



HAL
open science

A bijection for triangulations, quadrangulations, pentagulations, etc.

Olivier Bernardi, Eric Fusy

► **To cite this version:**

Olivier Bernardi, Eric Fusy. A bijection for triangulations, quadrangulations, pentagulations, etc.. 2011. hal-00498615v4

HAL Id: hal-00498615

<https://hal.science/hal-00498615v4>

Preprint submitted on 22 Jan 2011 (v4), last revised 20 Aug 2011 (v6)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A BIJECTION FOR TRIANGULATIONS, QUADRANGULATIONS, PENTAGULATIONS, ETC.

OLIVIER BERNARDI* AND ÉRIC FUSY†

ABSTRACT. A d -angulation is a planar map with faces of degree d . We present for each integer $d \geq 3$ a bijection between the class of d -angulations of girth d and a class of decorated plane trees. Each of the bijections is obtained by specializing a “master bijection” which extends an earlier construction of the first author. Bijections already existed for triangulations ($d = 3$) and for quadrangulations ($d = 4$). As a matter of fact, our construction unifies a bijection by Fusy, Poulalhon and Schaeffer for triangulations and a bijection by Schaeffer for quadrangulations. For $d \geq 5$, both the bijections and the enumerative results are new.

We also extend our bijections so as to enumerate p -gonal d -angulations, that is, d -angulations with a simple boundary of length p . We thereby recover bijectively the results of Brown for p -gonal triangulations and quadrangulations and establish new results for $d \geq 5$.

A key ingredient in our proofs is a class of orientations characterizing d -angulations of girth d . Earlier results by Schnyder and by De Fraisseyx and Ossona de Mendez showed that triangulations of girth 3 and quadrangulations of girth 4 are characterized by the existence of orientations having respectively indegree 3 and 2 at each inner vertex. We extend this characterization by showing that a d -angulation has girth d if and only if the graph obtained by duplicating each edge $d - 2$ times admits an orientation having indegree d at each inner vertex.

1. INTRODUCTION

The enumeration of planar maps (connected planar graphs embedded in the sphere) has received a lot of attention since the seminal work of Tutte in the 60’s [Tut63]. Tutte’s method for counting a class of maps consists in translating a recursive description of the class (typically obtained by deleting an edge) into a functional equation satisfied by the corresponding generating function. The translation usually requires to introduce a “catalytic” variable, and the functional equation is solved using the so-called “quadratic method” [GJ83, sec.2.9] or its extensions [BMJ06]. The final result is, for many classes of maps, a strikingly simple counting formula. For instance, the number of maps with n edges is $\frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$. Tutte’s method has the advantage of being systematic, but is quite technical in the way of solving the equations and does not give a combinatorial understanding of the simple-looking enumerative formulas. Other methods for the enumeration of

*Dept. of Mathematics, MIT, 77 Massachusetts Avenue, Cambridge MA 02139, USA, bernardi@math.mit.edu. Supported by the French ANR project A3 and the European project ExploreMaps – ERC StG 208471

†LIX, École Polytechnique, Palaiseau, fusy@lix.polytechnique.fr. Supported by the European project ExploreMaps – ERC StG 208471.

maps were later developed based either on matrices integrals, representations of the symmetric group, or bijections.

The bijective approach to maps has the advantage of giving deep insights to the combinatorial properties of maps. This approach was used to solve several statistical physics models (Ising, hard particles) on random lattices [BMS02, BFG07], and to investigate the metric properties of random maps [CS04, BFG04]. It also has nice algorithmic applications (random generation and asymptotically optimal encoding in linear time). The first bijections appeared in [CV81], and later in [Sch98] where a more transparent construction was proposed and applied to several classes of maps. Typically bijections are from a class of “decorated” plane trees to a class of maps and operate on trees by progressively closing facial cycles. However, even if the bijective approach was successfully applied to many classes of maps, it is not, up to now, as systematic as Tutte’s recursive method. Indeed, for each class of maps, one has to guess (using counting results) the corresponding family of decorated trees, and then to invent a mapping between trees and maps.

In this article, we contribute to making the bijective approach more systematic by showing that a single “master bijection” can be specialized in various ways so as to obtain bijections for several classes of maps. The master bijection Φ presented in Section 3 is an extension of a construction by the first author [Ber07]. It was already shown in [BC], that the bijection Φ can be specialized so as to obtain a famous bijection by Bouttier, Di Francesco, and Guitter [BFG04] between bipartite maps and certain decorated plane trees called *mobiles*. To be more exact, the bijection Φ is between a set \mathcal{O} of *oriented maps* and a set of mobiles. It turns out that for many classes of maps there is a canonical way of orienting the maps in the class \mathcal{C} , so that \mathcal{C} is identified with a subfamily $\mathcal{O}_{\mathcal{C}}$ of \mathcal{O} on which our bijection Φ restricts nicely. Typically, the oriented maps in $\mathcal{O}_{\mathcal{C}}$ are characterized by degree constraints which can be traced through our construction and yields a degree characterization of the associated mobiles. The mobile family is then specifiable by a decomposition grammar and amenable to the Lagrange inversion formula for counting. To summarize, the bijective approach presented here is more systematic, since it consists in specializing the master bijection Φ to several classes of maps. The problem of enumerating a class of map \mathcal{C} then reduces to guessing a family of “canonical” orientations (in \mathcal{O}) for \mathcal{C} (instead of guessing a family of decorated trees and a bijection, which is harder).

We apply our bijective strategy to an infinite number of classes of maps. More precisely, we consider for each integer $d \geq 3$ the class \mathcal{C}_d of *d-angulations* (planar maps with faces of degree d) of girth d (i.e., having no cycle of length less than d). The family \mathcal{C}_3 corresponds to the *simple* triangulations (those that have no loop nor multiple edges). The family \mathcal{C}_4 also correspond to the class of *simple* quadrangulations because quadrangulations are bipartite (hence never have cycles of odd length). We show in Section 4 that each class \mathcal{C}_d is amenable to our bijective strategy: we obtain a bijection between the class \mathcal{C}_d and a class of mobiles by specializing the bijection Φ . The class of mobiles is characterized by certain degree conditions, hence the generating function of the class \mathcal{C}_d is characterized and shown to be algebraic. Bijections already existed for the class \mathcal{C}_3 of simple triangulations [PS06, FPS08] and the class \mathcal{C}_4 of simple quadrangulations [Fus07, Sch98]. Our approach actually coincides (and unify) the bijections presented respectively

in [FPS08, Theo.4.10] for triangulations and in [Sch98, Sec.2.3.3] for quadrangulations.

In Section 5, we show that our bijective strategy applies even more to generally d -angulations with boundary of length $p \geq d$. More precisely, we give bijections for the classes of p -gonal d -angulations (maps with inner vertices of degree d and outer-face of degree p which is simple) of girth d . In particular, we recover bijectively the enumerative results obtained by Brown (via a recursive approach *à la Tutte*) for the cases $d = 3, 4$ [Bro64, Bro65]. That is, we show that the numbers $t_{p,n}$ of p -gonal simple triangulations with $p + n$ vertices and the numbers $q_{p,n}$ of $2p$ -gonal simple quadrangulations with $2p + n$ vertices are

$$t_{p,n} = \frac{2(2p-3)!}{(p-1)!(p-3)!} \frac{(4n+2p-5)!}{n!(3n+2p-3)!}, \quad q_{p,n} = \frac{3(3p-2)!}{(p-2)!(2p-1)!} \frac{(3n+3p-4)!}{n!(2n+3p-2)!}.$$

A bijection (different from ours) for simple triangulations with boundary was already presented in [PS06], but the proof of the bijectivity was based on the results of Brown (only injectivity was proved and the bijectivity followed by a cardinality argument).

Our counting results are new for $d \geq 5$. Indeed, due to the constraint on the girth, it seems tedious to apply a recursive method *à la Tutte* to d -angulations for $d \geq 5$. Even though there does not seem to be closed-form counting formulas for these cases (for $d = 5$ the counting coefficients have large prime factors), the generating functions follow an amazingly simple uniform algebraic pattern, see Propositions 24 and 25.

Before closing this introduction, we point out one of the key ingredients used in our bijective strategy. As explained above, for a class \mathcal{C} of maps, our bijective strategy requires to specify a canonical orientation in order to identify \mathcal{C} with a subclass $\mathcal{O}_{\mathcal{C}} \subset \mathcal{O}$ of oriented maps. For d -angulations with a simple outer face, a simple calculation based on the Euler relation shows that the numbers n and m of inner vertices and edges are related by $m = \frac{d}{d-2}n$. This suggests a candidate canonical orientation for a d -angulation G : just ask for an orientation (without counterclockwise directed cycle) with indegree $\frac{d}{d-2}$ at each inner vertex. . . or more reasonably an orientation of the graph $(d-2) \cdot G$ (the graph obtained from G replacing each edge by $(d-2)$ parallel edges) with indegree d at each inner vertex. We show that such an orientation exists for a d -angulation if and only if it has girth d (and use these orientations to identify the class \mathcal{C}_d with a subfamily $\mathcal{O}_{\mathcal{C}_d} \subset \mathcal{O}$). This extends earlier results by Schnyder for triangulations [Sch89] and by de Fraysseix and Ossona de Mendez for quadrangulations [dFOdM01].

2. MAPS AND ORIENTATIONS

In this section we gather our definitions about maps and orientations.

Maps. A (planar) *map* is a connected planar graph embedded in the oriented sphere and considered up to continuous deformation. The *faces* are the connected components of the complementary of the graph. A *plane tree* is a map without cycles (it has a unique face). Cutting an edge e at its middle point gives two *half-edges*, each incident to an endpoint of e (they are both incident to the same vertex if e is a loop). We shall also consider some maps decorated with dangling

half-edges called *buds* (see e.g. Figure 3). A *corner* is the angular section between two consecutive half-edges around a vertex. The degree of a vertex or face is the number of incident corners. A *d-angulation* is a map such that every face has degree d . *Triangulations* and *quadrangulations* correspond to the cases $d = 3$ and $d = 4$ respectively. The *girth* of a map is the minimal length of its cycles. Obviously a d -angulation has girth at most d (except if it is a tree). A map is *simple* if it has girth at least 3 (i.e., no loop nor multiple edge).

The numbers v , e , f of vertices, edges and faces of a map are related by the *Euler relation*: $v - e + f = 2$. We shall also use the following result.

Lemma 1. *If a map with v vertices and e edges has a face f_0 of degree p and the other faces of degree at least d , then*

$$(d - 2)(e - p) \leq d(v - p) + p - d,$$

with equality if and only if the faces other than f_0 have degree exactly d .

Proof. The incidence relation between faces and edges gives $2e \geq p + d(f - 1)$, with equality if the faces other than f_0 have degree exactly d . Combining this with the Euler relation (so as to eliminate f) gives the claim. \square

A map is said to be *vertex-rooted* if a vertex is marked, *face-rooted* if a face is marked, and *corner-rooted* if a corner is marked¹. The marked vertex, face or corner are called the *root-vertex*, *root-face* or *root-corner*. For a corner-rooted map, the marked corner is indicated by a dangling half-edge pointing to that corner; see Figure 8. A corner-rooted map is said to *induce* the vertex-rooted map (resp. face-rooted map) obtained by keeping the root-vertex (resp. root-face) as marked, but otherwise forgetting the root-corner. Given a face-rooted (or corner-rooted) map, vertices and edges are said to be *outer* or *inner* depending on whether they are incident to the root-face or not.

Orientations, bi-orientations. Let G be a map. An *orientation* of G is a choice of a direction for each edge of G . A *bi-orientation* of G is a choice of a direction for each half-edge of G : each half-edge can be either *ingoing* or *outgoing*. For $i = 0, 1, 2$ we call *i-way* an edge with exactly i ingoing half-edges. In figures, 0-way edges are represented by directing the two half-edges toward the middle of the edge, the 1-way edges are represented by directing the ingoing half-edge toward the incident vertex, and the 2-way edges are represented by directing the two ingoing half-edges toward their incident vertices; see Figure 1(a). Clearly, orientations can be seen as a special kind of bi-orientations, in which every edge is 1-way. The *indegree* (resp. *outdegree*) of a vertex v is the number of ingoing (resp. outgoing) half-edges incident to v . The *clockwise-degree* of a face f is the number of outgoing half-edges h incident to f having the face f on their right (the orientation of h is toward the midpoint of the edge). A *directed path* in a bi-orientation is a path going through some vertices v_1, \dots, v_{k+1} in such a way that for all $i = 1, \dots, k$ the edge between v_i and v_{i+1} is either 2-ways or 1-way oriented toward v_{i+1} . A *circuit* is a directed cycle.

¹Corner-rooted map are usually simply called *rooted maps* in the literature. A face-rooted map can be thought as a *plane map* (a connected graph embedded in the plane) by thinking of the root-face as the infinite face.

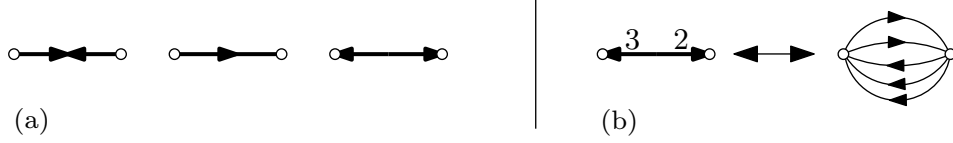


FIGURE 1. (a) Bi-orientations: a 0-way edge (left), a 1-way edge (middle) and a 2-ways edge (right). (b) Correspondence between \mathbb{N} -weighted orientations of a map M and ordinary orientations of the associated map M' (obtained by replacing the edge e by several edges in parallel).

A *planar bi-orientation* is a planar map with a bi-orientation. A *vertex-rooted bi-orientation* (resp. *face-rooted bi-orientation*, *corner-rooted bi-orientation*) is a planar bi-orientation whose underlying map is vertex-rooted (resp. face-rooted, corner-rooted). A bi-orientation is *accessible from a vertex v* if any vertex is reachable from v by a directed path. A vertex-rooted (or corner-rooted) bi-orientation is said to be *accessible* if it is accessible from the root-vertex. A circuit of a face-rooted (or corner-rooted) bi-orientation is *clockwise* if the root-face is on its left and *counterclockwise* otherwise. The bi-orientation is *minimal* if it has no counterclockwise circuit; it is *clockwise-minimal* if in addition the root-face is a (clockwise) circuit.

Fractional Orientations. A *weighted bi-orientation* (resp. *inner-weighted*) is a bi-orientation together with a *weight* $w(h)$ associated to each half-edge h (resp. to each half-edge on inner edges). The *weight* of an edge is the sum of the weights of the two half-edges. The *weight* of a vertex v is the sum of the weights of the incident ingoing half-edges. The *weight* of a face f is the sum of the weights of the outgoing half-edges incident to f and having f on their right.

We call *\mathbb{N} -weighted bi-orientation* a weighted, or inner-weighted, bi-orientation such that the weight of every outgoing half-edge is 0, and the weight of every ingoing half-edge is a positive integer. Observe that the weights of a \mathbb{N} -weighted bi-orientation uniquely determine the bi-orientation. Moreover, ordinary orientations identify with \mathbb{N} -weighted bi-orientation such that the weight of every edge is 1 (because in this case every edge is 1-way with weights 0 and 1 respectively on outgoing and ingoing half-edges). Let $G = (V, E)$ be a map, let α be a function from the vertex set V to $\mathbb{N} = \{0, 1, 2, \dots\}$ and let β be a function from the edge set E to \mathbb{N} . We call *α/β -orientation* an \mathbb{N} -weighted bi-orientation such that any vertex v has weight $\alpha(v)$ and any edge e has weight $\beta(e)$. We now give a criterion for the existence (and uniqueness) of a minimal α/β -orientation.

Lemma 2. *Let M be a map with vertex set V and edge set E , let α be a function from V to \mathbb{N} , and let β be a function from E to \mathbb{N} . If there exists an α/β -orientation of M , then there exists a unique minimal one.*

Lemma 3. *Let M be a map with vertex set V and edge set E , let α be a function from V to \mathbb{N} , and let β be a function from E to \mathbb{N} . The map M admits an α/β -orientation if and only if*

- (i) $\sum_{v \in V} \alpha(v) = \sum_{e \in E} \beta(e)$,
- (ii) for all subset S of vertices, $\sum_{v \in S} \alpha(v) \geq \sum_{e \in E_S} \beta(e)$ where E_S is the set of edges with both ends in S .

Moreover, α -orientations are accessible from a vertex u if and only if

$$(iii) \text{ for all subset } S \text{ of vertices not containing } u, \sum_{v \in S} \alpha(v) \geq \sum_{e \in E_S} \beta(e).$$

Lemmas 2 and 3 are both known for the special case where the function β takes value 1 on each edge of G (this case corresponds to ordinary orientations and the weight of a vertex is equal to its indegree); see [Fel04]. The proofs below are simple reductions to the case $\beta = 1$.

Proof of Lemmas 2 and 3. We call β -fractional orientations the \mathbb{N} -weighted orientations of M such that every edge e has weight $\beta(e)$. Let M' be the map obtained by replacing each edge e of M by $\beta(e)$ edges in parallel. We call *locally minimal* the (ordinary) orientations of M' such that for any edge e of M , the corresponding $\beta(e)$ parallel edges of M' do not contain any counterclockwise circuit. Clearly, the β -fractional orientations of M are in bijection with the locally minimal orientations of M' ; see Figure 1(b). Thus, there exists an α/β -orientation of M if and only if there exists an ordinary orientation of M' with indegree $\alpha(v)$ at each vertex. Therefore, Conditions (i),(ii) of Lemma 3 for M immediately follow from the same conditions for the particular case $\beta = 1$ applied to the map M' (the case $\beta = 1$ is treated in [Fel04]). Moreover an α/β -orientation of M is accessible from a vertex u if and only if the corresponding orientation of M' is. Hence Condition (iii) also follows from the particular case $\beta = 1$ applied to the map M' . Lastly, a β -fractional orientation of M is minimal if and only if the corresponding orientation of M' is minimal. Hence Lemma 3 immediately follows from the existence and uniqueness of a minimal orientation in the particular case $\beta = 1$ applied to the map M' . \square

We now define several important classes of bi-orientations. A face-rooted bi-orientation is called *clockwise-minimal* if it is minimal and its root-face is a clockwise circuit. A clockwise minimal bi-orientation is called *accessible* if it is accessible from one of the outer vertices (in this case, O is in fact accessible from any outer vertex because the root-face is a circuit). We denote by \mathcal{B}_d the set of weighted clockwise-minimal accessible bi-orientations such that the root-face has degree d . We denote by $\tilde{\mathcal{B}}_d \subset \mathcal{B}_d$ the set of inner-weighted clockwise-minimal accessible bi-orientations such that the root-face has degree d and every outer vertex has indegree 1. Observe that the outer face of a bi-orientation in $\tilde{\mathcal{B}}_d$ is a simple cycle made of 1-way edges oriented clockwise and that the inner half-edges incident to the outer vertices are all outgoing.

For any positive integer k , we call *k-fractional orientation* a \mathbb{N} -weighted bi-orientation such that the weight of any edge is k . Thus, k -fractional orientations do not have 0-way edges, their 1-way edges have weight 0 and k respectively on the outgoing and ingoing half-edges, and their 2-way edges have positive weights on both half-edges summing to k . The following classes of k -fractional orientations will play an important role.

Definition 4. *Let d, k be positive integers. We denote by \mathcal{O}_d^k the set of clockwise-minimal accessible k -fractional orientations with root-face of degree d . We denote by $\tilde{\mathcal{O}}_d^k \subset \mathcal{O}_d^k$ the subset of these orientations such that every outer vertex has weight k . Lastly we omit the superscript k when $k = 1$ (non-fractional orientations), that is, we denote $\mathcal{O}_d = \mathcal{O}_d^1$ and $\tilde{\mathcal{O}}_d = \tilde{\mathcal{O}}_d^1$.*

Examples of orientations in \mathcal{O}_5 , $\tilde{\mathcal{O}}_5$, \mathcal{O}_4^3 and $\tilde{\mathcal{O}}_4^3$ are represented in Figures 3 and 6.

Remark 5. The set \mathcal{O}_d^k is the subset of \mathbb{N} -weighted bi-orientations in $\tilde{\mathcal{B}}_d$ such that each edge has weight k . Similarly, up to forgetting the weights on the outer half-edges, the set $\tilde{\mathcal{O}}_d^k$ identifies with the subset of \mathbb{N} -weighted bi-orientations in $\tilde{\mathcal{B}}_d$ such that each inner edge has weight k . Indeed, the weight condition on the outer vertices of the orientations in $\tilde{\mathcal{O}}_d^k$, forces the outer face to be a simple cycle made of 1-way edges oriented clockwise (with weight 0 and k respectively on the outgoing and ingoing half-edges), and the inner half-edges incident to the outer vertices to be outgoing.

3. MASTER BIJECTIONS BETWEEN ORIENTED MAPS AND MOBILES

In this section we describe the two “master bijections” Φ_- , Φ_+ . These bijections will be specialized respectively in Sections 4 and 5 in order to obtain bijections for d -angulations without and with boundary.

3.1. Master bijections Φ_+ , Φ_- for ordinary orientations.

A *properly bicolored mobile* is a non-rooted plane tree with vertices colored black and white, with every edge joining a black and a white vertex. A (properly bicolored) *blossoming mobile* is a mobile together with some *outgoing buds* (dangling outgoing half-edges) incident to black vertices. An example is shown in Figure 3 (right). The *excess* of a blossoming mobile is the number of edges minus the number of outgoing buds. A mobile with excess δ is called a δ -mobile.

We now define the mappings Φ_{\pm} using a local operation performed around each edge.

Definition 6. Let e be an edge of a planar bi-orientation, made of half-edges h and h' incident to vertices v and v' respectively. Let c and c' be the corners preceding h and h' in clockwise order around v and v' respectively, and let f and f' be the faces containing these corners. Let $b_f, b_{f'}$ be some vertices placed inside the faces f and f' (with $b_f = b_{f'}$ if $f = f'$). If e is a 1-way edge with h being the ingoing half-edge (as in Figure 2), then the local transformation of e consists in creating an edge joining the black vertex b_f to the vertex v in the corner c , gluing an outgoing bud to $b_{f'}$ in the direction of c' , and deleting the edge e .

The local transformation of a 1-way edge is illustrated in Figure 2 (ignore the weights w, w' for the time being).

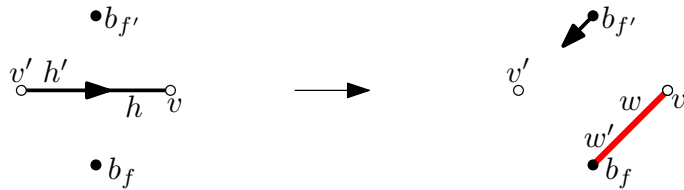


FIGURE 2. The local transformation of a 1-way edge.

Definition 7. Let O be a face-rooted orientation in \mathcal{O}_d with root-face f_0 . We view the vertices of O as white and place a black vertex b_f in each face f of O .

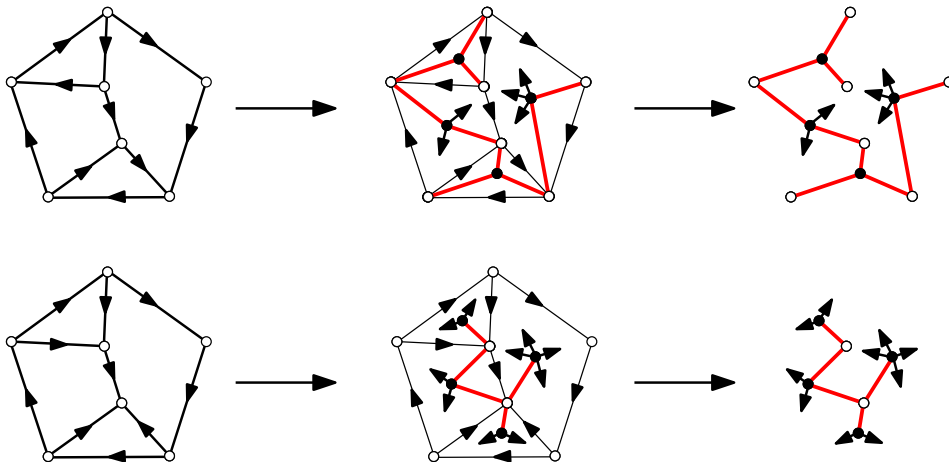


FIGURE 3. Top: Mapping Φ_+ applied to an orientation in \mathcal{O}_5 .
Bottom: Mapping Φ_- applied to an orientation in $\tilde{\mathcal{O}}_5$. The vertex b_{f_0} and the incident buds or edges are not represented.

- The embedded graph $\Phi_+(O)$ is obtained by performing the local transformation of each edge of O , and then deleting the black vertex b_{f_0} and all the incident buds (the vertex b_{f_0} is incident to no edge).
- If O is in $\tilde{\mathcal{O}}_d$, the embedded graph $\Phi_-(O)$ with black and white vertices is obtained by first returning all the edges of the outer-face (which is a clockwise circuit), then performing the local transformation of each edge, and lastly deleting the black vertex b_{f_0} , the white outer vertices of O , and the edges between them (no other edge or bud is incident to these vertices).

The mappings Φ_+ , Φ_- are illustrated in Figure 3. The following theorem is partly based on the results in [Ber07] and its proof is delayed to Section 7.

Theorem 8. *Let d be a positive integer.*

- The mapping Φ_+ is a bijection between the set \mathcal{O}_d of clockwise-minimal accessible orientations and the set of properly colored d -mobiles.
- The mapping Φ_- is a bijection between the subset $\tilde{\mathcal{O}}_d \subseteq \mathcal{O}_d$ and the set of properly colored $(-d)$ -mobiles.

3.2. Master bijections Φ_+ , Φ_- for weighted bi-orientations. In this subsection we extend the bijections Φ_{\pm} to weighted bi-orientations (with in mind the particular case of k -fractional orientations).

A *mobile* is a plane tree with vertices colored either black or white (the coloring is not necessarily proper). A *blossoming mobile* is a mobile together with some outgoing buds incident to black vertices. A blossoming mobile is *weighted* if a weight is associated to each non-bud half-edge. The *indegree* of a vertex v is the number of incident non-bud half-edges, and its *weight* is the sum of weight of the incident non-bud half-edges. The *outdegree* of a black-vertex b is the number of incident outgoing buds.

We now define the mappings Φ_{\pm} for weighted bi-orientations using local operations performed around each edge. We consider an edge e of a weighted bi-orientation and adopt the notations $h, h', v, v', c, c', f, f', b_f, b_{f'}$ of Definition 6. We also denote by w and w' respectively the weights of the half-edges h and h' . The Definition 6 of the *local transformation* of a 1-way edge is supplemented with weights attributed to the two halves of the edge created between v and b_f : the half-edge incident to v receives weight w and the half-edge incident to b_f receives weight w' . If the edge e is 0-way, the *local transformation* of e consists in creating an edge between b_f and $b_{f'}$ with weight w' for the half-edge incident to b_f and weight w for the half-edge incident to $b_{f'}$, and then deleting edge e ; see Figure 4. If the edge e is 2-ways, the *local transformation* of e consists in creating outgoing buds incident to b_f and $b_{f'}$, respectively in the direction of the corners c and c' , and leaving intact the weighted edge e .

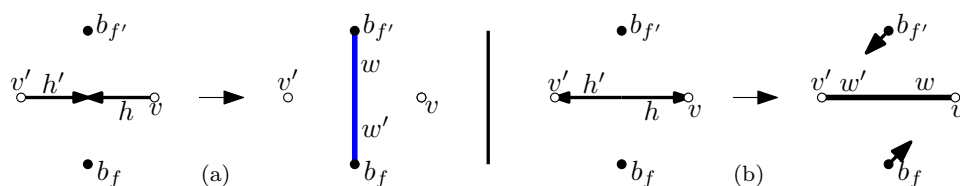


FIGURE 4. Local transformation of a 0-way edge (left) and of a 2-ways edge (right).

With these definitions of local transformations, Definition 7 of the mappings Φ_{-} and Φ_{+} is extended without change to the case of weighted bi-orientations in \mathcal{B} and $\tilde{\mathcal{B}}$ respectively. The mapping Φ_{+} is illustrated in Figure 5.

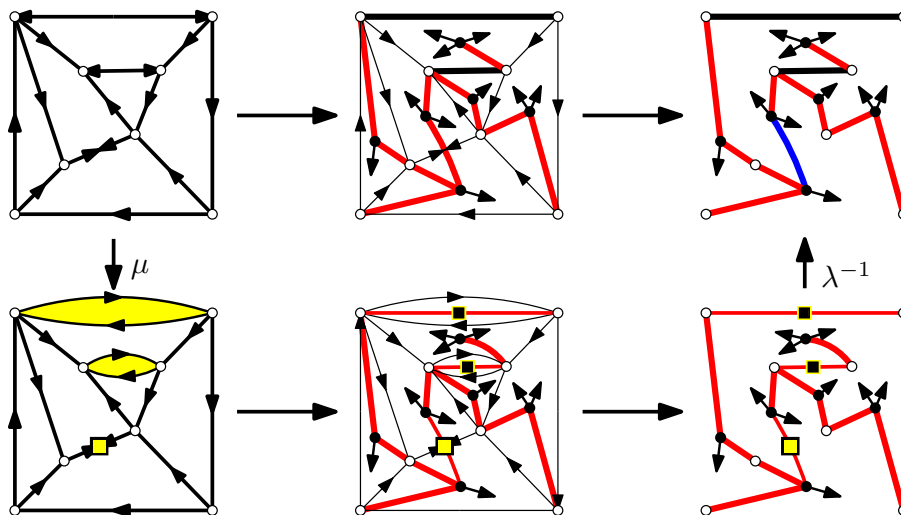


FIGURE 5. Top: the bijection Φ_{+} applied to a bi-orientation in \mathcal{B} (weights are omitted). Bottom: how to reduce the case of bi-orientations to the case of ordinary orientations.

Theorem 9. *Let d be a positive integer.*

- *The mapping Φ_+ is a bijection between the set \mathcal{B}_d of weighted clockwise-minimal accessible bi-orientations and the set of weighted d -mobiles.*
- *The mapping Φ_- is a bijection between the subset $\tilde{\mathcal{B}}_d$ of inner-weighted clockwise-minimal accessible bi-orientations and the set of weighted $(-d)$ -mobiles.*

Remark 10. *There are many obvious parameter-correspondences for the bijections Φ_{\pm} . For a start, the vertices, inner faces, 0-way edges, 1-way edges, 2-ways edges in a bi-orientation O are in natural bijection with the white vertices, black vertices, black-black edges, black-white edges, white-white edges of the mobile $\Phi_+(O)$. The indegree and weight of a vertex of O are equal to the indegree and weight of the corresponding white vertex of $\Phi_+(O)$. The degree, clockwise degree and weight of an inner face of O are equal to the degree, indegree and weight of the corresponding black vertex of $\Phi_+(O)$. Similar correspondences exist for Φ_- , and we do not list them all as they follow directly from the definitions.*

Before proving Theorem 9, we state its consequences for k -fractional orientations. We call *k -fractional mobile* a weighted mobile without black-black edge such that

- for white-white edges, the weights of the two half-edges are positive integers summing to k ,
- for black-white edges the weights of the half-edges incident to the black and white vertices are respectively 0 and k .

Observe that 1-fractional mobile can be identified with unweighted properly-colored mobiles. We now specialize Theorem 9 to k -fractional orientations. Recall from Remark 5 that the subset \mathcal{O}_d^k and $\tilde{\mathcal{O}}_d^k$ can be seen as subsets of \mathcal{B}_d and $\tilde{\mathcal{B}}_d$ respectively. Given the obvious parameter-correspondence of the master bijections Φ_{\pm} (see Remark 10) it is clear that specializing Theorem 9 to k -fractional orientations gives the following extension of Theorem 8.

Theorem 11. *Let d, k be positive integers.*

- *The mapping Φ_+ induces a bijection between the set \mathcal{O}_d^k of k -fractional clockwise-minimal accessible orientations and the set of k -fractional d -mobiles.*
- *The mapping Φ_- is a bijection between the subset $\tilde{\mathcal{O}}_d^k$ of k -fractional clockwise-minimal accessible orientations and the set of k -fractional $(-d)$ -mobiles.*

Moreover if the orientation O is in \mathcal{O}_d^k (resp. $\tilde{\mathcal{O}}_d^k$) and $T = \Phi_+(O)$ (resp. $T = \Phi_-(O)$), then the degree of the inner faces of O correspond to the degree of the black vertices of T . Moreover, the weight of vertices (resp. inner vertices) of O correspond to the weight of the white vertices of T .

The mappings Φ_+ and Φ_- are applied to some fractional orientations in Figure 6.

Proof of Theorem 9. We will prove Theorem 9 about weighted bi-orientations by a reduction to the case of ordinary unweighted orientations. The idea of the reduction is illustrated in Figure 5. First of all, observe that there is no need to consider weights in the proof. Indeed, through the mapping Φ_+ (resp. Φ_-) the weight of a half-edge (resp. inner half-edge) of a bi-orientation O in \mathcal{B}_d (resp. $\tilde{\mathcal{B}}_d$) is given to exactly one half-edge of the corresponding mobile (so that there is absolutely no weight constraint on the mobiles).

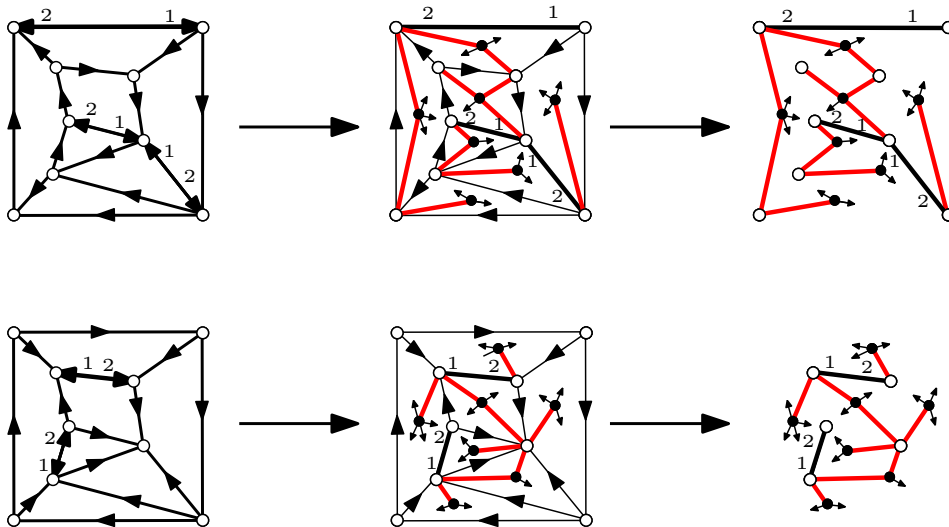


FIGURE 6. Top: Master bijection Φ_+ applied to a 3-fractional orientation in \mathcal{O}_4^3 . Bottom: Φ_- applied to an orientation in $\tilde{\mathcal{O}}_4^3$. The weights are indicated only for 2-ways edges. The vertex b_{f_0} and the incident buds or edges are not represented.

It remains to prove that the mapping Φ_+ (resp. Φ_-) is a bijection between unweighted bi-orientations in \mathcal{B}_d (resp. $\tilde{\mathcal{B}}_d$) and mobiles with excess d (resp. $-d$). We call *bi-marked orientation* an orientation with some marked vertices of degree 2 and indegree 2 and some marked faces of degree 2 and clockwise degree 2. Observe that two marked vertices cannot be adjacent, two marked faces cannot be adjacent, and moreover a marked vertex cannot be incident to a marked face. Given a bi-orientation O we denote by $\mu(O)$ the bi-marked orientation obtained by replacing every 0-way edge by two edges in series directed toward a marked vertex of degree 2 (and indegree 2), and replacing every 2-ways edges by two edges in parallel directed clockwise around a marked face of degree 2 (and clockwise degree 2). The mapping μ illustrated in Figure 5 (left) is clearly a bijection between bi-orientations and bi-marked orientations. We call *bi-marked mobile* a properly bicolored mobile with some set of non-adjacent marked vertices of degree 2. Given a mobile T , we denote by $\lambda(T)$ the bi-marked mobile obtained by inserting a marked black vertex at the middle of each white-white edge, and inserting a marked white vertex at the middle of each black-black edge. The mapping λ illustrated in Figure 5 (right) is clearly a bijection between mobiles and bi-marked mobiles. Moreover it is clear from the definitions that the mapping Φ_+ (resp. Φ_-) for bi-orientations is equal to the mapping $\lambda \circ \Phi \circ \mu$ where the mapping Φ is the restriction of Φ_+ (resp. Φ_-) to ordinary orientations. Thus Theorem 5 implies that Φ_+ (resp. Φ_-) is a bijection on \mathcal{B} (resp. on $\tilde{\mathcal{B}}$). \square

Before closing this section we state an additional claim which is useful for counting purposes. For any bi-orientation O in $\tilde{\mathcal{B}}_d$, we call *exposed* the outgoing buds of the mobile $T = \Phi_-(O)$ created by applying the local transformation to the outer edges of O (which have preliminarily been returned).

Claim 12. *Let O be a bi-orientation in $\tilde{\mathcal{B}}_d$ and let $T = \Phi_-(O)$. There is a bijection between the set \vec{O} of corner-rooted maps inducing the face-rooted map O (i.e., the maps obtained by choosing a root-corner in the root-face of O), and the set \vec{T} of mobiles obtained from T by marking an exposed bud. Moreover, there is a bijection between the set T_{\rightarrow} of mobiles obtained from T by marking a non-exposed bud, and the set $T_{\rightarrow\circ}$ of mobiles obtained from T by marking a half-edge incident to a white vertex.*

We point out that there is a little subtlety in the Claim 12 related to the possible symmetries (for instance, it could happen that the d corners of the root-face of O give less than d different corner-rooted maps).

Proof. The natural bijection γ between the d exposed buds of T and the d corners in the root-face of O (in which an exposed bud b points toward the vertex incident to the corner $\gamma(b)$) does not require any “symmetry breaking” convention. Hence the bijection γ respects the possible symmetries of O and T , and therefore it induces a bijection between \vec{T} and \vec{O} . We now prove the second bijection. Let B be the set of non-exposed bud of T , and let H be the set of half-edges incident to a white vertex. Using the fact that the local transformation of each inner i -way edge of O gives i buds in B and i half-edges in H (and that all buds in B , and all half-edges in H are obtained in this way), one can easily define a bijection γ' between B and H without using any “symmetry breaking” convention. Since the bijection γ' respects the eventual symmetries of T it induces a bijection between T_{\rightarrow} and $T_{\rightarrow\circ}$. \square

4. BIJECTIONS FOR d -ANGULATIONS OF GIRTH d

In this section, we prove the existence of a canonical class of fractional orientations for d -angulations of girth d . This allow us to identify the class \mathcal{C}_d of d -angulations of girth d with a class of orientations in $\tilde{\mathcal{O}}_d$. We then obtain a bijection between \mathcal{C}_d and a class of blossoming mobiles by specializing of the master bijection Φ_- to these orientations.

4.1. Fractional orientations for d -angulations of girth d .

Let D be a face-rooted d -angulation. A $d/(d-2)$ -orientation of D is a $(d-2)$ -fractional orientation such that each inner vertex has weight d and each outer vertex has weight $d-2$. Thus $d/(d-2)$ -orientation are particular cases of α/β -orientations where $\alpha(v) = d$ for inner vertices, $\alpha(v) = d-2$ for outer vertices and $\beta(e) = d-2$ for each edge. By Lemma 2, if D admits a $d/(d-2)$ -orientation, then there exists a unique minimal one. A minimal $d/(d-2)$ -orientation is represented in Figure 7 (top-left).

Theorem 13. *A face-rooted d -angulation admits a $d/(d-2)$ -orientation if and only if it has girth d . In this case, the minimal $d/(d-2)$ -orientation is in $\tilde{\mathcal{O}}_d^{d-2}$.*

Proof. We first prove the necessity of the girth d . If D has not girth d , then it is either a tree (its girth is infinite), or it has a cycle of length $c < d$. If $D = (V, E)$ is a tree, then $|E| = (|V| - 1)$. This prevents the existence of a $d/(d-2)$ -orientation for D because in such an orientation the sum of weights of the edges $(d-2)|E|$ should equal the sum of weights of the vertices $(d-2)|V|$. We now suppose that D has a simple cycle C of length $c < d$. Let D' be the submap of D enclosed by C which

does not contain the root-face of D . Let V', E' be respectively the set of vertices and edges strictly inside C . By Lemma 1, $(d-2)|E'| = d|V'| + c - d < d|V'|$. This prevents the existence of a $d/(d-2)$ -orientation for D because in such an orientation $(d-2)|E'|$ corresponds to the sum of weights of the edges in E' , while $d|V'|$ corresponds to the sum of weights of the vertices in V' .

We now consider a d -angulation D of girth d and want to prove that D has a $d/(d-2)$ -orientation. We use Lemma 3 with the map $D = (V, E)$, the function α taking value d on inner vertices and $d-2$ on outer vertices, and the function β taking value d on every edge. We first check Condition (i). First observe that D has d distinct outer vertices (otherwise the outer-face would create a cycle of length less than d). Hence $\sum_{v \in V} \alpha(v) = d(|V| - d) + (d-2)d$. This is equal to $(d-2)|E| = \sum_{e \in E} \beta(e)$ by Lemma 1 (with $p = d$), hence Condition (i) holds. We now check Condition (ii). Let S be a subset of vertices. We first suppose that the induced subgraph $G_S = (S, E_S)$ is connected. If this is the case, either G_S is a tree and (ii), (iii) clearly hold (since $\sum_{v \in S} \alpha(v) \geq (d-2)|S| > (d-2)|E_S|$), or it is a map with faces of degree at least d (since the girth of G is d). In the second case, Lemma 1 implies $(d-2)|E_S| \leq d(|S| - p) + p - d + p(d-2) \leq d|S| - 2d$, where $p \geq d$ is the degree of the outer face of G_S . Moreover, $\sum_{v \in S} \alpha(v) \geq d|S| - 2d$ with equality only if S contains every outer vertex of D . Thus (ii) holds and (iii) holds for any outer vertex v_0 . Now, if the induced subgraph $G_S = (S, E_S)$ is not connected, knowing that (ii), (iii) holds for every connected component of G_S ensures that it holds for G_S (since both sides of the inequalities are additive over components). This proves that D admits some $d/(d-2)$ -orientations and that they are all accessible from any outer vertex.

We now consider the minimal $d/(d-2)$ -orientations O of D . Since every outer vertex of O has weight $d-2$, its outer-face is a simple cycle made of 1-way edges oriented clockwise (this is the only way of avoiding a counterclockwise circuit). Thus O is clockwise-minimal and every outer vertex has indegree 1. The orientation O is also accessible from any outer vertex, hence O is in $\tilde{\mathcal{O}}_d^{d-2}$. \square

4.2. Bijection for d -angulations of girth d .

By Theorem 13, the class \mathcal{C}_d of d -angulations of girth d can be identified with the subset E_d of orientations in $\tilde{\mathcal{O}}_d^{d-2}$ such that every face has degree d and every inner vertex has weight d (the weight of the outer vertices is $d-2$ for any orientation in $\tilde{\mathcal{O}}_d^{d-2}$). We now characterize the mobiles that are in bijection with the subset E_d .

We call d -branching mobile a blossoming $(d-2)$ -fractional mobile such that every black vertex has degree d and every white vertex has weight d . A 5-branching mobile is shown in Figure 7 (top-right). By Theorem 11 the master bijection Φ_- induces a bijection between the orientations in E_d with n inner faces and the d -branching mobiles of excess $-d$ with n black vertices. The additional condition of having excess $-d$ is in fact redundant (hence can be omitted) as claimed below.

Claim 14. *Any d -branching mobile has excess $-d$.*

Proof. Let e, e', o be respectively the number of edges, white-white edges and outgoing buds of a d -branching mobile. By definition, the excess is $\delta = e + e' - o$. Let b and w be respectively the number of black and white vertices. The degree condition for black vertices gives $e - e' + o = db$, so that $\delta = 2e - db$. The weight condition on white vertices gives $(d-2)e = dw$ and combining this relation with $b + w = e + 1$ (so as to eliminate w) gives $b = \frac{2}{d}e + 1$, hence $\delta = -d$. \square

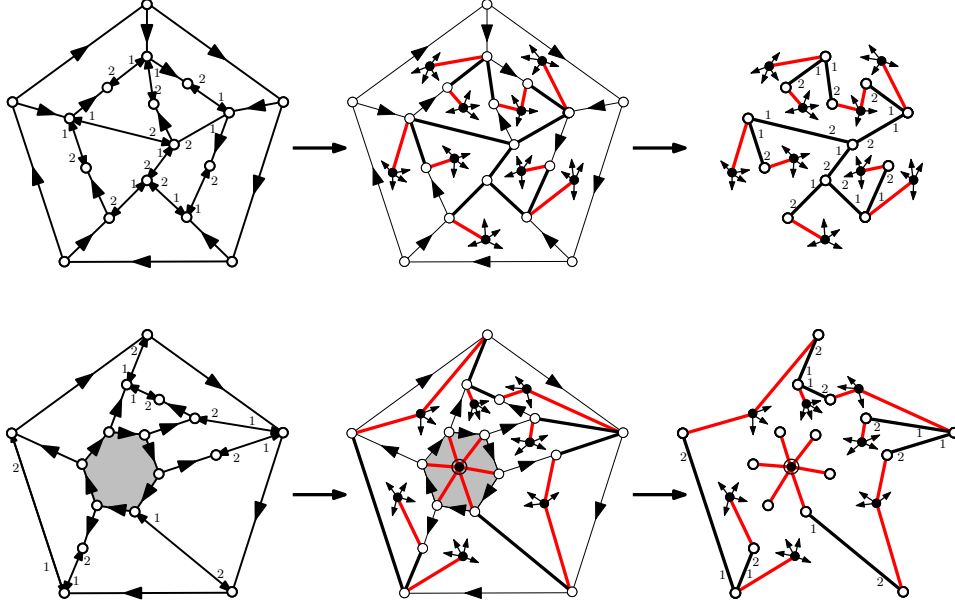


FIGURE 7. Top: a 5-angulation D of girth 5 and the 5-branching mobile $\Phi_-(D)$. Bottom: a non-separated 6-annular 5-angulation A and the $(6,5)$ -branching mobile $\Phi_+(A)$. The weights are only indicated on 2-ways edges.

From Claim 14 and the preceding discussion we obtain:

Theorem 15. *For $d \geq 3$ and $n \geq 0$, face-rooted d -angulations of girth d with n inner faces are in bijection with d -branching mobiles with n black vertices.*

Theorem 15 is illustrated by Figure 7 (top-part). Before closing this section we mention that a slight simplification appears in the definition of d -branching mobiles when d is even.

Proposition 16. *If $d = 2b$ is even, then every half-edge of the d -branching mobile has an even weight (in $\{0, \dots, d-2\}$). In particular, the 4-branching mobiles are properly-bicolored mobiles with weight 0 on half-edges incident to black colored and weight 2 on half-edges incident to white vertices. Similarly the minimal $d/(d-2)$ -orientation of any d -angulation has only even weights. In particular, for $d = 4$ the minimal $d/(d-2)$ -orientation does not have 2-ways edges.*

Proof. Let d be even and let M be a d -branching mobile. Edges of M have either two half-edges with even weights or two half-edges with odd weights, in which case we call them *odd*. We suppose by contradiction that the set of odd edges is non-empty. In this case, there exists a vertex v incident to exactly one odd edge (since the set of odd edges form a non-empty forest). Hence, the weight of v is odd, which is impossible since the weight of black vertices is 0 and the weight of white vertices is d .

Similarly, the edges of a $d/(d-2)$ -orientation have either two half-edges with even weights or two half-edges with odd weights, in which case we call them *odd*.

Moreover odd edges are 2-ways hence they form a forest in the minimal $d/(d-2)$ -orientation (otherwise there would be a counterclockwise circuit). Hence, if there are odd edges, there is a vertex v incident to only 1 odd edge, contradicting the requirement that every vertex has weight d . \square

5. BIJECTIONS FOR d -ANGULATIONS OF GIRTH d WITH A p -GONAL BOUNDARY

In this section we deal with d -angulations with a boundary. For any integers $p \geq d \geq 3$, we call p -gonal d -angulation a map having one face of degree p whose contour is simple (its vertices are all distinct) and all the other faces of degree d . Our bijective strategy for p -gonal d -angulations is similar to the case without boundary: we first exhibit a canonical orientation for these maps and then characterize the bijection induced by a master bijection. However, an additional difficulty arises: one needs to factorize p -gonal d -angulations into two parts, one being a d -angulation without boundary and the other being a *non-separated* p -gonal d -angulations. The master bijections Φ_- and Φ_+ then gives bijections for each part.

5.1. Orientations for p -gonal d -angulations.

Let A be a p -gonal d -angulation. The face of A of degree p is called *boundary-face* and the p incident vertices are called *boundary vertices*. A *pseudo $d/(d-2)$ -orientation* of A is a $(d-2)$ -fractional orientation such that non-boundary vertices have weight d and the boundary-face is a circuit of 1-way edges. By Lemma 1, the number v of vertices and e of edges of A satisfy $(d-2)e = d(v-p) + dp - p - d$. Since $d(v-p)$ is the sum of the weights of the non-boundary vertices, this proves the following claim.

Claim 17. *The sum of the weights of the boundary vertices in a pseudo $d/(d-2)$ -orientation of a p -gonal d -angulation is $dp - p - d$.*

We now prove that the existence of a pseudo $d/(d-2)$ -orientation characterizes the p -gonal d -angulations of girth d .

Lemma 18. *Let $p \geq d \geq 3$ be integers. A p -gonal d -angulation A admits a pseudo $d/(d-2)$ -orientation if and only if it has girth d . In this case, every pseudo $d/(d-2)$ -orientation is accessible from any boundary vertex.*

Proof. The proof of the necessity of girth d is the same as in the proof of Theorem 13. We now consider a p -gonal d -angulation A of girth d and prove that it admits a pseudo $d/(d-2)$ -orientation. From A we construct a d -angulation D of girth d as follows. First we insert vertices and edges inside the boundary-face f_b so as to d -angulate the interior of f_b while keeping the girth equal to d . This yields a d -angulation D of girth d . We can suppose that one of the faces created inside f_b is not incident to f_b and we choose such a face as the root-face of D . By Theorem 13, the face-rooted d -angulation D admits a minimal $d/(d-2)$ -orientation O . The orientation O induces an orientation O_A of A in which every non-boundary vertex has weight d . Let O'_A be the orientation of A obtained from O_A by reorienting the edges incident to the boundary-face f_b into a circuit of 1-way edges. By definition, O'_A is a pseudo $d/(d-2)$ -orientation.

It remains to show that any pseudo $d/(d-2)$ -orientation of A is accessible from any boundary vertex. We first show that the property holds for the orientation O'_A . Recall from Theorem 13 that the $d/(d-2)$ -orientation O is accessible from any outer

vertex of D . Thus, for any vertex v of A , there is a directed path of O from an outer vertex of D to v . Hence, there is a directed path of O'_A from a boundary vertex to v . Since the boundary-face of O'_A is a circuit, it means that there is a directed path of O'_A from any boundary vertex to v . Thus, O'_A is accessible from any boundary vertex. We now consider another pseudo $d/(d-2)$ -orientation O''_A of A . We want to show that O''_A is also accessible from the boundary vertices. Suppose the contrary and consider the set U of vertices u such that there is no directed path from a boundary vertex to u . Clearly any edge of O''_A from a vertex u in U to a vertex not in U is 1-way and is oriented away from u . Thus, in O''_A the sum of weights of the vertices in U is $(d-2)e_U$, where e_U is the number of edges with both ends in U . Since U contains no boundary vertex the sum of weights of the vertices in U is the same for O'_A and O''_A . Thus the sum of weights of the vertices in U is $(d-2)e_U$ in O'_A , implying that any edge of O'_A from a vertex u in U to a vertex not in U is 1-way and is oriented away from u . This contradicts the fact that the vertices in U are reachable from the boundary vertices by a directed path of O'_A . \square

We call *p-annular d-angulation* a face-rooted p -gonal d -angulation of girth d whose root-face is not the boundary-face. For a p -annular d -angulation, a cycle of length d is said *separating* if one of the boundary vertex lies strictly on one side of C and one of the outer vertex lies strictly on the other side of C . A p -annular d -angulation is *non-separated* if it has no separating cycle.

Proposition 19. *Let $p > d \geq 3$. Any p -annular d -angulation A admits a unique minimal pseudo $d/(d-2)$ -orientation. This orientation is clockwise-minimal accessible (i.e., in \mathcal{O}_d^{d-2}) if and only if A is non-separated.*

Proof. Let A be a p -annular d -angulation. The map A admits some pseudo $d/(d-2)$ -orientations by Lemma 18. The existence and uniqueness of a minimal one is not directly granted by Lemma 2 because the definition of pseudo $d/(d-2)$ -orientations includes the requirement that the boundary-face is a circuit of 1-way edges. Instead, we consider the map \hat{A} obtained from A by contracting into a vertex, denoted v_b , the boundary-face of A (together with the p incident edges). The mapping σ which associates to a pseudo $d/(d-2)$ -orientations of A the induced orientation of \hat{A} is a bijection between the (non-empty) set S of pseudo $d/(d-2)$ -orientations of A with a boundary-face directed clockwise (there exists such orientations by Lemma 18) and the set \hat{S} of orientations of \hat{A} such that every vertex other than v_b has weight d (while v_b has weight $p-d$). Moreover, an orientation $O \in S$ is minimal if and only if $\sigma(O)$ is minimal (indeed, the fact that the boundary-face of O is a circuit implies that any counterclockwise circuit of $\sigma(O)$ is part of a counterclockwise circuit of O). Thus, applying Lemma 2 on the set \hat{S} of orientations implies that A admits a unique minimal pseudo $d/(d-2)$ -orientation.

We now prove second assertion of the proposition. Suppose first that A has a separating cycle C : a boundary vertex u_b lies on one side and an outer vertex u_o lies on the other side. We now prove that the minimal pseudo $d/(d-2)$ -orientation O is not accessible from u_o . Let D be the d -gonal d -angulation made of C and all the edges and vertices on the side of u_b (the side containing the boundary-face). By Lemma 1, the numbers e of edges and v of vertices in D satisfy $(d-2)e = d(v-p) + dp - p - d$. Moreover the right-hand-side is the sum of the weight of the vertices in D (by claim 17). Thus every edge incident to C and lying in the region containing u_o are 1-way edges oriented away from C . Thus u_o cannot reach

the vertices of D by a directed path. The orientation O is not accessible from the outer vertex u_o , hence O is not clockwise-minimal accessible.

Suppose conversely that A is not separating. We first prove that the minimal pseudo $d/(d-2)$ -orientation O is accessible from any outer vertex. We reason by contradiction and suppose that O is not accessible from one of the outer vertex u_o . In this case, u_o cannot reach the boundary vertices by a directed path (since Lemma 18 ensures that the boundary vertices can reach all the other vertices). We consider the set U of vertices of A that can reach the boundary vertices by a directed path. This set contains all the boundary vertices but not the outer vertex u_o . Let $A_U = (U, E_U)$ be the map induced by the vertices in U (it is clear from the definitions that A_U is connected). Clearly, every edge between a vertex u in U and a vertex not in U is oriented away from U . Thus $(d-2)|E_U|$ equals the sum of the weight of the vertices in U . By claim 17 we get $(d-2)|E_U| = d(|U| - p) + pd - p - d$ hence

$$(1) \quad (d-2)(|E_U| - p) = d(|U| - p) + p - d.$$

Now, since A has girth d , the non-boundary faces of A_U have degree at least d . Therefore by Lemma 1, the equality (1) ensures that every face of A_U has degree d . In particular, the cycle C corresponding to the contour of the face of A_U in which the outer vertex u_o lies has length d . This cycle C is separating: a contradiction.

It remains to prove that the outer face of O is a clockwise cycle. Suppose the contrary. In this case, there is an outer edge e which is 1-way and has the root-face on its right. Let v, v' the end and origin of O . By accessibility from v there is a simple directed path P from v to a boundary vertex u . By accessibility from boundary vertices, there is a path simple directed P' from u to v' . The paths P and P' do not contain the 1-way edge e . Thus, there is circuit C containing e in the concatenation of P, P' and e . Since e has the root-face on its right, the circuit C is counterclockwise, contradicting the minimality of O . \square

5.2. Bijection for p -gonal d -angulations of girth d .

We call (p, d) -branching mobile a $(d-2)$ -fractional blossoming mobile such that:

- every black vertex has degree d except for one, called the *special vertex* s , which has degree p and is incident to no bud,
- every white vertex which is not a neighbor of s has weight d , and the weights of the neighbors of s add up to $pd - p - d$.

An example of $(6, 5)$ -branching mobile is represented in Figure 7(bottom-right).

By Proposition 19, the non-separated p -annular d -angulations can be identified with the class $\mathcal{O}_{p,d}$ of orientations in \mathcal{O}_d^{d-2} such that

- every face has degree d except for one non-root face of degree p , called boundary-face, whose contour is a clockwise circuit of 1-way edges
- every non-boundary vertex has weight d (and the sum of weights of the boundary vertices is $pd - p - d$ by Claim 17).

By Theorem 11 (and the definition of Φ_+ which implies that an inner face of degree p whose contour is a clockwise circuit of 1-way edges corresponds to a black vertex of the mobile incident to p edges and no bud), the class $\mathcal{O}_{p,d}$ of orientations are in bijection with the class of (p, d) -branching mobiles of excess d . As in the case

of d -angulations without boundary, the additional requirement on the excess is redundant as claimed below.

Claim 20. *Any (p, d) -branching mobile has excess d .*

Proof. By definition, the excess is $\delta = e + e' - o$, where e, e', o are respectively the number of edges, white-white edges and outgoing buds. Let b and w be the number of black and white vertices. The condition on black vertices gives $e - e' + o = d(b - 1) + p$, so that $\delta = 2e - db + d - p$. The condition on white vertices gives $(d - 2)e = d(w - p) + pd + -p - d$ and combining this relation with $b + w = e + 1$ (so as to eliminate w) gives $b = \frac{2}{d}e + \frac{p}{d}$, hence $\delta = d$. \square

To summarize, by specialization of the master bijection Φ_+ in the $(d - 2)$ -fractional setting, Proposition 19 gives the following result.

Theorem 21. *Let $p > d \geq 3$ and n be positive integers. The non-separated p -annular d -angulations (of girth d) with n inner faces are in bijection with (p, d) -branching mobiles with n black vertices.*

Theorem 21 is illustrated in Figure 7 (bottom). As in the case without boundary, a slight simplification appears in the definition of (p, d) -branching mobiles when d is even. First observe that when d is even the p -gonal d -angulations are bipartite (since the inner faces generate all cycles). Thus, there is no p -gonal d -angulations when p is odd and d even. Similarly there is no (p, d) -branching mobiles with p odd and d even. There is a further simplification:

Proposition 22. *If $d = 2b$ is even, then for any $p \geq b$, the weights of half-edges of $(2p, d)$ -branching mobiles are even. Similarly, the weights of the half-edges in minimal pseudo $d/(d - 2)$ -orientations are even.*

Proof. The proof is the same as the proof of Proposition 16. \square

We now explain how to deal bijectively with general (possibly separated) p -annular d -angulations. We call *pseudo-separating* a cycle of a p -annular d -angulations which is either separating or the contour of the root-face. A pseudo-separating cycle C of A defines:

- a p -annular d -angulation, denoted A_C , corresponding to the map on the side of C containing the boundary face (C is the contour of the root-face),
- a d -annular d -angulation, denoted A'_C , corresponding to the map on the side of C containing the root-face (C is the contour of the boundary face).

In order to make the decomposition $(A, C) \mapsto (A_C, A'_C)$ injective we consider *marked maps*. We call a p -annular d -angulation *marked* if a boundary vertex is marked. We now consider an arbitrary convention for each marked p -annular d -angulations by distinguishing one of the outer-vertices as *co-marked* (the co-marked vertex is entirely determined by the marked vertex). We then define for a marked p -annular d -angulation A^\bullet and a pseudo-separating cycle C , the marked annular d -angulations A_C^\bullet and $A_C^{\bullet'}$ obtained by marking A_C at the marked vertex of A^\bullet and marking $A_C^{\bullet'}$ at the co-marked vertex of A_C^\bullet .

Proposition 23. *Let $p > d \geq 3$. Any p -annular d -angulation A has a unique pseudo-separating cycle C such that A_C is non-separated. Moreover, the mapping Δ which associates to a marked p -annular d -angulation A^\bullet the pair $(A_C^\bullet, A_C^{\bullet'})$ is*

a bijection between marked p -annular d -angulations and pairs made of a marked non-separated p -annular d -angulation and a marked d -annular d -angulation.

Proposition 23 gives a bijective approach for general p -annular d -angulations: apply the master bijection Φ_+ (Theorem 21) to the non-separated p -annular d -angulation A_C , and the master bijection Φ_- (Theorem 15) to the d -angulation A'_C .

Proof. The set \mathcal{C} of pseudo-separating cycles of A is non-empty (it contains the cycle corresponding to the contour of the root-face) and is partially ordered by the relation \preceq defined by setting $C \preceq C'$ if $A_C \subseteq A_{C'}$ (here the inclusion is in terms of the vertex set and the edge set). Moreover, A_C is non-separated if and only if C is a minimal element of (\mathcal{C}, \preceq) . Observe also that for all $C, C' \in \mathcal{C}$, the intersection $A_C \cap A_{C'}$ is non-empty because it contains the boundary vertices and edges. Thus, if C, C' are not comparable for \preceq , then the cycles C, C' must intersect. Since they have both length d and A has no cycle of length less than d , the only possibility is that $A_C \cap A_{C'}$ is delimited by a separating cycle C'' of length d : $A_C \cap A_{C'} = A_{C''}$. This shows that (\mathcal{C}, \preceq) has a unique minimal element, or equivalently, that A has a unique pseudo-separating cycle C such that A_C is non-separated.

We now prove the second statement. The injectivity of Δ is clear: the markings allow to glue back the maps $A_C^\bullet, A_{C'}^\bullet$ by identifying the root-face C of A_C^\bullet with the boundary-face of $A_{C'}^\bullet$, and identifying the co-marked vertex of A_C^\bullet with the marked vertex of $A_{C'}^\bullet$. To prove surjectivity we only need to observe that gluing a non-separated p -annular d -angulation and a marked d -annular d -angulation as described above preserves the girth d (hence the glued map A^\bullet is a p -annular d -angulation) and makes C a pseudo-separating cycle of A^\bullet . \square

6. COUNTING RESULTS

In this section, we establish equations for the generating functions of d -angulations without and with boundary (Subsections 6.2 and 6.1). We then obtain closed formulas in the cases $d = 3$ and $d = 4$. We call *generating function*, or *GF* for short, of a class \mathcal{C} counted according to a parameter P the formal power series $C(x) = \sum_{n \in \mathbb{N}} c_n x^n$, where c_n is the number of objects $c \in \mathcal{C}$ satisfying $P(c) = n$. We also denote $[x^n]G(x)$ the coefficient of x^n in a formal power series $G(x)$.

6.1. Counting rooted d -angulations of girth d .

By Theorem 15, counting d -angulations of girth d reduces to counting d -branching mobiles. We will characterize the generating function of d -branching mobiles by a system of equations (obtained by a straightforward recursive decomposition). We first need a few notations. For a positive integer j we define the polynomial h_j in the variables w_1, w_2, \dots by:

$$(2) \quad h_j(w_1, w_2, \dots) := [t^j] \frac{1}{1 - \sum_{i>0} t^i w_i} = \sum_{r>0} \sum_{\substack{i_1, \dots, i_r > 0 \\ i_1 + \dots + i_r = j}} w_{i_1} \cdots w_{i_r}.$$

In other words, h_j is the (polynomial) generating function of integer compositions of j where the variable w_i is conjugated to the number of parts of size i .

We call *planted d -branching mobile* a $(d-2)$ -fractional blossoming mobile with a marked leaf (vertex of degree 1) such that non-marked black vertices have degree d

and non-marked white vertices have weight d . For $i \in \{0, \dots, d-2\}$, we denote by \mathcal{W}_i the family of planted d -branching mobiles where the marked leaf has weight i (the case $i = 0$ correspond to a black marked leaf). We also denote by $W_i \equiv W_i(x)$ the generating function of \mathcal{W}_i counted according to the number of non-marked black vertices.

For $i = d-2$, the marked leaf of a mobile in \mathcal{W}_i is a white vertex connected to a black vertex b . The $d-1$ other half-edges incident to b are either buds or belong to an edge e leading to white vertex w . In the second case the sub-mobile planted at b and containing w belongs to \mathcal{W}_0 . This decomposition gives the equation $W_{d-2} = x(1 + W_0)^{d-1}$.

For $i \in \{0, \dots, d-3\}$, the marked leaf of a mobile in \mathcal{W}_i is connected to a white vertex w . Let v_1, \dots, v_r the other neighbors of w . For $j = 1 \dots, r$ the sub-mobile planted at w and containing v_j belongs to one of the classes $\mathcal{W}_{\alpha(j)}$, and the sum of the weights $\sum_{j=1}^r \alpha(j)$ is $d - (d-2-i) = i+2$. This decomposition gives

$$W_i = \sum_{r>0} \sum_{\substack{i_1, \dots, i_r > 0 \\ i_1 + \dots + i_r = i+2}} W_{i_1} \cdots W_{i_r} = h_{i+2}(W_1, \dots, W_{d-2}).$$

Hence, the series W_0, W_1, \dots, W_{d-2} satisfy the following system of equations:

$$(3) \quad \begin{cases} W_{d-2} &= x(1 + W_0)^{d-1}, \\ W_i &= h_{i+2}(W_1, \dots, W_{d-2}) \quad \forall i \in \{0, \dots, d-3\}, \\ W_i &= 0 \quad \forall i > d-2. \end{cases}$$

Observe that the system (3) determines W_0, W_1, \dots, W_{d-2} uniquely as formal power series. Indeed, it is clear that any solutions W_0, W_1, \dots, W_{d-2} of this system have zero constant coefficient. And from this observation it is clear that the other coefficients are uniquely determined by induction.

The following table shows the system for the first values, $d = 3, 4, 5$:

$d = 3$	$d = 4$	$d = 5$
$W_1 = x(1 + W_0)^2$ $W_0 = W_1^2$	$W_2 = x(1 + W_0)^3$ $W_0 = W_1^2 + W_2$ $W_1 = W_1^3 + 2W_1W_2$	$W_3 = x(1 + W_0)^4$ $W_0 = W_1^2 + W_2$ $W_1 = W_1^3 + 2W_1W_2 + W_3$ $W_2 = W_1^4 + 3W_1^2W_2 + 2W_1W_3 + W_2^2$

In the case where d is even, $d = 2b$, one easily checks that $W_i = 0$ for i odd (this follows from the fact that, for odd i , all monomials in $h_{i+2}(W_1, \dots, W_{d-2})$ contain a W_j with odd j , and is related to Proposition 16). Hence the system can be simplified. The series $V_i := W_{2i}$ satisfy the system:

$$(4) \quad \begin{cases} V_{b-1} &= x(1 + V_0)^{2b-1}, \\ V_i &= h_{i+1}(V_1, \dots, V_{b-1}) \quad \forall i \in \{0, \dots, b-2\}, \\ V_i &= 0 \quad \forall i > b-1. \end{cases}$$

The following table shows the system for the first values, $b = 2, 3, 4$:

$b = 2$	$b = 3$	$b = 4$
$V_1 = x(1 + V_0)^3$ $V_0 = V_1$	$V_2 = x(1 + V_0)^5$ $V_0 = V_1$ $V_1 = V_1^2 + V_2$	$V_3 = x(1 + V_0)^7$ $V_0 = V_1$ $V_1 = V_1^2 + V_2$ $V_2 = V_1^3 + 2V_1V_2 + V_3$

Let \mathcal{M}_d be the family of d -branching mobiles rooted at a corner incident to a black vertex, and let $M_d(x)$ be the GF of \mathcal{M}_d counted according to the number of non-root black vertices. Since each of the d half-edges incident to the root-vertex is either a bud or is connected to a planted mobile in \mathcal{W}_0 we have

$$M_d(x) = (1 + W_0)^d.$$

Proposition 24 (Counting d -angulations of girth d). *For $d \geq 3$, the generating function $F_d(x)$ of corner-rooted d -angulations of girth d counted according to the number of inner faces has the following expression:*

$$F_d = W_{d-2} - \sum_{i=0}^{d-3} W_i W_{d-2-i},$$

where the series W_0, \dots, W_{d-2} are the unique power series solutions of the system (3) (which simplifies to (4) in the even case $d = 2b$). Therefore F_d is algebraic. Moreover it satisfies

$$F'_d(x) = (1 + W_0)^d.$$

Proof. Theorem 15 and Claim 12 (first part) imply that F_d is the series counting d -branching mobiles rooted at an exposed outgoing bud. Call B (resp. H) the GF of d -branching mobiles (counted according to the number of black vertices) with a marked outgoing bud (resp. marked half-edge incident to a white vertex). Claim 12 (second part) yields $F_d = B - H$. A mobile with a marked outgoing bud identifies to a planted mobile in \mathcal{W}_{d-2} (planted mobile where the vertex of attachment is black), hence $B = W_{d-2}$. A mobile with a marked half-edge incident to a white vertex identifies with an ordered pair of planted mobiles (w, w') in $\mathcal{W}_i \times \mathcal{W}_{d-2-i}$ for some i in $\{0, \dots, d-3\}$, hence $H = \sum_{i=0}^{d-3} W_i W_{d-2-i}$.

For the expression of $F'_d(x)$, observe that this is the GF of corner-rooted d -angulations of girth d with a secondary marked inner face. Equivalently, $F'_d(x)$ counts face-rooted d -angulations of girth d with a secondary marked corner not incident to the root-face. By the master bijection Φ_- , marking a corner not incident to the root-face in the d -angulation is equivalent to marking a corner incident to a black vertex in the corresponding mobile. Hence, Theorem 15 gives $F'_d(x) = M_d(x) = (1 + W_0)^d$. \square

The cases $d = 3$ and $d = 4$ correspond to simple triangulations and simple quadrangulations; we recover (see Section 6.3) exact-counting formulas due to Brown [Bro64, Bro65]. For $d \geq 5$ the counting results are new (to the best of our knowledge). For $d = 5$ and $d = 6$, one gets $F_5 = x + 5x^3 + 121x^5 + 4690x^7 + 228065x^9 + O(x^{11})$, and $F_6 = x + 3x^2 + 17x^3 + 128x^4 + 1131x^5 + 11070x^6 + O(x^7)$.

For any $d \geq 3$ a simple analysis based on the Drmota-Lalley-Wood theorem [FS09, VII.6] ensures that for odd d , the coefficient $[x^{2n}]F'_d(x)$ (from the Euler relation the odd coefficients of $F'_d(x)$ are zero) is asymptotically $c_d \gamma_d^n n^{-3/2}$ for some computable positive constants c_d and γ_d depending on d ; and for even d , the coefficient $[x^n]F'_d(x)$ is asymptotically $c_d \gamma_d^n n^{-3/2}$ again with c_d and γ_d computable constants. Since $[x^n]F_d(x) = \frac{1}{n}[x^{n-1}]F'_d(x)$, the number of corner-rooted d -angulations of girth d follows (up to the parity requirement for odd d) the asymptotic behavior $c \gamma^n n^{-5/2}$ which is universal for planar map families.

6.2. Counting rooted p -gonal d -angulations of girth d .

We start by characterizing the generating function of (p, d) -branching mobiles. A (p, d) -branching mobile is said *marked* if one of the p corners incident to the special vertex is marked. Let $\mathcal{M}_{p,d}$ be the class of marked (p, d) -branching mobiles, and let $M_{p,d}(x)$ be the GF of this class counted according to the number of non-special black vertices. Given a mobile in $\mathcal{M}_{p,d}$, we consider the (white) neighbors v_1, \dots, v_p of the special vertex s . For $i = 1 \dots p$ the sub-mobiles M_1, \dots, M_{r_i} planted at v_i (and not containing s) belong to some classes $\mathcal{W}_{\alpha(i,1)}, \dots, \mathcal{W}_{\alpha(i,r_i)}$, where $d - 2 + \sum_{j=1}^{r_i} \alpha(i, j)$ is the weight of v_i . Moreover, by definition of (p, d) -branching mobiles, the sum of the weights of v_1, \dots, v_p is $pd - p - d = (d - 2)p + p - d$. This decomposition gives for all $p \geq d \geq 3$

$$(5) \quad M_{p,d}(x) = h_{p-d}^{(p)}(W_1, \dots, W_{d-2}),$$

where for $j \geq 0$ the polynomial $h_j^{(p)}$ is the generating function of p -tuples of compositions of integers i_1, \dots, i_p with $i_1 + \dots + i_p = j$, where w_i marks the number of parts of size i . It is clear that $h_j^{(1)} = h_j$ and that $h_j^{(p)}$ satisfies:

$$(6) \quad h_j^{(p)}(w_1, w_2, \dots) := [t^j] \frac{1}{(1 - \sum_{i>0} t^i w_i)^p}.$$

Observe that in the special case $p = d$, there is a unique (d, d) -branching mobile, hence $M_{d,d}(x) = 1$ (which is coherent with (5) since $h_0^{(p)} = 1$).

We now turn to annular d -angulations. For $p \geq d \geq 3$, we denote by $\mathcal{A}_{p,d}$ the family of *marked* p -annular d -angulations of girth d (a boundary vertex is marked, or equivalently a corner of the boundary face is marked), and we denote by $\mathcal{N}_{p,d}$ the subfamily of those that are non-separated. Let $A_{p,d}(x)$ and $N_{p,d}(x)$ be the GF of these classes counted according to the number of non-boundary inner faces. By Theorem 21, the marked non-separated p -annular d -angulations are in bijection by Φ_+ with the marked (p, d) -branching mobiles (since marking the mobile is equivalent to marking the p -annular d -angulation on a boundary vertex) for $p > d \geq 3$. Thus, $N_{p,d}(x) = M_{p,d}(x)$ for all $p > d \geq 3$. Moreover, Proposition 23 directly implies:

$$A_{p,d}(x) = M_{p,d}(x)F'_d(x),$$

since the series $F'_d(x)$ (defined in Proposition 24) counts marked d -annular d -angulations. This relations also holds for $p = d$ since $M_{d,d}(x) = 1$. We summarize:

Proposition 25 (Counting p -gonal d -angulations of girth d). *For $p \geq d \geq 3$, the generating function $F_{p,d}(x)$ of p -gonal d -angulations of girth d rooted at a corner incident to the boundary-face, counted according to the number of non-boundary faces satisfies*

$$F'_{p,d}(x) = h_{p-d}^{(p)}(W_1, \dots, W_{d-2}) \cdot (1 + W_0)^d,$$

where $h_j^{(p)}$ is defined in (6) and the series W_1, \dots, W_{d-2} are specified by (3).

In the bipartite case, $d = 2b$ the expression simplifies to

$$F'_{2p,2b}(x) = h_{p-b}^{(2p)}(V_1, \dots, V_{b-1}) \cdot (1 + V_0)^{2b},$$

where the series V_1, \dots, V_{b-1} are specified by (4).

Proof. By definition, a p -annular d -angulation is a p -gonal d -angulation of girth d with a secondary marked face of degree d . Hence, $F_{p,d}'(x) = A_{p,d}(x) = M_{p,d}(x)F'_d(x)$.

Moreover $F'_d(x) = (1 + W_0)^d$ by Proposition 24 and $M_{p,d}(x) = h_{p-d}^{(p)}(W_1, \dots, W_{d-2})$ by (5). \square

6.3. Exact-counting formulas for triangulations and quadrangulations.

In the case of triangulations and quadrangulations ($d = 3, 4$) the system of equations given by Proposition 25 takes a form amenable to the Lagrange inversion formula. We thus recover bijectively the counting formulas established by Brown [Bro64, Bro65] for simple triangulations and quadrangulations with boundary.

Proposition 26 (Counting simple p -gonal triangulations). *For $p \geq 3$, $n \geq 0$, let $t_{p,n}$ be the number of corner-rooted p -gonal triangulations with $n + p$ vertices which are simple (i.e., have girth 3) and have the root-corner in the p -gonal face (no restriction on the root-corner for $p = 3$). The generating function $T_p(x) = \sum_{n \geq 0} (2n + p - 2) t_{p,n} x^n$ satisfies*

$$T_p(x) := \binom{2p-4}{p-3} u^{2p-3}, \quad \text{where } u = 1 + xu^4.$$

Consequently, the Lagrange inversion formula gives:

$$t_{p,n} = \frac{2(2p-3)!}{(p-1)!(p-3)!} \frac{(4n+2p-5)!}{n!(3n+2p-3)!}.$$

Proof. By the Euler relation, a p -gonal triangulation with $n + p$ vertices has $2n + p$ faces. Hence $T_p(x^2)x^{p-3} = F'_{p,3}(x)$. By proposition 25,

$$F_{p,3}'(x) = \binom{2p-4}{p-3} W_1^{p-3} (1 + W_0)^3 = x^{p-3} \binom{2p-4}{p-3} (1 + W_0)^{2p-3},$$

where W_0, W_1 are specified by $\{W_1 = x(1 + W_0)^2, W_0 = W_1^2\}$. Thus, $T_p(x^2) = \binom{2p-4}{p-3} (1 + W_0)^{2p-3}$. Moreover, $1 + W_0 = 1 + W_1^2 = 1 + x^2(1 + W_0)^4$, hence $1 + W_0(x) = u(x^2)$, where $u \equiv u(x)$ is the series specified by $u = 1 + xu^4$. Thus, $T_p(x) = \binom{2p-4}{p-3} u(x)^{2p-3}$. \square

Proposition 27 (Counting simple $2p$ -gonal quadrangulations). *For $p \geq 2$, $n \geq 0$, let $q_{p,n}$ be the number of corner-rooted $2p$ -gonal quadrangulations with $n + 2p$ vertices which are simple (i.e., have girth 4) and have the root-corner incident to the $2p$ -gonal face (no restriction on the root-corner for $p = 2$). The generating function $Q_p(x) = \sum_{n \geq 0} (n + p - 1) q_{p,n} x^n$ satisfies*

$$Q_p(x) = \binom{3p-3}{p-2} u^{3p-2}, \quad \text{where } u = 1 + xu^3.$$

Consequently, the Lagrange inversion formula gives:

$$q_{p,n} = \frac{3(3p-2)!}{(p-2)!(2p-1)!} \frac{(3n+3p-4)!}{n!(2n+3p-2)!}.$$

Proof. By the Euler relation, a $2p$ -gonal quadrangulation with $n + 2p$ vertices has $n + p$ faces. Hence $Q_p(x)x^{p-2} = F'_{2p,4}(x)$. By Proposition 25,

$$F'_{2p,4}(x) = \binom{3p-3}{p-2} V_1^{p-2} (1 + V_0)^4 = x^{p-2} \binom{3p-3}{p-2} (1 + V_0)^{3p-2},$$

where V_0, V_1 are specified by $\{V_1 = x(1 + V_0)^3, V_0 = V_1\}$. Thus, $Q_p(x) = \binom{3p-3}{p-2} (1 + V_0)^{3p-2}$. Moreover, $1 + V_0 = 1 + V_1 = 1 + x(1 + V_0)^3$, hence $1 + V_0(x) = u(x)$, where $u \equiv u(x)$ is the series specified by $u = 1 + xu^3$. Thus, $Q_p(x) = \binom{3p-3}{p-2} u(x)^{3p-2}$. \square

7. PROOF THAT THE MAPPINGS Φ_+ , Φ_- ARE BIJECTIONS

In this section, we prove Theorem 8 thanks to a reduction to a bijection Ψ described in [Ber07]. In this section, *the orientations are non-fractional, and the mobiles are properly bicolored*. The relation between Ψ and the master bijections Φ_+ , Φ_- involves *duality*.

Duality. The *dual* O^* of an orientation O is obtained by the process represented in Figure 8(a):

- place a vertex v_f of O^* in each face f of O ,
- for each edge e of O having a face f on its left and f' on its right, draw a *dual edge* e^* of O^* oriented from v_f to $v_{f'}$ across e .

Note that the duality (as defined above) is not an involution (applying duality twice inverse the direction of every edge). Duality maps vertex-rooted orientations to face-rooted orientations. It is easy to check that a face-rooted orientation O is minimal if and only if O^* is *accessible*, that is, accessible from the root-vertex. Similarly, a vertex-rooted orientation O is accessible if and only if O^* is *maximal*, that is, has no clockwise circuit. We denote by \mathcal{S}_d and $\tilde{\mathcal{S}}_d$ respectively the set of orientations which are the image by duality of the sets \mathcal{O}_d and $\tilde{\mathcal{O}}_d$. Thus, \mathcal{S}_d is the set of vertex-rooted orientations which are accessible, have a root-vertex of indegree 0, and which are maximal for one of the faces incident to the root-vertex (it is maximal for each of these faces in this case); see Figure 8(b). The set $\tilde{\mathcal{S}}_d$ is the subset of \mathcal{S}_d made of the orientations such that each of the faces incident to the root-vertex have *counterclockwise degree* 1 (only one edge is oriented in counterclockwise direction around these faces).

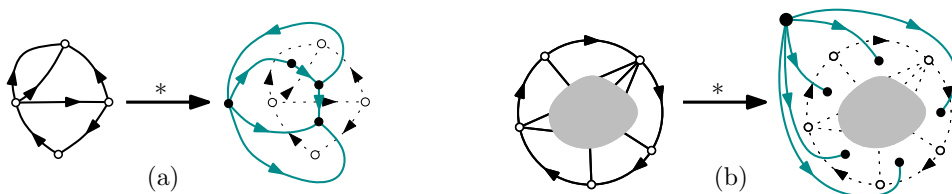
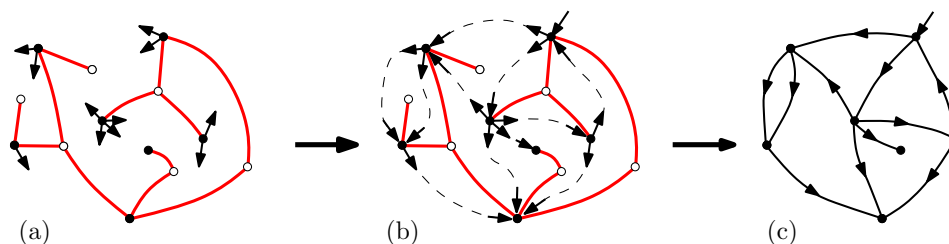


FIGURE 8. The dual of an orientation.

Partial closure and partial opening.

We now present the bijection Ψ from [Ber07] between corner-rooted maximal accessible orientations and mobile of excess 1. The presentation of Ψ below is in fact taken from [BC, Section 7].

Let M be a blossoming mobile with p edges and q outgoing buds (hence excess $\delta = p - q$). The corresponding *fully blossoming* mobile M' is obtained from M by inserting *ingoing bud* (dangling ingoing half-edge) in each of the p corners of M following an edge in counterclockwise direction around a black vertex. A fully blossoming mobile is represented in solid lines in Figure 9(b). A *counterclockwise walk* around M' (with the edges of the mobile M' on the left of the walker) sees a succession of outgoing buds and ingoing buds. Associating an opening parenthesis to outgoing buds and a closing parenthesis to ingoing buds, one obtains a cyclic

FIGURE 9. The rooted-closure of a mobile of excess $\delta = 1$.

binary word with q opening and p closing parentheses. This yields in turn a partial matching of the buds (an outgoing bud is matched with the next free ingoing bud counterclockwise around the mobile), leaving $|\delta|$ buds unmatched (the unmatched buds are ingoing if $\delta > 0$ and outgoing if $\delta < 0$).

The *partial closure* C of the blossoming mobile M is obtained by forming an oriented edge out of each matched pair of buds. Clearly the oriented edges can be formed in a planar way, and this process is uniquely defined (recall that all our maps are on the sphere). The partial closure is represented in Figure 9(a)-(b). We consider the partial closure C as a planar map with two types of edges (those of the mobile, which are non-oriented, and the new formed edges, which are oriented) and $|\delta|$ buds which are all incident to the same face, which we take to be the root-face of C . Note that, if $\delta \geq 0$, there are δ white corners incident to the root-face of C , because initially the number of such corners is equal to the number of edges of the mobile, and then each matched pair of buds decreases this number by 1. These corners, which stay incident to the root-face throughout the partial closure, are called *exposed white corners*.

Let O be an oriented map. The *partial opening* of O is the map C with vertices colored black or white and with two types of edges (oriented and non-oriented) obtained as follows.

- Color in black the vertices of O and insert a white vertex in each face of O .
- Around each vertex v of O draw a non-oriented edge from any corner c which follows an ingoing edge in clockwise order around v to the white vertex in the face containing c .

If O is corner-rooted, then the bud indicating the root-corner is interpreted as an ingoing half-edge and gives rise to an edge of C . For instance, the partial opening of the corner-rooted map in Figure 9(c) is the map in Figure 9(b).

The rooted-, positive- and negative- openings and closures

We now recall and extend the results given in [Ber07] about closures and openings. Observe that the partial closure C of a 1-mobile M has one dangling ingoing bud. The *rooted-closure* of M , denoted $\Psi(M)$, is obtained from the partial closure C by erasing every white vertex and every edge of the mobile; see Figure 9(b)-(c). The embedded graph $\Psi(M)$ is always connected (hence a map) as explained below and the dangling ingoing bud is considered as indicating the root-corner of $\Psi(M)$. The *rooted-opening* of a corner-rooted orientation O is obtained from its partial opening C by erasing all the ingoing half-edges of O (this leaves only the non-oriented edges

of C and some outgoing buds incident to black vertices). The following result was proved in [Ber07] (see also [BC]).

Theorem 28. *The rooted-closure Ψ is a bijection between 1-mobiles and corner-rooted maximal accessible orientations. The rooted-opening is the inverse mapping.*

We now present the mappings Ψ_+ and Ψ_- defined respectively on blossoming mobiles of positive and negative excess; see Figure 10.

Definition 29. *Let M be a blossoming mobile of excess $\delta \neq 0$ and let C be its partial closure.*

- *If $\delta > 0$, then C has δ ingoing buds (incident to the root-face). The positive-closure of M , denoted $\Psi_+(M)$, is obtained from C by first creating a root-vertex v of $\Psi_+(M)$ in the root-face of C and connecting it to each ingoing bud (buds thus become part of an edge of O oriented away from v); second erasing the edges and white vertices of the mobile.*
- *If $\delta < 0$, then C has δ outgoing buds (incident to the root-face). The negative-closure of M , denoted $\Psi_-(M)$, is obtained from C by first creating a root-vertex v of $\Psi_-(M)$ in the root-face of C and connecting it to each outgoing bud and then reorienting these edges (buds thus become part of an edge of $\Psi_-(M)$ oriented away from v); second erasing the edges and white vertices of the mobile.*

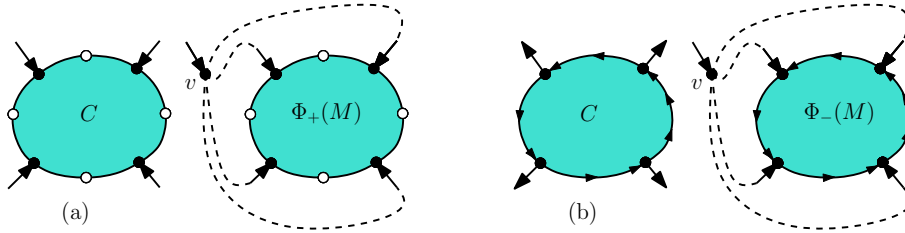


FIGURE 10. (a) Positive-closure Ψ_+ . (b) Negative-closure Ψ_- .

Theorem 30. *Let d be a positive integer.*

- *The positive-closure Ψ_+ is a bijection between the set of d -mobiles and the set \mathcal{S}_d of orientations. Moreover, the mapping defined on \mathcal{O}_d by first applying duality (thereby obtaining an orientation in \mathcal{S}_d) and then applying the inverse mapping Ψ_+^{-1} is the mapping Φ_+ defined in Definition 7. Thus, Φ_+ is a bijection between \mathcal{O}_d and the set of d -mobiles.*
- *The negative-closure Ψ_- is a bijection between the set of $(-d)$ -mobiles and the subset $\tilde{\mathcal{S}}_d$ of orientations. Moreover, the mapping defined on $\tilde{\mathcal{O}}_d$ by first applying duality (thereby obtaining an orientation in $\tilde{\mathcal{S}}_d$) and then applying the inverse mapping Ψ_-^{-1} is the mapping Φ_- defined in Definition 7. Thus, Φ_- is a bijection between $\tilde{\mathcal{O}}_d$ and the set of $(-d)$ -mobiles.*

Theorem 30 clearly implies Theorem 8 (indeed the correspondence of parameters mentioned in Theorem 8 is obvious from the definitions). It only remains to prove Theorem 30. Before doing so, we mention that a result can be stated for 0-mobiles:

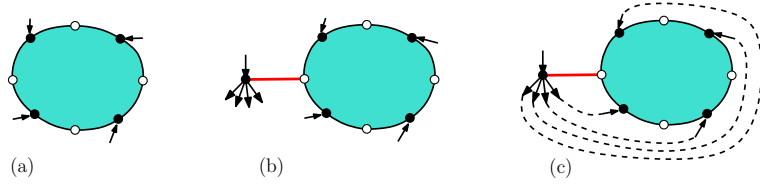


FIGURE 11. Formulation of the positive-closure Ψ_+ by reduction to the rooted-closure Ψ . Figure (a) shows generically the partial closure of a d -mobile for $d = 4$. In (b) one creates a black vertex b with d outgoing buds, and connects it to an exposed white corner. In (c) one performs the remaining matchings of buds to complete the positive-closure.

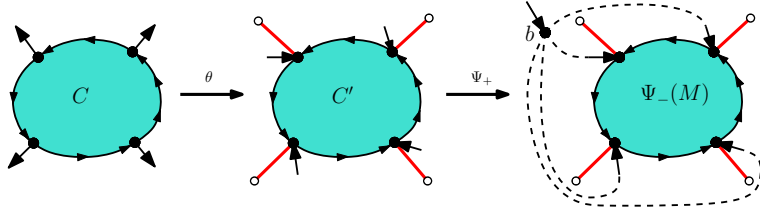


FIGURE 12. Formulation of the negative-closure Ψ_- , by reduction to the positive-closure Ψ_+ .

the θ -closure obtained by performing the partial closure and then erasing the edges of the mobile is a bijection between 0-mobiles and the set $\bigcup_{d \in \mathbb{N}} \mathcal{S}_d$.

Proof. •We first treat the case of the positive-closure Ψ_+ . We begin by showing that the positive-closure of a d -mobile M is in \mathcal{S}_d . Let C be be partial closure of M and let $O = \Psi_+(M)$ be its positive-closure. As observed above, the mobile M has $d > 0$ exposed white corners. Let M' be the blossoming mobile obtained from M by creating a new black vertex b , joining b to an exposed white corner, and adding d outgoing buds to b ; see Figure 11. The excess of M' is 1, hence by Theorem 28 the rooted-closure $O' = \Psi(M')$ is maximal and accessible. Moreover, it is easily seen (Figure 11) that b is the root-vertex of O' (because the ingoing bud incident to b is not matched during the partial closure) and has indegree 0. Thus, the orientation O is in \mathcal{S}_d .

We also make the following observation (useful for the negative-closure):

Fact. the positive closure $O = \Psi_+(M)$ is in $\tilde{\mathcal{S}}_d$ if and only if each of the exposed white corners of M is incident to a (white) leaf of M .

Indeed, a white vertex w_f of M has an exposed white corner if and only if it corresponds to a face f of O incident to the root-vertex b . Moreover, the clockwise degree of f is (as always) the degree of w_f .

We now prove that the positive-closure is a bijection by defining the inverse mapping. Let O be a vertex-rooted orientation in \mathcal{S}_d . We define the *positive-opening* of O , as the embedded graph with buds M obtained by applying the partial opening of O , and then erasing every ingoing half-edge of O as well as the root-vertex b (and

the incident outgoing half-edges). In order to prove that M is a blossoming mobile (i.e. a tree with buds), we consider a minimal accessible orientation O' obtained from O by choosing a root-corner for O among the corners incident to the root-vertex b . By Theorem 28, the rooted-opening of O' gives a blossoming mobile M' . It is clear from definitions that M is obtained from M' by erasing the root-vertex b . Moreover, b is a leaf of M' (since b is incident to no ingoing half-edge except the bud indicating the root-corner of O), hence M is a mobile. Moreover, since the rooted-closure and rooted-opening are inverse bijections, it is clear that positive-closure and positive-opening are inverse bijections.

Lastly, it is clear from the definitions that taking an orientation O in \mathcal{O}_d , and applying the positive-opening Ψ_+^{-1} to the dual orientation O^* gives the mobile $\Phi^+(O)$ as defined in Definition 7.

•We now treat the case of the negative-closure Ψ_- . Let M be a $(-d)$ -mobile. We denote by $\theta(M)$ the d -mobile obtained from M by transforming each of its d unmatched outgoing buds into an edge connected to a new white leaf. It is clear (Figure 12) that the positive-closure of $\theta(M)$ is equal to the negative-closure of M , hence $\Psi_- = \Psi_+ \circ \theta$. Moreover, θ is clearly a bijection between the set of $(-d)$ -mobiles and the set of d -mobiles such that every exposed white corner belongs to a leaf (the inverse mapping θ^{-1} replaces each edge incident to an exposed leaf by an outgoing bud). By the fact underlined above, this shows that $\Psi_- = \Psi_+ \circ \theta$ is a bijection between the set of $(-d)$ -mobiles and the set $\tilde{\mathcal{S}}_d$.

Lastly, by denoting $*$ the duality mapping, one gets from the above $\Psi_-^{-1} \circ * = \theta^{-1} \circ \Psi_+^{-1} \circ * = \theta^{-1} \circ \Phi_+$. In other words, taking an orientation O in $\tilde{\mathcal{O}}_d$, and applying the negative-opening Ψ_-^{-1} to O^* , is the same as applying the mapping Φ_+ and then replacing each edge of the mobile incident to an outer vertex by an outgoing bud. By definition, this is the same as applying the mapping Φ_- to O . This completes the proof of Theorem 30 and of Theorem 8. \square

8. ADDITIONAL REMARKS

We have presented a general bijective strategy for planar maps that relies on certain orientations with no counterclockwise circuit. We have applied the approach to an infinite collection $(\mathcal{C}_{d,p})_{d \geq 3, p \geq d}$ of families planar maps, where $\mathcal{C}_{d,p}$ denotes the set of d -angulations of girth d with a boundary of size p . For this purpose we have introduced the $d/(d-2)$ -orientations on d -angulations of girth d . In future work we shall further explore and exploit the properties of these orientations:

$d/(d-2)$ -orientations and Schnyder decompositions. Schnyder showed in [Sch89] that the 3-orientations of triangulations (case $d = 3$) can be given more structure by adding a coloring of the edges. More precisely, for each 3-orientation, it is possible to color the edges with three colors in such a way that each color defines a spanning tree of the triangulation oriented away from the outer vertices. Moreover, there is a precise rule for the crossing of the 3 spanning-trees making the correspondence between the orientations and the colorings bijective. As detailed in an article to come [BFc], a similar property holds for any $d \geq 3$. More precisely, it is shown there that given an $d/(d-2)$ -orientation of a d -angulation G (seen as an orientation of the graph $(d-2) \cdot G$) it is possible to color the edges of $(d-2) \cdot G$ with d colors, in

such a way that each color is a spanning tree oriented away from the outer vertices. Moreover, there is a precise rule for the crossing of the d spanning-trees making the correspondence between the fractional-orientations and the colorings bijective.

Extension of the bijections to planar maps of fixed girth. For each integer $d \geq 3$, we have presented a bijection for the class of d -angulations of girth d . In the articles [BFa, BFb] we show that this bijection can be extended to the class \mathcal{G}_d of all maps of girth d . The maps in \mathcal{G}_d have faces of degree at least d , and our bijection allows for a counting of the maps in \mathcal{C}_d according to the number of faces of each degree. The strategy in the articles [BFa, BFb] parallel the one initiated here: we characterize the maps in \mathcal{G}_d by certain (generalized) orientations and then obtain a bijection by specialization of the master bijections Φ_-, Φ_+ . The bijection obtained for \mathcal{G}_d associates to any map of girth d a mobile such that an inner face of degree i in the map corresponds to a black vertex of degree i in the mobile.

REFERENCES

- [BC] O. Bernardi and G. Chapuy. A bijection for covered maps, or a shortcut between Harer-Zagier's and Jackson's formulas. Submitted. ArXiv:1001.1592.
- [Ber07] O. Bernardi. Bijective counting of tree-rooted maps and shuffles of parenthesis systems. *Electron. J. Combin.*, 14(1):R9, 2007.
- [BFa] O. Bernardi and É. Fusy. Bijective counting of maps by girth and degrees I: restricted boundary conditions. Submitted.
- [BFb] O. Bernardi and É. Fusy. Bijective counting of maps by girth and degrees II: general boundary conditions. In preparation.
- [BFc] O. Bernardi and É. Fusy. Schnyder decompositions for regular plane graphs and application to drawing. Submitted.
- [BFG04] J. Bouttier, P. Di Francesco, and E. Guitter. Planar maps as labeled mobiles. *Electron. J. Combin.*, 11(1):R69, 2004.
- [BFG07] J. Bouttier, P. Di Francesco, and E. Guitter. Blocked edges on eulerian maps and mobiles: Application to spanning trees, hard particles and the Ising model. *J. Phys. A*, 40:7411–7440, 2007.
- [BMJ06] M. Bousquet-Mélou and A. Jehanne. Polynomial equations with one catalytic variable, algebraic series and map enumeration. *J. Combin. Theory Ser. B*, 96(5):623 – 672, 2006.
- [BMS02] M. Bousquet-Mélou and G. Schaeffer. The degree distribution in bipartite planar maps: applications to the Ising model, 2002. arXiv:math.CO/0211070.
- [Bro64] W.G. Brown. Enumeration of triangulations of the disk. *Proc. London Math. Soc.*, 14(3):746–768, 1964.
- [Bro65] W.G. Brown. Enumeration of quadrangular dissections of the disk. *Canad. J. Math.*, 21:302–317, 1965.
- [CS04] Philippe Chassaing and Gilles Schaeffer. Random planar lattices and integrated superBrownian excursion. *Probab. Theory Related Fields*, 128(2):161–212, 2004.
- [CV81] R. Cori and B. Vauquelin. Planar maps are well labeled trees. *Canad. J. Math.*, 33(5):1023–1042, 1981.
- [dFOdM01] H. de Fraysseix and P. Ossona de Mendez. On topological aspects of orientations. *Discrete Math.*, 229:57–72, 2001.
- [Fel04] S. Felsner. Lattice structures from planar graphs. *Electron. J. Combin.*, 11(1), 2004.
- [FPS08] É. Fusy, D. Poulalhon, and G. Schaeffer. Dissections, orientations, and trees, with applications to optimal mesh encoding and to random sampling. *Transactions on Algorithms*, 4(2):Art. 19, 2008.
- [FS09] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, New York, NY, USA, 2009.

- [Fus07] É. Fusy. *Combinatoire des cartes planaires et applications algorithmiques*. PhD thesis, École Polytechnique, 2007.
- [GJ83] I. P. Goulden and D. M. Jackson. *Combinatorial Enumeration*. John Wiley, New York, 1983.
- [PS06] D. Poulalhon and G. Schaeffer. Optimal coding and sampling of triangulations. *Algorithmica*, 46(3-4):505–527, 2006.
- [Sch89] W. Schnyder. Planar graphs and poset dimension. *Order*, 5(4):323–343, 1989.
- [Sch98] G. Schaeffer. *Conjugaison d'arbres et cartes combinatoires aléatoires*. PhD thesis, Université Bordeaux I, 1998.
- [Tut63] W. T. Tutte. A census of planar maps. *Canad. J. Math.*, 15:249–271, 1963.