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POLISH TOPOMETRIC GROUPS

ITAI BEN YAACOV, ALEXANDER BERENSTEIN, AND JULIEN MELLERAY

ABSTRACT. We define and study the notion of *ample metric generics* for a Polish topological group, which is a weakening of the notion of ample generics introduced by Kechris and Rosendal in [KR07]. Our work is based on the concept of a *Polish topometric group*, defined in this article. Using Kechris and Rosendal's work as a guide, we explore consequences of ample metric generics (or, more generally, ample generics for Polish topometric groups). Then we provide examples of Polish groups with ample metric generics, such as the isometry group $\text{Iso}(\mathbb{U}_1)$ of the bounded Urysohn space, the unitary group $\mathcal{U}(\ell_2)$ of a separable Hilbert space, and the automorphism group $\text{Aut}([0, 1], \lambda)$ of the Lebesgue measure algebra on $[0, 1]$. We deduce from this and earlier work of Kittrell and Tsankov that this last group has the automatic continuity property, i.e any morphism from $\text{Aut}([0, 1], \lambda)$ into a separable topological group is continuous.

1. INTRODUCTION

This paper presents a technique to study “large” Polish groups¹. These groups, which typically appear as automorphism groups of highly homogeneous structures, may have some surprising properties; an example could be the small index property, which states that any subgroup of index strictly less than the continuum is open. Also remarkable is the Bergman property: whenever G acts by isometries on a metric space, all the orbits are bounded. Or one could think of extreme amenability (all continuous actions on compact sets have fixed points), the Steinhaus property (which implies that any morphism from G to a separable group H is continuous)... The list could go on and on.

In their paper [KR07], Kechris and Rosendal continued work initiated by Hodges, Hodkinson, Lascar and Shelah in [HHLS93] and introduced the notion of *ample generics* for a Polish group G : G has ample generics if for any n there exists $(g_1, \dots, g_n) \in G$ such that

$$\{(kg_1k^{-1}, \dots, kg_nk^{-1}) : k \in G\}$$

is comeagre in G^n . One could express this fact by saying that G has comeagre diagonal conjugacy classes.

Kechris and Rosendal showed that this notion provides a unified approach to some of the properties discussed above, as any Polish group with ample generics must have the small index property, the Bergman property and the Steinhaus property (among others).

Thus, having ample generics is an extremely strong condition; it is hence not surprising that Polish groups with ample generics are relatively rare. Frustratingly, they are even scarcer than one would hope: at the moment all Polish groups known to have ample generics are subgroups of \mathfrak{S}_∞ , the permutation group on the set of integers.

While there are many such groups, they are far from covering the whole variety of Polish groups; actually, closed subgroups of \mathfrak{S}_∞ are automorphism groups of discrete objects, since any such group is

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¹for definitions of most concepts discussed in this abstract and in the Introduction, see section 2 below.

isomorphic to the automorphism group of some countable first-order relational structure. If we want to tackle automorphism groups of non-discrete structures, it is then natural to use *continuous logic*, a generalization of first-order logic introduced by Ben Yaacov and Usvyatsov in [BU]. Indeed, Polish groups are exactly the automorphism groups of relational Polish metric structures, which are the generalization to this setting of the countable relational structures of first-order logic.

When making the move to continuous logic, one needs to deal with the fact that the underlying universe M of the structure \mathcal{M} is no longer a discrete set, but a metric space. The automorphism group of \mathcal{M} may then be endowed with two natural topologies: that of pointwise convergence and that of uniform convergence. Both are group topologies, the former is Polish whenever M is separable, the latter refines the former and is induced by a bi-invariant distance. So one is led to study Polish groups (G, τ) , with the addition of a bi-invariant distance ∂ that refines τ ; below we call such triplets (G, τ, ∂) (that satisfy an additional technical condition) *Polish topometric groups*. Note that any Polish group G admits a coarsest bi-invariant distance ∂_u that refines the topology τ of G , and it is easy to check that (G, τ, ∂_u) is a Polish topometric group. Below, we will be particularly interested in topometric groups of this form; ideally, we want to use ∂_u to retrieve information about (G, τ) .

One can use the metric to define topometric versions of classical properties of topological groups; loosely speaking, consider statements of the form: “whenever A is a subset of (G, τ, ∂) with some property (P), A is large in G ”, where the notion of largeness is related to the topology (dense, open, comeagre, comeagre in some nonempty open set...). Then one can relax this statement to a statement of the form “whenever A is a subset of G with some property (P), the ∂ -closure of A is large in G ”, where the notion of largeness is *still topological*.

At first glance, it may seem useless to consider this notion in large Polish groups: those have very many uniformly closed sets, so the uniform closure of a small set seems bound to be a small set itself. Surprisingly, the opposite is sometimes true: for instance, below we provide examples of groups G such that all diagonal conjugacy classes are known to be meagre, but G has diagonal conjugacy classes whose uniform closure is topologically comeagre. Such groups include the isometry group of the universal Urysohn space, the group of measure-preserving automorphisms of the unit interval, and the unitary group of a complex Hilbert space, each endowed with its natural Polish topology and uniform distance ∂_u .

We say that a Polish topometric group (G, τ, ∂) has *ample generics* if it admits diagonal conjugacy classes whose ∂ -closure is τ -comeagre. If (G, τ, ∂_u) has ample generics we say that (G, τ) has *ample metric generics*.

We use this notion to obtain results similar to those in [KR07]. Our main automatic continuity result is the following.

Theorem. *Let (G, τ, ∂) be a Polish topometric group with ample generics, H a separable topological group and $\varphi: G \rightarrow H$ a morphism such that $\varphi: (G, \partial) \rightarrow H$ is continuous. Then $\varphi: (G, \tau) \rightarrow H$ is continuous.*

This theorem, along with work of Kittrell and Tsankov and the fact that the group $\text{Aut}([0, 1], \lambda)$ of measure preserving transformations of the unit interval has ample metric generics, yields the following result.

Theorem. *Let $\varphi: \text{Aut}([0, 1], \lambda) \rightarrow H$ be a morphism, with H a separable topological group. Then φ is continuous when $\text{Aut}([0, 1], \lambda)$ is endowed with its usual Polish topology. Consequently, $\text{Aut}([0, 1], \lambda)$ has a unique Polish topology compatible with its group structure.*

In the end of the paper, we try to understand the structure of the set of invariant distances which turn a Polish group (G, τ) into a Polish topometric group with ample generics. We obtain the basic theorem below.

Theorem. *Let (G, τ) be a Polish group; denote by ∂_u the coarsest bi-invariant distance that refines τ , and assume that (G, τ, ∂_u) has ample generics. Then there exists a finest distance ∂ such that (G, τ, ∂) is a topometric group with ample generics. This distance may be strictly finer than ∂_u - actually it may be discrete even in cases when ∂_u is not.*

The structure of most of our arguments is very similar to those of Kechris and Rosendal; in some proofs in sections 3 and 4 of the paper, whenever one replaces ε by 0 one obtains an argument that appears verbatim in [KR07]. For the convenience of the reader, we still present the proofs in full detail.

2. PRELIMINARIES

2.1. Polish spaces and groups.

Recall that a *Polish space* is a topological space X which is homeomorphic to a complete, separable metric space; in other words, it is a second countable topological space whose topology admits a compatible complete distance. Most of the time, we want the distance on X to be fixed; by a *Polish metric space* we mean a complete separable metric space (X, d) .

Whenever (X, d) is a metric space, one can consider its isometry group $\text{Iso}(X, d)$, i.e the group of distance-preserving bijections from X onto itself. When (X, d) is a Polish metric space, we endow its isometry group with the pointwise convergence topology, which turns it into a *Polish group*, i.e a topological group which is Polish as a topological space. Note that, by the Birkhoff-Kakutani theorem, any Polish group admits a left-invariant distance defining its topology.

A major tool in the study of Polish spaces is the Baire category theorem; recall that a subset A of a Polish space is *meagre* if it is contained in a countable union of closed sets with empty interior. A is *comeagre* if its complement is meagre, equivalently, if A contains a countable intersection of dense open sets. The Baire category theorem asserts that, in a Polish space, any comeagre set is dense.

This notion is particularly useful in Polish groups, because of Pettis' theorem, which asserts that whenever G is a Polish group and A is a subset of G that is comeagre in some nonempty open set then $A.A^{-1}$ contains a neighborhood of 1_G . Below we will make frequent use of this theorem. Let us simply point out two well-known consequences of Pettis' theorem: the first is that if H is a subgroup of a Polish group G which is comeagre in some nonempty open subset of G , then H is open in G . The second consequence is that, if $\varphi: G \rightarrow H$ is a continuous, bijective morphism between two Polish groups, then φ is an isomorphism of topological groups (i.e φ^{-1} is also continuous).

We need to recall some other classical notions of descriptive set theory. If X is a Polish space and $A \subseteq X$, one says that A is *Baire-measurable* if there exists an open subset O of X such that $A \Delta O$ is meagre. These sets form a σ -algebra that contains the Borel subsets of X . The point here is that whenever a Baire-measurable set is non-meagre it must be comeagre in some nonempty open set, and this fact is frequently used below, often in conjunction with Pettis' theorem.

The σ -algebra of Baire-measurable sets of a Polish space X contains (if X is uncountable) sets that are non Borel. In particular, any *analytic* subset of X is Baire-measurable. Recall that a subset A of X is analytic if there exists a Polish space Y and a Borel subset B of $X \times Y$ such that A is equal to the projection of B on the first coordinate. In what follows, we will use two facts about analytic sets, besides their Baire-measurability: the intersection of countably many analytic subsets of a Polish space X is still analytic; and whenever X, Y are Polish metric spaces, $A \subseteq X$ is analytic and $f: X \rightarrow Y$ is continuous, the set $f(A)$ is an analytic subset of Y .

For information about Polish spaces and groups, and about descriptive set theory, the reader is invited to consult [Gao09], [Kec95] and the bibliographical references therein.

2.2. Relational metric structures.

Let us begin with a convention that we will follow throughout the paper: whenever (M, d) is a metric

space and $n \in \mathbb{N}$, we endow M^n with the product topology and the supremum metric, which we denote respectively by τ and d .

Definition 2.1. A *relational metric structure* \mathcal{M} is a complete metric space (M, d) of diameter at most 1, along with a family $(P_i)_{i \in I}$ of *predicates*, i.e uniformly continuous maps from M^{k_i} to $[0, 1]$ (where $k_i \in \mathbb{N}$); we always assume that the distance function $d: M^2 \rightarrow [0, 1]$ is included in our list of predicates. The structure is said to be *Polish* if the underlying metric space is, that is, if M is separable.

A *morphism* from \mathcal{M} to \mathcal{M} is simply a distance-preserving map from M to M that also preserves all the predicates; it is an *automorphism* if it is also onto.

If \mathcal{M} is a relational Polish metric structure, then its automorphism group, when endowed with the pointwise convergence topology τ , is a closed subgroup of $\text{Iso}(M, d)$ and so it is a Polish group.

Definition 2.2. We say that two tuples (a_1, \dots, a_n) and (b_1, \dots, b_n) in M^n have *the same quantifier-free type* if for all $\{j_1, \dots, j_k\} \subseteq \{1, \dots, n\}$ and all $i \in I$ with $k_i = k$ one has $P_i(a_{j_1}, \dots, a_{j_k}) = P_i(b_{j_1}, \dots, b_{j_k})$. A relational metric structure \mathcal{M} is *approximately ultrahomogeneous* if for any n -tuples (a_1, \dots, a_n) , (b_1, \dots, b_n) with the same quantifier-free type and any $\varepsilon > 0$ there exists $g \in \text{Aut}(\mathcal{M})$ such that $d(g(a_i), b_i) \leq \varepsilon$ for all $i = 1, \dots, n$.

One reason why we are interested in these structures is the following proposition which extends to continuous logic a basic fact of the theory of automorphism groups of countable structures in classical first-order logic.

Proposition 2.3. (see [Mel])

Any Polish group is isomorphic (as a topological group) to the automorphism group of some approximately ultrahomogeneous relational Polish metric structure.

Sketch of proof.

Let G be a Polish group, and d be a left-invariant distance compatible with the topology of G . We denote by \hat{G} the metric completion of (G, d) , and let G act on \hat{G} by left-translation, and on \hat{G}^n by the diagonal product of this action. For any n , and any G -orbit \mathcal{O} in \hat{G}^n , we let $R_{\mathcal{O}}: \hat{G}^n \rightarrow [0, 1]$ be defined by

$$R_{\mathcal{O}}(\bar{x}) = d(\bar{x}, \mathcal{O}) .$$

This is a uniformly continuous function on \hat{G} , and one can consider the relational Polish metric structure

$$\mathcal{G} = (\hat{G}, d, \{R_{\mathcal{O}}\}) .$$

It is then easy to check that G is the automorphism group of \mathcal{G} , and that \mathcal{G} is approximately ultrahomogeneous. \square

2.3. Topometric groups.

A source of complexity in the theory of Polish groups is that complete distances inducing the topology need not be compatible with the group operations, i.e one cannot ask for the existence of, say, a complete left-invariant distance, without restricting dramatically the class of groups under consideration. Still, whenever (G, τ) is a Polish group, one can consider the bi-invariant uniformity generated by τ ; this is a metric uniformity. To see this, let d be a left-invariant metric on G (not complete a priori) and define

$$\partial_u(g, h) = \sup\{d(gk, hk) : k \in G\} .$$

One can check that ∂_u is complete, bi-invariant, refines the topology of G , and any distance with the same properties has to refine ∂_u . In general, ∂_u is not separable. Below, we will abuse notation somewhat and, whenever G is a Polish group and ∂ is a distance that generates the coarsest bi-invariant uniformity refining the topology of G , we will write $\partial = \partial_u$, even though of course there are many different distances

that generate this uniformity.

Note. It is important to point out that some distances (and most bi-invariant distances) we consider below are *not* separable; to emphasize the fact that these are not the distances one is used to considering when dealing with Polish spaces and groups, we use throughout the paper the notation ∂ to denote distances that are not a priori separable.

Definition 2.4. We say that a triplet (X, τ, ∂) is a *topometric space* if X is a set, τ is a topology on X and ∂ is a distance such that the topology induced by ∂ refines τ in such a way that ∂ is τ -lower semicontinuous, i.e the set $\{(x, y) : \partial(x, y) \leq r\}$ is closed for all r .

A *topometric group* is a triplet (G, τ, ∂) which is a topometric space and is such that ∂ is bi-invariant and (G, τ) is a topological group.

Topometric spaces were originally introduced by the first author of this paper in [Ben08].

Example. Let \mathcal{M} be a relational Polish metric structure with universe (M, d) , and denote by G its automorphism group. As we saw earlier, G , when endowed with the pointwise convergence topology, is a Polish group; the topology of G is naturally refined by the bi-invariant distance $\partial_u^{\mathcal{M}}$, defined by

$$\partial_u^{\mathcal{M}}(g, h) = \sup\{d(g(x), h(x)) : x \in M\} .$$

A word of caution is in order here: there are a priori many ways to turn a given Polish group (G, τ) into the automorphism group of some relational Polish metric structure \mathcal{M} . Attached to each such structure \mathcal{M} comes a topometric structure for (G, τ) , induced by the metric of uniform convergence on M . *Different structures may induce different metrics on G .* So it is interesting to note that, in the structure obtained in Proposition 2.3, the uniform distance generates the coarsest bi-invariant uniformity that refines the topology of G .

An interesting aspect of the metric in topometric spaces is that it enables us to enlarge sets slightly: if (X, τ, ∂) is a topometric space, $A \subseteq X$ and $\varepsilon > 0$ then we define

$$(A)_\varepsilon = \{x \in X : \partial(x, A) < \varepsilon\} .$$

We now specify the type of actions we will consider in this paper.

Definition 2.5. A *topometric action* of a topometric group (G, τ_G, ∂_G) on a topometric space (X, τ, ∂) is an action of G on X such that each map $x \mapsto g.x$ is a τ -homeomorphism and a ∂ -isometry, and the map $(g, x) \mapsto g.x$ is continuous both from $(X, \tau) \times (G, \tau_G)$ into (X, τ) and from $(X, \partial) \times (G, \partial_G)$ into (X, ∂) .

Whenever G acts topometrically on X , we will consider the diagonal action of G on X^n , which is also naturally a topometric action on X^n (X^n being endowed with the product topology and the sup-metric).

In the following, we focus on Polish topometric spaces and groups, that is to say we only consider topometric spaces and groups whose topology is Polish. In this setting, our technical assumption on ∂ implies in particular that it is a Borel function from $X \times X$ to X , and so it is easy to check that, whenever $A \subseteq X$ is analytic and $\varepsilon > 0$, $(A)_\varepsilon$ is also analytic. Indeed, $(A)_\varepsilon$ is equal to the projection on the first coordinate of the following subset of $X \times X$:

$$\{(x, y) \in X \times X : x \in X \text{ and } y \in A \text{ and } \partial(x, y) < \varepsilon\}$$

This set is analytic, and the projection of an analytic subset of $X \times X$ is an analytic subset of X .

Consequently, the uniform closure of any analytic subset of X is again analytic.

For information about metric structures, we refer to [BBHU08]; we will not need to go into the logic of these structures in this paper, the basic vocabulary presented above should be enough to understand what follows.

3. AMPLE GENERICS FOR TOPOMETRIC GROUPS

Definition 3.1. Let G be a topometric group acting on a topometric space X . The action has *ample generics* if for any $n \in \mathbb{N}$ and any $\varepsilon > 0$ there exists some $\bar{x} \in X^n$ such that $(G.\bar{x})_\varepsilon$ is comeagre in X^n .

Note that one can invert the quantifiers above, that is to say, an action has ample generics iff for any $n \in \mathbb{N}$ there exists $\bar{x} \in X^n$ such that $(G.\bar{x})_\varepsilon$ is comeagre for any $\varepsilon > 0$: indeed, if the action has ample generics then for any n and m the set $\{\bar{x} \in X^n : (G.\bar{x})_{1/m} \text{ is comeagre}\}$ is comeagre, and any element \bar{x} in the intersection of these sets as m ranges over \mathbb{N} has a G -orbit satisfying the desired property.

This shows that having ample generics is also equivalent to having diagonal conjugacy classes whose ∂ -closure is comeagre; while this may be a shorter definition to write down, in our proofs we will always use the version above.

We will not really be interested in general topometric actions of Polish topometric groups; rather, we will focus on the action of G on itself by conjugacy.

Definition 3.2. Let (G, τ, ∂) be a Polish topometric group. We say that (G, τ, ∂) has *ample generics* if the topometric action of G on itself by conjugacy has ample generics.

If $\partial = \partial_u$, we simply say that (G, τ) has *ample metric generics*.

We now begin explaining how one can extend the techniques of Kechris and Rosendal to the topometric setting. We first need to establish two technical lemmas; these are the extension to Polish topometric groups of lemmas 6.6 and 6.7 of [KR07]. The proofs are very similar, but one needs to deal with some ε 's to account for our new setting.

In the following two lemmas, G is a Polish topometric group acting on a Polish topometric space X , and we assume that the action has ample generics; we say that some tuple $\bar{x} \in X^n$ is *generic* if $(G.\bar{x})_\varepsilon$ is comeagre for any $\varepsilon > 0$. Recall that the notation $\forall^* y P(y)$ means that the set $\{y : P(y)\}$ is comeagre. Below we repeatedly use without details some classical results of descriptive set theory; see Section 2.1 for definitions and references.

Lemma 3.3. *Let $A, B \subseteq X$ be such that A is not meagre and B is not meagre in any nonempty open set. Then if $\bar{x} \in X^n$ is generic and V is an open neighborhood of the identity of G , there exist for any $\varepsilon > 0$ some $y_0 \in (A)_\varepsilon$, $y_1 \in (B)_\varepsilon$, $h \in V$ such that (\bar{x}, y_0) and (\bar{x}, y_1) are generic and $\partial(h.(\bar{x}, y_0), (\bar{x}, y_1)) < \varepsilon$.*

Proof. Fix an open neighborhood V of 1_G , $n \in \mathbb{N}$, and a generic $\bar{x} \in X^n$.

Let C be the ∂ -closure of the orbit of some (equivalently: any) generic element in X^{n+1} ; consider the set

$$D = \{\bar{z} \in X^n : \forall^* y \in X (\bar{z}, y) \in C\} .$$

Claim. $\bar{x} \in D$.

Proof of the Claim. Let us first show that D is ∂ -closed.

To show this, let (\bar{z}_n) be a sequence of elements of D and $\bar{z} \in X^n$ be such that $\partial(\bar{z}_n, \bar{z}) \rightarrow 0$. Then we have

$$\forall n \forall^* y \in X (\bar{z}_n, y) \in C .$$

Using the Baire category theorem, we then see that

$$\forall^* y \in X \forall n (\bar{z}_n, y) \in C .$$

Since C is ∂ -closed, this implies that

$$\forall^* y \in X (\bar{z}, y) \in C .$$

The set D is also G -invariant, and the Kuratowski-Ulam theorem tells us that D is comeagre. Since \bar{x} is generic in X^n the ∂ -closure of $G.\bar{x}$ must meet D , and so be contained in D . From this we get that

$\bar{x} \in D$, as claimed. □ Claim

In the following, we let $C_{\bar{x}} = \{y : (\bar{x}, y) \in C\}$. Note that if $y \in C_{\bar{x}}$ then (\bar{x}, y) is generic. Indeed, there is some (\bar{x}_0, y_0) such that C is the ∂ -closure of $G.(\bar{x}_0, y_0)$; since $(\bar{x}, y) \in C$ we get that there exists a sequence (g_n) of elements of G such that $\partial(g_n.(\bar{x}_0, y_0), (\bar{x}, y)) \rightarrow 0$, hence $\partial(g_n^{-1}.(\bar{x}, y), (\bar{x}_0, y_0)) \rightarrow 0$. This shows that C is also the ∂ -closure of $G.(\bar{x}, y)$.

Now fix $\varepsilon > 0$, and pick some $y_0 \in (A)_\varepsilon \cap C_{\bar{x}}$; in the following we denote

$$G_{\bar{x}, \varepsilon} = \{g \in G : \partial(g.\bar{x}, \bar{x}) < \varepsilon\} .$$

Remember that C is the ∂ -closure of $G.(\bar{x}, y_0)$; hence $z \in C_{\bar{x}}$ iff for any $\varepsilon > 0$ there exists $g \in G$ such that $\partial(g.\bar{x}, \bar{x}) < \varepsilon$ and $\partial(g.y_0, z) < \varepsilon$. From this, we obtain that

$$C_{\bar{x}} = \bigcap_{\varepsilon > 0} (G_{\bar{x}, \varepsilon}.y_0)_\varepsilon .$$

Also, one can check that

$$(G_{\bar{x}, \varepsilon}.y_0)_\varepsilon = \bigcup_{g \in G_{\bar{x}, \varepsilon}} ((g.V \cap G_{\bar{x}, \varepsilon}).y_0)_\varepsilon = \left(\left(\bigcup_{g \in G_{\bar{x}, \varepsilon}} g.V \cap G_{\bar{x}, \varepsilon} \right).y_0 \right)_\varepsilon$$

From this we obtain the existence of a sequence (g_n) of elements of $G_{\bar{x}, \varepsilon}$ such that

$$(G_{\bar{x}, \varepsilon}.y_0)_\varepsilon = \bigcup_n ((g_n.V \cap G_{\bar{x}, \varepsilon}).y_0)_\varepsilon .$$

So there must exist some n such that $((g_n.V \cap G_{\bar{x}, \varepsilon}).y_0)_\varepsilon$ is non-meagre, and hence $((V \cap G_{\bar{x}, 2\varepsilon}).y_0)_\varepsilon$ is also non-meagre. Since this set is analytic it must be Baire-measurable and so it is comeagre in some nonempty open subset of X , so it must intersect $(B)_\varepsilon \cap C_{\bar{x}}$; pick some $y_1 \in ((V \cap G_{\bar{x}, 2\varepsilon}).y_0)_\varepsilon \cap (B)_\varepsilon \cap C_{\bar{x}}$ and let $h \in V \cap G_{\bar{x}, 2\varepsilon}$ be such that $\partial(h.y_0, y_1) < \varepsilon$.

We have that (\bar{x}, y_0) and (\bar{x}, y_1) are generic, $h \in V$, $\partial(h.\bar{x}, \bar{x}) < 2\varepsilon$ and $\partial(h.y_0, y_1) < \varepsilon$. Since ε was arbitrary, this concludes the proof of the lemma. □

Lemma 3.4. *Let $(A_n), (B_n)$ be two sequences of subsets of X such that for any n A_n is not meagre and B_n is not meagre in any nonempty open set. Let also (r_n) be a sequence of nonnegative reals. There exists a continuous map $a \mapsto h_a$ from 2^ω into G such that if $a_{|n} = b_{|n}$, $a(n) = 0$, $b(n) = 1$ then $\partial(h_a.A_n, h_b.B_n) < r_n$.*

Proof. First, fix a sequence of nonnegative reals (ε_n) such that for all n

$$3 \sum_{m=n}^{+\infty} \varepsilon_m < r_n .$$

Let d be a complete distance inducing the topology of G . we denote by $2^{<\omega}$ the set of finite binary sequences; for $s \in 2^{<\omega}$ we denote its length by $|s|$.

Claim 1. We can define for any $s \in 2^{<\omega} \setminus \{\emptyset\}$ elements $f_s \in G$, and $x_s \in X$ in such a way that, denoting $h_s = f_{s_{|1}} \dots f_s$ ($s \neq \emptyset$; we let $h_\emptyset = 1_G$) we have

- (1) $\bar{x}_s = (x_{s_{|1}}, \dots, x_s)$ is generic.
- (2) $\partial(x_{s \smallfrown 0}, A_{|s|}) < \varepsilon_{|s|}$ and $\partial(x_{s \smallfrown 1}, B_{|s|}) < \varepsilon_{|s|}$
- (3) $f_{s \smallfrown 0} = 1_G$
- (4) $d(h_s, h_s f_{s \smallfrown 1}) < 2^{-|s|}$
- (5) $\partial(f_{s \smallfrown 1}.\bar{x}_{s \smallfrown 1}, \bar{x}_{s \smallfrown 0}) < \varepsilon_{|s|}$.

To do this, we begin by applying lemma 3.3 to find x_0, x_1, f_0, f_1 ; we let $h_\emptyset = 1_G$.

Now assume that f_s and \bar{x}_s are given.

Applying lemma 3.3 again, we can find $x_{s\smallfrown 0}, x_{s\smallfrown 1}, f_{s\smallfrown 1}$ such that $\bar{x}_{s\smallfrown 0}, \bar{x}_{s\smallfrown 1}$ are generic, $f_{s\smallfrown 1} \in \{f \in G : d(h_s, h_s f) < 2^{-|s|}\}$, $\partial(x_{s\smallfrown 0}, A_{|s|}) < \varepsilon_{|s|}$, $\partial(x_{s\smallfrown 1}, B_{|s|}) < \varepsilon_{|s|}$, and finally $\partial(f_{s\smallfrown 1} \cdot \bar{x}_{s\smallfrown 1}, \bar{x}_{s\smallfrown 0}) < \varepsilon_{|s|}$.

□Claim 1

Assume now that sequences satisfying the properties above have been built; keeping the notations above, we can let for any $a \in 2^\omega$ $h_a = \lim h_{a|_n}$ and the map $a \mapsto h_a$ is continuous.

Claim 2. Pick $a \in 2^\omega$ and $m \in \mathbb{N}$. Then

$$\partial(h_a \cdot \bar{x}_{a|_m}, h_{a|_m} \cdot \bar{x}_{a|_m}) \leq \sum_{k=m+1}^{+\infty} \varepsilon_k .$$

Indeed, let n be any integer $> m$; points (3) and (5) of our construction ensure that we have

$$\partial(f_s \cdot \bar{x}_{s|_m}, \bar{x}_{s|_m}) < \varepsilon_{|s|} .$$

Then we have $h_a(x_{a|_m}) = \lim_{n \rightarrow +\infty} h_{a|_n} \cdot x_{a|_m}$ and so the above inequality gives us

$$\partial(h_a \cdot \bar{x}_{a|_m}, h_{a|_m} \cdot \bar{x}_{a|_m}) \leq \sum_{k=m+1}^{+\infty} \varepsilon_k .$$

This proves Claim 2. □Claim 2

Now we can finish the proof: assume that $a|_n = b|_n$, $a(n) = 0$ and $b(n) = 1$. Let $s = a|_n$. We have $h_{s\smallfrown 0} \cdot \bar{x}_{s\smallfrown 0} = h_s \cdot \bar{x}_{s\smallfrown 0}$, and

$$\partial(h_{s\smallfrown 1} \cdot \bar{x}_{s\smallfrown 1}, h_s \cdot \bar{x}_{s\smallfrown 0}) \leq \varepsilon_{n+1}$$

From this we obtain that

$$\partial(h_{s\smallfrown 0} \cdot \bar{x}_{s\smallfrown 0}, h_{s\smallfrown 1} \cdot \bar{x}_{s\smallfrown 1}) \leq \varepsilon_{n+1}$$

Using the second claim above and the triangle inequality, we get

$$\partial(h_a \cdot \bar{x}_{s\smallfrown 0}, h_b \cdot \bar{x}_{s\smallfrown 1}) \leq \varepsilon_{n+1} + 2 \sum_{k>n} \varepsilon_k .$$

And since $x_{s\smallfrown 0}, x_{s\smallfrown 1}$ have been chosen close to A_n, B_n we finally obtain

$$\partial(h_a \cdot A_n, h_b \cdot B_n) \leq 2 \sum_{k>n} \varepsilon_k + \varepsilon_{n+1} + 2\varepsilon_n < r_n .$$

This concludes the proof of the lemma. □

4. CONSEQUENCES OF AMPLE GENERICS

4.1. Automatic continuity for topometric groups.

Recall the following classical notion from topological group theory.

Definition 4.1. If G is a group, we say that $A \subseteq G$ is σ -*syndetic* if G is covered by countably many left-translates of A .

A topological group G has the *Steinhaus property* if there exists some integer k such that, whenever A is symmetric, σ -syndetic and contains 1_G , we have $1_G \in \text{Int}(A^k)$.

In the topometric setting, this becomes:

Definition 4.2. A topometric group has the *Steinhaus property* if there exists some integer k such that, whenever A is symmetric, σ -syndetic and contains 1_G , we have $1 \in \text{Int}((A^k)_\varepsilon)$ for all $\varepsilon > 0$. If G and k are as above, we shall say that G is Steinhaus with exponent k .

Proposition 4.3. *Let (G, τ_G, ∂_G) be a Polish topometric group with the Steinhaus property, and (H, τ) be a second countable topological group. Assume $\varphi: (G, \partial_G) \rightarrow H$ is a continuous morphism; then $\varphi: (G, \tau_G) \rightarrow H$ is continuous.*

Proof. Let k be an integer witnessing that G is Steinhaus; let V be a τ -neighborhood of 1_H and pick some symmetric neighborhood W of 1_H such that $W^{k+1} \subseteq V$.

There exists some $\varepsilon > 0$ such that $(1_G)_\varepsilon \subseteq \varphi^{-1}(W)$. Also, since $\varphi^{-1}(W)$ is σ -syndetic, we know that

$$1_G \in \text{Int}((\varphi^{-1}(W)^k)_\varepsilon) \subseteq \text{Int}((\varphi^{-1}(W^k))_\varepsilon) =: U .$$

We have that U is open, contains 1_G , and for any $g \in U$ one can find g_0, g_1 such that $g = g_0 g_1$, with $\varphi(g_0) \in W^k$ and $g_1 \in (1_G)_\varepsilon$; hence $\varphi(g) \in W^{k+1} \subseteq V$ for any $g \in U$, and this proves that $\varphi: (G, \tau_G) \rightarrow H$ is continuous. \square

Note, as a corollary of the above proposition, that if (G, τ, ∂) is a Polish topometric group with the Steinhaus property then for any Polish topology τ' such that (G, τ', ∂) is also a topometric group we must have $\tau = \tau'$.

The link between ample generics and the Steinhaus property is made clear by the following theorem.

Theorem 4.4. *Let G be a topometric Polish group with ample generics. Then G is Steinhaus with exponent 10.*

This is a corollary of the following result (again an analogue of a theorem due to Kechris and Rosendal), which we will also use in section 4.3.

Theorem 4.5. *Let G be a Polish topometric group with ample generics. If $\{k_i A_i\}_{i < \omega}$ is a covering of G then there is some i such that for any $\varepsilon > 0$ the set*

$$(A_i^{-1} A_i A_i^{-1} A_i^{-1} A_i A_i^{-1} A_i A_i A_i^{-1} A_i)_\varepsilon$$

contains an open neighborhood of the identity.

Proof. Enumerate all non-meagre A_i 's in a list $(B_n)_{n \in \mathbb{N}}$ in which each of them appears infinitely many times. If there is some n such that $(B_n^{-1} B_n B_n B_n^{-1} B_n)_\varepsilon$ is comeagre in some nonempty open set for any $\varepsilon > 0$, then using Pettis' theorem gives us the desired result.

So we may pick a countable sequence (r_n) such that $r_n > 0$ and each $(B_n^{-1} B_n B_n B_n^{-1} B_n)_{r_n}$ is not comeagre in any nonempty open set. We let C_n denote the set $G \setminus (B_n^{-1} B_n B_n B_n^{-1} B_n)_{r_n}$, and we apply Lemma 3.4 to $(B_n), (C_n), (r_n)$ to obtain a continuous map $a \mapsto h_a$ from 2^ω into G . This map is injective, since if $a, b \in 2^\omega$ are different and n is the smallest integer such that $a(n) \neq b(n)$ then (assuming w.l.o.g that $a(n) = 0, b(n) = 1$) one has $\partial(h_a B_n h_a^{-1}, h_b C_n h_b^{-1}) < r_n$; since $\partial(B_n, C_n) \geq r_n$ and conjugacy by h_a (resp. h_b) is a ∂ -isometry (because ∂ is bi-invariant) we see that $h_a \neq h_b$.

We denote by H the set $\{h_a : a \in 2^\omega\}$; it is homeomorphic to 2^ω . Note that $B = \cup_{i,n} k_i B_n$ contains the union of all non-meagre $k_i A_i$, in particular it is comeagre. Since the map $(g, h) \mapsto g^{-1} h$ is open and continuous from $G \times H$ into G , and the inverse image of a comeagre set by such a mapping is comeagre, we see that

$$\forall^* g \in G \forall^* h \in H \ g^{-1} h \in B .$$

Thus we can fix $g \in G$ such that

$$\forall^* h \in H \ h \in g B .$$

Further, we can find some l, n such that the set $\{a \in 2^\omega : h_a \in gk_l B_n\}$ is non-meagre. In particular, its closure cannot have empty interior and so this set must be dense in $V_t = \{a \in 2^\omega : t \subseteq a\}$ for some $t \in 2^{<\omega}$. Choose $m > |t|$ such that $B_n = B_m$, and let a and b be such that $a_{|m} = b_{|m}$, $a(m) = 0$, $b(m) = 1$ and $h_a, h_b \in gk_l B_n = gk_l B_m$. Set $h_a = gk_l h_1$ and $h_b = gk_l h_2$. Then one checks that

$$h_b^{-1} h_a B_m h_a^{-1} h_b = h_2^{-1} h_1 B_m h_1^{-1} h_2 \subseteq B_m^{-1} B_m B_m B_m^{-1} B_m .$$

Thus, $\partial(h_b^{-1} h_a B_m h_a^{-1} h_b, C_m) > r_n$, and this contradicts the definition of h_a, h_b . \square

As far as automatic continuity theorems go, one can actually do better. Remember that a group G has *uniform Souslin number* κ if κ is the least cardinal such that, for any open neighborhood V of 1_G , G can be covered by less than κ many left translates of V . Equivalently, κ is the least cardinal such that for any open neighborhood V of 1_H there do not exist κ disjoint left translates of V .

Then the techniques of [KR07] yield the following theorem.

Theorem 4.6. *Let (G, τ, ∂) be a Polish topometric group with ample generics, H a topological group with uniform Souslin number $< 2^{\aleph_0}$, and $\varphi: G \rightarrow H$ a morphism that is continuous from (G, ∂) into H . Then φ is continuous from (G, τ) to H .*

Proof. Let G, H, φ be as in the statement of the theorem. Let W be a neighborhood of 1_H ; choose some symmetric open neighborhood V of 1_H such that $V^{21} \subseteq W$. Pick also $\varepsilon > 0$ such that

$$\partial(g, id) \leq 2\varepsilon \Rightarrow \varphi(x) \in V .$$

Finally, let $A = \varphi^{-1}(V^2)$.

Note first that the condition on the uniform Suslin number of H implies that A covers G by less than 2^{\aleph_0} left translates. If A were meagre then there would be, by the Kuratowski-Mycielski theorem (see [Kec95], 19.1), a Cantor set $C \subseteq G$ such that for any $g \neq h \in C$ $g^{-1}h \notin A$. Since

$$\varphi(g)V \cap \varphi(h)V = \emptyset \Leftrightarrow \varphi(g^{-1}h) \in V.V^{-1} \Leftrightarrow g^{-1}h \in A ,$$

there would be uncountably many left translates of V in H , contradicting the assumption on its uniform Suslin number.

If we prove that the set $(A^5)_\varepsilon$ is comeagre in some nonempty open set, then we will obtain as a consequence of Pettis' theorem that $(A^{10})_{2\varepsilon}$ contains a neighborhood O of 1_G . By choice of V and ε , one has

$$\varphi(O) \subseteq \varphi(A^{10}).V \subseteq V^{21} \subseteq W .$$

To conclude the proof, it is thus enough to show that $(A^5)_\varepsilon$ is comeagre in some nonempty open set. If such is not the case then, applying lemma 3.4 to $A, G \setminus (A^5)_\varepsilon, \varepsilon$, we can find $h_a \in G$, with $a \in 2^\omega$, such that if $a_{|n} = b_{|n}$, $a(n) = 0$, $b(n) = 1$ then

$$\partial(h_a A h_a^{-1}, h_b (G \setminus (A^5)_\varepsilon) h_b^{-1}) < \varepsilon .$$

Equivalently,

$$\partial(h_b^{-1} h_a A h_a^{-1} h_b, G \setminus (A^5)_\varepsilon) < \varepsilon .$$

Since A covers G by less than 2^{\aleph_0} right translates, there is an uncountable $B \subseteq 2^\omega$ and some $g \in G$ such that for all $a \in B$ one has $h_a \in g.A$. Then one can find $a, b \in B$ and $n \in \mathbb{N}$ such that $a_{|n} = b_{|n}$, $a(n) = 0$ and $b(n) = 1$. Let $g_a, g_b \in A$ be such that $h_a = g g_a$, $h_b = g g_b$. Then

$$h_b^{-1} h_a A h_a^{-1} h_b = g_b^{-1} g_a A g_a^{-1} g_b \subseteq A^5 .$$

Hence we see that $\partial(A^5, G \setminus (A^5)_\varepsilon) < \varepsilon$, and this is of course a contradiction. \square

4.2. A weak small index property for topometric groups.

Recall that a Polish group G has the *small index property* if any subgroup $H \leq G$ of index strictly less than 2^{\aleph_0} is open. Kechris and Rosendal proved that a Polish group with ample generics must have the small index property. In our context, their techniques yield the following result.

Proposition 4.7. *Assume (G, τ, ∂) is a Polish topometric group with ample generics. Then any ∂ -closed subgroup of G of index strictly less than 2^{\aleph_0} is open.*

Said differently, the ∂ -closure of a subgroup of index strictly less than 2^{\aleph_0} is open (hence clopen). This result is not very useful, for most groups we have in mind are connected, and thus do not have any open subgroups. So, for these groups, the above Proposition only says that they have no ∂ -closed subgroups of index $< 2^{\aleph_0}$. Since the proof is a straightforward adaptation of Kechris and Rosendal's proof, we don't include it and content ourselves with stating the above Proposition for the record.

4.3. Bergman property for groups with a bi-invariant metric.

Definition 4.8. An abstract group G has the *Bergman property* if whenever $W_0 \subseteq W_1 \subseteq \dots \subseteq W_n \dots$ is an increasing, exhaustive sequence of subsets of G there exist n and k such that $G = W_n^k$. If k above can be chosen independently of the sequence (W_n) , then we say that G is *k-Bergman*.

In [Ber06], G. Bergman proved that \mathfrak{S}_∞ has the Bergman property; it was noticed by a number of authors that the Bergman property is equivalent to each of the following statements:

- (a) Any left-invariant pseudometric on G is bounded
- (b) Any action by isometries of G on a metric space X has bounded orbits.

In [Ros09b], C. Rosendal introduced a variant of the Bergman property for topological groups, which he dubbed *property (OB)*, and which turns out to be equivalent to each of the following two statements:

- (a) Any *continuous* left-invariant pseudometric on G is bounded.
- (b) Any *continuous* action by isometries of G on a metric space X has bounded orbits.

Let us now introduce a new, and closely related, property.

Definition 4.9. Let G be a group endowed with a bi-invariant (and, as usual, not necessarily separable) distance ∂ . We say that G has the *metric Bergman property* if whenever (W_n) is an increasing, exhaustive sequence of subsets of G then for any $\varepsilon > 0$ there exists n and k such that $(W_n^k)_\varepsilon = G$. If k above can be chosen independently from the sequence (W_n) and ε , then we say that (G, ∂) is *metrically k-Bergman*.

In the topometric context, the equivalences above become:

Theorem 4.10. *Let G be a group endowed with a bi-invariant distance ∂ . Then the following are equivalent:*

- (a) (G, ∂) has the *metric Bergman property*.
- (b) Whenever d is a left-invariant pseudometric on G such that there is ε with the property that $\{g: \partial(g, id) \leq \varepsilon\}$ is d -bounded, d must be bounded on G .
- (c) Whenever (G, ∂) acts by isometries on a metric space (X, d) in such a way that for all x there exists $\varepsilon > 0$ and M with $\partial(g, 1_G) \leq \varepsilon \Rightarrow d(x, g(x)) \leq M$, all G -orbits are bounded in (X, d) .

Note that the metric Bergman property for (G, ∂) implies property (OB) for (G, ∂) and is weaker than the Bergman property for G .

Proof. It is easy to check that (b) and (c) are equivalent, so we only show that (a) and (b) are equivalent. (a) \Rightarrow (b): Let d be as in the statement of (b), and $\varepsilon > 0$, M such that $\partial(g, id) \leq \varepsilon \Rightarrow d(g, 1_G) \leq M$.

Let W_n denote the closed ball or radius n for the distance d ; by the metric Bergman property for (G, ∂) there is some n and k such that $G = (W_n^k)_\varepsilon$. Now, note that, by the triangle inequality and an easy induction using the left-invariance of d , one sees that if $g = g_1 \dots g_k$, where each $d(g_i, 1_G)$ is less than n , then

$$d(g, 1_G) \leq d(g_1 \dots g_k, g_1) + d(g_1, 1_G) \leq d(g_2 \dots g_k, 1_G) + d(g_1, 1_G) \leq \dots \leq n.k$$

Also, if $d(g, 1_G) \leq n.k$, $d(h, 1_G) \leq M$ then $d(gh, 1_G) \leq n.k + M$. So, the fact that $G = (W_n^k)_\varepsilon$ means that for any $g \in G$ one has $d(g, 1_G) \leq nk + M$.

(b) \Rightarrow (a): Let (W_i) be an increasing, exhaustive sequence of subsets of G ; then $(W_i \cap W_i^{-1})$ is still exhaustive, and so one can assume without loss of generality that each W_i is symmetric and $W_0 = \{1_G\}$. Then fix $\varepsilon > 0$; one can define a left-invariant pseudometric d on G by setting

$$d(f, g) = \inf\{k_1 + \dots + k_n : f = gh_1 \dots h_n, h_i \in (W_{k_i})_\varepsilon\}.$$

By definition, $d(f, 1_G) \leq 1$ for all $f \in (1_G)_\varepsilon$ and so d is bounded. This is only possible if there is some k and some n such that $G = (W_k)_\varepsilon^n$, which shows that (G, ∂) has the metric Bergman property. \square

Definition 4.11. Let G act continuously by isometries on some Polish metric space M . We say that the action is *approximately oligomorphic* if for any $n \in \mathbb{N}$ and any $\varepsilon > 0$ there exist $\bar{b}_1, \dots, \bar{b}_k \in M^n$ such that for each $\bar{x} \in M^n$ there exists $j \in \{1, \dots, k\}$ and $g \in G$ satisfying $d(g.\bar{b}_j, \bar{x}) \leq \varepsilon$.

The action of the isometry group of the Urysohn space \mathbb{U}_1 of diameter 1 on \mathbb{U}_1 is an example of approximately oligomorphic action².

We then have the following analogue of Theorem 6.19 of [KR07].

Theorem 4.12. *Let (G, τ) be a Polish group that admits a continuous, oligomorphic action by isometries on a Polish metric space (X, d) . Let ∂_∞ denote the uniform distance on G associated to this action (note that it may be strictly finer than ∂_u). Assume additionally that $(G, \tau, \partial_\infty)$ has ample generics. Then (G, ∂_∞) is metrically 21-Bergman.*

Proof. Assume that (W_n) is an exhaustive, increasing sequence of subsets of G , and fix $\varepsilon > 0$. Then, by Proposition 4.5, there is some n such that $(W_n^{10})_\varepsilon$ contains a neighborhood of 1_G . Since the action of G on X is continuous, one can find $n, \delta > 0$ and $\bar{x} = (x_1, \dots, x_n) \in X^n$ such that

$$\{g \in G : d(g(\bar{x}_i), \bar{x}_i) < 2\delta\} \subseteq (W_n^{10})_\varepsilon.$$

Since the action of G on X is approximately oligomorphic, there exist for any $\eta > 0$ some $\bar{a}_1, \dots, \bar{a}_k, \bar{b}_1, \dots, \bar{b}_k$ such that

$$\{(\bar{x}, h.\bar{x}) : h \in G\} \subseteq (G.\{(\bar{a}_i, \bar{b}_i)\})_\eta.$$

Applying this with $\eta = \delta/2$, and using the triangle inequality we see that there exist $\bar{b}_1, \dots, \bar{b}_k$ such that

$$\{(\bar{x}, h.\bar{x}) : h \in G\} \subseteq (G.\{(\bar{x}_i, \bar{b}_i)\})_\delta.$$

Now, let $f \in G$. There exists some i and some $g \in G$ such that $d(f\bar{x}, g\bar{b}_i) < \delta$ and $d(g.\bar{x}_i, \bar{x}) < \delta$. Note that i is such that there is some $h_i \in G$ with $d(\bar{b}_i, h_i\bar{x}) < \delta$ and so we may assume that for all $i \in \{1, \dots, k\}$ there is some $h_i \in G$ with this property. Now let l be large enough that $h_1, \dots, h_k \in W_l$ and $l \geq n$.

Then one has for all $f \in G$ some $g \in G$ and i such that $d(f\bar{x}, g\bar{b}_i) < \delta$ and $d(g.\bar{x}_i, \bar{x}) < \delta$. In particular, $d(f\bar{x}, gh_i\bar{x}) < 2\delta$.

But then $h_i^{-1}g^{-1}f \in (W_l^{10})_\varepsilon$, and since $g \in (W_l^{10})_\varepsilon$ and $h_i \in W_l$ we finally obtain that $f \in (W_l^{21})_{2\varepsilon}$.

Since ε was arbitrary, this concludes the proof. \square

²See the next section for a definition and a brief discussion of Urysohn spaces.

5. EXAMPLES

Our most pressing concern at this point is to provide examples of topometric Polish groups with ample generics (besides the known examples of Polish groups with ample generics, of course).

5.1. A criterion for ample metric generics.

So far all the examples we know come, in a sense, from completions of countable structures whose automorphism group has ample generics. Theorem 5.3 below is the only tool we have at the moment to show that an automorphism group has ample metric generics. Before stating this theorem, we state and prove a simpler version.

Proposition 5.1. *Assume the following.*

- \mathcal{M} is a Polish metric structure, $N \subseteq M$ is countable, dense, and there is a first-order structure \mathcal{N} with universe N such that any automorphism of \mathcal{N} extends to an automorphism of \mathcal{M} .
- $\text{Aut}(\mathcal{N})$ is dense in $\text{Aut}(\mathcal{M})$ and $\text{Aut}(\mathcal{N})$ has ample generics.
- for any $\varepsilon > 0$, any partial automorphism h of \mathcal{N} with domain $\{n_1, \dots, n_k\}$ and any automorphism g of \mathcal{M} such that $d(g(n_i), h(n_i)) < \varepsilon$ for all i there is an automorphism \tilde{h} of \mathcal{N} that extends h and is such that $\partial_u(g, \tilde{h}) < \varepsilon$.

Then $(\text{Aut}(\mathcal{M}), \tau, \partial_u^{\mathcal{M}})$ has ample generics.

To prove this, we need to use the characterization of comeagre sets via Banach-Mazur games; recall that, whenever X is a topological space and A is a subset of X , the Banach-Mazur game G_A for A is the game where two players I, II take turns playing nonempty open sets U_i, V_i in such a way that $U_{i+1} \subseteq V_i \subseteq U_i$ for all integer i . Player II wins a run of this game is $\bigcap U_i \subseteq A$, else player I wins. Schematically, a run of G_A may be represented as follows:

$$\begin{array}{ccccccc} \text{I} & U_0 & U_1 \subseteq V_0 & \dots & U_n & U_{n+1} \subseteq V_n & \dots \\ \text{II} & & V_0 \subseteq U_0 & \dots & & V_n \subseteq U_n & \dots \end{array}$$

A strategy σ for one of the players (say, player II), is what one would expect: σ gives player II a unique way to respond to any move by player I; σ is *winning* if player II wins any run of the game where he has followed σ .

The reason these games are useful here is the following theorem: if X is Polish and $A \subseteq X$, then player II has a winning strategy in G_A if, and only if, A is comeagre in X . This is what we will use below. We refer the reader to Chapter I, section 8 of [Kec95] for more background on Banach-Mazur games.

Proof. To simplify notation, we simply denote the uniform distance $\partial_u^{\mathcal{M}}$ by ∂_u . We pick a comeagre set A in $\text{Aut}(\mathcal{N})^n$, fix $\varepsilon > 0$, and set about proving that $(A)_\varepsilon$ is comeagre in $\text{Aut}(\mathcal{M})^n$. We assume that $n = 1$ (the general case is similar).

We pick a winning strategy σ_N for player II in the Banach-Mazur game G_N for A in $\text{Aut}(\mathcal{N})$; we assume additionally that II only plays basic open sets of vanishing diameter, and use σ_N to define a winning strategy σ_M for player II in the Banach-Mazur game G_M for $(A)_\varepsilon$ in $\text{Aut}(\mathcal{M})$, thus proving that $(A)_\varepsilon$ is comeagre.

Let us describe a run of the game where II follows σ_M . Player I opens the game by playing an open set U of $\text{Aut}(\mathcal{M})$; we can, without loss of generality, assume that U is of the form

$$\{g \in \text{Aut}(\mathcal{M}) : \forall i \in \{1, \dots, p_0\} \ d(g(n_i), g_0(n_i)) < \varepsilon_0\}$$

for some automorphism g_0 of \mathcal{M} and some $\varepsilon_0 < \varepsilon$. Using the lemma's assumptions, we may find an automorphism h_0 of \mathcal{N} that belongs to U and, reducing ε_0 if necessary, assume that

$$\{g \in \text{Aut}(\mathcal{M}) : \forall i \in \{1, \dots, p_0\} \, d(g(n_i), h_0(n_i)) < \varepsilon_0\} \subseteq U .$$

Then we may tell II that I has played (in G_N) the open set

$$\tilde{U} = \{h \in \text{Aut}(\mathcal{N}) : \forall i \in \{1, \dots, p_0\} \, h(n_i) = h_0(n_i)\}$$

From this, σ_N gives player II a basic open set \tilde{V} of $\text{Aut}(\mathcal{N})$ that is contained in \tilde{U} , and so is of the form

$$\tilde{V} = \{h \in \text{Aut}(\mathcal{N}) : \forall i \in \{1, \dots, p_0 + p_1\} \, h(n_i) = m_i\}$$

where $m_i = h_0(n_i)$ for all $i \leq p_0$. Pick some $h'_0 \in \tilde{V}$, define $f_0 = h'_0 h_0^{-1}$ and note that $f_0(h_0(n_i)) = h'_0(n_i) = h_0(n_i)$ for all $i \leq p_0$ since $h'_0 \in \tilde{V}$.

Then we have that, for all $i \in \{1, \dots, p_0\}$,

$$d(f_0 g_0(n_i), h_0(n_i)) = d(f_0 g_0(n_i), f_0 h_0(n_i)) = d(g_0(n_i), h_0(n_i)) < \varepsilon_0 .$$

So $f_0 g_0 \in U$, and we may pick some ε_1 such that $\varepsilon_0 + \varepsilon_1 < \varepsilon$ and the open set V of $\text{Aut}(\mathcal{M})$ defined by

$$V = \{g \in \text{Aut}(\mathcal{M}) : \forall i \in \{1, \dots, p_0 + p_1\} \, d(g(n_i), f_0 g_0(n_i)) < \varepsilon_1\}$$

is contained in U . The strategy σ_M tells II to play this V in G_M .

This ends the first step of the game; to explain how to go on, assume that I has answered to II playing V by playing some basic open set U_1 contained in V , and pick $g_1 \in U_1$. Now, denote by h_1 the partial automorphism of \mathcal{N} defined by the restriction of h'_0 to $\{n_i\}_{i \leq p_0 + p_1}$ and note that for all $i \leq p_0 + p_1$ we have, by the choice of V and the definition of h_1 , that

$$d(g_1(n_i), h_1(n_i)) \leq d(g_1(n_i), f_0 g_0(n_i)) + d(f_0 g_0(n_i), f_0 h_0(n_i)) < \varepsilon_1 + \varepsilon_0 < \varepsilon .$$

Observe that the fact that $d(f_0 g_0(n_i), f_0 h_0(n_i)) < \varepsilon_0$ is a direct consequence of f_0 being an isometry and h_0 having been chosen in such a way that $\partial_u(g_0, h_0) < \varepsilon_0$.

We now know that we may extend h_1 to an automorphism of \mathcal{N} , still denoted by h_1 , which is such that $\partial_u(g_1, h_1) < \varepsilon_0 + \varepsilon_1$. Note that we took care of ensuring that $h_1 \in \tilde{V}$: thus II may now apply the same procedure as in the first step.

This should be enough to describe the strategy σ_M - player II translates the moves of player I in G_M to moves in G_N which he knows how to respond to. Now consider a run of the game in which I plays basic open sets U_n of vanishing diameter (in $\text{Aut}(\mathcal{M})$) and II follows σ_M . The run produces two sequences $(g_n), (h_n)$ such that $g_n \in U_n$ and $h_n \in \tilde{U}_n$ and $\partial_u(g_n, h_n) < \varepsilon$ for all n ; since the diameters are vanishing, $g_n \rightarrow g$ in $\text{Aut}(\mathcal{M})$ and $h_n \rightarrow h \in \text{Aut}(\mathcal{N})$, a fortiori $h_n \rightarrow h \in \text{Aut}(\mathcal{M})$. Also, we must have $h \in A$ since σ_N is winning. We are finally done: indeed, $\partial_u(g_n, h_n) \leq \varepsilon$ for all n and, since ∂_u is lower semicontinuous, this shows that $\partial_u(g, h) \leq \varepsilon$. Thus $g \in (A)_\varepsilon$ and II has won in G_M too. \square

The result above is not quite sufficient for some of the applications we have in mind. We can improve it a little bit as follows.

Definition 5.2. Let \mathcal{M} be a Polish metric structure, and \mathcal{N} be a countable first-order structure whose universe N is a subset of M .

We say that a finite set $A \subseteq N$ is ε -*extendable* if, whenever $g \in \text{Aut}(\mathcal{M})$ and a partial automorphism h of \mathcal{N} with domain A are such that $\sup\{d(g(a), h(a)) : a \in A\} < \varepsilon$, there exists some automorphism h_1 of \mathcal{N} that extends h and is such that $\partial_u(g, h_1) < \varepsilon$.

With this definition in hand, we may check that to make the proof of Proposition 5.1 work we only need to assume that any finite set of N is contained in a finite ε -extendable subset. We thus obtain our sharpest tool to prove that a Polish group has ample metric generics.

Theorem 5.3. *Assume the following.*

- \mathcal{M} is a Polish metric structure, $N \subseteq M$ is countable, dense, and there is a first-order structure \mathcal{N} with universe N such that any automorphism of \mathcal{N} extends to an automorphism of \mathcal{M} .
- $\text{Aut}(\mathcal{N})$ is dense in $\text{Aut}(\mathcal{M})$ and $\text{Aut}(\mathcal{N})$ has ample generics.
- There exists a sequence (ε_i) that decreases to 0 and is such that any finite subset of \mathcal{N} is contained in a finite ε_i -extendable set.

Then $(\text{Aut}(\mathcal{M}), \tau, \partial_u^{\mathcal{M}})$ has ample generics.

Let us discuss three examples of situations where one can use the above theorem. In the three examples below, conjugacy classes are known to be meagre: this is a theorem of Kechris for $\text{Iso}(\mathbb{U}_1)$, a result variously attributed to Rohlin or Del Junco for $\text{Aut}([0, 1], \lambda)$, and a theorem of Nadkarni for $U(\ell_2)$. Chapter I.2 of the book [Kec10] contains a proof that conjugacy classes in both $\text{Aut}([0, 1], \lambda)$ and $U(\ell_2)$ are meagre, while a proof that conjugacy classes in $\text{Iso}(\mathbb{U}_1)$ are meagre may be found in [Ros09a].

5.2. The isometry group of the bounded Urysohn space.

Recall that the *Urysohn space of diameter 1* \mathbb{U}_1 is the unique, up to isometry, metric space of diameter 1 which is both *universal* for Polish metric spaces of diameter at most 1 and *ultrahomogeneous*, i.e. isometries between finite subsets extend to isometries of the whole space. For information about this space, we refer to the volume [LPR⁺08]. We simply recall that a formula for ∂_u is

$$\partial_u(\varphi, \psi) = \sup\{d(\varphi(x), \psi(x)) : x \in \mathbb{U}_1\}.$$

The countable counterpart of \mathbb{U}_1 is the *rational Urysohn space of diameter 1* $\mathbb{Q}\mathbb{U}_1$, which is both ultrahomogeneous and universal for countable metric spaces with rational distances and diameter at most 1.

The relationship between these two spaces is simple: \mathbb{U}_1 is the completion of $\mathbb{Q}\mathbb{U}_1$; in particular, $\mathbb{Q}\mathbb{U}_1$ is dense in \mathbb{U}_1 and any isometry of $\mathbb{Q}\mathbb{U}_1$ extends to an isometry of \mathbb{U}_1 . A theorem due to S. Solecki [Sol05], combined with the techniques of Kechris and Rosendal, implies that the automorphism group of $\mathbb{Q}\mathbb{U}_1$ has ample generics (this is pointed out in [KR07]). Finally, the usual techniques when dealing with isometries of the Urysohn space enable one to check that the final condition of Proposition 5.1 is satisfied by the embedding of $\mathbb{Q}\mathbb{U}_1$ into \mathbb{U}_1 . Hence, we see that $\text{Iso}(\mathbb{U}_1)$ has ample metric generics.

Let us point out an application of this before moving on to the next example: our earlier results, along with the compact homogeneity of \mathbb{U}_1 , imply that $(\text{Iso}(\mathbb{U}_1), d_u)$ has the metric Bergman property. If we denote by \mathbb{U} the unbounded Urysohn space, then, as its isometry group $\text{Iso}(\mathbb{U})$ acts transitively on \mathbb{U} , it is clear that $(\text{Iso}(\mathbb{U}), d_u)$ does not have the metric Bergman property. Thus we obtain that $(\text{Iso}(\mathbb{U}), d_u)$ and $(\text{Iso}(\mathbb{U}_1), d_u)$ are not isomorphic topological groups. A natural question is then: are $\text{Iso}(\mathbb{U})$ and $\text{Iso}(\mathbb{U}_1)$ isomorphic as abstract groups? To show that this is not the case, one might try to prove that $\text{Iso}(\mathbb{U}_1)$ is simple.

5.3. The unitary group of a separable Hilbert space.

The situation is a bit more complicated in $\mathcal{U}(\ell_2)$, the unitary group of the complex Hilbert space $\ell_2(\mathbb{N})$. In this case, the topology is the weak-operator topology (given, say, by pointwise convergence on the unit ball) and the uniform distance ∂_u is simply given by the operator norm, i.e. whenever S, T are two unitaries of ℓ_2 one has $\partial_u(S, T) = \|S - T\|$.

Fortunately, C. Rosendal has already worked out everything we need here (see [Ros09b]): let \mathcal{Q} denote the algebraic closure of \mathbb{Q} , and consider the countable subset $\mathcal{Q}\ell_2$ of $\ell_2(\mathbb{N})$ made up of the sequences with a finite support and all coordinates in \mathcal{Q} . The crucial fact here is that, as the norm of an element of $\mathcal{Q}\ell_2$ belongs to \mathcal{Q} , one can carry out the usual constructions of Hilbert space geometry (Gram-Schmidt, etc.) inside $\mathcal{Q}\ell_2$. This is used in [Ros09b] to prove that the automorphism group of $\mathcal{Q}\ell_2$ has ample generics;

the other conditions of Proposition 5.1 are relatively easy to check, and so we see that $\mathcal{U}(\ell_2)$ has ample metric generics.

5.4. The group of measure-preserving bijections of $[0, 1]$.

Denote by λ the Lebesgue measure on the unit interval $[0, 1]$; we see its automorphism group $\text{Aut}([0, 1], \lambda)$ as the automorphism group of the Polish metric structure $(MALG, d, d(\cdot, \emptyset))$, where $MALG$ denotes the measure algebra on $[0, 1]$ and $d(A, B) = \lambda(A \Delta B)$ (see [Kec95]). A possible formula for ∂_u is

$$\partial_u(S, T) = \sup\{\lambda(T(A) \Delta S(A)) : A \in MALG\} .$$

In ergodic theory, one often uses instead of this ∂_u the distance ∂'_u defined by

$$\partial'_u(S, T) = \lambda(\text{support}(ST^{-1}))$$

It is well-known that these two distances are equivalent, indeed that $\partial_u \leq \partial'_u \leq 2\partial_u$.

This time, the approximating substructure is the countable measure algebra generated by dyadic intervals. Kechris and Rosendal proved in [KR07] that its automorphism group has ample generics; any automorphism of the dyadic algebra extends to a measure-preserving automorphism of $[0, 1]$. However, the final condition of Proposition 5.1 is not satisfied in this case; fortunately, it is easy to check that the final condition of Theorem 5.3 is satisfied. Hence $\text{Aut}([0, 1], \lambda)$, with its usual Polish topology, has ample metric generics.

It is interesting to point out that the techniques used by J. Kittrell and T. Tsankov in [KT] to prove an automatic continuity theorem for the full group of a countable, measure-preserving equivalence relation readily apply to show the following:

Theorem. (essentially Kittrell-Tsankov [KT])

Any morphism from $(\text{Aut}([0, 1], \lambda), \partial_u)$ into a separable topological group is continuous.

Combining this with our above result that $(\text{Aut}([0, 1], \lambda)$ has ample metric generics, we obtain the following theorem.

Theorem 5.4. *Any morphism from $\text{Aut}([0, 1], \lambda)$, endowed with its usual Polish topology, into a separable topological group is continuous.*

Consequently, there is a unique Polish topology on $(\text{Aut}([0, 1], \lambda)$ that is compatible with its algebraic structure.

It would be interesting to identify other Polish groups with ample metric generics. In the classical first-order case, it was proved by Hodges, Hodkinson, Lascar and Shelah that the group of automorphisms of a countable model of an \aleph_0 -categorical, \aleph_0 -stable first order theory has ample generics. Does the same result hold (in the continuous logic setting) for the separable model of a separably categorical \aleph_0 -stable metric theory? In particular, does $\text{Aut}(L^p([0, 1]))$ have ample generics?

6. TRYING TO UNDERSTAND AMPLE METRIC GENERICS

Now, let (G, τ) be a Polish group such that (G, τ, ∂_u) has ample metric generics. It seems natural to wonder whether (G, τ, ∂) can have ample generics for a distance ∂ which strictly refines ∂_u ; or one could ask for a weaker result, namely, whether there exists a finest distance ∂ such that (G, τ, ∂) has ample generics. As we shall see shortly, the second question has a positive answer, and there exist Polish groups with ample generics and a non-discrete ∂_u .

An example of a Polish group with ample generics and a non-discrete ∂_u is simply $\mathfrak{S}_\infty^{\mathbb{N}}$. This group has ample generics because \mathfrak{S}_∞ has ample generics, and because of the following fact.

Proposition 6.1. *Let $(G_i)_{i \in I}$ be an at most countable family of Polish groups, and $G = \prod G_i$. Then G has ample generics if, and only if, each G_i has ample generics.*

This adapts to the topometric context: let $(G_i, \tau_i, \partial_i)$ be an at most countable family of Polish topometric groups, and $G = \prod G_i$ endowed with the product topology τ and the metric

$$\partial((g_i), (h_i)) = \sup\{\partial_i(g_i, h_i) : i \in I\} .$$

Then (G, τ, ∂) has ample generics iff $(G_i, \tau_i, \partial_i)$ has ample generics for all i .

Proof. We only give the proof of the first statement, since the topometric version is a straightforward adaptation.

Note first that, since coordinate projections induce surjective open morphisms from each G^n onto G_i^n which map diagonal conjugacy classes onto diagonal conjugacy classes, the existence of comeagre diagonal conjugacy classes in G^n immediately implies the existence of comeagre diagonal conjugacy classes in each G_i^n .

Now, assume that each G_i has comeagre diagonal conjugacy classes. For our purposes, it is enough to note the following easy fact: whenever $(X_i)_{i \in I}$ is an at most countable family of Polish spaces, and $(A_i)_{i \in I}$ are such that each A_i is comeagre in X_i , $\prod A_i$ is comeagre in $\prod X_i$. Applying this with $X_i = G_i^n$ and A_i equal to the comeagre diagonal conjugacy class in G_i^n , we obtain the desired result. \square

Proposition 6.1 implies in particular that $\mathfrak{S}_\infty^{\mathbb{N}}$ has ample generics, yet in that case ∂_u is non discrete: for instance, if σ is any element of $\mathfrak{S}_\infty \setminus \{id\}$, then the sequence σ^n defined by

$$\sigma_i^n = \begin{cases} id & \text{if } i \leq n \\ \sigma & \text{if } i > n \end{cases}$$

is a non-trivial sequence that ∂_u -converges to the identity of $\mathfrak{S}_\infty^{\mathbb{N}}$.

So we see that there may be different distances such that (G, τ, ∂) has ample generics. We now prove that there always exists a finest bi-invariant distance with ample generics.

Theorem 6.2. *Let (G, τ) be a Polish group such that (G, τ, ∂_u) has ample generics. Then there exists a finest distance ∂ such that (G, τ, ∂) is a topometric group with ample generics. This distance may be strictly finer than ∂_u - actually it may be discrete even in cases when ∂_u is not.*

Proof. To see this, let (G, τ) be a Polish group such that (G, τ, ∂_u) has ample generics. Note that, by the Baire category theorem, the supremum ∂ of a countable set of distances (∂_i) such that each (G, τ, ∂_i) has ample generics still has ample generics (∂ is obviously τ -lower semicontinuous, as the supremum of any family of τ -lower semicontinuous maps is τ -lower semicontinuous).

Now consider a family $(\partial_\alpha)_{\alpha < \omega_1}$ of bi-invariant distances such that each $(G, \tau, \partial_\alpha)$ has ample generics. We let ∂ denote the supremum of this family; this is a bi-invariant and τ -lower semicontinuous distance. We prove below that (G, τ, ∂) has ample generics. Note that, by replacing each ∂_α by $\tilde{\partial}_\alpha = \sup_{\beta \leq \alpha} \partial_\beta$, one obtains an increasing ω_1 -sequence of distances such that each $(G, \tau, \tilde{\partial}_\alpha)$ has ample generics and whose supremum is still ∂ . So we may assume that the sequence (∂_α) is increasing.

For each $\varepsilon > 0$ and each $\alpha < \omega_1$, let

$$F_{\alpha, \varepsilon} = \{g \in G : \partial_\alpha(g, 1_G) \leq \varepsilon\} .$$

For $\varepsilon > 0$ fixed, these sets form a decreasing family of τ -closed subsets of G . The fact that (G, τ) is Polish then implies that for each $\varepsilon > 0$ there exists $\alpha_\varepsilon < \omega_1$ such that $F_{\beta, \varepsilon} = F_{\alpha_\varepsilon, \varepsilon}$ whenever $\beta \geq \alpha_\varepsilon$. Define now

$$\alpha = \sup\{\alpha_{1/n} : n < \omega\} .$$

From the definition of ∂ we have that

$$\forall n \in \mathbb{N} \quad \{g \in G : \partial(g, 1_G) \leq \frac{1}{n}\} = \{g \in G : \partial_\alpha(g, 1_G) \leq \frac{1}{n}\} .$$

This shows that (G, τ, ∂) still has ample generics.

Now let ∂_∞ denote the supremum of all distances ∂ such that (G, τ, ∂) has ample generics³. The reasoning above shows that for any $n \in \mathbb{N}$ one cannot produce an uncountable set of distances (∂_i) with bi-invariant distances such that the ball $B_{\partial_\infty}(1_G, 1/n)$ is strictly smaller than each $\bigcap_{j < i} B_{\partial_j}(1_G, 1/n)$. So, for any $n > 0$, ∂_∞ has the same balls of radius $1/n$ around 1_G as the supremum of some *countable* set of distances with bi-invariant distances. Hence $(G, \tau, \partial_\infty)$ has ample metric generics. \square

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³There may be more than \aleph_1 such distances!

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