An Extension of Kelvin and Bredt Formulas

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Abstract: In dell'Isola and Ruta, and dell'Isola and Rosa is suggested a "perturbative approach" (Nayfeh; Trabucho and Viaño) to the Saint-Venant problem for thin cross sections, however, the papers deal with closed cross sections or with cross sections of constant thickness only (see also Wheeler and Horgan). Here we generalize the proposed procedure by giving a method for treating the case of open or closed sections of variable thickness. We find all the known formulas (Trabucho and Viaño; Feodosyev; Chase and Chilver; Baldacci) due to Kelvin and Bredt as first non-vanishing-terms of our perturbative development and give the corrections to these formulas, too. This seems to be a first step toward solving the open problem formulated in Trabucho and Viaño, pp. 162-164.

I. INTRODUCTION

In applying Saint-Venant torsion theory to thin-walled (linear elastic) beams, Bredt [9] suggested a way to solve approximately, in the case of negligible thickness, the Prandtl [10] elliptic boundary value problem for general cross sections. Here, we extend these results by giving a systematic procedure for deducing and generalizing formulas useful in technical applications. The cross section $\mathcal{D}$ of the Saint-Venant cylinder will be represented as follows: $\mathcal{D} = \mathcal{D}_1 \setminus \mathcal{D}_0$, where $\mathcal{D}_i$, $i = 0, 1$, are simply connected domains, $\mathcal{D}_0 \subset \mathcal{D}_1$ and $\partial\mathcal{D}_0 \cap \partial\mathcal{D}_1 = \emptyset$. We will consider two cases: $\mathcal{D}_0 \neq \emptyset$ and $\mathcal{D}_0 = \emptyset$. In the first case we have (so called) *closed cross sections*, while in the second we have *open cross sections*.

For reasons expounded in dell'Isola and Ruta [1], we state the Saint-Venant torsion problem in terms of the Prandtl stress function ϕ:

$$\Delta\phi + 2 = 0 \quad \text{in } \mathcal{D} \subset \Pi, \tag{1}$$

$$*\nabla\phi \cdot n = 0 \quad \text{on } \partial\mathcal{D}_i, \tag{2}$$

$$\oint_{\partial\mathcal{D}_0} \nabla\phi \cdot n = -2A_0. \tag{3}$$

Here, Π is the plane of the section, $*$ is the $\pi/2$-rotation operator in Π, Δ is the Laplace operator, ∇ is the gradient operator, n is the outer normal of the domain $\mathcal{D}$ and A_0 is the

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area of $\mathcal{D}_0$. Because of (2), ϕ has constant values on $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_0$. As it is defined up to a constant, we can choose

$$\phi(\partial\mathcal{D}_1) = 0, \tag{4}$$

while

$$c := \phi(\partial\mathcal{D}_0) \tag{5}$$

is determined by the integral condition (3). We explicitly remark that condition (3) is equivalent to requiring that the warping function is not polydrome (see Fraeijs de Veubeke[11]).

For open sections, condition (1) implies (3) so, because $\partial\mathcal{D} \equiv \partial\mathcal{D}_1$ we have to satisfiy condition (4) only (see Section 4.3).

Let the function $\bar{\phi} : \mathcal{D}_1 \to \mathbb{R}$ be defined by

$$\bar{\phi}(x) = \begin{cases} c \text{ if } x \in \mathcal{D}_0 \\ \phi(x) \text{ if } x \in \mathcal{D} \end{cases} . \tag{6}$$

Once the problem for the Prandtl function is solved, we can calculate the torsional rigidity R, the warping w and the tangent stress t by the following formulas (Clebsh [12], Sokolnikoff [13], and Love [14])

$$R = 2G\int_{\mathcal{D}_1} \bar{\phi}, \qquad \nabla w(y) = -\tau(*\nabla\phi(y) + *(y-o)), \qquad t = -G\tau *\nabla\phi \tag{7}$$

where $o \in \Pi$, $y \in \mathcal{D}$, G is the shear modulus of elasticity and τ is the angle of twist.

II. PERTURBATIVE DEVELOPMENT OF THE PRANDTL FUNCTION

In this paper, we will consider an annular region Axler, Bourdon and Ramey [15]) A ($\partial A =: \gamma_0 \cup \gamma_1$) and a one parameter family of C^2-diffeomorphisms from A to Π (Section 3):

$$\chi_\varepsilon : A \to \Pi.$$

Let $\mathcal{D}_\varepsilon = \chi_\varepsilon(\mathcal{A})$, $\partial\mathcal{D}_\varepsilon = \chi_\varepsilon(\gamma_0) \cup \chi_\varepsilon(\gamma_1) =: \Gamma_0^\varepsilon \cup \Gamma_1^\varepsilon$. We will consider Prandtl problem (1) through (3) for the family of domains $\mathcal{D}_\varepsilon$, and we will call $\tilde{\phi}_\varepsilon$ the corresponding solution, $\tilde{\phi}_\varepsilon : \mathcal{D}_\varepsilon \to \mathbb{R}$. Let $\phi_\varepsilon := \tilde{\phi}_\varepsilon \circ \chi_\varepsilon$. We will assume for ϕ_ε the following formal expansion in terms of $\varepsilon > 0$ (ε measures the thickness of the section and $\varphi_k : A \to \mathbb{R}$):

$$\phi_\varepsilon = \sum_{k=0}^{\infty} \varphi_k \varepsilon^k. \tag{8}$$

In particular, for the image c of φ_ε on γ_0, we assume the following development:

$$c_\varepsilon = \sum_{k=0}^{\infty} c_k \varepsilon^k. \tag{9}$$

It turns out that $\varphi_k's$ are solutions of a hierarchy of ordinary differential equations easily solved by quadratures, therefore, (Nayfeh [3]) we have a singular perturbation problem. In the sequel, we do not discuss convergence of the proposed formal expansion because our principal aim is to give a procedure for deducing formulas useful in technical applications. However, we supply formal expansions of all other quantities appearing in Saint-Venant torsion theory in terms of the small parameter ε thus obtaining all known formulas, due to Kelvin and Bredt, as terms of first order in ε. We then can find all higher order terms and in this paper, we give the next non-vanishing corrections to Kelvin and Bredt formulas.

III. THE GEOMETRY OF THE PROBLEM

We identify a point $p \in A$ with the couple (s,z) where $s \in \gamma_0$ and $z \in [0,1]$, so that we have

$$A \simeq \gamma_0 \times [0,1].$$

Moreover, we use the notation $f_{,x}$ to mean the derivative of f with respect to x.

Let Γ_0 be a curve whose parametrization is $r_0 : \gamma_0 \to \Pi$,

$$r_0 : s \mapsto r_0(s) \tag{10}$$

(when not misleading, we will identify s with the arc length of Γ_0: this means that the length of an arc $\gamma \in \gamma_0$ is equal to the length of $r_0(\gamma) \in \Gamma_0$). Let δ be a scalar field such that $\delta : \gamma_0 \to \mathbb{R}^+$ and Γ_z ($z \in [0,1]$) the z-lifted curve from Γ_0 whose parametric representation is given by

$$r(s,z) = r_0(s) - z\varepsilon\delta(s) * r_{0,s}(s). \tag{11}$$

We define

$$(\forall \varepsilon)\left(\mathcal{D}_\varepsilon := \bigcup_{z\in[0,1]} \Gamma_z.\right)$$

Therefore $\varepsilon\delta(s)$ can be regarded as the thickness of the section at the point s along Γ_0.

The couple (A, χ_ε) is a chart in D_ε to which are associated the following:

(i) holonomic basis

$$e_1(s,z) = \frac{\partial r}{\partial s} = (1 + z\varepsilon\delta(s)K(s))r_{0,s} - z\varepsilon\delta_{,s}(s) * r_{0,s}(s) \tag{12}$$

$$e_2(s,z) = \frac{\partial r}{\partial z} = -\varepsilon\delta(s) * r_{0,s}(s) \tag{13}$$

($K(s)$ is the curvature of Γ_0),

(ii) metric-tensor (when not misleading we will omit the explicit s-dependence of various functions):

$$g^{ij} = \frac{1}{\varepsilon^2\delta^2(1+z\varepsilon\delta K)^2}\begin{pmatrix} \varepsilon^2\delta^2 & -z\varepsilon^2\delta\delta_{,s} \\ -z\varepsilon^2\delta\delta_{,s} & (1+z\varepsilon\delta K)^2 + z^2\varepsilon^2\delta_{,s}^2 \end{pmatrix}, \tag{14}$$

as $\mathcal{D}_\varepsilon$ has to be regarded as a Riemannian submanifold of Π.

For the sake of completeness we give expression for the gradient and the Laplacian, which will be needed below (Weatherburn [16], and Germain [17]):

$$\nabla\phi = g^{ij}\phi_{,i}e_j, \quad \Delta\phi = g^{ij}\left(\frac{\partial^2\phi}{\partial x^i\partial x^j} - \frac{\partial\phi}{\partial x^h}\left\{\begin{matrix} h \\ i\ j \end{matrix}\right\}\right) = \frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial x_i}\sqrt{g}g^{ij}\phi_{,i}\right), \tag{15}$$

$\left\{\begin{matrix} h \\ i\ j \end{matrix}\right\}$ are the Christoffel symbols and g is the determinant of the metric tensor.

IV. FORMAL EXPANSION OF THE PRANDTL FUNCTION

Using (14) and the expansion (8) and (9), equations (1), (4) and (5) become:

$$\begin{aligned}
\Sigma_{n=0}^{\infty} \quad & \Big\{\varepsilon^n\varphi_{n,zz} + \varepsilon^{n+1}(3z\delta K\varphi_{n,zz} + \delta K\varphi_{n,z}) \\
& + \varepsilon^{n+2}\Big[\delta^2\varphi_{n,ss} + z^2(3\delta^2K^2 + \delta_{,s}^2)\varphi_{n,zz} - 2z\delta\delta_{,s}\varphi_{n,zs} \\
& + (2z\delta_{,s}^2 - z\delta\delta_{,ss} + 2z\delta^2K^2)\varphi_{n,z}\Big] \\
& + \varepsilon^{n+3}\Big[z\delta^3K\varphi_{n,ss} + z^3(\delta^3K^3 + \delta\delta_{,s}^2K)\varphi_{n,zz} - 2z^2\delta^2\delta_{,s}K\varphi_{n,zs} - z\delta^3K_{,s}\varphi_{n,s} \\
& + \left(2z^2\delta\delta_{,s}^2K - z^2\delta^2\delta_{,ss}K + z^2\delta^3K^3 - z^2\delta^2\delta_{,s}K_{,s}\right)\varphi_{n,z}\Big]\Big\} \\
& = -2\varepsilon^2\delta^2(1+\varepsilon z\delta K)^3.
\end{aligned} \tag{16}$$

$$\Sigma_{n=0}^{\infty} \quad \varphi_n|_{z=1}\varepsilon^n = 0 \tag{17}$$

$$\Sigma_{n=0}^{\infty} \quad (c_n - \varphi_n|_{z=0})\,\varepsilon^n = 0. \tag{18}$$

A. Closed Sections

We explicitly show here only the first three terms of the ε-expansion for the Prandtl function. Noting that, because of the geometry of considered sections, $\nabla\phi\cdot n|_{z=0} = \frac{1}{\varepsilon\delta}\phi_{,z}$, condition (3) becomes

$$\oint_{\Gamma_0}\left[\frac{\varphi_{0,z}}{\delta} + \left(\frac{\varphi_{1,z}}{\delta} + 2A_0\right)\varepsilon + \sum_{n=2}^{\infty}\frac{\varphi_{n,z}}{\delta}\varepsilon^n\right] = 0. \tag{19}$$

Using (16),(17),(18) and, in the case of closed sections, (19), we get

$$\left(I := \oint_{\Gamma_0}\delta^{-1},\ I(s) := \int_0^s\delta^{-1}\right):$$

$$\varphi_0(s,z) = 0 \tag{20}$$

$$\varphi_1(s,z) = \frac{2A_0}{I}(1-z) \tag{21}$$

$$\varphi_2(s,z) = \frac{(1-z)}{I}\left\{\left[\oint_{\Gamma_0}\delta - \frac{2\pi A_0}{I}\right] + z\delta^2\left[-\frac{KA_0}{\delta} + I\right]\right\} \tag{22}$$

$$\varphi_3(s,z) = \frac{(1-z)}{2I}\oint_{\Gamma_0}\left(\frac{\beta_1}{3} + \beta_2\right) + (z-z^3)\frac{\beta_1}{6} + (z-z^2)\frac{\beta_2}{2} \tag{23}$$

with

- $\beta_1(s) = \frac{2A_0}{I}\delta_{,ss} - 2\frac{\delta_{,s}^2}{\delta} - 2\delta^2 K$
- $\beta_2(s) = K\left(-\delta^2 + \frac{A_0}{I}\delta K - \frac{1}{I}\oint\delta + \frac{2\pi A_0}{I^2(\Gamma_0)}\right)$.

Note that the π appearing in the expressions of φ_2 and φ_3 is due to the Gauss-Bonnet theorem, Kobayashi and Nomizu [18].

B. Torsional Rigidity, Warping and Shearing Stress

Using formulas (7) and the expansions $R = \sum_{n=0}^{\infty} R_n\varepsilon^n$, $w(s,z) = \sum_{n=0}^{\infty} w_n(s,z)\varepsilon^n$ and $t(s,z) = \sum_{n=0}^{\infty} t_n(s,z)\varepsilon^n$, we get:

$$R_0 = 0, \tag{24}$$

$$R_1 = \frac{4GA_0^2}{I}, \tag{25}$$

$$R_2 = \frac{4GA_0}{I^2}\left\{I\oint_{\Gamma_0}\delta - \pi A_0\right\} \tag{26}$$

$$\frac{w_0(s,z)}{\tau} = 2A_0\frac{I(s)}{I} + \int_0^s r_0 \times r_{0,s} \tag{27}$$

$$\frac{w_1(s,z)}{\tau} = \left[\frac{1}{I}\oint_{\Gamma_0}\delta - \frac{2\pi A_0}{I^2(\Gamma_0)}\right]I(s) - \int_0^s \delta + \frac{A_0}{I}\int_0^s K, \tag{28}$$

and finally for the shearing stress

$$\frac{t_0(s,z)}{G\tau} = \left(\frac{t_{0s}}{G\tau}, \frac{t_{0z}}{G\tau}\right) = \left(2A_0\frac{I}{\delta}, 0\right) \tag{29}$$

$$\left(\frac{t_{1s}}{G\tau}, \frac{t_{1z}}{G\tau}\right) = \left(-\frac{1}{\delta I}\left[\oint_{\Gamma_0}\delta - \frac{2\pi A_0}{I}\right] + \delta(1-2z) - \frac{KA_0}{I}, 0\right) \tag{30}$$

$$\begin{aligned}\left(\frac{t_{2s}}{G\tau}, \frac{t_{2z}}{G\tau}\right) = {} & \left(\frac{1}{2\delta I}\oint_{\Gamma_0}\left(\frac{\beta_1}{3} + \beta_2\right) + (3z^2-1)\frac{\beta_1}{6} + (2z-1)\frac{\beta_2}{2}\right. \\ & \left. + z\delta K\left[-\frac{1}{I}\left[\oint_{\Gamma_0}\delta - \frac{2\pi A_0}{I}\right] + \delta(\delta - K\frac{A_0}{I})(2z-1)\right], z\delta_{,s}\frac{2A_0}{I}\right).\end{aligned} \tag{31}$$

Note that the first non-vanishing z-component of the shearing stress is of second order in ε. Thus our procedure illustrates one of the assumptions made by Bredt in deducing his formulas.

The values of R_1, $t_0(s,z)$ and $w_0(s,z)$ are the usual ones reported in the literature (Feodosyev [6], Chase and Chilver [7], and Baldacci [8]); they are due to Bredt [9]. We want to emphasize how easily we got them and the expressions giving the next-leading corrections.

C. Open Sections

Since our asymptotic expansion is singular, we cannot always satisfy all of the boundary conditions on $\partial\mathcal{D}$. More precisely, boundary conditions for the Prandtl functions cannot be satisfied at all points that are z-lifted from $r_0(0)$ and $r_0(l)$. Consequently, our development has to be considered as approximating the solution at points far away from the ending edges of $\mathcal{D}$. If we want to study φ in these terminal regions, we can use, for example, an asymptotic-matching technique [3], Trabucho and Viaño [4]. Remembering that in this case condition (3) is automatically satisfied and using (16) and (2), we get for the Prandtl function

$$\varphi_0(s,z) = 0, \quad \varphi_1(s,z) = 0, \tag{32}$$

$$\varphi_2(s,z) = \delta^2(s)\left(z - z^2\right), \tag{33}$$

$$\varphi_3(s,z) = \delta^3(s)K(s)\left(\frac{z}{6} - \frac{z^2}{2} + \frac{z^3}{3}\right). \tag{34}$$

D. Torsional Rigidity, Warping and Shearing Stress

As a consequence of the fact that for an open section we do not consider the effects of the two extrema of the section, we introduce an error in our estimate of the torsional rigidity. This should not affect lower order terms in the development for R: R_0, R_1, R_2 and R_3, but could change the values of R_4. Further investigations will clear up this point.

$$R_0 = 0\,, \quad R_1 = 0\,, \quad R_2 = 0 \tag{35}$$

$$R_3 = \frac{G}{3}\int_0^s \delta^3 \tag{36}$$

$$R_4 = \frac{G}{6}\int_0^s \delta^4 K \tag{37}$$

$$\frac{w_0(s,z)}{\tau} = \int_0^s r_0 \times r_{0,s} \tag{38}$$

$$\frac{w_1(s,z)}{\tau} = z\delta r_0 \cdot r_{0,s} - \int_0^s \delta \tag{39}$$

$$t_0(s,z) = (t_{0s}, t_{0z}) = (0,0) \tag{40}$$

$$(t_{1s}, t_{1z}) = G\tau\delta(1-2z,0) \tag{41}$$

$$(t_{2s}, t_{2z}) = -G\tau\delta^2 K\left(z^2 - \frac{1}{6}K, 0\right). \tag{42}$$

Timoshenko and Goodier [20] credit (36), (38) and (41) to Lord Kelvin.

V. CONCLUSIONS AND PERSPECTIVES

It seems to us that the method proposed herein allows one in a rational and simple way to get a hierarchy of formulas generalizing those of Kelvin and Bredt. Moreover, we get a new iterative method for solving the Prandtl problem. Particularly interesting is the second order correction to the torsion rigidity. In fact, it is easy to verify that the ratio R_2/R_1 can assume sensible values, choosing, for example, (a case of a certain technical relevance), for Γ_0 a circle of radius r_0 and $\varepsilon\delta = \varepsilon\delta_0 + \varepsilon\delta_1 \sin\frac{2k\pi}{l}s$, with $l = 2\pi r_0$ and $\varepsilon\delta_1 \leq \varepsilon\delta_0 \ll r_0$, we have

$$\frac{R_2\varepsilon^2}{R_1\varepsilon} = \frac{\varepsilon\delta_0}{r_0}\left(2 - \frac{1}{2}\sqrt{1 - \left(\frac{\delta_1}{\delta_0}\right)^2}\right) \sim \frac{\varepsilon\delta_0}{r_0}\left(\frac{3}{2} + \frac{1}{4}\left(\frac{\delta_1}{\delta_0}\right)^2 + O\left(\left(\frac{\delta_1}{\delta_0}\right)^4\right)\right). \tag{43}$$

If $\delta_1 = 0$ we have $\frac{R_2\varepsilon^2}{R_1\varepsilon} = \frac{3\varepsilon\delta_0}{2r_0}$, a result that can be obtained from the well-known (exact) formula for the torsion rigidity of circular sections.

With $\frac{\varepsilon\delta_0}{r_0} = \frac{1}{10}$ and $\frac{\delta_1}{\delta_0} = \frac{1}{2}$, we get $\frac{R_2\varepsilon^2}{R_1\varepsilon} \sim 16\%$. Moreover, in the singular case in which $\delta_1 = \delta_0$, $R_2 = R_1 = 0$ we must determine the following correction to R.

Finally, we state two limitations of our approach and suggest ways of getting around these:

(i) The main limitation of this method comes from the way of generating the section from the inner curve. For example, it is not easy to handle the problem when Γ_0 and Γ_1 are two confocal ellipses. Thus we are led to generalize the z-lifting as follows

$$r_0(s) \mapsto r_0(s) + z(r_{0,s}\delta_1(s) + *r_{0,s}(s)\delta_2(s)).$$

Maybe the most natural procedure for constructing thin sections is by means of Conformal Mapping Theory, Caratheodory [19].

(ii) Within this approach it is difficult to study the convergence properties of the proposed series. This could be better understood by using the theory of Padé approximants.

(iii) Developing the ideas exposed in Wheeler and Horgan [21], we expect to be able to apply our procedure also to the analysis of flexure.

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