_n)\chi_{\Lambda_1(n)}$ (here we think of φ as extended by a constant onto all of $\mathbb{R}^d$). Note that $1/C \leq \phi \leq C$ for some $C > 0$ uniformly in ω.

This leads to van Hove behavior of the IDS at $E_0 = 0$, i.e.

$$\frac{1}{C} E^{d/2} \leq N(E) \leq C E^{d/2}$$

for $E > 0$, which follows with the same argument as provided for a closely related example in Section 3 of [20].

It would be interesting to know if this can generate non-trivial transport (and thus prevent dynamical localization). This is the case in dimension $d = 1$. Starting with ϕ, the reduction of order argument provides a second, linearly independent, solution which grows at most linearly. Thus, the transfer matrix also grows linearly. By Corollary 2.1 in [6], which is based on work in [8], this implies that the time-averaged moments $|X|^p$ of suitable solutions of the time-dependent Schrödinger equation are bounded below by $CT^{(p-5)/2}$. This rules out dynamical localization in the sense of (54) for $p > 5$. Clearly, there are multiple obstacles to extending these methods to higher dimension.

We finally remark that the methods of [8] and [6] do not suffice to obtain non-trivial transport under the sub-exponential growth bounds on the transfer matrix discussed in remark (iii) above.

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References

[1] J. Baker, M. Loss, and G. Stolz, Minimizing the ground state energy of an electron in a randomly deformed lattice, *Comm. Math. Phys.* **283**, 397–415 (2008)

[2] J. Baker, M. Loss, and G. Stolz, Low energy properties of the random displacement models, *J. Funct. Anal.* **256**, 2725–2740 (2009)

[3] M. Sh. Birman and D. R. Yafaev, The spectral shift function. The work of M. G. Krein and its further development, *St. Petersburg Math. J.* **4**, 833-870 (1993)

[4] D. Buschmann and G. Stolz, Two-parameter spectral averaging and localization properties for non-monotonic random Schrödinger operators, *Trans. Am. Math. Soc.* **353**, 635–653 (2000)

[5] J. M. Combes, P. D. Hislop, S. Nakamura, The L^p-theory of the spectral shift function, the Wegner estimate, and the integrated density of states for some random operators, *Comm. Math. Phys.* **218**, 113–130 (2001)

[6] D. Damanik, D. Lenz and G. Stolz , Lower transport bounds for one-dimensional continuum Schrödinger operators, *Math. Ann.* **336**, 361-389 (2006)
[7] D. Damanik, R. Sims and G. Stolz, Localization for one-dimensional continuum, Bernoulli-Anderson models, *Duke Math. J.* **114**, 59–100 (2002)
[8] D. Damanik and S. Tcheremchantsev, Power-law bounds on transfer matrices and quantum dynamics in one dimension, *Comm. Math. Phys.* **236**, 513-534 (2003)
[9] R. Fukushima, Brownian survival and Lifshitz tail in perturbed lattice disorder, *J. Funct. Anal.* **256**, 2867–2893 (2009)
[10] R. Fukushima and N. Ueki, Classical and quantum behavior of the integrated density of states for a randomly perturbed lattice, preprint 2009, available at www.math.h.kyoto-u.ac.jp/$\sim$ueki/Frenkel-Noncpt.pdf
[11] F. Germinet, P. D. Hislop and A. Klein, Localization for Schrödinger operators with Poisson random potential, *J. Eur. Math. Soc.* **9**, 577-607 (2007)
[12] F. Germinet, P. D. Hislop and A. Klein, Localization at low energies for attractive Poisson random Schrödinger operators, 153-165, *CRM Proc. Lecture Notes* **42**, Amer. Math. Soc., Providence, RI, 2007
[13] F. Germinet and A. Klein, Bootstrap multiscale analysis and localization in random media, *Comm. Math. Phys.*, **222**, 415–448 (2001)
[14] F. Ghribi and F. Klopp, Localization for the random displacement model at weak disorder, *Ann. Henri Poincaré*, **11**, 127-149 (2010).
[15] P. D. Hislop, F. Klopp, The integrated density of states for some random operators with nonsign definite potentials, *J. Funct. Anal.* **195**, 12-47 (2002)
[16] S. Jitomirskaya, H. Schulz-Baldes and G. Stolz, Delocalization in random polymer models, *Comm. Math. Phys.* **233**, 27-48 (2003)
[17] T. Kato, *Perturbation Theory for Linear Operators*, 2nd Ed., Springer, Berlin, 1976
[18] A. Klein, Multiscale analysis and localization of random operators. Random Schröinger operators, 121–159, *Panor. Synthèses*, **25**, Soc. Math. France, Paris, 2008
[19] F. Klopp, Localization for semiclassical continuous random Schrödinger operators. II. The random displacement model, *Helv. Phys. Acta* **66**, 810-841 (1993)
[20] F. Klopp, S. Nakamura, Spectral extrema and Lifshitz tails for non monotonous alloy type models. *Comm. Math. Phys.* **287**, 1133–1143 (2009)
[21] F. Klopp and S. Nakamura, Lifshitz tails for generalized alloy type random Schrödinger operators, to appear in *Analysis and PDE*, http://arxiv.org/abs/0903.2105, 2009.
[22] F. Klopp, S. Nakamura, F. Nakano and Y. Nomura, Anderson localization for 2D discrete Schrödinger operators with random magnetic fields, *Ann. H. Poincaré* **4**, 795-811 (2003)
[23] B. Simon, Spectral averaging ad the Krein spectral shift, *Proc. Am. Math. Soc.* **126**, 1409-1413 (1998)
[24] R. Sims, Localization in one-dimensional models of disordered media, PhD Thesis, University of Alabama at Birmingham, 2001
[25] P. Stollmann, *Caught by Disorder: Bound States in Random Media*, *Progress in Mathematical Physics* **20**, Birkhäuser, Boston, MA, 2001.
[26] D. R. Yafaev, *Mathematical Scattering Theory: General Theory*, Translations of Mathematical Monographs **105**, Providence, RI: American Mathematical Society, 1992.

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