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Abstract. We present a new method for estimating the frontier of a multidimensional sample when the conditional distribution function decreases at a polynomial rate to zero in the neighborhood of the frontier. The estimator is based on a kernel regression on high moments. It is assumed that the order of the moments goes to infinity while the bandwidth of the kernel goes to zero. We give conditions on these two parameters to obtain the asymptotic normality of the estimator. The good performance of the estimator is illustrated on some finite sample situations.

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1 Introduction

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be n independent copies of a random pair (X, Y) such that their common density has a support defined by

$$S = \{(x, y) \in \Omega \times \mathbb{R};\ 0 \leq y \leq g(x)\},$$

where Ω is a compact subset of $\mathbb{R}^d$. The unknown function g is called the frontier. We address the problem of estimating g. In Girard and Jacob (2008), a new estimator is introduced based upon kernel regression on high power-transformed data. In the particular case where Y given $X = x$ is uniformly distributed it is proved that this estimator is asymptotically Gaussian with the minimax rate of convergence for Lipschitzian frontiers. From the practical point

of view, compared to the numerous extreme-value based estimators (Gardes (2002), Geffroy (1964), Girard and Jacob (2003a, 2003b, 2004), Girard and Menneteau (2005), Menneteau (2008)), projection estimators (Jacob and Suquet (1995)), or piecewise polynomial estimators (Korostelev and Tsybakov (1993), Korostelev *et al* (1995), Härdle *et al* (1995)) this estimator does not require a partition of the support S. When the conditional distribution of Y given X is not uniform, this estimator is still convergent (Girard and Jacob (2008), Theorem 1) but suffers from a strong bias on finite sample situations (Girard and Jacob (2008), Table 1). Under monotonicity assumptions, the frontier can also be interpreted as the endpoint of Y given $X \leq x$. Specific estimation techniques have been developed in this context, see for instance Deprins *et al.* (1984), Farrel (1957), Gijbels *et al.* (1999) or Aragon *et al.* (2005), Cazals *et al.* (2002), Daouia and Simar (2005) for the definition of robust estimators.

In this paper, we investigate the more general case where the conditional distribution function of Y given $X = x$ is

$$F(y|x) = 1 - (1 - y/g(x))^{\alpha(x)}, \quad \forall\, y \in [0,\, g(x)].$$

Here, α is a positive function driving the behavior of the distribution tail of Y given $X = x$ in the neighborhood of its endpoint $g(x)$. If $\alpha(x) < 1$ then the density of Y given $X = x$ tends to infinity as $y \to g(x)$ whereas it tends to 0 in the case $\alpha(x) > 1$. The intermediate case $\alpha(x) = 1$ corresponds to the uniform distribution already investigated in Girard and Jacob (2008) where the density has a jump at the endpoint. In the general context of extreme-value theory, the conditional distribution of Y given $X = x$ is said to belong to the Weibull max-domain of attraction with extreme-value index $-1/\alpha(x)$. Our aim is to estimate the frontier g when α is unknown. In the next section, we introduce an estimator based on a kernel regression on high moments of the variable of interest Y. Similarly to Girard and Jacob (2008), this technique enables us to avoid the partitioning of S. In Section 3 the estimator is proved to be asymptotically Gaussian. Some simulations are proposed in Section 4 to illustrate the efficiency of our estimator and to compare it with the one of Girard and Jacob (2008). Our main result is proved in Section 5 and auxiliary results are postponed to the Appendix.

2 Construction of the estimator

To motivate the construction of our estimator, first consider the unconditional case where $\alpha(x)$ is constant, but not necessarily 1 and g is also constant equal to θ. The distribution function F can thus be rewritten as

$$F_{\theta,\,\alpha}(y) = 1 - (1 - y/\theta)^{\alpha}, \quad \forall\, y \in [0,\, \theta].$$

If Z is a random variable with distribution function $F_{\theta,\,\alpha}$, then

$$\mu_p := \mathbb{E}|Z|^p = p\int_0^\infty t^{p-1}\mathbb{P}(|Z|>t)\,dt = \alpha\;\theta^p\;B(p+1,\,\alpha)$$

where $B(x,\,y) = \displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ is the Beta function. Therefore

$$\frac{\mu_{p+1}}{\mu_p} = \theta\;\frac{p+1}{p+1+\alpha}$$

which leads to the equation

$$\frac{1}{\theta} = (p+2)\frac{\mu_{p+1}}{\mu_{p+2}} - (p+1)\frac{\mu_p}{\mu_{p+1}}$$

or more generally, for $N \in \mathbb{N}\setminus\{0\}$, to:

$$\begin{aligned}\frac{1}{\theta} &= \frac{1}{N}\sum_{k=1}^{N}\left[(p+k+1)\frac{\mu_{p+k}}{\mu_{p+k+1}} - (p+k)\frac{\mu_{p+k-1}}{\mu_{p+k}}\right]\\ &= \frac{1}{N}\left[(p+N+1)\frac{\mu_{p+N}}{\mu_{p+N+1}} - (p+1)\frac{\mu_p}{\mu_{p+1}}\right].\end{aligned}$$

Coming back to the conditional case, the idea of our estimator is the following: let (p_n) be a sequence tending to $+\infty$, (h_n) a positive one converging to 0 and $x \in \mathbb{R}^d$. Then

$$\frac{1}{\widehat{g}_n(x)} = \frac{1}{r_n}\left[(p_n+r_n+1)\frac{\widehat{\mu}_{p_n+r_n}(x)}{\widehat{\mu}_{p_n+r_n+1}(x)} - (p_n+1)\frac{\widehat{\mu}_{p_n}(x)}{\widehat{\mu}_{p_n+1}(x)}\right] \tag{1}$$

is an estimator of $1/g(x)$ where (r_n) is a positive sequence of real numbers greater than or equal to 1 and

$$\widehat{\mu}_{p_n}(x) = \frac{1}{n}\sum_{k=1}^{n} Y_k^{p_n}\,K_{h_n}(x-X_k)$$

is a kernel estimator of $\mu_{p_n}(x) = \mathbb{E}(Y^{p_n}|X=x)$. Here $K_h(u) = \dfrac{1}{h^d}\cdot K\left(\dfrac{u}{h}\right)$ where K is a probability density on $\mathbb{R}^d$ and h a window-width. Intuitively, the use of the window-width h_n allows to select the pairs $(X_k,\,Y_k)$ such that X_k is close to x while the use of the high power p_n gives more weight to the Y_k close to $g(x)$. Let us also highlight that, compared to the estimator suggested in the further work of Girard and Jacob (2008), Section 6, our proposition (1) does not require the knowledge of $\alpha(x)$. Moreover, it benefits from an explicit formulation which is not the case of estimators defined by optimization problems (Girard *et al.* (2005)) such as local polynomial estimators (Hall *et al.* (1998), Hall and Park (2004), Knight (2001)). Finally, although our estimator $\widehat{g}_n(x)$ is based on kernel regressions, classical results do not apply (see for instance Ferraty and Vieu (2005), Theorem 6.11) since the condition $p_n \to \infty$ induces technical difficulties.

3 Main result

In all the sequel r_n will be chosen as $r_n = ap_n + b$ with $a > 0$ or $a = 0$ and $b \geq 1$. Let us denote by f the density function of X. To establish the asymptotic normality of $\widehat{g}_n(x)$, the following regularity assumptions are needed:

(A_1) $\exists \delta_f,\, \varepsilon_f,\, \eta_f > 0 \;\; \forall\, x,\, y \in \mathbb{R}^d :\; \|x-y\| < \delta_f \Rightarrow |f(x)-f(y)| \leq \varepsilon_f \|x-y\|^{\eta_f}$.

(A_2) $\exists \delta_\alpha,\, \varepsilon_\alpha,\, \eta_\alpha > 0 \;\; \forall\, x,\, y \in \mathbb{R}^d :\; \|x-y\| < \delta_\alpha \Rightarrow |\alpha(x)-\alpha(y)| \leq \varepsilon_\alpha \|x-y\|^{\eta_\alpha}$.

(A_3) $\exists \delta_g,\, \varepsilon_g,\, \eta_g > 0 \;\; \forall\, x,\, y \in \mathbb{R}^d :\; \|x-y\| < \delta_g \Rightarrow |g(x)-g(y)| \leq \varepsilon_g \|x-y\|^{\eta_g}$.

Let B the unit ball of $\mathbb{R}^d$. We shall also assume that

(A_4) The support of K is included in B and $\displaystyle\int_B K^3(x)\,dx < \infty$.

Note that (A_4) implies that $\forall\, q \in \{1,2,3\}$, $\displaystyle\int_B K^q(x)\,dx < \infty$.

Our main result can now be stated:

Theorem 1. *Suppose (A_1-A_4) hold. For all $x \in \mathbb{R}^d$ such that $f(x) > 0$, $n\, p_n^{-\alpha(x)}\, h_n^d \longrightarrow +\infty$ and $p_n\, h_n^{\eta_g} \longrightarrow 0$ as $n \to \infty$, we have:*

$$\sqrt{n}\, p_n^{-\alpha(x)/2+1}\, h_n^{d/2} \left(\frac{\widehat{g}_n(x)}{g(x)} - 1 \right) \xrightarrow{d} \mathcal{N}\left(0,\, \frac{\|K\|_2^2\, V(\alpha(x),\, a)}{f(x)} \right), \quad as \quad n \to \infty$$

where $\|K\|_2^2 = \displaystyle\int_B K^2(x)\,dx$ and

1. *If $a > 0$,*

$$V(\alpha(x),\, a) = \frac{\alpha(x)+1}{a^2\ \Gamma(\alpha(x))} \left[2^{-\alpha(x)-2} - 2\frac{(1+a)^{\alpha(x)+1}}{(2+a)^{\alpha(x)+2}} + 2^{-\alpha(x)-2}(1+a)^{\alpha(x)} \right].$$

2. *If $a = 0$,*

$$V(\alpha(x),\, 0) = \lim_{a \to 0} V(\alpha(x),\, a) = \frac{2^{-\alpha(x)-4}}{\Gamma(\alpha(x))} \cdot (\alpha^3(x) + 2\alpha^2(x) + 3\alpha(x) + 2).$$

Note that, if we set $x \in \Omega$, the sequences $h_n(x) = n^{-1/(d+\eta_g \alpha(x))}$ and $p_n(x) = \varepsilon_n\, n^{\eta_g/(d+\eta_g \alpha(x))}$ can be chosen, where (ε_n) is an arbitrary sequence of positive real numbers tending to 0. However, this requires a precise knowledge of the function α which is often unrealistic. Remarking that Ω is a compact subset of $\mathbb{R}^d$ and that α is continuous under the assumption (A_2), one can define $\alpha_\infty = \max_\Omega \alpha$ and set $h_n = n^{-1/(d+\eta_g \alpha_\infty)}$ and $p_n = \varepsilon_n\, n^{\eta_g/(d+\eta_g \alpha_\infty)}$. In the uniform case (that is, when α is constant equal to 1), with such choices, the rate of convergence of the estimator is then $\varepsilon_n^{1/2}\, n^{\eta_g/(d+\eta_g)}$, which is also, up to the factor ε_n, the optimal rate of convergence for the estimator presented in Girard and Jacob (2008). In the next section, we shall thus use such sequences to compare Girard and Jacob's estimator to the moment estimator.

The asymptotic variance of the estimator also involves the multiplicative factor $V(\alpha(x),\, a)$. The choice of an "optimal" value for a by minimization of $V(\alpha(x),\, a)$ is a difficult task since it depends on the unknown value of $\alpha(x)$. One can observe on Figure 1 that, for $\alpha(x) \leq 2$, $V(\alpha(x),\, .)$ is a decreasing function and thus large values of a should be preferred. These results are now illustrated on finite sample situations.

4 Numerical experiments

In this section, we limit ourselves to unidimensional standard uniform random variables X. ($\Omega = [0,1]$ and $d = 1$). The behavior of the estimators is illustrated on samples with size $n = 500$ on four situations obtained by combining two models for both g and α. The first frontier

$$g_1(x) = \left| \begin{array}{ll} 1 + \exp\left(-60\left(x - \frac{1}{4}\right)^2\right) & \text{if } x \in \left[0,\ \frac{1}{3}\right] \\ 1 + \exp\left(-\frac{5}{12}\right) & \text{if } x \in \left]\frac{1}{3},\ \frac{2}{3}\right] \\ 1 + 5\exp\left(-\frac{5}{12}\right) - 6\exp\left(-\frac{5}{12}\right)x & \text{if } x \in \left]\frac{2}{3},\ \frac{5}{6}\right] \\ 6x - 4 & \text{if } x \in \left]\frac{5}{6},\ 1\right] \end{array} \right.$$

is continuous but not differentiable at $x = 1/3$, $x = 2/3$ and $x = 5/6$. The second one

$$g_2(x) = \left(\frac{1}{10} + \sin(\pi x)\right)\left[\frac{11}{10} - \frac{1}{2}\exp\left(-64\left(x - \frac{1}{2}\right)^2\right)\right]$$

is infinitely differentiable. The first function $\alpha(x)$ considered is the constant $\alpha_1 = 1.25$ and the second one is $\alpha_2(x) = 1.25 + |\cos(4\pi x)|$.

The uniform kernel is chosen:

$$K(x) = \frac{1}{2}1_{[-1,\,1]}(x)$$

with associated bandwidth $h_n^{(m)} = 2\widehat{\sigma}(X)/n^{1/(1+\alpha_\infty)}$ and $p_n^{(m)} = n^{1/(1+\alpha_\infty)}/\sqrt{\ln(n)}$, where $\widehat{\sigma}(X)$ is the empirical standard deviation of X. These sequences are chosen according to the remark below Theorem 1. Note that the multiplicative constant $\widehat{\sigma}(X)$ has been suggested by Girard and Jacob (2008), whereas the constant 2 was empirically chosen. Two versions of the moment estimators are considered. The first one is based on a constant number of moments, $r_n = 35$ ($a = 0$, $b = 35$), and is denoted by $\widehat{g}_n^{(m,\,c)}$. The second one is based on a variable number of moments, $r_n = 15\,p_n^{(m)}$ ($a = 15$, $b = 0$), and is denoted by $\widehat{g}_n^{(m,\,v)}$. The constants have been chosen after intensive simulations. These two estimators are compared to $\widehat{g}_n^{(gj)}$ proposed by Girard and Jacob (2008) with $h_n^{(gj)} = 4\widehat{\sigma}(X)/\sqrt{n}$ and $p_n^{(gj)} = \sqrt{n/\ln(n)}$. On each of the four considered situations, 500 replications of the sample are simulated which permits to compute the minimum, maximum and mean L^1- errors associated to each estimator. Results are reported in Table 1. It appears that, as soon as α increases, performance of all these estimators decrease, since the simulated points are getting more and more distant from the frontier. Let us highlight that, in all the considered situations, the chosen constants give satisfactory results. Moreover, in all cases, both moment estimators $\widehat{g}_n^{(m,\,c)}$ and $\widehat{g}_n^{(m,\,v)}$ yield better results than $\widehat{g}_n^{(gj)}$, the estimator of Girard and Jacob (2008). These results are illustrated on Figure 2–5. On each of the upper panels, the best estimates, *i.e.* the estimates that yield the smallest L_1- errors in Table 1, $\widehat{g}_n^{(gj)}$, $\widehat{g}_n^{(m,\,c)}$ and $\widehat{g}_n^{(m,\,v)}$ are superimposed to the

frontier g. Similarly, the worst estimates are depicted on the lower panels, *i.e.* the estimates that yield the largest L_1- errors in Table 1.

Note also that in three situations over four, the best estimator is $\widehat{g}_n^{(m,\,v)}$, the estimator with a variable number of moments whose L^1 error is smaller than $\widehat{g}_n^{(m,\,c)}$. The essential advantage of $\widehat{g}_n^{(m,\,v)}$ is that, for $\alpha \leq 2$, its asymptotic variance is smaller than that of $\widehat{g}_n^{(m,\,c)}$ (for all a), as illustrated on Figure 1. Thus, taking $a = 15$ allows us to build pointwise confidence intervals for the frontier that are significantly tighter in the case where α is small, but just a little larger in the extreme cases.

5 Proof of Theorem 1

Our goal is to prove that the random variable

$$\xi_n(x) = \frac{g(x)}{\|K\|_2}\sqrt{\frac{f(x)}{V(\alpha(x),\,a)}}\cdot\sqrt{n}\,p_n^{-\alpha(x)/2+1}\,h_n^{d/2}\cdot\left(\frac{1}{\widehat{g}_n(x)}-\frac{1}{g(x)}\right)$$

converges in distribution to a standard Gaussian random variable. Theorem 1 is then a simple consequence of this result. The proof of this result is divided into two parts whether $a = 0$ or not.

1. Let us consider the case $a > 0$. The first step consists to use Lemma 3 in order to linearize $\xi_n(x)$:

$$\begin{aligned}\xi_n(x) &= \left[\zeta_n^{(1)}(x)+\left(\frac{\mu_{p_n+1}(x)}{\widehat{\mu}_{p_n+1}(x)}-1\right)\zeta_n^{(2)}(x)+\left(1+\frac{r_n}{p_n+1}\right)\left(\frac{\mu_{p_n+r_n+1}(x)}{\widehat{\mu}_{p_n+r_n+1}(x)}-1\right)\zeta_n^{(3)}(x)\right]\\ &\times \frac{u_{n,\,a}(x)}{r_n}(1+\mathrm{o}(1)).\end{aligned}$$

Now, Lemma 2 yields

$$\xi_n(x) = \frac{u_{n,\,a}(x)}{r_n}\left[\zeta_n^{(1)}(x)+\mathrm{o}_{\mathbb{P}}(\zeta_n^{(2)}(x))+\mathrm{o}_{\mathbb{P}}(\zeta_n^{(3)}(x))\right](1+\mathrm{o}(1))$$

and to conclude the proof, it is sufficient to establish that

$$\frac{u_{n,\,a}(x)}{r_n}\zeta_n^{(1)}(x)\xrightarrow{d}\mathcal{N}(0,\,1), \tag{2a}$$

$$C^{(2)}\frac{u_{n,\,a}(x)}{r_n}\zeta_n^{(2)}(x)\xrightarrow{d}\mathcal{N}(0,\,1), \tag{2b}$$

$$C^{(3)}\frac{u_{n,\,a}(x)}{r_n}\zeta_n^{(3)}(x)\xrightarrow{d}\mathcal{N}(0,\,1), \tag{2c}$$

where $C^{(2)}$ and $C^{(3)}$ are suitable constants. Note that in fact, (2b) and (2c) are stronger than what is necessary, but their proofs are similar to (2a). In all the sequel, the following notations will be used

$$\begin{aligned}Z_k^{(n,\,c,\,j)}(x) &= Y_k^{cp_n+j}\,K_{h_n}(x-X_k),\\ \mu_{cp_n+j}(x) &= \mathbb{E}(Z^{(n,\,c,\,j)}(x)),\\ \nu_p(x) &= \widehat{\mu}_p(x)-\mu_p(x).\end{aligned}$$

To prove (2a), remark that $\zeta_n^{(1)}(x)$ can be expanded as the sum of independent and centered random variables: $\zeta_n^{(1)}(x) = \sum_{k=1}^n S_{n,\,k}^{(1,\,a)}(x)$ with

$$\begin{aligned}
S_{n,\,k}^{(1,\,a)}(x) &= \frac{1}{n}\left[Z_k^{(n,\,1,\,0)}(x),\ Z_k^{(n,\,1,\,1)}(x),\ Z_k^{(n,\,1+a,\,b)}(x),\ Z_k^{(n,\,1+a,\,b+1)}(x)\right] A_n^{(1)}(x), \quad (3)\\
A_n^{(1)}(x) &= \left[a_{n,\,0}^{(1)}(x),\ a_{n,\,1}^{(1)}(x),\ a_{n,\,2}^{(1)}(x),\ a_{n,\,3}^{(1)}(x)\right]^t,\\
a_{n,\,0}^{(1)}(x) &= -1,\\
a_{n,\,1}^{(1)}(x) &= \frac{\mu_{p_n}(x)}{\mu_{p_n+1}(x)},\\
a_{n,\,2}^{(1)}(x) &= \left(1+\frac{r_n}{p_n+1}\right)\frac{\mu_{p_n+1}(x)}{\mu_{p_n+r_n+1}(x)},\\
a_{n,\,3}^{(1)}(x) &= -\left(1+\frac{r_n}{p_n+1}\right)\frac{\mu_{p_n+1}(x)\,\mu_{p_n+r_n}(x)}{\mu_{p_n+r_n+1}^2(x)},
\end{aligned}$$

where A^t stands for the transposed matrix of A. In order to use Lyapounov's central limit theorem (see e.g. Billingsley, 1979, p. 312), it remains to prove that

$$\sum_{k=1}^n \frac{1}{[\mathrm{Var}(\zeta_n^{(1)}(x))]^{3/2}}\mathbb{E}|S_{n,\,k}^{(1,\,a)}(x)|^3 \xrightarrow[n\to\infty]{} 0, \quad (4)$$

which requires to control $\mathrm{Var}(\zeta_n^{(1)}(x))$ and $\mathbb{E}|S_{n,\,k}^{(1,\,a)}(x)|^3$. The variance can be rewritten as

$$\begin{aligned}
\mathrm{Var}(\zeta_n^{(1)}(x)) &= \frac{1}{n}\left[w(p_n,\,p_n)(x) + \left(1+\frac{r_n}{p_n+1}\right)^2\frac{\mu_{p_n+1}^2(x)}{\mu_{p_n+r_n+1}^2(x)}\,w(p_n+r_n,\,p_n+r_n)(x)\right.\\
&\qquad \left. -2\left(1+\frac{r_n}{p_n+1}\right)\frac{\mu_{p_n+1}(x)}{\mu_{p_n+r_n+1}(x)}\,w(p_n,\,p_n+r_n)(x)\right]
\end{aligned}$$

where

$$w(sp_n+t,\,up_n+v)(x) = \left[-1,\ \frac{\mu_{sp_n+t}(x)}{\mu_{sp_n+t+1}(x)}\right] M_n(s,\,t,\,u,\,v)(x)\left[-1,\ \frac{\mu_{up_n+v}(x)}{\mu_{up_n+v+1}(x)}\right]^t$$

and $M_n(s,\,t,\,u,\,v)(x)$ is the 2×2 covariance matrix defined by

$$M_n(s,\,t,\,u,\,v)(x) = \begin{bmatrix} \mathbb{E}(Z^{(n,\,s,\,t)}(x)\,Z^{(n,\,u,\,v)}(x)) & \mathbb{E}(Z^{(n,\,s,\,t)}(x)\,Z^{(n,\,u,\,v+1)}(x)) \\ \mathbb{E}(Z^{(n,\,s,\,t+1)}(x)\,Z^{(n,\,u,\,v)}(x)) & \mathbb{E}(Z^{(n,\,s,\,t+1)}(x)\,Z^{(n,\,u,\,v+1)}(x)) \end{bmatrix}.$$

Therefore, since Lemma 1 provides an asymptotic expansion of the matrix $M_n(s,\,t,\,u,\,v)(x)$, it is sufficient to compute an asymptotic expansion of $\mu_{sp_n+t}(x)/\mu_{sp_n+t+1}(x)$. The use of Lemma 1 and tedious computations lead to

$$\mathrm{Var}(\zeta_n^{(1)}(x)) \sim a^2\|K\|_2^2\,f(x)\,\Gamma^2(\alpha(x)+1)V(\alpha(x),\,a)\,\frac{1}{n}\,\frac{1}{h_n^d}\,g^{2p_n}(x)\,p_n^{-\alpha(x)-2}. \quad (5)$$

Now, focusing on the third moment, Hölder's inequality yields

$$\begin{aligned}
n^3\,\mathbb{E}|S_{n,\,1}^{(1,\,a)}(x)|^3 &\le 4\cdot\mathbb{E}|a_{n,\,0}^{(1)}(x)\,Z_1^{(n,\,1,\,0)}(x) + a_{n,\,1}^{(1)}(x)\,Z_1^{(n,\,1,\,1)}(x)|^3\\
&+ 4\cdot\mathbb{E}|a_{n,\,2}^{(1)}(x)\,Z_1^{(n,\,1+a,\,b)}(x) + a_{n,\,3}^{(1)}(x)\,Z_1^{(n,\,1+a,\,b+1)}(x)|^3.
\end{aligned}$$

The next step consists in applying Lemma 4 to each term of the right-hand side of the inequality. To this end, let us consider the functions

$$\begin{aligned}
H_{n,\,0}^{(1,\,a)}(u) &= -1,\\
H_{n,\,1}^{(1,\,a)}(u) &= \alpha(x)u,\\
H_{n,\,2}^{(1,\,a)}(u) &= \left(1+\frac{r_n}{p_n+1}\right) g^{r_n}(x)\,\frac{\mu_{p_n+1}(x)}{\mu_{p_n+r_n+1}(x)},\\
H_{n,\,3}^{(1,\,a)}(u) &= -\left(1+\frac{r_n}{p_n+1}\right) g^{r_n}(x)\,\frac{\mu_{p_n+1}(x)}{\mu_{p_n+r_n+1}(x)}\cdot\frac{\alpha(x)u}{1+a},
\end{aligned}$$

and note that there exist two sequences of measurable functions $(\chi_{n,\,1}^{(a)})$ and $(\chi_{n,\,2}^{(a)})$ uniformly convergent to 0 on [0, 1] such that

$$\begin{aligned}
\max_{u\in B}\left|a_{n,\,0}^{(1)}(x)+a_{n,\,1}^{(1)}(x)\,g(x-h_nu)\,y\right| &\le |H_{n,\,0}^{(1,\,a)}(y)|(1-y)+\frac{|H_{n,\,1}^{(1,\,a)}(y)|}{p_n}+\frac{\chi_{n,\,1}^{(a)}(y)}{p_n},\\
\max_{u\in B}\left|a_{n,\,2}^{(1)}(x)+a_{n,\,3}^{(1)}(x)\,g(x-h_nu)\,y\right| &\le \frac{1}{g^{r_n}(x)}\left[|H_{n,\,2}^{(1,\,a)}(y)|(1-y)+\frac{|H_{n,\,3}^{(1,\,a)}(y)|}{p_n}+\frac{\chi_{n,\,2}^{(a)}(y)}{p_n}\right].
\end{aligned}$$

Since

$$g^{r_n}(x)\,\frac{\mu_{p_n+1}(x)}{\mu_{p_n+r_n+1}(x)}\sim(1+a)^{\alpha(x)},$$

the functions $H_{n,\,j}^{(1,\,a)}$, $j\in\{0,1,2,3\}$ are bounded on [0, 1], uniformly in n, and thus Lemma 4 entails that

$$\mathbb{E}|S_{n,\,1}^{(1,\,a)}(x)|^3=\mathrm{O}(n^{-3}\cdot g^{3p_n}(x)\cdot p_n^{-3-\alpha(x)}\cdot h_n^{-2d}). \tag{6}$$

Combining (5) and (6), convergence (4) follows from the condition $np_n^{-\alpha(x)}h_n^d\to\infty$ and therefore (2a) holds.

Proofs of (2b) and (2c) are completely similar since $\zeta_n^{(2)}$ and $\zeta_n^{(3)}$ can be rewritten as

$$\begin{aligned}
\zeta_n^{(2)}(x) &= \frac{1}{n}\sum_{k=1}^{n}\left[Z_k^{(n,\,1,\,0)}(x),\ Z_k^{(n,\,1,\,1)}(x)\right]\left[a_{n,\,0}^{(2)}(x),\ a_{n,\,1}^{(2)}(x)\right]^t,\\
\zeta_n^{(3)}(x) &= \frac{1}{n}\sum_{k=1}^{n}\left[Z_k^{(n,\,1+a,\,b)}(x),\ Z_k^{(n,\,1+a,\,1+b)}(x)\right]\left[a_{n,\,0}^{(3)}(x),\ a_{n,\,1}^{(3)}(x)\right]^t,
\end{aligned}$$

with clear definitions of the sequences $a_{n,\,i}^{(j)}(x)$, $i=0,1$, $j=2,3$. Applying Lemma 4 with

$$\begin{aligned}
H_{n,\,0}^{(2,\,a)}(u) &= -1,\\
H_{n,\,1}^{(2,\,a)}(u) &= \alpha(x)u,\\
H_{n,\,0}^{(3,\,a)}(u) &= g^{r_n}(x)\,\frac{\mu_{p_n+1}(x)}{\mu_{p_n+r_n+1}(x)},\\
H_{n,\,1}^{(3,\,a)}(u) &= -g^{r_n}(x)\,\frac{\mu_{p_n+1}(x)}{\mu_{p_n+r_n+1}(x)}\cdot\frac{\alpha(x)u}{1+a}
\end{aligned}$$

yields $\mathbb{E}|S_{n,\,1}^{(j,\,a)}(x)|^3=\mathrm{O}(n^{-3}\cdot g^{3p_n}(x)\cdot p_n^{-3-\alpha(x)}\cdot h_n^{-2d})$, $j=2,3$. Lyapounov's central limit theorem then gives the convergence. The first part of Theorem 1 is therefore established.

2. Now, we shift to the case $a = 0$ which is analogous to the previous one, but this time the asymptotic expansions have to be more precise. The computations are therefore much more complicated, but all the details can be found in Girard *et al.* (2010). At first sight, it can appear somewhat odd as the case $a = 0$ corresponds to the simple case of a constant value of the sequence r_n. However, heuristically, in the case $a > 0$, our estimator is a Césaro mean of the estimators considered in the case $a = 0$. As a consequence, the convergence of the latter is expected to be harder to study than that of the former.

In the case $a = 0$, we obtain, using Lemma 3,

$$\begin{aligned}
\xi_n(x) &= \frac{u_{n,\,0}(x)}{b}\left[\zeta_n^{(1)}(x) + \left(\frac{\mu_{p_n+1}(x)}{\widehat{\mu}_{p_n+1}(x)} - 1\right)\widetilde{\zeta}_n^{(2)}(x)\right](1 + \mathrm{o}(1)) \\
&+ \frac{u_{n,\,0}(x)}{b}\left[\left(\frac{\mu_{p_n+b+1}(x)}{\widehat{\mu}_{p_n+b+1}(x)} - \frac{\mu_{p_n+1}(x)}{\widehat{\mu}_{p_n+1}(x)}\right)\zeta_n^{(3)}(x)\right](1 + \mathrm{o}(1)) \\
&+ \frac{u_{n,\,0}(x)}{p_n + 1}\left(\frac{\mu_{p_n+b+1}(x)}{\widehat{\mu}_{p_n+b+1}(x)} - 1\right)\zeta_n^{(3)}(x)(1 + \mathrm{o}(1)).
\end{aligned}$$

We shall therefore establish that

$$\frac{u_{n,\,0}(x)}{b}\,\zeta_n^{(1)}(x) \xrightarrow{d} \mathcal{N}(0,\,1), \tag{7a}$$

$$u_{n,\,0}(x)\left[\frac{\mu_{p_n+1}(x)}{\widehat{\mu}_{p_n+1}(x)} - 1\right]\widetilde{\zeta}_n^{(2)}(x) \xrightarrow{\mathbb{P}} 0, \tag{7b}$$

$$u_{n,\,0}(x)\left[\frac{\mu_{p_n+b+1}(x)}{\widehat{\mu}_{p_n+b+1}(x)} - \frac{\mu_{p_n+1}(x)}{\widehat{\mu}_{p_n+1}(x)}\right]\zeta_n^{(3)}(x) \xrightarrow{\mathbb{P}} 0, \tag{7c}$$

$$u_{n,\,0}(x)\,\frac{1}{p_n+1}\left[\frac{\mu_{p_n+b+1}(x)}{\widehat{\mu}_{p_n+b+1}(x)} - 1\right]\zeta_n^{(3)}(x) \xrightarrow{\mathbb{P}} 0. \tag{7d}$$

Using representation (3), $\mathrm{Var}(\zeta_n^{(1)}(x))$ has to be expanded up to order $\mathrm{o}(p_n^{-4-\alpha(x)})$. Indeed, comparing the two versions of Lemma 3 whether $a = 0$ or not, it appears that in the latter the right-hand side of Lemma 3 is divided by r_n instead of b, a constant, when $a = 0$. This explains why in this case, the variance has to be expanded to a higher order. To this aim, remark that

$$\mathrm{Var}(\zeta_n^{(1)}(x)) = \frac{1}{n} A_n^{(1)}(x)^t\, B_n(x)\, A_n^{(1)}(x)$$

where $B_n(x)$ is the 4×4 covariance matrix defined by

$$B_n(x) = \begin{bmatrix} M_n(1,\,0,\,1,\,0)(x) & M_n(1,\,0,\,1,\,b)(x) \\ M_n(1,\,b,\,1,\,0)(x) & M_n(1,\,b,\,1,\,b)(x) \end{bmatrix}.$$

From (8) in the proof of Lemma 1, Stirling's formula (see e.g. Abramowitz and Stegun, 1965, p. 257) and Tricomi and Erdelyi (1951), we deduce that

$$\mathrm{Var}(\zeta_n^{(1)}(x)) \sim \|K\|_2^2\, f(x)\, \Gamma^2(\alpha(x)+1) V(\alpha(x),\,0)\, b^2 \frac{1}{n}\frac{1}{h_n^d}\, g^{2p_n}(x)\, p_n^{-\alpha(x)-4}.$$

These tedious computations are detailed in Girard *et al.* (2010). Like in the case $a > 0$, our aim is now to apply Lyapounov's central limit theorem which requires the order of $\mathbb{E}|S_{n,\,k}^{(1,\,0)}(x)|^3$. This can be obtained from Lemma 4. Setting

$$
\begin{aligned}
H_{n,\,0}^{(1,\,0)}(y) &= -\frac{1-y^b}{1-y},\\
H_{n,\,1}^{(1,\,0)}(y) &= \left[\alpha(x) + \frac{\displaystyle\int_B L_n(x,\,u)\Big[J_n(x,\,u) + p_n\,G_n(x,\,u)\Big]K(u)\,du}{\displaystyle\int_B L_n(x,\,u)K(u)\,du}\right]\left[y\frac{1-y^b}{1-y} + by^b\right],\\
&\quad + p_n \max_{u\in B}\left|\frac{g^b(x-h_nu)-g^b(x)}{g^b(x)}\right| y^b + p_n \max_{u\in B}\left|\frac{g(x-h_nu)-g(x)}{g(x)}\right| y\frac{1-y^b}{1-y} + by^b\\
H_{n,\,2}^{(1,\,0)}(y) &= -\alpha(x)y + \left[\frac{(b-1)b}{2}\alpha^2(x) + \left[b(b-1) - \frac{b(b+1)}{2}\right]\alpha(x) - b\right]y^b\\
&\quad - \left[\frac{b(b+1)}{2}\alpha^2(x) + \left[b^2 - \frac{(b+1)(b+2)}{2}\right]\alpha(x) - b\right]y^{b+1},
\end{aligned}
$$

there exists a sequence of measurable functions $(\chi_n^{(1,\,0)})$ uniformly convergent to 0 on $[0,1]$ such that

$$
\max_{u\in B}\left|a_{n,\,0}^{(1)}(x) + a_{n,\,1}^{(1)}(x)\,g(x-h_nu)\,y + a_{n,\,2}^{(1)}(x)\,g^b(x-h_nu)\,y^b + a_{n,\,3}^{(1)}(x)\,g^{b+1}(x-h_nu)\,y^{b+1}\right|
$$

$$
\le |H_{n,\,0}^{(1,\,0)}(y)|(1-y)^2 + \frac{|H_{n,\,1}^{(1,\,0)}(y)|}{p_n}(1-y) + \frac{|H_{n,\,2}^{(1,\,0)}(y)|}{p_n^2} + \frac{\chi_n^{(1,\,0)}(y)}{p_n^2}.
$$

For all $b \ge 1$, the function $y \mapsto (1-y^b)/(1-y)$ is bounded on [0, 1], so that the functions $H_{n,\,j}^{(1,\,0)}$ are bounded on [0, 1] uniformly in n. Lemma 4 then yields

$$
\mathbb{E}|S_{n,\,1}^{(1,\,0)}(x)|^3 = \mathrm{O}(n^{-3}\,g^{3p_n}(x)\,p_n^{-6-\alpha(x)}\,h_n^{-2d})
$$

and consequently (7a) holds.

Let us come back to the proof of (7b)–(7d) and remark that $\widetilde{\zeta}_n^{(2)}(x) = \sum_{k=1}^n S_{n,\,k}^{(2,\,0)}(x)$ where

$$
S_{n,\,k}^{(2,\,0)}(x) = \frac{1}{n}\left[Z_k^{(n,\,1,\,0)}(x),\ Z_k^{(n,\,1,\,1)}(x),\ Z_k^{(n,\,1,\,b)}(x),\ Z_k^{(n,\,1,\,b+1)}(x)\right]A_n^{(2)}(x),
$$

and

$$
A_n^{(2)}(x) = \left[a_{n,\,0}^{(2)}(x),\ a_{n,\,1}^{(2)}(x),\ a_{n,\,0}^{(3)}(x),\ a_{n,\,1}^{(3)}(x)\right]^t.
$$

Similarly to the case $a > 0$, direct computations lead to

$$
\mathrm{Var}(\widetilde{\zeta}_n^{(2)}(x)) \sim \|K\|_2^2\,f(x)\,\Gamma^2(\alpha(x)+1)\widetilde{V}(\alpha(x),\,0)\,b^2\,\frac{1}{n}\,\frac{1}{h_n^d}\,g^{2p_n}(x)\,p_n^{-\alpha(x)-4}
$$

with

$$
\widetilde{V}(\alpha(x),\,0) = \frac{1}{\Gamma(\alpha(x))}\cdot 2^{-\alpha(x)-4}\cdot(\alpha^3(x) - 2\alpha^2(x) + 3\alpha(x) + 6).
$$

To apply Lyapounov's central limit theorem, the order of $\mathbb{E}|S_{n,\,1}^{(2,\,0)}(x)|^3$ has to be determined.

To this aim, Lemma 4 is used with

$$
\begin{aligned}
H_{n,\,0}^{(2,\,0)}(y) &= -\frac{1-y^b}{1-y},\\
H_{n,\,1}^{(2,\,0)}(y) &= \left[\alpha(x) + \frac{\displaystyle\int_B L_n(x,\,u)\Big[J_n(x,\,u) + p_n\,G_n(x,\,u)\Big]K(u)\,du}{\displaystyle\int_B L_n(x,\,u)K(u)\,du}\right]\left[y\frac{1-y^b}{1-y} + by^b\right]\\
&\quad + p_n \max_{u\in B}\left|\frac{g^b(x-h_nu)-g^b(x)}{g^b(x)}\right| y^b + p_n \max_{u\in B}\left|\frac{g(x-h_nu)-g(x)}{g(x)}\right| y\frac{1-y^b}{1-y},\\
H_{n,\,2}^{(2,\,0)}(y) &= -\alpha(x)y + \left[\frac{(b-1)b}{2}\alpha^2(x) - \left[\frac{b(b+1)}{2} + b\right]\alpha(x)\right] y^b\\
&\quad - \left[\frac{b(b+1)}{2}\alpha^2(x) - \left[b + \frac{(b+1)(b+2)}{2}\right]\alpha(x)\right] y^{b+1},
\end{aligned}
$$

after remarking that there exists a measurable sequence $(\chi_n^{(2,\,0)})$ uniformly convergent to 0 on $[0,1]$ such that

$$
\max_{u\in B}\left|a_{n,\,0}^{(2)}(x) + a_{n,\,1}^{(2)}(x)\,g(x-h_nu)\,y + a_{n,\,0}^{(3)}(x)\,g^b(x-h_nu)\,y^b + a_{n,\,1}^{(3)}(x)\,g^{b+1}(x-h_nu)\,y^{b+1}\right|
$$

$$
\le |H_{n,\,0}^{(2,\,0)}(y)|(1-y)^2 + \frac{|H_{n,\,1}^{(2,\,0)}(y)|}{p_n}(1-y) + \frac{|H_{n,\,2}^{(2,\,0)}(y)|}{p_n^2} + \frac{\chi_n^{(2,\,0)}(y)}{p_n^2}.
$$

It follows that $\mathbb{E}|S_{n,\,1}^{(2,\,0)}(x)|^3 = \mathrm{O}(n^{-3}\,g^{3p_n}(x)\,p_n^{-6-\alpha(x)}\,h_n^{-2d})$ which yields (7b). Proofs of (7c) and (7d) use the same method and are therefore omitted. Finally, since

$$
\frac{\mu_{p_n+1}(x)}{p_n+1}\,\frac{u_{n,\,0}(x)}{b} \sim \frac{g(x)}{\|K\|_2}\sqrt{\frac{f(x)}{V(\alpha(x),\,0)}}\,\sqrt{n}\,p_n^{-\alpha(x)/2+1}\,h_n^{d/2},
$$

the second part of Theorem 1 is established. ■

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Appendix A: Auxiliary results

The first result of this section is technical and only useful for the proof of our next lemmas and of the main theorem.

Lemma 1. *For all $x \in \mathbb{R}^d$ such that $f(x) > 0$, $u \in B$ and $n \in \mathbb{N}^*$, let*

$$\begin{aligned} J_n(x,\, u) &= \alpha(x - h_n u) - \alpha(x) \\ G_n(x,\, u) &= 1 - \frac{g(x - h_n u)}{g(x)} \\ L_n(x,\, u) &= \frac{f(x - h_n u)\,\Gamma(\alpha(x - h_n u) + 1)}{f(x)\,\Gamma(\alpha(x) + 1)} \exp\left[-p_n G_n(x,\, u) - \ln(p_n) J_n(x,\, u)\right]. \end{aligned}$$

Suppose $(A_1 - A_4)$ hold and $p_n\, h_n^{\eta_g} \longrightarrow 0$ as $n \to \infty$. Then, there exist two real numbers δ_1 and δ_2 such that for all $q \in [1,\, 3]$,

$$\begin{aligned} h_n^{d(q-1)} \frac{\mathbb{E}(Y^{p_n}\, K_{h_n}^q(X - x))}{\Gamma(\alpha(x) + 1)\, f(x)\, g^{p_n}(x)\, p_n^{-\alpha(x)}} &= \int_B L_n(x,\, u) \left[1 + \frac{\delta_1}{p_n} - \frac{p_n}{2} G_n^2(x,\, u)\right] K^q(u)\, du \\ &- \frac{1}{p_n} \int_B L_n(x,\, u) \frac{J_n(x,\, u)}{2} \left[\alpha(x - h_n u) + \alpha(x) + 1\right] K^q(u)\, du \\ &+ \frac{\delta_2}{p_n^2} \int_B K^q(u)\, du + \mathrm{o}(p_n^{-2}). \end{aligned}$$

Most of the time, in the proofs, a weaker result, which is a corollary of the previous lemma, is enough:

Corollary 1. *Let us denote, for all integers i and j,*

$$\begin{aligned} \varphi^{(n,\, c,\, i)}(x) &= \alpha(x)\, g^{cp_n + i}(x)\, B(cp_n + i + 1,\, \alpha(x)) \\ \psi^{(n,\, c,\, c',\, i,\, j)}(x) &= \alpha(x)\, g^{(c+c')p_n + i + j}(x)\, B((c + c')p_n + i + j + 1,\, \alpha(x)) \end{aligned}$$

Suppose $(A_1 - A_4)$ *hold and* $p_n\, h_n^{\eta_g} \longrightarrow 0$ *as* $n \to \infty$. *Then, for all* $x \in \mathbb{R}^d$ *such that* $f(x) > 0$, *we have*

$$\begin{aligned}
\mathbb{E}(Z^{(n,\,c,\,i)}(x)) &\sim \varphi^{(n,\,c,\,i)}(x)\, f(x) \\
\mathbb{E}(Z^{(n,\,c,\,i)}(x)\, Z^{(n,\,c',\,j)}(x)) &\sim \frac{1}{h_n^d}\, \psi^{(n,\,c,\,c',\,i,\,j)}(x)\, f(x)\, \|K\|_2^2 \\
\mathrm{Cov}(Z^{(n,\,c,\,i)}(x),\, Z^{(n,\,c',\,j)}(x)) &\sim \Gamma(\alpha(x)+1) f(x) \|K\|_2^2 \frac{g^{(c+c')p_n+i+j}(x)\, ((c+c')p_n)^{-\alpha(x)}}{h_n^d}.
\end{aligned}$$

The quantity $\widehat{\mu}_{cp_n+k}(x)/\mu_{cp_n+k}(x)$ appears at several places in the linear expansion of $\xi_n(x)$ given in Lemma 3 below. The corollary we have just stated allows us to prove its convergence in probability to 1.

Lemma 2. *Suppose* $(A_1 - A_4)$ *hold. For all* $x \in \mathbb{R}^d$ *such that* $f(x) > 0$, $n\, p_n^{-\alpha(x)}\, h_n^d \longrightarrow +\infty$ *and* $p_n\, h_n^{\eta_g} \longrightarrow 0$ *as* $n \to \infty$, *we have:*

$$\frac{\widehat{\mu}_{cp_n+k}(x)}{\mu_{cp_n+k}(x)} = 1 + \mathrm{o}_{\mathbb{P}}(1).$$

Our next lemma consists in linearizing the quantity $\xi_n(x)$.

Lemma 3. *Let* (p_n) *be an arbitrary sequence of positive real numbers tending to* $+\infty$. *Two cases are considered.*

i) If $a > 0$ *then*

$$\begin{aligned}
\xi_n(x) &= \left[\zeta_n^{(1)}(x) + \left(\frac{\mu_{p_n+1}(x)}{\widehat{\mu}_{p_n+1}(x)} - 1\right)\zeta_n^{(2)}(x) + \left(1 + \frac{r_n}{p_n+1}\right)\left(\frac{\mu_{p_n+r_n+1}(x)}{\widehat{\mu}_{p_n+r_n+1}(x)} - 1\right)\zeta_n^{(3)}(x)\right] \\
&\times \frac{u_{n,\,a}(x)}{r_n}(1 + \mathrm{o}(1))
\end{aligned}$$

where

$$\begin{aligned}
\zeta_n^{(1)}(x) &= -\nu_{p_n}(x) + \frac{\mu_{p_n}(x)}{\mu_{p_n+1}(x)}\nu_{p_n+1}(x) \\
&\quad + \left[1 + \frac{r_n}{p_n+1}\right] \cdot \left[\frac{\mu_{p_n+1}(x)}{\mu_{p_n+r_n+1}(x)}\, \nu_{p_n+r_n}(x) - \frac{\mu_{p_n+1}(x)\, \mu_{p_n+r_n}(x)}{\mu_{p_n+r_n+1}^2(x)}\nu_{p_n+r_n+1}(x)\right] \\
\zeta_n^{(2)}(x) &= -\nu_{p_n}(x) + \frac{\mu_{p_n}(x)}{\mu_{p_n+1}(x)}\nu_{p_n+1}(x) \\
\zeta_n^{(3)}(x) &= \frac{\mu_{p_n+1}(x)}{\mu_{p_n+r_n+1}(x)}\, \nu_{p_n+r_n}(x) - \frac{\mu_{p_n+1}(x)\, \mu_{p_n+r_n}(x)}{\mu_{p_n+r_n+1}^2(x)}\, \nu_{p_n+r_n+1}(x) \\
\text{and } u_{n,\,a}(x) &= \frac{1}{\|K\|_2\, \Gamma(\alpha(x)+1) g^{p_n}(x)} \sqrt{\frac{1}{f(x)\, V(\alpha(x),\, a)}}\, \sqrt{n}\, p_n^{\alpha(x)/2+1}\, h_n^{d/2}.
\end{aligned}$$

ii) If $a = 0$ then $r_n = b$ and

$$\begin{aligned}\xi_n(x) &= \frac{u_{n,\,0}(x)}{b}\left[\zeta_n^{(1)}(x) + \left(\frac{\mu_{p_n+1}(x)}{\widehat{\mu}_{p_n+1}(x)} - 1\right)\widetilde{\zeta}_n^{(2)}(x)\right](1+\mathrm{o}(1)) \\ &+ \frac{u_{n,\,0}(x)}{b}\left[\left(\frac{\mu_{p_n+b+1}(x)}{\widehat{\mu}_{p_n+b+1}(x)} - \frac{\mu_{p_n+1}(x)}{\widehat{\mu}_{p_n+1}(x)}\right)\zeta_n^{(3)}(x)\right](1+\mathrm{o}(1)) \\ &+ \frac{u_{n,\,0}(x)}{p_n+1}\left(\frac{\mu_{p_n+b+1}(x)}{\widehat{\mu}_{p_n+b+1}(x)} - 1\right)\zeta_n^{(3)}(x)(1+\mathrm{o}(1))\end{aligned}$$

where

$$\widetilde{\zeta}_n^{(2)}(x) = -\nu_{p_n}(x) + \frac{\mu_{p_n}(x)}{\mu_{p_n+1}(x)}\nu_{p_n+1}(x) + \frac{\mu_{p_n+1}(x)}{\mu_{p_n+b+1}(x)}\left[\nu_{p_n+b}(x) - \frac{\mu_{p_n+b}(x)}{\mu_{p_n+b+1}(x)}\nu_{p_n+b+1}(x)\right]$$

and $$u_{n,\,0}(x) = \frac{1}{\|K\|_2\,\Gamma(\alpha(x)+1)g^{p_n}(x)}\sqrt{\frac{1}{f(x)\,V(\alpha(x),\,0)}}\,\sqrt{n}\,p_n^{\alpha(x)/2+2}\,h_n^{d/2}.$$

Finally, the following result provides the asymptotic order of the third moments appearing in the proofs.

Lemma 4. *Suppose $(A_1 - A_4)$ are satisfied and $p_n\,h_n^{\eta_g} \longrightarrow 0$ as $n \to \infty$. Let $k \in \mathbb{N}$, $(b_{n,\,j})_{n\in\mathbb{N}^*,\,0\le j\le k}$ be real numbers and $x \in \mathbb{R}^d$ such that there exist an integer m, sequences of measurable functions $(H_{n,\,j})$, $0 \le j \le m$, uniformly bounded on $[0,\,1]$ and a sequence of measurable functions (χ_n) uniformly convergent to 0 on $[0,\,1]$ with*

$$\max_{u\in B}\left|\sum_{j=0}^{k} b_{n,\,j}\,g^j(x-h_nu)\,y^j\right| \le \sum_{j=0}^{m}\frac{H_{n,\,j}(y)}{p_n^j}(1-y)^{m-j} + \frac{\chi_n(y)}{p_n^m}$$

Let us consider

$$S_n(x) = \sum_{j=0}^{k} b_{n,\,j}\,Z_1^{(n,\,c,\,j)}(x).$$

Then, $\mathbb{E}|S_n(x)|^3 = \mathrm{O}(n^{-3}\,g^{3cp_n}(x)\,p_n^{-3m-\alpha(x)}\,h_n^{-2d})$.

Appendix B: Proof of the auxiliary results

In all the proofs, $C(x)$ will be a generic notation for constants.

Proof of Lemma 1. By definition of the Beta function,

$$h_n^{d(q-1)}\frac{\mathbb{E}(Y^{p_n}\,K_{h_n}^q(X-x))}{\alpha(x)g^{p_n}(x)B(p_n+1,\alpha(x))f(x)\displaystyle\int_B K^q(u)\,du}$$

$$= \int_B Q_n(x,\,u)\frac{\Gamma(p_n+1+\alpha(x))}{\Gamma(p_n+1+\alpha(x-h_nu))}\,\frac{g^{p_n}(x-h_nu)}{g^{p_n}(x)}\,\frac{K^q(u)}{\displaystyle\int_B K^q(u)\,du}\,du \qquad (8)$$

where $Q_n(x,\,u) = \dfrac{f(x-h_nu)\,\Gamma(\alpha(x-h_nu)+1)}{f(x)\,\Gamma(\alpha(x)+1)}$. Since $h_n^{\eta_g} p_n \to 0$ as $n \to \infty$, we have

$$\sup_{u\in B} \ln(p_n)\,|J_n(x,\,u)| \leq \varepsilon_\alpha\, h_n^{\eta_\alpha}\,|\ln p_n| = \varepsilon_\alpha \left[h_n^{\eta_g} p_n\right]^{\eta_\alpha/\eta_g} \frac{|\ln p_n|}{p_n^{\eta_\alpha/\eta_g}} \to 0$$

and

$$\sup_{u\in B} p_n\,|G_n(x,\,u)| \leq p_n\,\varepsilon_g \frac{h_n^{\eta_g}}{g(x)} \to 0$$

as $n \to \infty$, which implies that $J_n(x,\,u) = \mathrm{o}(1/\ln(p_n))$ and $G_n(x,\,u) = \mathrm{o}(1/p_n)$ uniformly in $u \in B$. Applying Stirling's formula, simple calculations yield

$$\frac{\Gamma(p_n+1+\alpha(x))}{\Gamma(p_n+1+\alpha(x-h_nu))} = e^{-\ln(p_n)J_n(x,\,u)}\left[1-\frac{J_n(x,\,u)}{p_n}\left[\frac{1}{2}+\frac{\alpha(x-h_nu)+\alpha(x)}{2}\right]\right] + \mathrm{o}(p_n^{-2}),$$

uniformly in $u \in B$. We then deduce that

$$\begin{aligned}\frac{g^{p_n}(x-h_nu)}{g^{p_n}(x)} &= \exp(p_n \ln(1-G_n(x,\,u))) \\ &= \exp(-p_nG_n(x,\,u))\,\left[1-\frac{p_n}{2}G_n^2(x,\,u)\right] + \mathrm{o}(p_n^{-2})\end{aligned}$$

uniformly in $u \in B$. Now, according to Tricomi and Erdélyi (1951), for all κ and ι, there exist two real numbers $\delta_1(\kappa,\,\iota)$ and $\delta_2(\kappa,\,\iota)$ such that

$$\frac{\Gamma(x+\kappa)}{\Gamma(x+\iota)} = x^{\kappa-\iota}\left[1+\frac{\delta_1(\kappa,\,\iota)}{x}+\frac{\delta_2(\kappa,\,\iota)}{x^2}+\mathrm{o}\left(\frac{1}{x^2}\right)\right].$$

Consequently, setting $\delta_1 = \delta_1(1,\,\alpha(x)+1)$ and $\delta_2 = \delta_2(1,\,\alpha(x)+1)$, we have

$$B(p_n+1,\,\alpha(x)) = \Gamma(\alpha(x))\,p_n^{-\alpha(x)}\left[1+\frac{\delta_1}{p_n}+\frac{\delta_2}{p_n^2}+\mathrm{o}\left(\frac{1}{p_n^2}\right)\right].$$

Since Γ is lipschitz on any compact set in $\mathbb{R}_+ \setminus \{0\}$, the function $y \mapsto \Gamma(\alpha(y)+1)$ is locally Hölderian with exponent η_α. This is also the case for the function $y \mapsto f(y)\,\Gamma(\alpha(y)+1)$ with exponent $\min(\eta_f,\,\eta_\alpha)$. Condition $h_n^{\eta_g} p_n \to 0$ and the results above on G_n and J_n hence imply that $L_n(x,\,u) = 1 + \mathrm{o}(1/\ln(p_n))$ uniformly in $u \in B$. It is then sufficient to use the asymptotic expansion above in (8), and to use the equality

$$\int_B L_n(x,\,u)\frac{\delta_2}{p_n^2}\,K^q(u)\,du = \frac{\delta_2}{p_n^2}\int_B K^q(u)\,du + \mathrm{o}\left(\frac{1}{p_n^2}\right)$$

to conclude the proof of Lemma 1. ■

Proof of Corollary 1. The proof is a direct consequence of Lemma 1 and is therefore omitted. ■

Proof of Lemma 2. Note that $\frac{\widehat{\mu}_{cp_n+k}(x)}{\mu_{cp_n+k}(x)} - 1$ is a centered random variable. Therefore to establish its convergence in probability to 0, it is sufficient to prove that its variance tends to 0. Corollary 1 yields

$$\mathrm{Var}\left(\frac{\widehat{\mu}_{cp_n+k}(x)}{\mu_{cp_n+k}(x)}-1\right) = \frac{\mathrm{Var}(\widehat{\mu}_{cp_n+k}(x))}{\mu^2_{cp_n+k}(x)} \sim C(x)\cdot\frac{1}{n\,p_n^{-\alpha(x)}\,h_n^d}$$

which tends to 0 under the condition $n\, p_n^{-\alpha(x)}\, h_n^d \to \infty$.

Proof of Lemma 3. Let us first remark that, from Corollary 1,

$$\xi_n(x) = \frac{\mu_{p_n+1}(x)}{p_n+1}\, u_{n,\,a}(x) \cdot \left(\frac{1}{\widehat{g}_n(x)} - \frac{1}{g(x)}\right)(1+\mathrm{o}(1)). \tag{9}$$

If $a > 0$, we have

$$\begin{aligned}
\frac{1}{\widehat{g}_n(x)} - \frac{1}{g(x)} &= \frac{1}{r_n}\left[(p_n+r_n+1)\frac{\widehat{\mu}_{p_n+r_n}(x)\,\mu_{p_n+r_n+1}(x) - \mu_{p_n+r_n}(x)\,\widehat{\mu}_{p_n+r_n+1}(x)}{\widehat{\mu}_{p_n+r_n+1}(x)\,\mu_{p_n+r_n+1}(x)}\right.\\
&\qquad \left. -(p_n+1)\frac{\widehat{\mu}_{p_n}(x)\,\mu_{p_n+1}(x) - \mu_{p_n}(x)\,\widehat{\mu}_{p_n+1}(x)}{\widehat{\mu}_{p_n+1}(x)\,\mu_{p_n+1}(x)}\right]\\
&=: \frac{1}{r_n}[\Delta_n^{(1)}(x) - \Delta_n^{(2)}(x)]
\end{aligned}$$

with

$$\begin{aligned}
\Delta_n^{(1)}(x) &:= \frac{p_n+r_n+1}{\mu_{p_n+r_n+1}(x)} \cdot \frac{\mu_{p_n+r_n+1}(x)}{\widehat{\mu}_{p_n+r_n+1}(x)} \cdot \left(\nu_{p_n+r_n}(x) - \frac{\mu_{p_n+r_n}(x)}{\mu_{p_n+r_n+1}(x)}\,\nu_{p_n+r_n+1}(x)\right),\\
\Delta_n^{(2)}(x) &:= \frac{p_n+1}{\mu_{p_n+1}(x)} \cdot \frac{\mu_{p_n+1}(x)}{\widehat{\mu}_{p_n+1}(x)} \cdot \left(\nu_{p_n}(x) - \frac{\mu_{p_n}(x)}{\mu_{p_n+1}(x)}\,\nu_{p_n+1}(x)\right),
\end{aligned}$$

and consequently,

$$\begin{aligned}
\frac{\mu_{p_n+1}(x)}{p_n+1} \cdot \Delta_n^{(1)}(x) &= \left(1 + \frac{r_n}{p_n+1}\right)\frac{\mu_{p_n+r_n+1}(x)}{\widehat{\mu}_{p_n+r_n+1}(x)} \cdot \zeta_n^{(3)}(x),\\
\frac{\mu_{p_n+1}(x)}{p_n+1} \cdot \Delta_n^{(2)}(x) &= -\frac{\mu_{p_n+1}(x)}{\widehat{\mu}_{p_n+1}(x)} \cdot \zeta_n^{(2)}(x).
\end{aligned}$$

Replacing in (9) and remarking that

$$\zeta_n^{(1)}(x) = \zeta_n^{(2)}(x) + \left(1 + \frac{r_n}{p_n+1}\right)\zeta_n^{(3)}(x)$$

lead to the first part of Lemma 3. To prove the second part ($a = 0$), we only have to remark that $\widetilde{\zeta}_n^{(2)}(x) = \zeta_n^{(2)}(x) + \zeta_n^{(3)}(x)$. ■

Proof of Lemma 4. We only prove the case $c = 1$ since the result can be obtained by replacing p_n by cp_n. Let $\omega : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}_+$ be the function defined by

$$\omega(y,\, u) = \frac{\alpha(u)}{g(u)}\left(1 - \frac{y}{g(u)}\right)^{\alpha(u)-1} 1_{[0,\, g(u)]}(y).$$

We get

$$\begin{aligned}
\mathbb{E}|S_n(x)|^3 &= \frac{1}{n^3}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\left|\sum_{j=0}^{k} b_{n,\,j}\, y^{p_n+j}\, K_{h_n}(x-u)\right|^3 \omega(y,\, u)\, f(u)\, du\, dy\\
&= \frac{1}{n^3 h_n^{2d}}\int_B\int_0^1\left|\sum_{j=0}^{k} b_{n,\,j}\, y^{p_n+j}\, g^{p_n+j}(x-h_n u)\right|^3 K^3(u)\Psi(y,\, x-h_n u) f(x-h_n u)\, du\, dy
\end{aligned}$$

where $\Psi : [0,1] \times \mathbb{R}^d \to \mathbb{R}_+$ is the function defined by $\Psi(y,\, u) = \alpha(u)\,(1-y)^{\alpha(u)-1}$. From condition $h_n^{\eta_g} p_n \to 0$ as $n \to \infty$, we obtain

$$\frac{n^3 \mathbb{E}|S_n(x)|^3}{g^{3p_n}(x)} \leq \frac{c_n(x)}{h_n^{2d}} \int_B \int_0^1 \left| \sum_{j=0}^{k} b_{n,\,j}\, y^j\, g^j(x - h_n u) \right|^3 K^3(u)\, y^{3p_n}\, \Psi(y,\, x - h_n u)\, f(x - h_n u)\, du\, dy$$

where $c_n(x) \to 1$ as $n \to \infty$. From the assumptions on $b_{n,\,j}$ and Hölder's inequality, it follows

$$\max_{u \in B} \left| \sum_{j=0}^{k} b_{n,\,j}\, y^j\, g^j(x - h_n u) \right|^3 \leq (m+2)^2 \left[\sum_{j=0}^{m} \frac{|H_{n,\,j}(y)|^3}{p_n^{3j}} (1-y)^{3m-3j} + \frac{|\chi_n(y)|^3}{p_n^{3m}} \right].$$

Therefore, let us first consider, for $j \in \{0, \ldots, m\}$,

$$\begin{aligned}
& \int_B \int_0^1 \frac{|H_{n,\,j}(y)|^3}{p_n^{3j}} (1-y)^{3m-3j}\, K^3(u)\, y^{3p_n}\, \Psi(y,\, x - h_n u)\, f(x - h_n u)\, du\, dy \\
\leq\ & p_n^{-3j} \sup_{\substack{[0,\,1] \\ n \in \mathbb{N}^*}} |H_{n,\,j}|^3 \int_B B(3p_n + 1,\, 3m - 3j + \alpha(x - h_n u))\, \alpha(x - h_n u)\, K^3(u)\, f(x - h_n u)\, du \\
\leq\ & C(x)\, p_n^{-3j} \sup_{\substack{[0,\,1] \\ n \in \mathbb{N}^*}} |H_{n,\,j}|^3\, B(3p_n + 1,\, 3m - 3j + \alpha(x))\, \alpha(x)\, f(x) \int_B K^3(u)\, du \\
\sim\ & C(x)\, p_n^{-3m-\alpha(x)}
\end{aligned}$$

following the lines of the proof of Lemma 1. Secondly,

$$\begin{aligned}
& \int_B \int_0^1 \frac{|\chi_n(y)|^3}{p_n^{3m}} K^3(u)\, y^{3p_n}\, \Psi(y,\, x - h_n u)\, f(x - h_n u)\, du\, dy \\
\leq\ & p_n^{-3m} \max_{[0,\,1]} |\chi_n|^3 \int_B B(3p_n + 1,\, \alpha(x - h_n u))\, \alpha(x - h_n u)\, K^3(u)\, f(x - h_n u)\, du \\
\leq\ & C(x)\, p_n^{-3m} \max_{[0,\,1]} |\chi_n|^3\, B(3p_n + 1,\, \alpha(x))\, \alpha(x)\, f(x) \int_B K^3(u)\, du \\
\sim\ & C(x) \max_{[0,\,1]} |\chi_n|^3\, p_n^{-3m-\alpha(x)}.
\end{aligned}$$

As $\max_{[0,\,1]} |\chi_n| \to 0$ when $n \to +\infty$, it follows that $(n^3\, g^{-3p_n}(x)\, p_n^{3m+\alpha(x)}\, h_n^{2d}\, \mathbb{E}|S_n(x)|^3)$ is bounded and the result is proved. ■

Parameters	$\widehat{g}_n^{(m,\,c)}$	$\widehat{g}_n^{(m,\,v)}$	$\widehat{g}_n^{(gj)}$
$\alpha(x) = \alpha_1$			
Frontier g_1	0.082 [0.054, 0.137]	0.083 [0.051, 0.143]	0.091 [0.043, 0.161]
Frontier g_2	0.048 [0.036, 0.068]	0.047 [0.034, 0.068]	0.050 [0.033, 0.074]
$\alpha(x) = \alpha_2(x)$			
Frontier g_1	0.141 [0.091, 0.202]	0.140 [0.089, 0.203]	0.246 [0.175, 0.326]
Frontier g_2	0.099 [0.072, 0.128]	0.097 [0.072, 0.126]	0.101 [0.069, 0.142]

Table 1: Mean L^1- errors and [minimum, maximum] L^1- errors associated to the estimators in the four situations.

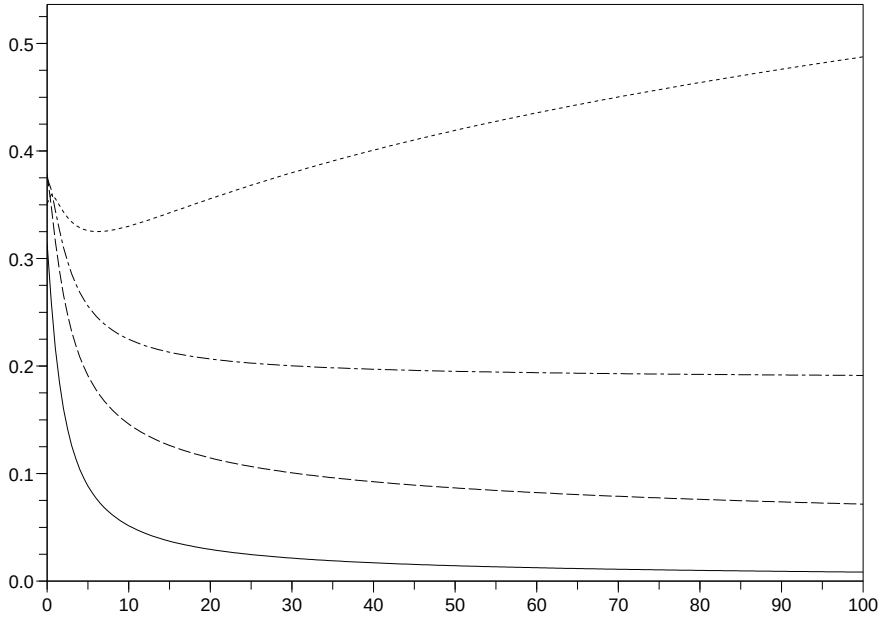

Figure 1: Graphs of the functions $a \to V(\alpha, a)$. Solid line $\alpha = 1.25$, dashed line $\alpha = 1.75$, dashed-dotted line $\alpha = 2$, dotted line $\alpha = 2.25$.

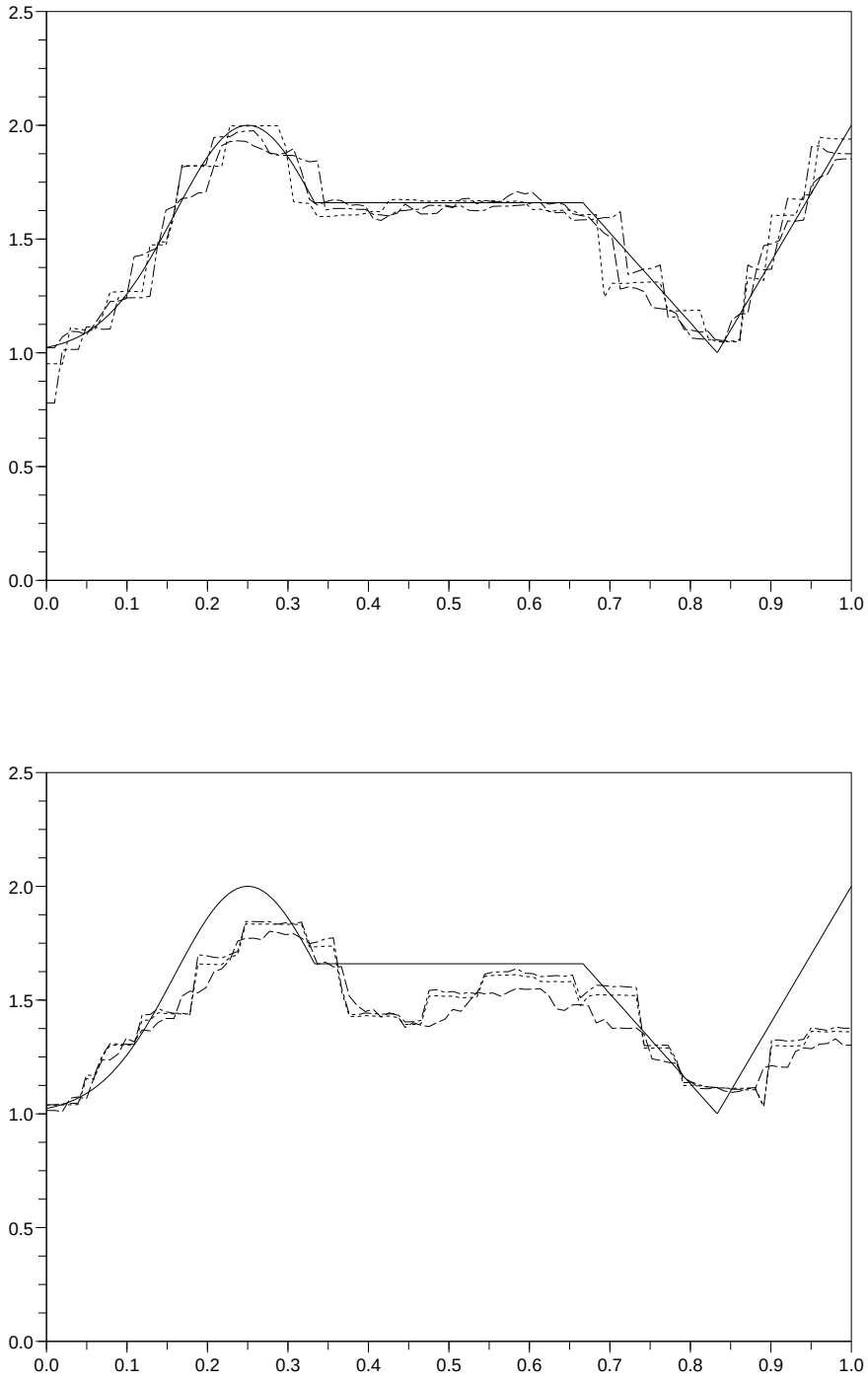

Figure 2: The frontier g_1 (solid line) and its estimates when $\alpha(x) = \alpha_1$. Top: best estimates, bottom: worst estimates. Dashed line: Girard & Jacob estimator $\widehat{g}_n^{(gj)}$, dashed-dotted line: moment estimator $\widehat{g}_n^{(m,\,c)}$ with $r_n = 35$, dotted line: moment estimator $\widehat{g}_n^{(m,\,v)}$ with $r_n = 15\,p_n^{(m)}$.

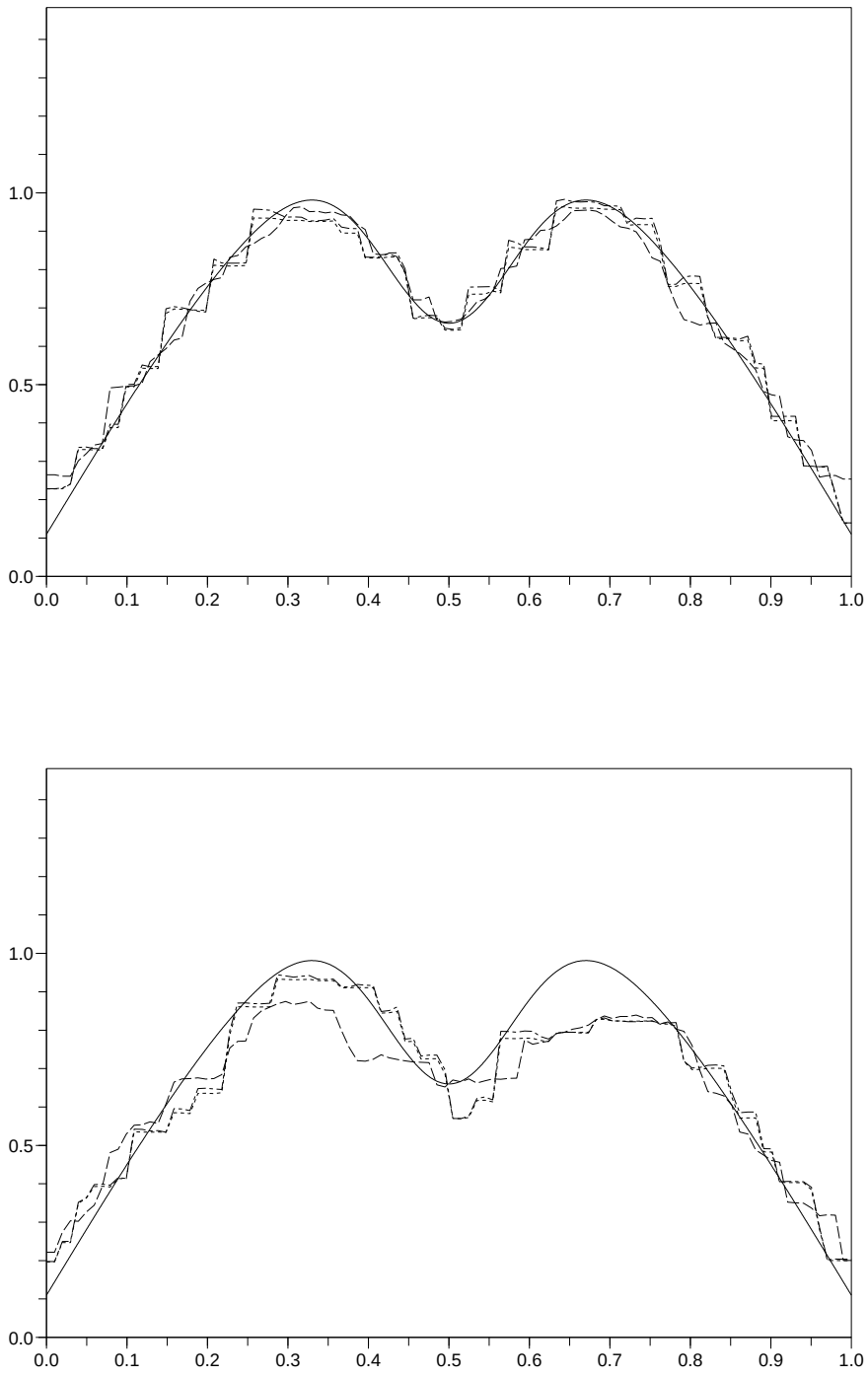

Figure 3: The frontier g_2 (solid line) and its estimates when $\alpha = \alpha_1$. Top: best estimates, bottom: worst estimates. Dashed line: Girard & Jacob estimator $\widehat{g}_n^{(gj)}$, dashed-dotted line: moment estimator $\widehat{g}_n^{(m,\,c)}$ with $r_n = 35$, dotted line: moment estimator $\widehat{g}_n^{(m,\,v)}$ with $r_n = 15\,p_n^{(m)}$.

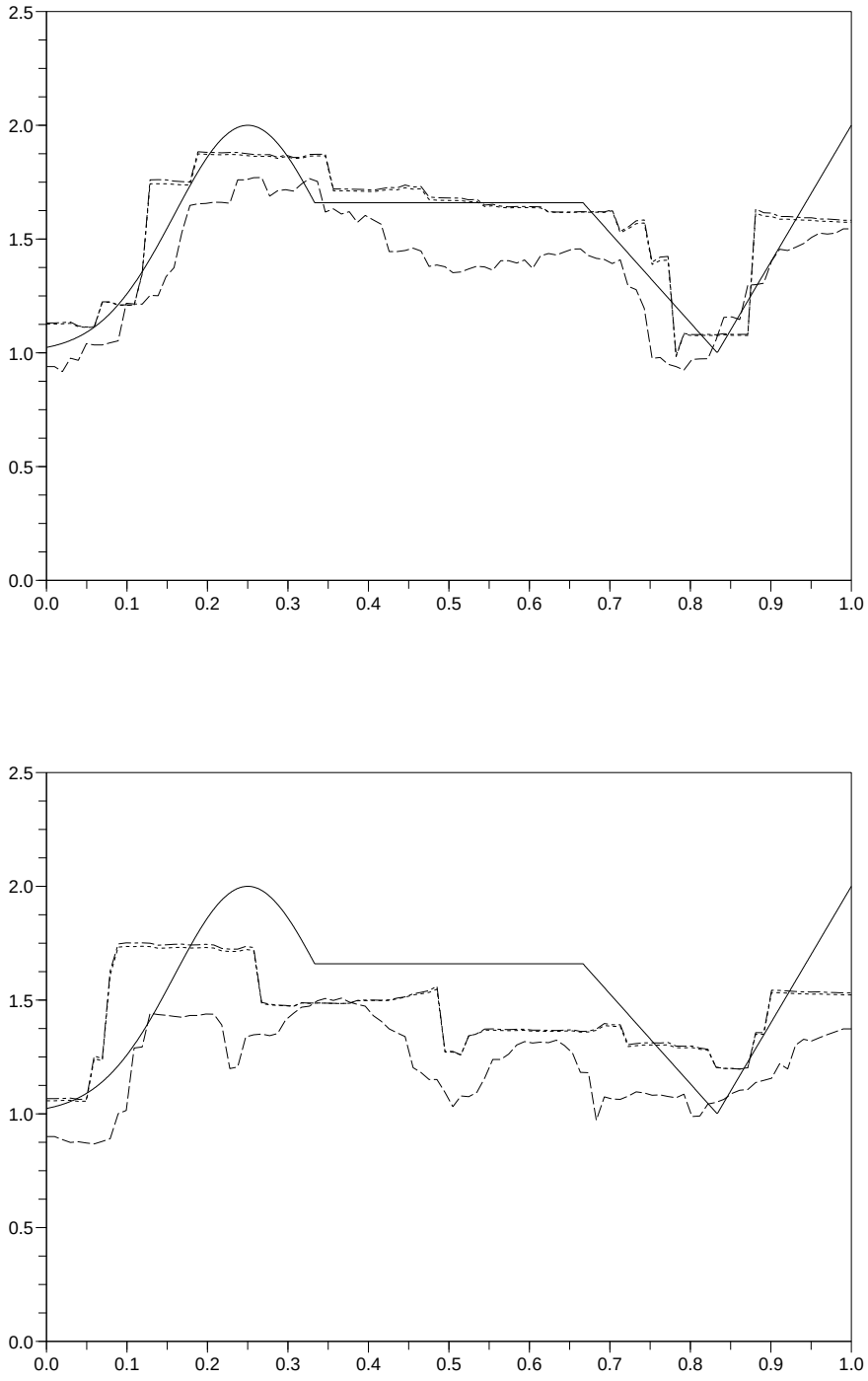

Figure 4: The frontier g_1 (solid line) and its estimates when $\alpha(x) = \alpha_2(x)$. Top: best estimates, bottom: worst estimates. Dashed line: Girard & Jacob estimator $\widehat{g}_n^{(gj)}$, dashed-dotted line: moment estimator $\widehat{g}_n^{(m,\,c)}$ with $r_n = 35$, dotted line: moment estimator $\widehat{g}_n^{(m,\,v)}$ with $r_n = 15\, p_n^{(m)}$.

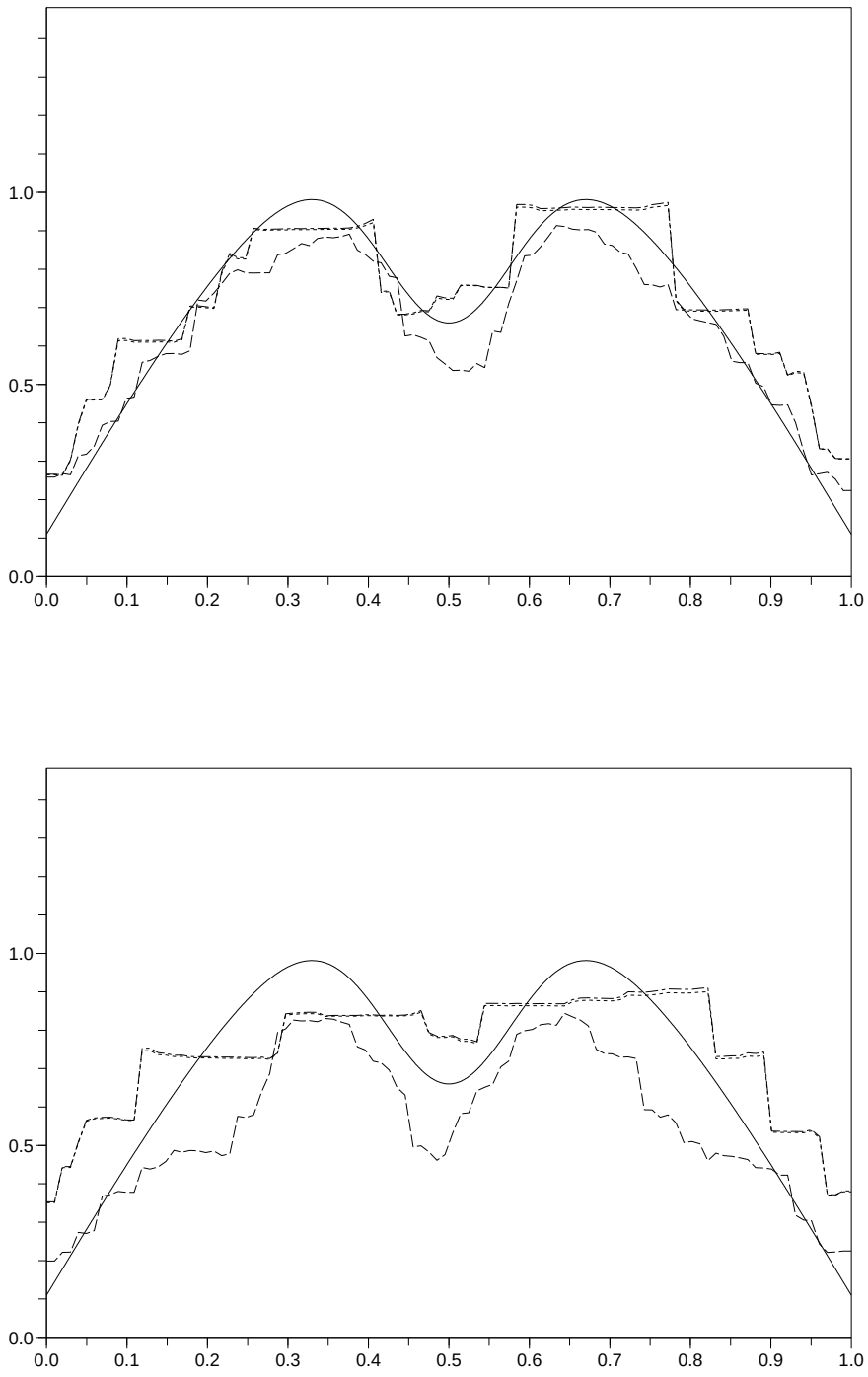

Figure 5: The frontier g_2 (solid line) and its estimates when $\alpha = \alpha_2(x)$. Top: best estimates, bottom: worst estimates. Dashed line: Girard & Jacob estimator $\widehat{g}_n^{(gj)}$, dashed-dotted line: moment estimator $\widehat{g}_n^{(m,\,c)}$ with $r_n = 35$, dotted line: moment estimator $\widehat{g}_n^{(m,\,v)}$ with $r_n = 15\,p_n^{(m)}$.