

# Numerical methods versus asymptotic expansion for torsion of hollow elastic beams

Kurt Frischmuth, Michael Hänler and Francesco dell'Isola

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## Abstract

We consider the torsion problem for a hollow elastic beam. In a series of papers dell'Isola et al.(1994, [5, 7]) a uniform method for the derivation of classical formulas for the torsional rigidity by Bredt, Prandtl and Vlasov was derived using an asymptotic expansion. We show that this expansion yields useful approximations for torsional rigidity if properly applied. Note that it does not converge in general. For the evaluation we use a numerical solution obtained by a Finite Difference Method. Finally, we examine the results of both methods for two example domains.

**Keywords:** Elliptic boundary value problems, torsion problems, asymptotic expansions

**AMS subject classification:** 35R30, 65N30

## 1 Introduction

In Saint Venant's theory of torsion there occurs a pair of 2D elliptic boundary value problems, one for the warping function  $u$ , another for the Prandtl function  $v$ . The warping function  $u$  describes the displacement along the axis of the beam, it satisfies Neumann type boundary conditions. For the Prandtl function  $v$  componentwise constant Dirichlet boundary conditions are posed. While in the case of a simply connected cross section the boundary value of the Prandtl function is meaningless, the choice of the different constants in the case of multiply connected cross sections is important. Conditions for the integrability of a first order pde for the warping function  $u$  – integrals of the tangential derivatives of  $u$  along contours surrounding each of the holes should vanish – fix the value of the flux of the gradient of Prandtl function. In this way it is uniquely determined up to an additive constant.

For closed hollow cross sections of annular type (cf. dell'Isola et al.(1994)) a transformation to arclength-thickness coordinates  $(s, z)$  proved useful. The pde for  $v$  contains a small parameter  $\epsilon$  describing the thickness of the section. An expansion with respect to powers of  $\epsilon$  reduces the 2D elliptic problem to a sequence of 1D boundary value problems in  $z$ -direction for each value of  $s$ . The four first approximations yield all known explicit solutions to special cases and the classical formulas for the torsional rigidity (dell'Isola et al., [7]). However, till now there was no proof of the convergence of the method.

The plan of the present paper is as follows. In Section 2 we develop a recurrence scheme for the members of the asymptotic expansion. In Section 3 we present a numerical method for the solution of the original boundary value problem with special emphasis on the integrability condition. In Section 4 we give an estimate for the partial sum of the expansion. Then we choose two test domains for which the asymptotic expansion (in the present form) does not converge and compare the results of both analytical and numerical approach.

## 2 Asymptotic expansion

We restrict ourselves to cross sections  $\Omega$  of the annular type. In this case we have  $\Omega = \Omega_1 \setminus \Omega_0$  with  $\Omega_0 \subset \Omega_1$ , both  $\Omega_0$  and  $\Omega_1$  simply connected, bounded domains, and the Prandtl function  $v$  is defined by the following bdivp

$$-\Delta v = 2 \quad \text{in } \Omega, \quad (1)$$

$$v = 0 \quad \text{on } \Gamma_1, \quad (2)$$

$$v = c \quad \text{on } \Gamma_0, \quad (3)$$

$$\oint_{\Gamma} \nabla v \cdot n \, ds = -2A. \quad (4)$$

Here we denote by  $\Gamma$  an arbitrary Jordan curve in  $\Omega$  not homotopic to a point,  $\Gamma_0 = \partial\Omega_0$ , and  $\Gamma_1 = \partial\Omega_1$ .  $A$  is the area inside  $\Gamma$ ,  $n$  is the outer normal unit vector to the domain inside  $\Gamma$ . We consider a family of plane domains for the Prandtl problem parametrized by  $\epsilon$ . The domain  $\Omega_\epsilon$  will be obtained as the union of  $z$ -lifted curves from the given curve  $\Gamma_0$ . The latter is defined by

$$r : s \mapsto r(s) \quad s \in [0, l], \quad r(s) \in \mathbb{R}^2 \quad (5)$$

where  $s$  is the arclength of the curve, i. e.,

$$\Omega_\epsilon = \{r(s, z) := r(s) - Qr'(s)z\epsilon\delta(s); \quad 0 < z < 1; \quad 0 \leq s < l\} \quad (6)$$

In our notation  $\delta(s)$  is the thickness of  $\Omega$  in the point of the coordinate  $s$  along  $\Gamma_0$  and  $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

is a  $-\pi/2$  rotation matrix which maps the tangent vector  $Qr'(s)$  onto the normal vector.

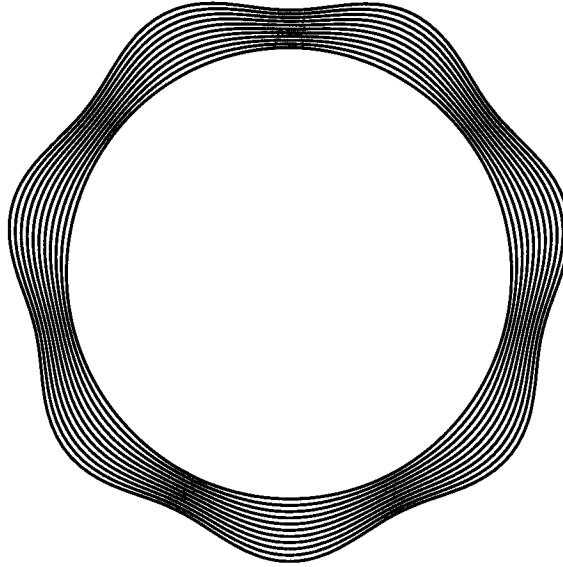


Figure 1: Test domain, cf. Sec. 4

Introducing a formal asymptotic expansion for the Prandtl function

$$v^\epsilon = \sum_{n=0}^{\infty} \epsilon^n v_n : \Omega_\epsilon \mapsto \mathbb{R} \quad (7)$$

and using the expansion of (1) computed in (dell'Isola et al., [5]) we get the following recurrence scheme to determine the second derivative with respect to  $z$  of the function  $v_n$ .

$$\begin{aligned} v_{n,zz} = - & \left\{ 2\varphi_n + \left( 3z\lambda v_{n-1,zz} + \lambda v_{n-1,z} \right) + \left( \delta^2 v_{n-2,ss} - 2z\delta\delta' v_{n-2,zs} \right. \right. \\ & + z^2(3\lambda^2 + \delta'^2) v_{n-2,zz} + \left. \left. (2z\delta'^2 - z\delta\delta'' + 2z\lambda^2) v_{n-2,z} \right) \right. \\ & + \left. \left( z\delta^2 \lambda v_{n-3,ss} - 2z^2\delta\delta' \lambda v_{n-3,zs} + z^3(\lambda^3 + \lambda\delta'^2) v_{n-3,zz} \right. \right. \\ & \left. \left. - z\delta^3 K' v_{n-3,s} + z^2(2\lambda\delta'^2 + \lambda^3 - \delta^2(\delta'K)') v_{n-3,z} \right) \right\}, \end{aligned} \quad (8)$$

where  $K = K(s)$  is the curvature along  $\Gamma_0$ ,  $\lambda = \delta K$  and

$$\varphi_n = \begin{cases} \binom{3}{n-2} \delta^n K^{n-2} z^{n-2} & \text{for } n = 2 \dots 5 \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

corresponds to the right-hand side of (1). The function  $v_n$  itself is obtained by making use of the expansion of the boundary conditions

$$\begin{aligned} v_n(s, 1) &= 0 \quad \forall s \in [0, l] \\ v_n(s, 0) &= c_n \quad \forall s \in [0, l] \\ \int_{\Gamma_0} \frac{\partial v_n}{\partial z} \frac{ds}{\delta(s)} &= 0 \quad \forall n \neq 1 \\ \int_{\Gamma_0} \frac{\partial v_1}{\partial z} \frac{ds}{\delta(s)} &= 2A \quad . \end{aligned} \quad (10)$$

Starting with  $v_n = 0 \forall n \leq 0$  we obtain that  $v_n$  is of polynomial structure in  $z$

$$v_n(s, z) = \sum_{i=0}^n a_{n,i}(s) z^i. \quad (11)$$

With the help of (8-11) we find the recurrence formula for the coefficients  $a_{n,i}(s)$

$$\begin{aligned}
a_{n+1,j+1} = \frac{-1}{j(j+1)} & \left\{ \delta_{nj} b_{n+1} + (3j-2)j\lambda a_{n,j} + \delta^2 a''_{n-1,j-1} - 2(j-1)\delta\delta' a'_{n-1,j-1} \right. \\
& + (j(j-1)\delta'^2 + (3j-4)(j-1)\lambda^2 - (j-1)\delta\delta'') a_{n-1,j-1} \\
& + \delta^2 \lambda a''_{n-2,j-2} - (\delta^3 K' + 2(j-2)\delta\delta'\lambda) a'_{n-2,j-2} \\
& \left. + ((j-2)^2\lambda^3 + (j-2)(j-1)\delta'^2\lambda - (j-2)\delta^2(\delta'K')) a_{n-2,j-2} \right\} \quad (12)
\end{aligned}$$

$$\forall n \geq 1, 1 \leq j \leq n$$

$$a_{n+1,0} = -\frac{1}{I_1} \int_{\Gamma_0} \sum_{j=2}^{n+1} a_{n+1,j}(s) \frac{ds}{\delta(s)} \quad \forall n \geq 1$$

$$a_{n+1,1} = -\sum_{j=2}^{n+1} a_{n+1,j} - a_{n+1,0} \quad \forall n \geq 1$$

with  $\delta_{nj}$  the Kronecker symbol,  $a_{n,j} = 0$  for  $n \leq 0 \vee j \leq 0$  and

$$b_n = \begin{cases} 2 \binom{3}{n-2} \delta^2 \lambda^{n-2} & n = 2 \dots 5 \\ 0 & \text{otherwise} \end{cases}.$$

We immediately get  $a_{1,1} = -a_{1,0} = -\frac{2A}{I_1}$  with

$$I_1 = \int_{\Gamma_0} \frac{ds}{\delta(s)}.$$

To compare this approach with numerical results the recurrence formula (12) is implemented in Mathematica. The results for two test domains will be discussed in Sec. 4.

### 3 Numerical solution

The determination of the constant  $c$  in (3) from the integral condition (4) is the most crucial problem connected with the numerical solution of the problem (1-4). With  $c$  given, the problem (1-3) does not cause serious trouble.

Hence, we solve the problem in three steps. Since the integral on the left-hand side of (4) is an affine scalar function in  $c$ , say  $\phi(c) = \phi_0 + \phi_1 c$  we determine the coefficients  $\phi_0, \phi_1$  of this function by calculating two suitable auxiliary functions  $v_0$  and  $v_1$ . Then we solve (4) for  $c$  and obtain the solution of (1-4) as a linear combination of  $v_0$  and  $v_1$ . The auxiliary problems are

$$-\Delta v_0 = 2 \quad \text{in } \Omega, \quad (13)$$

$$v_0 = 0 \quad \text{on } \partial\Omega, \quad (14)$$

and

$$-\Delta v_1 = 0 \quad \text{in } \Omega, \quad (15)$$

$$v_1 = 0 \quad \text{on } \Gamma_1, \quad (16)$$

$$v_1 = 1 \quad \text{on } \Gamma_0. \quad (17)$$

As we have

$$v = v_0 + cv_1, \quad (18)$$

and

$$\phi_0 = \oint_{\Gamma} \nabla v_0 \, dn, \quad \phi_1 = \oint_{\Gamma} \nabla v_1 \, dn. \quad (19)$$

for  $c$  we obtain the value

$$c = \frac{-2A - \phi_0}{\phi_1}. \quad (20)$$

**Remark 1** For the choice  $\Gamma = \Gamma_0$  we obtain as a matter of course the inequalities  $\phi_0 > 0$ ,  $\phi_1 < 0$ . Further,  $\phi_1$  is independent of the choice of  $\Gamma$ , as it is the flux (through  $\Gamma$ ) of the divergence free field  $\nabla v_1$ . Hence  $c$  is well defined and positive. It is also independent of  $\Gamma$ .

For the discretization of the above bvp's we introduce a uniform rectangular grid with stepsize  $h$ ,  $\Omega_h = \Omega \cap \mathbb{Z}_h \times \mathbb{Z}_h$ ,  $\mathbb{Z}_h = h\mathbb{Z} = \{hi \mid i \in \mathbb{Z}\}$ . Given any curve  $\gamma$ , we define  $\gamma_h$  by

$$x_{ij} = h(i, j) = (ih, jh) \in \gamma_h \quad \text{iff} \quad \exists x \in \gamma : \text{dist}(x_i, x) < h/2.$$

Here we use the metric  $\text{dist}(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|)$ .

That way we define  $\Gamma_h$ ,  $\Gamma_{0,h}$ , and  $\Gamma_{1,h}$ , and prescribe the boundary values to the discrete functions  $v_0$  and  $v_1$  on the corresponding gridded contours. For the discretization of the Laplacian we use the standard 5-point stencil

$$\begin{array}{ccc} & 1 & \\ & 1 & -4 & 1 \\ & & 1 & \end{array}$$

while the right-hand side is simply discretized by  $f_{ij} = 0$  or  $f_{ij} = -2$ , respectively. Since the right-hand side of (1) is constant we obtain the same system of equations by FDM and by linear finite elements on a standard triangulation.

The discrete equations are solved very effectively by iterative methods. We chose SOR with the overrelaxation parameter  $\omega = 1.85$ . Because of the special geometry of the considered domains and the constant boundary conditions we observed very quick convergence. The number of iterations needed to achieve a given error reduction is  $O(h)$  as typical in 1D-problems, rather than  $O(h^2)$ . We solved problems with up to  $10^5$  equations with about 250 iterations, the error bound for the remainder being  $10^{-10}$ .

Given a numerical solution  $v_h$  to (13,14) or (15-17), the integration (19) has to be carried out. It is not advisable to choose  $\Gamma$  as one of the components of the boundary because of the rough approximation there and the restriction to one-sided approximations for the normal derivative. Instead, we prefer a contour  $\Gamma_h$  composed entirely of inner points from  $\Omega_h$ .

Let  $x_{ij}$  and  $(x_{i',j'})$  be neighboring grid points on  $\Gamma_h$ . Then the four points  $x_{ij} \pm y$ ,  $x_{i',j'} \pm y$  with  $y = h(j' - j, -i' + i)$  lay in  $\Omega_h \cup \Gamma_i \cup \Gamma_o$ . We obtain a numerical approximation of the contour integral as the sum of the differences between the value of  $v_h$  at the first two and at the second two of them, added up over all of  $\Gamma_h$  and divided by four. By Gauss' theorem, we calculate  $2A$  by summing  $h^2(i(j' - j) - j(i' - i))$ . Thus the numerical calculation of  $c$  by (20) and finally  $v$  by (18) is now straightforward.

**Remark 2** In order to obtain reasonable results the stepsize  $h$  should be at least 10 times smaller than the minimum of the section thickness. Consequently, for larger perimeter-to-thickness ratio ( $> 100$ ) the above method might result in very large systems of equations. However, in those cases the solution is almost perfectly linear in the direction normal to the contour, and the classical Bredt formulas should be applied.

A FEM with much better approximation of the boundaries and considerably smaller systems of equations will be studied in a forthcoming paper.

## 4 (Non-)Convergence of $\epsilon$ -Expansion

We consider again the family  $\Omega_\epsilon$  of domains introduced in sec. 2. With  $w_n^\epsilon$  we denote the partial sum of the formal expansion of  $v^\epsilon$ , i.e.

$$w_n^\epsilon = \sum_{i=0}^n \epsilon^i v_i . \quad (21)$$

An estimate for the difference between partial sum and true solution of (1-4) gives the following

**Lemma 1** *Let  $\delta(s), K(s) \in C^{n+2}(\mathbb{R})$   $l$ -periodic and  $\Gamma_0$  smooth. Then it holds*

$$\|v^\epsilon - w_n^\epsilon\|_{C(\Omega_\epsilon)} \leq C_n \epsilon^{n+1} ,$$

where  $C_n = C_n(\delta, K, \Gamma_0)$ .

Proof: We denote  $\psi_n = v^\epsilon - w_n^\epsilon$ . Using the expansion of the Laplacian from (dell'Isola et al., [7]) we get

$$\begin{aligned} \Delta \psi_n &= \epsilon^n f_n && \text{in } \Omega_\epsilon \\ \psi_n &= 0 && \text{on } \partial\Omega_\epsilon \end{aligned} \quad (22)$$

with

$$\psi_n = \varphi_{n+1} + 3z\lambda v_{n,zz} + \lambda v_{n,z} + o(\epsilon)$$

where  $\varphi_n = 0$  for  $n \geq 5$  (see (8)). As it is easily seen from (12)  $v_n$  depends only on  $\Gamma_0$  and on  $\delta(s), K(s)$  resp. their derivatives up to order  $n$ . Thus we have under the given smoothness assumptions

$$\|\psi_n\|_{C(\Omega_\epsilon)} \leq C_n$$

with some suitable constant. This together with (22) implies immediately the above inequality. ■

**Remark 3** *The sequence of constants  $C_n$  of Lemma 1 may grow very fast. So for practical applications the above result is not useful since the thickness of a given domain is fix. For a class of thickness functions  $\delta(s)$  it is possible to show that the formal expansion will not converge for any fixed  $\epsilon$ .*

We consider domains of the following type:

$$\Omega = \left\{ (x, y) : r \leq \sqrt{x^2 + y^2} \leq r + \delta(s) \right\} , \quad (23)$$

with

$$\delta(s) = c_1 + c_2 \sin(ks/r); \quad c_1 > c_2 > 0; \quad k \in \mathbb{Z}^+ .$$

In this case  $\Gamma_0$  is simply a circle and  $K(s) = 1/r$  is a constant. The following picture shows the behaviour of the coefficients  $a_{n,j}(s)$  in the asymptotic expansion (7,11) for the domain  $\Omega$  with  $r = 0.9; k = 7; c_1 = 0.2; c_2 = 0.05$ .

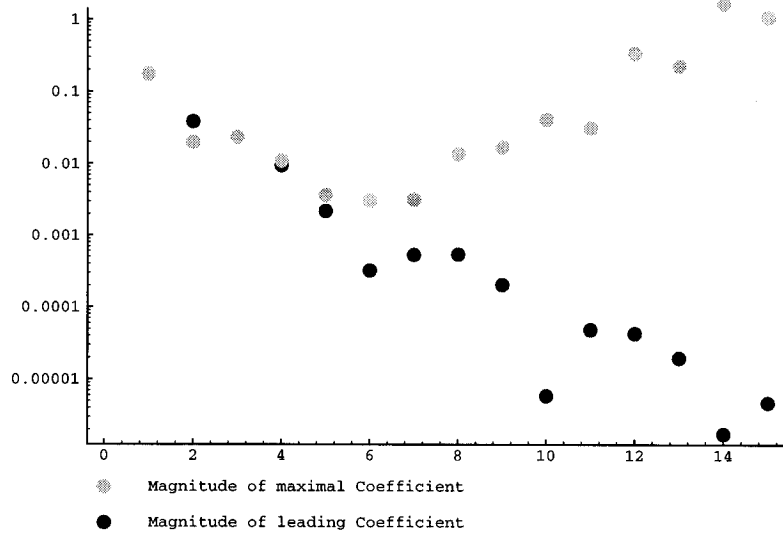


Figure 2: Logarithmic plot of the coefficients in the asymptotic for  $s = 0.2$

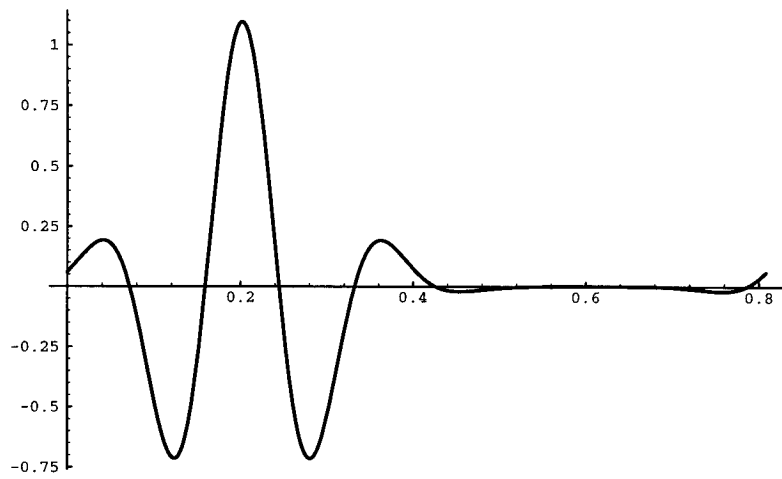


Figure 3: Coefficient  $a_{15,4}(s)$  of (11)

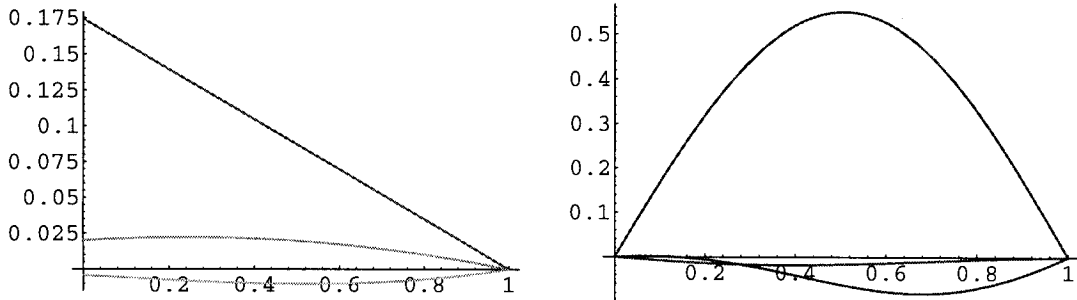


Figure 4: Functions  $v_n$  of the expansion series for  $s = 0.2$

In the left part we see a cut at arclength  $s = 0.2$  for the functions  $v_1, v_2, v_3$  and in the right part for  $v_{13}, v_{14}, v_{15}$ . We remark that the functions  $v_n$  are very small for  $n = 4, \dots, 12$ . For bigger  $n$  they grow rapidly.

Nevertheless, the next picture shows that we obtain from the  $\epsilon$ -Expansion for this domain useful results for torsional rigidity. The line represents the value obtained from the numerical method.

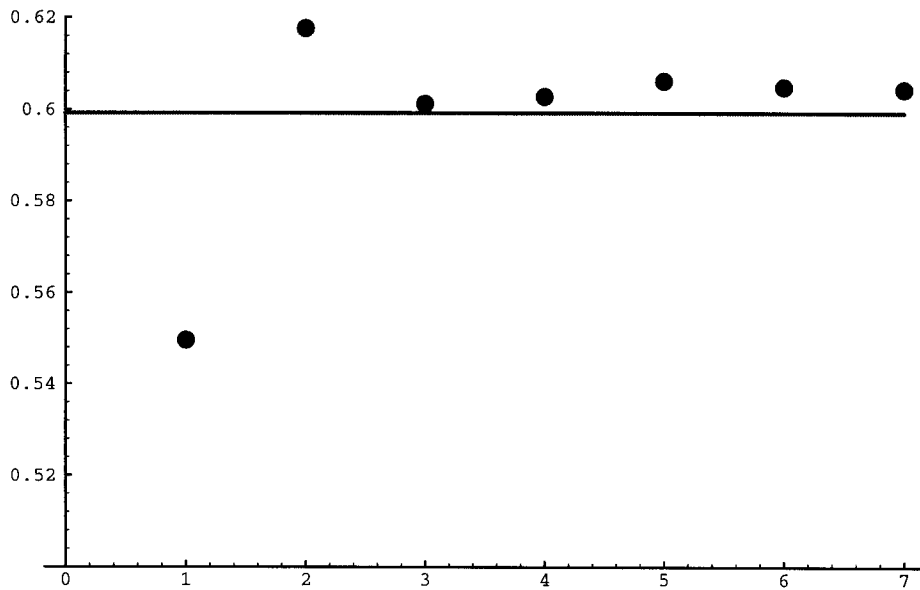


Figure 5: Torsional rigidity for  $k = 7$



For a domain with 13 humps, i. e.  $k = 13$ , we get the following picture:

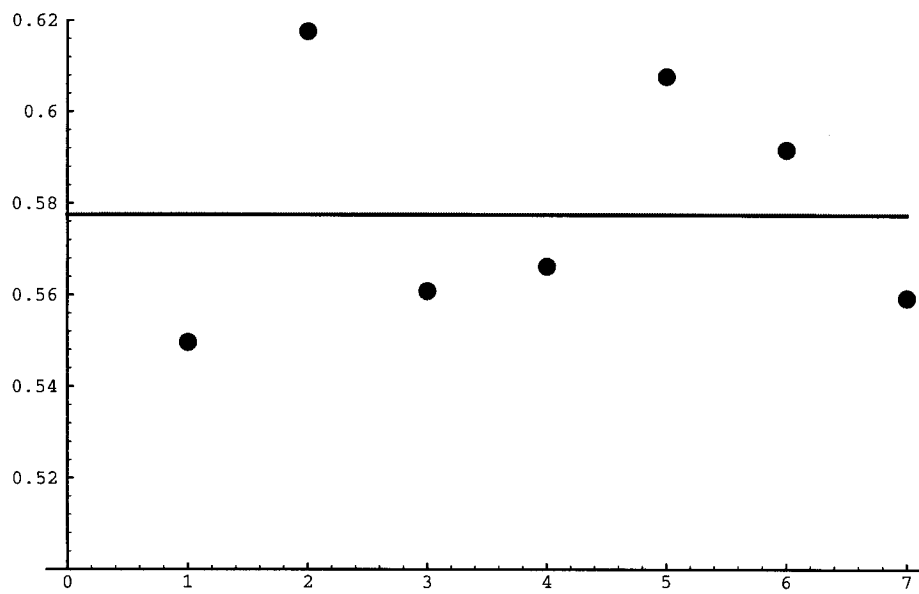


Figure 6: Torsional rigidity for  $k = 13$

Finally, we present the numerical solution for the domain  $\Omega$  with 7 humps.

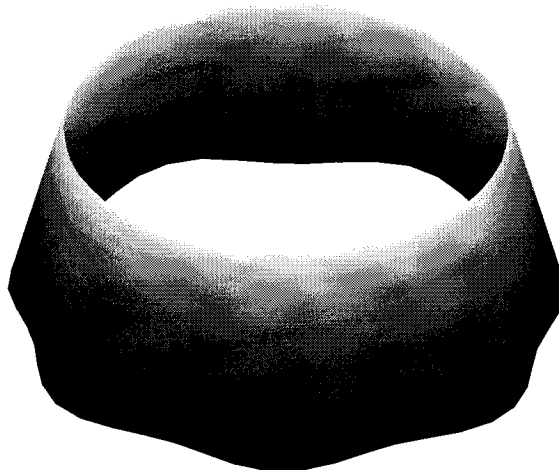


Figure 7: Numerical solution

## 5 Conclusions

We proved that also for not simply connected cross sections a direct numerical calculation of the Prandtl function is possible – contrary to assertions in Wang (1995). Even a very rough and simple approach gives accurate results at a reasonably low computational cost.

On the other hand, analytical formulas as derived by the asymptotic method are always very appealing. However, despite the elegance, the cost for deriving higher order formulas, and for calculating numerical results from them, often exceeds the cost of the direct numerical approach.

Further, it is essential to take the right number of terms in the power series. For “good natured domains” there is a first interval of apparent convergence, we can pick any small number, say 3, 4 or 5, to obtain a good approximation. For domains the thickness of which varies with high frequency, however, there is no stable behaviour at all.

Generally, the first term (Bredt formula) underestimates the torsional rigidity, the second usually overshoots the exact value. Considering cost and accuracy, we recommend the third as a reasonable compromise, if no numerical procedure is desired or at hand.

In such cases, e. g. for shape optimization, we strongly suggest to check for the given class of sections the behaviour of the chosen approximation with a numerical solution. Finally, we want to stress that our aim was not to diminish the meaning of asymptotic methods, but rather to learn about their behaviour in a simple test case with easily available numerical solution before applying them to a 3-D problem. In that case we assume that an advantage of the expansion method is quite possible.

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*Anschrift:* Dr. Kurt Frischmuth und Dr. Michael Hänler, Fachbereich Mathematik, Universität Rostock, D-18051 Rostock, Germany, e-mail: Frischmuth@sun2.math.uni-rostock.de, micha@alf.math.uni-rostock.de

Dr. Francesco dell’Isola, Dipartimento di Ingegneria Strutturale e Geotecnica, Università di Roma I ”La Sapienza”, Via Eudossiana 18, I-00184 Roma, Italy, e-mail: isola@scilla.ing.uniroma1.it