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# TRUNCATIONS OF HAAR UNITARY MATRICES, TRACES AND BIVARIATE BROWNIAN BRIDGE

C. DONATI-MARTIN AND A. ROUAULT

ABSTRACT. Let  $U$  be a Haar distributed unitary matrix in  $\mathbb{U}(n)$ . We show that after centering the two-parameter process

$$W^{(n)}(s, t) = \sum_{i \leq [ns], j \leq [nt]} |U_{ij}|^2$$

converges in distribution to the bivariate tied-down Brownian bridge. The proof relies on the notion of second order freeness.

## 1. INTRODUCTION

Let  $\sigma$  be a random permutation uniformly distributed on the symmetric group  $\mathcal{S}_n$ . Define for  $p, q \leq n$

$$X_{p,q}^{(n)} = \text{card}\{1 \leq i \leq p, \sigma(i) \leq q\}.$$

In [4], G. Chapuy proved that a suitable normalization of  $X_{p,q}^{(n)}$  converges in distribution to the bivariate tied-down Brownian bridge. Note that  $X_{p,q}^{(n)} = \text{Tr}(\Sigma_{p,q} \Sigma_{p,q}^*)$  where  $\Sigma_{p,q}$  is the truncated matrix of size  $p \times q$  of  $\Sigma$ , the permutation matrix associated to  $\sigma$ . In this paper, we prove a similar result when the symmetric group is replaced by the unitary group, equipped with the Haar measure.

Let  $U$  be a Haar distributed unitary matrix in  $\mathbb{U}(n)$ . We consider, for  $p \leq n$  and  $q \leq n$ , the upper-left  $p \times q$  submatrix  $V_{p,q}$  and the  $p \times p$  Hermitian matrix

$$A_{p,q} = V_{p,q} V_{p,q}^*.$$

We are interested in the asymptotic behavior of

$$T_{p,q} = \text{Tr} A_{p,q} = \sum_{i \leq p, j \leq q} |U_{i,j}|^2. \quad (1.1)$$

Let for  $p, q \leq n$

$$Y_{p,q}^{(n)} = T_{p,q} - \mathbb{E}T_{p,q}.$$

We define a sequence of two-parameter processes  $W^{(n)}$  by

$$W^{(n)} := \left( Y_{[ns], [nt]}^{(n)}, s, t \in [0, 1] \right)$$

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Chapuy used the space  $C([0, 1]^2)$  completing its process in such a way that it is continuous and affine on each closed "lattice triangle". We prefer using the multidimensional generalization of Skorokhod space  $D([0, 1]^2)$  given by [3]. It consists of functions from  $[0, 1]^2$  to  $\mathbb{R}$  which are at each point right continuous (with respect to the natural partial order of  $[0, 1]^2$ ) and admit limits in all "orthants". The space  $D([0, 1]^2)$  is endowed with the topology of Skorohod (see [3] for the definition).

Our main result is the following

**Theorem 1.1.** *The process  $W^{(n)}$  converges in distribution in  $D([0, 1]^2)$  to the tied-down Brownian bridge  $W^{(\infty)}$  which is a centered continuous Gaussian process on  $[0, 1]^2$  of covariance*

$$\mathbb{E}[W^{(\infty)}(s, t)W^{(\infty)}(s', t')] = (s \wedge s' - ss')(t \wedge t' - tt').$$

The proof of this theorem relies on the notion of second order freeness introduced by [14] and further developed in [15] in the case of unitary matrices. Roughly speaking, whereas the freeness, introduced by Voiculescu [20], provides the asymptotic behavior of expectation of traces of random matrices, the second order freeness describes the leading order of the fluctuations of these traces.

Previous works are related to our problem. First, Borel in 1906 shows that for a uniformly distributed point on the  $(n - 1)$ -dimensional (real) sphere, the scaled first coordinate converges in distribution to the standard normal. Since that time, many authors studied the entries and partial traces of matrices from the orthogonal and unitary group. In particular Diaconis and d'Aristotile ([8, 9]) proved that the sequence of one-parameter processes

$$\left\{ \sum_{i=1}^{\lfloor ns \rfloor} U_{ii}, s \in [0, 1] \right\}_n$$

converges in distribution to the complex Brownian motion. Besides, Silverstein [18] proved that for  $q$  fixed, the sequence of one-parameter processes

$$\left\{ n^{1/2} \left( \sum_{i=1}^{\lfloor ns \rfloor} |U_{iq}|^2 - s \right), s \in [0, 1] \right\}_n \quad (1.2)$$

converges in distribution to the Brownian bridge.

The rest of the paper is organized as follows. In section 2, we compute the two first moments of  $T_{p,q}$  from some relations obtained in [12] giving the multiple moments (up to order 4) of the matrix elements  $U_{ij}$  of a Haar distributed unitary matrix  $U$ . Section 3 is devoted to some combinatorics for the unitary group, giving a summary of the main results of [15]. In particular, we state a formula for the cumulants of variables of the form  $X = \text{Tr}(AUBU^*)$  for deterministic matrices  $A, B$  of size  $n$ . In Section 4, we apply the above formula to the computation of the second and fourth cumulant of  $T_{p,q}$ . Section 5 is devoted to the proof of Theorem 1.1. As usual, the proof is

divided in two parts: tightness of the distributions of  $W^{(n)}$  and convergence of the finite dimensional laws. To prove the tightness, we use a criterion of Bickel and Wichura for two-parameter processes, with the help of the estimates obtained in section 4. The convergence of the finite dimensional distributions to Gaussian distributions relies on the computations of their cumulants and their asymptotics, obtained in [15] and recalled in Section 3. In section 6, we give complementary remarks, connections with other problems and a conjecture of universality.

## 2. PRELIMINARY REMARKS: SOME MOMENTS

We have the following important relations (see [12], Proposition 4.2.3). If  $U_{i,j}$  is the generic element of  $U \in \mathbb{U}(n)$ , then the random variable  $|U_{i,j}|^2$  follows the beta distribution on  $[0, 1]$  with parameter  $(1, n-1)$  of density  $(n-1)(1-x)^{n-2}$ . Thus,

$$\mathbb{E}|U_{i,j}|^2 = \frac{1}{n}, \quad \mathbb{E}|U_{i,j}|^4 = \frac{2}{n(n+1)} \quad \text{Var } |U_{i,j}|^2 = \frac{n-1}{n^2(n+1)}, \quad (2.1)$$

and more generally

$$\mathbb{E}|U_{ij}|^{2k} = \frac{(n-1)!k!}{(n-1+k)!}. \quad (2.2)$$

If  $X = |U_{i,j}|^2$  and  $Y = |U_{i,k}|^2$  with  $k \neq j$ , then  $(X, Y)$  follows the Dirichlet distribution on  $\{0 \leq x, y, x+y \leq 1\}$  with parameters  $(1, 1, n-2)$  of density  $(n-1)(n-2)(1-x-y)^{n-3}$ . Thus

$$\mathbb{E}(|U_{i,j}|^2 |U_{i,k}|^2) = \frac{1}{n(n+1)} \quad (2.3)$$

Besides, if  $i \neq k, j \neq \ell$ ,

$$\mathbb{E}(|U_{i,j}|^2 |U_{k,\ell}|^2) = \frac{1}{n^2-1}. \quad (2.4)$$

From these relations, we can compute the first moments of  $T_{p,q}$ , defined in (1.1).

**Proposition 2.1.** *The mean and the variance of  $T_{p,q}$  are given by:*

$$\mathbb{E}T_{p,q} = \sum_{i \leq p, j \leq q} \mathbb{E}|U_{ij}|^2 = pq \mathbb{E}|U_{11}|^2 = \frac{pq}{n}. \quad (2.5)$$

and

$$\text{Var } T_{p,q} = pq \frac{n^2 - n(p+q) + pq}{n^2(n^2-1)}. \quad (2.6)$$

Assume that  $p/n \rightarrow s, q/n \rightarrow t$ , then,

$$\lim_n \frac{1}{n} \mathbb{E}T_{p,q} = st, \quad \lim_n \text{Var } T_{p,q} = st(1 - (s+t) + st) = st(1-s)(1-t).$$

**Proof:**

$$\begin{aligned}
\mathbb{E}T_{p,q}^2 &= \sum_{i,k \leq p, j,l \leq q} \mathbb{E}|U_{ij}|^2 |U_{kl}|^2 \\
&= \sum_{i \leq p, j \leq q} \mathbb{E}|U_{ij}|^4 + \sum_{i \leq p, j \neq l \leq q} \mathbb{E}|U_{ij}|^2 |U_{il}|^2 \\
&\quad + \sum_{i \neq k \leq p, j \leq q} \mathbb{E}|U_{ij}|^2 |U_{kj}|^2 + \sum_{i \neq k \leq p, j \neq l \leq q} \mathbb{E}|U_{ij}|^2 |U_{kl}|^2 \\
&= pq \frac{2}{n(n+1)} + pq(q-1) \frac{1}{n(n+1)} \\
&\quad + p(p-1)q \frac{1}{n(n+1)} + p(p-1)q(q-1) \frac{1}{n^2-1} \\
&= pq \left( \frac{p+q}{n(n+1)} + \frac{(p-1)(q-1)}{n^2-1} \right).
\end{aligned}$$

This yields (2.6).

**Remark 2.2.** *An easy consequence of the above Proposition is*

$$\lim_n \frac{1}{n} T_{[ns], [nt]} = st$$

*in probability. Actually the convergence is uniform in  $s, t \in [0, 1]$  (see section 6).*

### 3. COMBINATORICS FOR THE UNITARY GROUP

We recall in this section the notations and the main result of [15] that we shall use to prove our main theorem. Let  $U$  be a  $n \times n$  Haar distributed unitary matrix. The expectations of products of entries of  $U$  can be described by a special function, called the Weingarten function (see [5]) defined as follows:

$$\text{Wg}(n, \pi) = \mathbb{E}(U_{11} \dots U_{kk} \bar{U}_{1\pi(1)} \dots \bar{U}_{k\pi(k)}) \quad (3.1)$$

where  $\pi \in \mathcal{S}_k$  the set of permutations of  $k$  elements,  $k \leq n$ . Then, matrix integrals can be expressed as follows:

$$\begin{aligned}
&\mathbb{E}(U_{i'_1 j'_1} \dots U_{i'_k j'_k} \bar{U}_{i_1 j_1} \dots \bar{U}_{i_k j_k}) \\
&= \sum_{\alpha, \beta \in \mathcal{S}_k} \delta_{i'_1 i_{\alpha(1)}} \dots \delta_{i'_k i_{\alpha(k)}} \delta_{j_1 j_{\beta(1)}} \dots \delta_{j_k j_{\beta(k)}} \text{Wg}(n, \beta \alpha^{-1}),
\end{aligned}$$

(see [7] Cor. 2.4). The Weingarten functions for  $k = 1, 2$  are given by (see [5]):

$$\begin{aligned}
\text{Wg}(n, (1)) &= \frac{1}{n} \\
\text{Wg}(n, (1)(2)) &= \frac{1}{n^2-1}, \quad \text{Wg}(n, (12)) = -\frac{1}{n(n^2-1)}
\end{aligned} \quad (3.2)$$

From these equations, we can recover (2.3), (2.4).

**Notations:** For  $n$  a positive integer we set  $[[n]] := \{1, 2, \dots, n\}$ .

1) Let  $\epsilon : [[2l]] \mapsto \{-1, 1\}$  be such that  $\sum_{i=1}^{2l} \epsilon_i = 0$ . We write  $\epsilon^{-1}(1) = \{p_1 < p_2 < \dots < p_l\}$  and  $\epsilon^{-1}(-1) = \{q_1 < q_2 < \dots < q_l\}$ . Let  $\mathcal{S}_{2l}^{(\epsilon)}$  be the permutations  $\pi$  in  $\mathcal{S}_{2l}$  such that  $\pi$  sends  $\{p_1 < p_2 < \dots < p_l\}$  on  $\{q_1 < q_2 < \dots < q_l\}$  and conversely. For  $\mathcal{S}_{2l}^{(\epsilon)}$ , we can associate a permutation  $\tilde{\pi}$  in  $\mathcal{S}_l$  defined by

$$\pi^2(p_k) = p_{\tilde{\pi}(k)}.$$

For  $\pi \in \mathcal{S}_{2l}$ , a partition  $A = (A_1, \dots, A_k)$  of  $[[2l]]$  is called  $\pi$ -invariant if  $\pi$  leaves invariant each block  $A_i$ . To  $\pi \in \mathcal{S}_{2l}^{(\epsilon)}$  and  $A$  a partition  $\pi$ -invariant, we can associate a partition  $\tilde{A}$  on  $[[l]]$  which is  $\tilde{\pi}$ -invariant as follows: for a block  $A_i$  of  $A$ , we associate the block  $\tilde{A}_i = \{k \leq l, p_k \in A_i\}$ .

2) *Cumulants.* For  $r \geq 1$ ,  $\kappa_r$  denotes the classical cumulant of order  $r$  (see [17], [15]). It is a multilinear function of  $r$  variables defined as follows: if  $a_1, \dots, a_r$  are random variables,

$$\kappa_r(a_1, \dots, a_r) = \sum_{C \in \mathcal{P}(r)} \text{Möb}(C, 1_r) \mathbb{E}_C(a_1, \dots, a_r)$$

where  $\mathcal{P}(r)$  is the set of partitions of  $[[r]]$ , the Möbius function is given by  $\text{Möb}(C, 1_r) = (-1)^{k-1} (k-1)!$  where  $k$  is the number of blocks of  $C$  and for  $C = \{C_1, \dots, C_k\}$ ,

$$\mathbb{E}_C(a_1, \dots, a_r) = \prod_{i=1}^k \mathbb{E} \left( \prod_{j \in C_i} a_j \right).$$

If  $X_1, \dots, X_{2l}$  are random matrices, for  $\pi$  a permutation with cycle structure  $\pi = \pi_1 \times \dots \times \pi_r$  with  $\pi_i = (\pi_{i,1}, \dots, \pi_{i,\ell(i)})$

$$\kappa_\pi(X_1, \dots, X_{2l}) = \kappa_r \left( \text{Tr}(X_{\pi_{1,1}} \cdots X_{\pi_{1,\ell(1)}}), \dots, \text{Tr}(X_{\pi_{r,1}} \cdots X_{\pi_{r,\ell(r)}}) \right)$$

For  $A = \{A_1, \dots, A_k\}$  a  $\sigma$ -invariant partition of  $[[2l]]$  we can write  $\sigma = \sigma_1 \times \dots \times \sigma_k$  where  $\sigma_i = \sigma|_{A_i}$  is a permutation of the set  $A_i$ , we define

$$\kappa_{\sigma,A}(X_1, \dots, X_{2l}) = \kappa_{\sigma_1}(X_1, \dots, X_{2l}) \cdots \kappa_{\sigma_k}(X_1, \dots, X_{2l}).$$

**Remark 3.1.** When the matrices  $X_i$  are deterministic,  $\kappa_{\sigma,A}(X_1, \dots, X_{2l})$  is non zero only if  $A$  is the partition consisting in the cycles of  $\sigma$ .

3) *Limit distribution.* Let  $s$  be a fixed integer and let  $\{A_1, \dots, A_s\}_n$  be a sequence of  $n \times n$  deterministic matrices. We say that  $\{A_1, \dots, A_s\}_n$  has a limit distribution if there exists a non commutative probability space  $(\mathcal{A}, \varphi)$  and  $a_1, \dots, a_s \in \mathcal{A}$  such that for any polynomial  $p$  in  $s$  non commuting variables,

$$\lim_{n \rightarrow \infty} \text{tr}(p(A_1, \dots, A_s)) = \varphi(p(a_1, \dots, a_s)).$$

where  $\text{tr}$  denotes the normalized trace.

We can now state a proposition which is a particular case of [15, Theorem 3.10]. In the following,  $U^\epsilon = U$  if  $\epsilon = 1$  and  $U^*$  if  $\epsilon = -1$ .

**Proposition 3.2.** *Let  $(U_n)$  a sequence of Haar distributed unitary matrices of size  $n$  and  $\{A_1, \dots, A_s\}_n$  a sequence of deterministic matrices of size  $n$  which has a limit distribution. Let  $r > 1$  and  $\epsilon_1, \dots, \epsilon_{2r} \in \{-1, 1\}$  such that  $\sum \epsilon_i = 0$ . Consider  $p_1, \dots, p_{2r}$  polynomials in  $s$  non commuting variables. For  $i = 1, \dots, 2r$ , we set*

$$D_i = p_i(A_1, \dots, A_s)$$

and for  $1 \leq i \leq r$ ,

$$X_i = \text{Tr}(D_{2i-1}U^{\epsilon(2i-1)}D_{2i}U^{\epsilon(2i)}).$$

Then,

$$\kappa_r(X_1, \dots, X_r) = \sum_{\pi \in \mathcal{S}_{2r}^{(\epsilon)}} \sum_{A, B} C_{\pi, \bar{A}} \kappa_{\gamma\pi^{-1}, B}(D_1, \dots, D_{2r}) \quad (3.3)$$

where the second sum is taken over pairs of partitions of  $[[2r]]$  such that  $A$  is  $\pi$ -invariant,  $B$  is  $\gamma\pi^{-1}$ -invariant and  $A \vee B = 1_{2r}$  the one block partition,  $\gamma$  is given by the product of transpositions  $\prod_{i \leq r} (2i-1, 2i)$  and  $C_{\pi, A}$  are relative cumulants (see [5]) defined, for  $A$  a  $\pi$ -invariant partition, by

$$C_{\pi, A} = \sum_{C \in [\pi, A], C = \{V_1, \dots, V_k\}} \text{Möb}(C, A) \text{Wg}(\pi|_{V_1}) \dots \text{Wg}(\pi|_{V_k}). \quad (3.4)$$

Moreover, for  $r \geq 3$ ,

$$\lim_{n \rightarrow \infty} \kappa_r(X_1, \dots, X_r) = 0.$$

In the sequel, we shall apply this Proposition with  $\epsilon(2i-1) = 1$ ,  $\epsilon(2i) = -1$ .

#### 4. COMPUTATIONS OF THE SECOND AND FOURTH CUMULANTS OF $T_{p,q}$

4.1. **The covariance of  $T_{p,q}$ .** The fundamental remark is that

$$A_{p,q} = D_1 U D_2 U^*$$

with  $D_1 = I_p, D_2 = I_q$ , where  $I_k$  is the matrix of projection on the  $k$  first coordinates. Note that if  $p/n \rightarrow s, q/n \rightarrow t$ ,  $\{D_1, D_2\}$  are commuting projectors with a limit law  $a_1, a_2$  commuting projectors on  $(\mathcal{A}, \varphi)$  such that  $a_1 a_2 = a_1$  if  $s < t$  and  $= a_2$  if  $t < s$ , and  $\varphi(a_1) = s, \varphi(a_2) = t$ .

Let  $p, p', q, q' \leq n$ . We now give an application of Proposition 3.2 to the computation of  $\text{cov}(T_{p,q}, T_{p',q'}) = \kappa_2(T_{p,q}, T_{p',q'})$ . This can also be done, using the computations of Section 2.

We set  $D_3 = I_{p'}, D_4 = I_{q'}$  and apply formula (3.3) to  $X_1 = T_{p,q}, X_2 = T_{p',q'}$ ,  $r = 2$ . We have  $\epsilon(1) = \epsilon(3) = 1, \epsilon(2) = \epsilon(4) = -1$ . The elements of  $\mathcal{S}_4^{(\epsilon)}$  are the permutations of  $\mathcal{S}_4$  which send  $\{1, 3\}$  on  $\{2, 4\}$ , namely

$$\mathcal{S}_4^{(\epsilon)} = \{\sigma, \pi, \tau, \gamma\} \quad (4.1)$$

where

$$\sigma = (14)(23), \quad \pi = (1234), \quad \tau = \pi^{-1} = (1432), \quad \gamma = \gamma_{22} = (12)(34)$$

We have, successively

$$\gamma\sigma^{-1} = \gamma\sigma = (13)(24) , \quad \gamma\pi^{-1} = \gamma\tau = (13)(2)(4) , \quad \gamma\tau^{-1} = \gamma\pi = (1)(24)(3)$$

and, of course  $\gamma\gamma^{-1} = (1)(2)(3)(4)$ . The tilde permutations are obtained easily:

$$\begin{aligned} \sigma^2(1) &= 1 , \quad \tilde{\sigma} = (1)(2) , \quad \tau^2(1) = 3 , \quad \tilde{\tau} = (12) \\ \pi^2(1) &= 3 , \quad \tilde{\pi} = (12) , \quad \gamma^2(1) = 1 , \quad \tilde{\gamma} = (1)(2) . \end{aligned}$$

In (3.3), for given  $\pi$ ,  $B := B_\pi$  is determined as the partition given by the cycles of  $\gamma\pi^{-1}$  (see Remark 3.1). Thus,

$$B_\sigma = \{\{1, 3\}, \{2, 4\}\} , \quad B_\pi = \{\{1, 3\}, \{2\}, \{4\}\} , \quad B_\tau = \{\{1\}, \{2, 4\}, \{3\}\} ,$$

$$B_\gamma = \{\{1\}, \{2\}, \{3\}, \{4\}\} .$$

The corresponding  $A$  satisfying the conditions of (3.3) are

$$A_\pi = A_\tau = A_\gamma = \{\{1, 2, 3, 4\}\} \quad \text{and} \quad \tilde{A} = \{\{1, 2\}\}$$

and for  $\sigma$ , we have two choices

$$A_\sigma(1) = \{\{1, 2, 3, 4\}\} ; \quad A_\sigma(2) = \{\{1, 4\}, \{2, 3\}\} \quad \text{and} \quad \tilde{A}(2) = \{\{1\}, \{2\}\} .$$

The relative cumulants are given by (see (3.2), (3.4))

$$\begin{aligned} C_{\tilde{\pi}, \tilde{A}_\pi} &= C_{\tilde{\tau}, \tilde{A}_\tau} = -\frac{1}{n(n^2 - 1)} \\ C_{\tilde{\gamma}, \tilde{A}_\gamma} &= C_{\tilde{\sigma}, \widetilde{A_\sigma(1)}} = -\frac{1}{n^2} + \frac{1}{n^2 - 1} = \frac{1}{n^2(n^2 - 1)} \\ C_{\tilde{\sigma}, \widetilde{A_\sigma(2)}} &= \frac{1}{n^2} \\ \sum_{p=1}^2 C_{\tilde{\sigma}, \widetilde{A_\sigma(p)}} &= \frac{1}{n^2 - 1} , \end{aligned}$$

and the  $D$ 's contribution (we omit the arguments) are

$$\begin{aligned} \kappa_{\gamma\sigma^{-1}} &= (\text{Tr } D_1 D_3) (\text{Tr } D_2 D_4) = (p \wedge p')(q \wedge q') \\ \kappa_{\gamma\pi^{-1}} &= (\text{Tr } D_1 D_3) (\text{Tr } D_2) (\text{Tr } D_4) = (p \wedge p')qq' \\ \kappa_{\gamma\tau^{-1}} &= (\text{Tr } D_1) (\text{Tr } D_3) (\text{Tr } D_2 D_4) = pp'(q \wedge q') \\ \kappa_{\gamma\gamma^{-1}} &= (\text{Tr } D_1) (\text{Tr } D_2) (\text{Tr } D_3) (\text{Tr } D_4) = pp'qq' \end{aligned}$$

Plugging into (3.3), we get

$$\begin{aligned} \kappa_2(T_{p,q}, T_{p',q'}) &= \tag{4.2} \\ &= \frac{(p \wedge p')(q' \wedge q')}{n^2 - 1} - \frac{(p \wedge p')qq'}{n(n^2 - 1)} - \frac{pp'(q \wedge q')}{n(n^2 - 1)} + \frac{pp'q'q'}{n^2(n^2 - 1)} . \end{aligned}$$

In the limit  $p/n \rightarrow s, q/n \rightarrow t, p'/n \rightarrow s', q'/n \rightarrow t'$ , we get

$$\begin{aligned} & \lim_n \kappa_2(T_{p,q}, T_{p',q'}) \\ &= (s \wedge s')(t \wedge t') - (s \wedge s')tt' - ss'(t \wedge t') + ss'tt' \\ &= (s \wedge s' - ss')(t \wedge t - tt'). \end{aligned} \quad (4.3)$$

**4.2. The fourth cumulant.** We now give an estimate for  $\kappa_4(T_{p,q})$ . From (3.3),

$$\kappa_4 = \sum_{\pi \in \mathcal{S}_8^{(\epsilon)}} \sum_{A, B} C_{\tilde{\pi}, \tilde{A}} \kappa_{\gamma\pi^{-1}, B}(D_1, \dots, D_8) \quad (4.4)$$

where  $\mathcal{S}_8^{(\epsilon)}$  is the subset of  $\mathcal{S}_8$  which sends  $\{1, 3, 5, 7\}$  onto  $\{2, 4, 6, 8\}$  and conversely,  $\gamma = (12)(34)(56)(78) \in \mathcal{S}_8$ ,  $A$  and  $B$  are partitions of  $[[8]]$  such that  $A$  is  $\pi$ -invariant,  $B$  is  $\gamma\pi^{-1}$ -invariant,  $A \vee B = 1_{[[8]]}$ , and finally

$$D_1 = D_3 = D_5 = D_7 = I_p, \quad D_2 = D_4 = D_6 = D_8 = I_q.$$

Since the matrices  $D_i$  are deterministic,  $\kappa_{\sigma, B}(D)$  is non-zero only if  $B$  is the partition consisting in the cycles of  $\sigma$ . If  $\gamma\pi^{-1} = \tau_1 \times \dots \times \tau_h$ , then

$$\kappa_{\gamma\pi^{-1}, B}(D_1, \dots, D_8) = \kappa_1 \left( \text{Tr}(D_{\tau_{1,1}} \cdots D_{\tau_{1, \ell(1)}}) \right) \cdots \kappa_1 \left( \text{Tr}(D_{\tau_{h,1}} \cdots D_{\tau_{h, \ell(h)}}) \right)$$

Note that  $\gamma\pi^{-1}$  sends even (resp. odd) integers to even (resp. odd) integers, thus  $h \geq 2$ .

First assume  $p < q$ . For each block  $B_j$  of  $B$  corresponding to a cyclic permutation  $\tau_j$ ,

$$\text{Tr}(D_{\tau_{j,1}} \cdots D_{\tau_{j, \ell(j)}}) = \begin{cases} p & \text{if } \exists k : \tau_j(k) \in \{1, 3, 5, 7\}, \\ q & \text{otherwise.} \end{cases}$$

Let  $N$  be the number of blocks  $B_j$  of  $B$  such that  $\exists k : \tau_j(k) \in \{1, 3, 5, 7\}$ .

We have now

$$\kappa_{\gamma\pi^{-1}, B}(D_1, \dots, D_8) = p^N q^{\#(B)-N}.$$

In [5, Cor. 2.9], Collins proved that the order of  $C_{\tilde{\pi}, \tilde{A}}$  is at most  $n^{-8-\#(\pi)+2\#(A)}$  where  $\#(\pi)$  denotes the number of cycles of  $\pi$ . Finally

$$C_{\tilde{\pi}, \tilde{A}} \kappa_{\gamma\pi^{-1}, B}(D_1, \dots, D_8) = O \left( n^{-8-\#(\pi)+2\#(A)} p^N q^{\#(B)-N} \right)$$

From equation (20) in [15], we see that

$$2\#(A) + \#(B) - \#(\pi) \leq 6$$

so that

$$\begin{aligned} n^{-8-\#(\pi)+2\#(A)} p^N q^{\#(B)-N} &\leq p^N q^{\#(B)-N} n^{-2-\#(B)} \\ &\leq (q/n)^{\#(B)-2} q^{2-N} p^N n^{-4} \\ &\leq q^{2-N} p^N n^{-4} \\ &\leq p^2 q^2 n^{-4}. \end{aligned}$$

It is straightforward that this result holds also if  $p \geq q$ . We conclude that

$$\kappa_4 = O(p^2 q^2 n^{-4}) \quad (4.5)$$

## 5. PROOF OF THEOREM 1.1

**5.1. Tightness.** According to Bickel and Wichura [3, Theorem 3], since our processes are null on the axes, the tightness of the distributions of  $W^{(n)}$  is in force as soon as the condition  $\mathcal{C}(\beta, \gamma)$  with  $\beta > 1$  is satisfied (see (2), (3) in [3]):

$$\mathbb{E}(|W^{(n)}(B)|^{\gamma_1} |W^{(n)}(C)|^{\gamma_2}) \leq (\mu(B))^{\beta_1} (\mu(C))^{\beta_2} \quad (5.1)$$

where  $\gamma = \gamma_1 + \gamma_2 > 0$  and  $\beta = \beta_1 + \beta_2 > 1$ ,  $B$  and  $C$  are two adjacent blocks in  $[0, 1]^2$  and  $W^{(n)}(B)$  denotes the increment of  $W^{(n)}$  around  $B$ , given by

$$W^{(n)}(B) = W_{s',t'}^{(n)} - W_{s',t}^{(n)} - W_{s,t'}^{(n)} + W_{s,t}^{(n)}$$

for  $B = ]s, s'] \times ]t, t']$ ,  $\mu$  is a finite positive measure on  $[0; 1]^2$  with continuous marginals.

From Cauchy-Schwarz inequality, (5.1) is implied by

$$\mathbb{E}(|W^{(n)}(B)|^{2\gamma_1}) \leq (\mu(B))^{2\beta_1}. \quad (5.2)$$

Moreover, it is enough to consider blocks whose corners points are in  $T^n = \{\frac{p}{n}, 0 \leq p \leq n\} \times \{\frac{q}{n}, 0 \leq q \leq n\}$  (see [3], p. 1665.)

Let  $p \leq p' \leq n$  and  $q \leq q' \leq n$  and  $B = ]\frac{p}{n}, \frac{p'}{n}] \times ]\frac{q}{n}, \frac{q'}{n}]$

$$\begin{aligned} W^{(n)}(B) &:= \Delta_{p,q}^{(n)}(p', q') = Y_{p',q'}^{(n)} - Y_{p',q}^{(n)} - Y_{p,q'}^{(n)} + Y_{p,q}^{(n)} \\ &= \sum_{p+1 \leq i \leq p'} \sum_{q+1 \leq j \leq q'} |U|_{i,j}^2 - \mathbb{E}(|U|_{i,j}^2). \end{aligned}$$

If we show that there exists a constant  $C$ , such that

$$\sup_n \mathbb{E} \left[ \left( \Delta_{p,q}^{(n)}(p', q') \right)^4 \right] \leq C \frac{(p' - p)^2 (q' - q)^2}{n^4}, \quad (5.3)$$

then (5.2) is satisfied with  $\gamma_1 = 2$ ,  $\beta_1 = 1$  and  $\mu$  is the Lebesgue measure. Since  $\Delta_{p,q}^{(n)}(p', q')$  has the same distribution as  $Y_{p'-p, q'-q}^{(n)}$ , it is enough to show

$$\mathbb{E} \left[ \left( Y_{p,q}^{(n)} \right)^4 \right] = O(p^2 q^2 n^{-4}). \quad (5.4)$$

If  $X$  is a real random variable, an elementary computation gives

$$\mathbb{E}(X - \mathbb{E}X)^4 = \kappa_4 + 3\kappa_2^2, \quad (5.5)$$

where  $\kappa_r$  is the  $r$ -th cumulant of  $X$ . Taking  $X = T_{p,q} = \text{Tr } D_1 U D_2 U^*$ , we saw above in (2.6) that

$$\kappa_2 = \text{Var } T_{p,q} \leq 2 \frac{pq}{n^2}. \quad (5.6)$$

Gathering (5.5), (5.6) and (4.5) we get that (5.4) is checked, which proves the tightness.

**5.2. Finite-dimensional laws.** Let  $(a_i)_{i \leq k} \in \mathbb{R}$ ,  $(s_i, t_i)_{i \leq k} \in [0, 1]^2$ . We must prove the convergence in distribution of  $X^{(n)} := \sum_{i=1}^k a_i W_{s_i, t_i}^{(n)}$  to a Gaussian distribution.

Let us denote  $p_i = \lfloor ns_i \rfloor$ ,  $q_i = \lfloor nt_i \rfloor$ . Then

$$\begin{aligned} X^{(n)} &= \sum_{i=1}^k a_i Y_{p_i, q_i}^{(n)} \\ &= \sum_{i=1}^k a_i [\text{Tr}(D_{2i-1} U D_{2i} U^*) - \mathbb{E}(\text{Tr}(D_{2i-1} U D_{2i} U^*))] \end{aligned}$$

where  $D_{2i-1} = I_{p_i}$ ,  $D_{2i} = I_{q_i}$ .

$\{D_{2i-1}, D_{2i}, i = 1, \dots, k\}$  are commuting projectors with a limit distribution  $\{q_{2i-1}, q_{2i}, i = 1, \dots, k\}$  on a probability space  $(\mathcal{A}, \phi_1)$  with  $\phi_1(q_{2i-1}) = s_i$ ,  $\phi_1(q_{2i}) = t_i$  and  $q_i q_j = q_i$  if  $u_i \leq u_j$  (and  $= q_j$  otherwise) where  $u_i = s_i$  for  $i$  odd and  $u_i = t_i$  for  $i$  even.

Let  $r \geq 3$ , then

$$\begin{aligned} \kappa_r(X^{(n)}, \dots, X^{(n)}) &= \sum_{i_1, \dots, i_r=1}^k a_{i_1} \dots a_{i_r} \kappa_r(Y_{p_{i_1}, q_{i_1}}^{(n)}, \dots, Y_{p_{i_r}, q_{i_r}}^{(n)}) \\ &= \sum_{i_1, \dots, i_r=1}^k a_{i_1} \dots a_{i_r} \kappa_r(X_{i_1}, \dots, X_{i_r}) \end{aligned}$$

where  $X_{i_p} = \text{Tr}(D_{2i_p-1} U D_{2i_p} U^*)$ . From Proposition 3.2

$$\lim_{n \rightarrow \infty} \kappa_r(X_{i_1}, \dots, X_{i_r}) = 0. \quad (5.7)$$

Now, the second cumulant is given by

$$\kappa_2(X^{(n)}, X^{(n)}) = \sum_{i, j=1}^k a_i a_j \kappa_2(\text{Tr}(D_{2i-1} U D_{2i} U^*), \text{Tr}(D_{2j-1} U D_{2j} U^*)).$$

From (4.3)

$$\begin{aligned} \lim_n \kappa_2(\text{Tr}(D_{2i-1} U D_{2i} U^*), \text{Tr}(D_{2j-1} U D_{2j} U^*)) \\ = (s_i \wedge s_j - s_i s_j)(t_i \wedge t_j - t_i t_j). \end{aligned}$$

Thus, we get the convergence of  $X^{(n)}$  to a centered Gaussian distribution with variance

$$\sum_{i, j=1}^k a_i a_j (s_i \wedge s_j - s_i s_j)(t_i \wedge t_j - t_i t_j).$$

It follows that the finite-dimensional laws of the process  $W^{(n)}$  converge to the finite-dimensional laws of the tied-down Brownian bridge.  $\blacksquare$

## 6. COMPLEMENTARY REMARKS

1) Since the sup norm is continuous for the Skorokhod topology, Theorem 1.1 implies that

$$\sup_{s,t \in [0,1]} |W^{(n)}(s,t)| \rightarrow \sup_{s,t \in [0,1]} |W^{(\infty)}(s,t)|$$

in distribution, which implies that

$$\sup_{s,t \in [0,1]} \left| \frac{1}{n} T_{[ns],[nt]} - st \right| \rightarrow 0 \quad (6.1)$$

in probability.

2) The definition of our process focusses on the trace of a random matrix  $A_{p,q}$ . This trace is a linear statistic of its empirical spectral distribution, i.e.

$$T_{p,q} = \text{Tr } A_{p,q} = p \int x d\mu^{(p)}(x),$$

where

$$\mu^{(p)} = \frac{1}{p} \sum_{k=1}^p \delta_{\lambda_k^{(p)}},$$

and the  $\lambda_k^{(p)}$ 's are the eigenvalues of  $A_{p,q}$ . If we are interested only in marginals ( $p = [ns], q = [nt]$ , with  $s, t \in (0, 1)$  fixed), we can look directly at the asymptotic behavior of  $\mu^{(p)}$ , as  $n \rightarrow \infty$ . It is known that the random matrix  $A_{p,q}$  belongs to the Jacobi unitary ensemble ([6], [11]) and we can deduce results from the continuity of the mapping  $\mu \mapsto \int x d\mu(x)$  on  $\mathcal{M}_1([0, 1])$ . The sequence of empirical spectral distributions converges to the Kesten-McKay distribution of density which can be parametrized by  $s, t$  or by the endpoints of its support  $(u_-, u_+)$  with  $0 \leq u_- < u_+ \leq 1$ :

$$\pi_{u_-, u_+}(x) = C_{u_-, u_+} \frac{\sqrt{(x - u_-)(u_+ - x)}}{2\pi x(1 - x)} \quad (6.2)$$

where

$$C_{u_-, u_+}^{-1} := \frac{1}{2} \left[ 1 - \sqrt{u_- u_+} - \sqrt{(1 - u_-)(1 - u_+)} \right].$$

The relation between  $(s, t)$  and  $u_{\pm}$  is

$$u_{\pm} = \left[ \sqrt{s(1-t)} \pm \sqrt{(1-s)t} \right]^2.$$

By continuity, we recover a weak form of (6.1), i.e.

$$\lim_n \frac{1}{n} T_{[ns],[nt]} = s \int x \pi_{u_-, u_+}(x) dx = st,$$

in probability.

It could also be possible to recover the fluctuation result for the marginal distribution, i.e.

$$T_{[ns],[nt]} - \mathbb{E}T_{[ns],[nt]} \xrightarrow{\text{law}} \mathcal{N}(0, s(1-s)t(1-t))$$

from the known results on the fluctuations of linear statistics of  $\mu^{(p)}$ . Actually, the result of Johansson [13] is not specific of the Jacobi ensemble, but uses a model of random matrices invariant by conjugation, with polynomial external field. Here, the ensemble is invariant but the potential is logarithmic. The result is a Gaussian limit with the good variance.

At another level, in the same asymptotics as above, Hiai and Petz [11] proved that the family  $(\mu^{(p)})$  satisfies the Large Deviation Principle in  $\mathcal{M}_1([0, 1])$  with scale  $n^{-2}$  and good rate function, which in the case  $s < t < 1/2$  is

$$\begin{aligned} \mathcal{I}(\nu) = & -s^2 \int \int \log |x - y| d\nu(x) d\nu(y) \\ & -s \int ((1 - s - t) \log(1 - x) + (t - s) \log x) d\nu(x) + I_0(s, t). \end{aligned}$$

where  $I_0(s, t)$  is some constant (the limiting free energy). Appealing again to the continuity of the mean, we deduce from the contraction principle that  $n^{-1}T_{[ns], [nt]}$  satisfies the LDP at scale  $n^{-2}$  with good rate function

$$I(c) = \inf \{ \mathcal{I}(\nu); \nu \in \mathcal{M}_1([0, 1]), \int_0^1 x d\nu(x) = c \}.$$

3) It would be interesting to develop the same study with the orthogonal group  $\mathbb{O}(n)$  and we will address this point in a separate work. Indeed, in multivariate (real) analysis of variance, the random variable  $T_{p,q}$  is known as the Bartlett-Nanda-Pillai statistics. The exact distribution of  $T_{p,q}$  is known by its Laplace transform which is an hypergeometric function of matrix argument ([16] p.479). Various asymptotic studies have been performed, essentially  $p, q$  fixed,  $n \rightarrow \infty$  (large sample framework), or high-dimensional framework with  $q$  fixed,  $n, p \rightarrow \infty$  and  $p/n \rightarrow s < 1$  (see for instance [10]). The asymptotic regime of the present paper ( $p/n \rightarrow s, q/n \rightarrow t$ ) is considered in Section 4.4 of the book [1] and a CLT for the statistic  $T_{p,q}$  may be deduced from Theorem 2.2 of [2].

4) In the paper [19], Silverstein discussed the similarity between the matrix of eigenvectors of a (real) sample covariance matrix and a Haar distributed orthogonal matrix, with a one-parameter parameter process analogous to (1.2). To extend this study, Djalil Chafai conjectures that if  $M$  is a  $n \times n$  matrix with i.i.d. entries having the same four first moments as the complex Gaussian standard and if  $\mathcal{U}$  denotes the matrix of eigenvectors of  $MM^*$ , then the sequence  $\mathcal{W}^{(n)}$  obtained by changing  $U_{ij}$  into  $\mathcal{U}_{ij}$  converges to the tied-down Brownian bridge as in Theorem 1.1.

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