



A bijection for triangulations, quadrangulations, pentagulations, etc.

Olivier Bernardi, Eric Fusy

► To cite this version:

Olivier Bernardi, Eric Fusy. A bijection for triangulations, quadrangulations, pentagulations, etc.. 2010. hal-00498615v3

HAL Id: hal-00498615

<https://hal.science/hal-00498615v3>

Preprint submitted on 1 Oct 2010 (v3), last revised 20 Aug 2011 (v6)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A BIJECTION FOR TRIANGULATIONS, QUADRANGULATIONS, PENTAGULATIONS, ETC.

OLIVIER BERNARDI* AND ÉRIC FUSY†

ABSTRACT. A d -angulation is a planar map with faces of degree d . We present for each integer $d \geq 3$ a bijection between the class of d -angulations of girth d and a class of decorated plane trees. Each of the bijections is obtained by specializing a “master bijection” which extends an earlier construction of the first author. Bijections already existed for triangulations ($d = 3$) and for quadrangulations ($d = 4$). As a matter of fact, our construction unifies a bijection by Fusy, Poulalhon and Schaeffer for triangulations and a bijection by Schaeffer for quadrangulations. For $d \geq 5$, both the bijections and the enumerative results are new.

We also extend our bijections so as to enumerate p -gonal d -angulations, that is, d -angulations with a simple boundary of length p . We thereby recover bijectively the results of Brown for p -gonal triangulations and quadrangulations and establish new results for $d \geq 5$.

A key ingredient in our proofs is a class of orientations characterizing d -angulations of girth d . Earlier results by Schnyder and by De Fraisseyx and Ossona de Mendez showed that triangulations of girth 3 and quadrangulations of girth 4 are characterized by the existence of orientations having respectively indegree 3 and 2 at each inner vertex. We extend this characterization by showing that a d -angulation has girth d if and only if the graph obtained by duplicating each edge $d - 2$ times admits an orientation having indegree d at each inner vertex.

1. INTRODUCTION

The enumeration of planar maps (connected planar graphs embedded in the sphere) has received a lot of attention since the seminal work of Tutte in the 60’s [Tut63]. Tutte’s method for counting a class of maps consists in translating a recursive description of the class (typically obtained by deleting an edge) into a functional equation satisfied by the corresponding generating function. The translation usually requires to introduce a “catalytic” variable, and the functional equation is solved using the so-called “quadratic method” [GJ83, sec.2.9] or its extensions [BMJ06]. The final result is, for many classes of maps, a strikingly simple counting formula. For instance, the number of maps with n edges is $\frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$. Tutte’s method has the advantage of being systematic, but is quite technical in the way of solving the equations and does not give a combinatorial understanding of the simple-looking enumerative formulas. Other methods for the enumeration of

*Dept. of Mathematics, MIT, 77 Massachusetts Avenue, Cambridge MA 02139, USA, bernardi@math.mit.edu. Supported by the French ANR project A3 and the European project ExploreMaps – ERC StG 208471

†LIX, École Polytechnique, Palaiseau, fusy@lix.polytechnique.fr. Supported by the European project ExploreMaps – ERC StG 208471.

maps were later developed based either on matrices integrals, representations of the symmetric group, or bijections.

The bijective approach to maps has the advantage of giving deep insights to the combinatorial properties of maps. This approach was used to solve several statistical physics models (Ising, hard particles) on random lattices [BMS02, BFG07], and to investigate the metric properties of random maps [CS04, BFG04]. It also has nice algorithmic applications (random generation and asymptotically optimal encoding in linear time). The first bijections appeared in [CV81], and later in [Sch98] where a more transparent construction was proposed and applied to several classes of maps. Typically bijections are from a class of “decorated” plane trees to a class of maps and operate on trees by progressively closing facial cycles. However, even if the bijective approach was successfully applied to many classes of maps, it is not, up to now, as systematic as Tutte’s recursive method. Indeed, for each class of maps, one has to guess (using counting results) the corresponding family of decorated trees, and then to invent a mapping between trees and maps.

In this article, we contribute to making the bijective approach more systematic by showing that a single “master bijection” can be specialized in various ways so as to obtain bijections for several classes of maps. The master bijection Φ presented in Section 3 is an extension of a construction by the first author [Ber07]. It was already shown in [BC], that the bijection Φ can be specialized so as to obtain a famous bijection by Bouttier, Di Francesco, and Guitter [BFG04] between bipartite maps and certain decorated plane trees called *mobiles*. To be more exact, the bijection Φ is between a set \mathcal{O} of *oriented maps* and a set of mobiles. It turns out that for many classes of maps there is a canonical way of orienting the maps in the class \mathcal{C} , so that \mathcal{C} is identified with a subfamily $\mathcal{O}_{\mathcal{C}}$ of \mathcal{O} on which our bijection Φ restricts nicely. Typically, the oriented maps in $\mathcal{O}_{\mathcal{C}}$ are characterized by degree constraints which can be traced through our construction and yields a degree characterization of the associated mobiles. The mobiles family is then specifiable by a decomposition grammar and amenable to the Lagrange inversion formula for counting. To summarize, the bijective approach presented here is more systematic, since it consists in specializing the master bijection Φ to several classes of maps. The problem of enumerating a class of map \mathcal{C} then reduces to guessing a family of “canonical” orientations (in \mathcal{O}) for \mathcal{C} (instead of guessing a family of decorated trees and a bijection, which is harder).

We apply our bijective strategy to an infinite number of classes of maps. More precisely, we consider for each integer $d \geq 3$ the class \mathcal{C}_d of *d-angulations* (planar maps with faces of degree d) of girth d (i.e., having no cycle of length less than d). The family \mathcal{C}_3 corresponds to the *simple* triangulations (those that have no loop nor multiple edges). The family \mathcal{C}_4 also correspond to the class of *simple* quadrangulations because quadrangulations are bipartite (hence never have cycles of odd length). We show in Section 4 that each class \mathcal{C}_d is amenable to our bijective strategy: we obtain a bijection between the class \mathcal{C}_d and a class of mobiles by specializing the bijection Φ . The class of mobiles is characterized by certain degree conditions, hence the generating function of the class \mathcal{C}_d is characterized and shown to be algebraic. Bijections already existed for the class \mathcal{C}_3 of simple triangulations [PS06, FPS08] and the class \mathcal{C}_4 of simple quadrangulations [Fus07, Sch98]. Our approach actually coincides (and unify) the bijections presented respectively

in [FPS08, Theo.4.10] for triangulations and in [Sch98, Sec.2.3.3] for quadrangulations.

In Section 5, we show that our bijective strategy applies even more generally d -angulations with boundary of length $p \geq d$. More precisely, we give bijections for the classes of p -gonal d -angulations (maps with inner vertices of degree d and outer-face of degree p which is simple) of girth d . In particular, we recover bijectively the enumerative results obtained by Brown (via a recursive approach *à la Tutte*) for the cases $d = 3, 4$ [Bro64, Bro65]. That is, we show that the numbers $t_{p,n}$ of p -gonal simple triangulations with $p + n$ vertices and the numbers $q_{p,n}$ of $2p$ -gonal simple quadrangulations with $2p + n$ vertices are

$$t_{p,n} = \frac{2(2p-3)!}{(p-1)!(p-3)!} \frac{(4n+2p-5)!}{n!(3n+2p-3)!}, \quad q_{p,n} = \frac{3(3p-2)!}{(p-2)!(2p-1)!} \frac{(3n+3p-4)!}{n!(2n+3p-2)!}.$$

A bijection (different from ours) for simple triangulations with boundary was already presented in [PS06], but the proof of the bijectivity was based on the results of Brown (only injectivity was proved and the bijectivity followed by a cardinality argument).

Our counting results are new for $d \geq 5$. Indeed, due to the constraint on the girth, it seems tedious to apply a recursive method *à la Tutte* to d -angulations for $d \geq 5$. Even though there does not seem to be closed-form counting formulas for these cases (for $d = 5$ the counting coefficients have large prime factors), the generating functions follow an amazingly simple uniform algebraic pattern, see Propositions 19 and 20.

Before closing this introduction, we point out one of the key ingredients used in our bijective strategy. As explained above, for a class \mathcal{C} of maps, our bijective strategy requires to specify a canonical orientation in order to identify \mathcal{C} with a subclass $\mathcal{O}_{\mathcal{C}} \subset \mathcal{O}$ of oriented maps. For d -angulations with a simple outer face, a simple calculation based on the Euler relation shows that the numbers n and m of inner vertices and edges are related by $m = \frac{d}{d-2}n$. This suggests a candidate canonical orientation for a d -angulation G : just ask for an orientation (without counterclockwise directed cycle) with indegree $\frac{d}{d-2}$ at each inner vertex... or more reasonably an orientation of the graph $(d-2) \cdot G$ (the graph obtained from G replacing each edge by $(d-2)$ parallel edges) with indegree d at each inner vertex. We show that such an orientation exists for a d -angulation if and only if it has girth d (and use these orientations to identify the class \mathcal{C}_d with a subfamily $\mathcal{O}_{\mathcal{C}_d} \subset \mathcal{O}$). This extends earlier results by Schnyder for triangulations [Sch89] and by De Fraisseyx and Ossona de Mendez for quadrangulations [dFOdM01].

2. MAPS AND ORIENTATIONS

In this section we gather our definitions about maps, orientations and duality.

Maps. A (planar) *map* is a connected planar graph embedded in the oriented sphere and considered up to continuous deformation. The *faces* are the connected components of the complementary of the graph. A *plane tree* is a map without cycles (it has a unique face). Cutting an edge e at its middle point gives two half-edges, each incident to an endpoint of e (they are both incident to the same vertex if e is a loop). We shall also consider some maps decorated with dangling

half-edges called *buds* (see e.g. Figure 1). A *corner* is the angular section between two consecutive half-edges around a vertex. The degree of a vertex or face is the number of incident corners. A *d-angulation* is a map such that every face has degree d . *Triangulations* and *quadrangulations* correspond to the cases $d = 3$ and $d = 4$ respectively. The *girth* of a map is the minimal length of its cycles. Obviously a *d-angulation* has girth at most d (except if it is a tree). A map is *simple* if it has girth at least 3 (i.e., no loop nor multiple edge).

The numbers v , e , f of vertices, edges and faces of a map are related by the *Euler relation*: $v - e + f = 2$. We shall also use the following result.

Lemma 1. *If a map with v vertices and e edges has a face f_0 of degree p and the other faces of degree at least d , then*

$$e - p \leq \frac{d}{d-2}(v - p) + \frac{p-d}{d-2},$$

with equality if and only if the faces other than f_0 have degree exactly d .

Proof. The incidence relation between faces and edges gives $2e \geq p + d(f-1)$, with equality if the faces other than f_0 have degree exactly d . Combining this with Euler relation (so as to eliminate f) gives the claim. \square

A map is said to be *vertex-rooted* if a vertex is marked, *face-rooted* if a face is marked, and *corner-rooted* if a corner is marked¹. The marked vertex, face or corner are called the *root-vertex*, *root-face* or *root-corner*. For a corner-rooted map, the marked corner is indicated by a dangling half-edge pointing to that corner; see Figure 6. A corner-rooted map is said to *induce* the vertex-rooted map (resp. face-rooted map) obtained by keeping the root-vertex (resp. root-face) as marked, but otherwise forgetting the root-corner. Given a face-rooted (or corner-rooted) map, vertices and edges are said to be *outer* or *inner* depending on whether they are incident to the root-face or not.

Orientations. A *planar orientation* O , or *orientation* for short, is a pair made of a map and of the choice of a direction for each edge of the map. An orientation is *vertex-*, *face-*, or *corner-rooted* if the underlying map is. The *indegree* of a vertex v , denoted $\text{in}(v)$, is the number of edges oriented toward v . The *clockwise-degree* of a face f , denoted $\text{cw}(f)$, is the number of edges incident to f having f on their right. An orientation is *accessible* from a vertex v if any vertex is reachable from v by a directed path. A *circuit* is a directed cycle. A circuit of a face-rooted (or corner-rooted) orientation is said *clockwise* if the root-face is on its left and *counterclockwise* otherwise. The orientation is *minimal* if it has no counterclockwise circuit; it is *clockwise-minimal* if in addition the root-face is a (clockwise) circuit.

Fractional Orientations. Let k be a positive integer and G be a map. A *k-fractional orientation* of G is an attribution of a *flow* in $\{0, \frac{1}{k}, \frac{2}{k}, \dots, 1\}$ to each direction of each edge in such a way that the two flows in opposite direction of an edge sum to 1. Observe that the 1-orientations of G can be identified with its non-fractional orientations by orienting each edge of G in the direction of the flow 1.

¹Corner-rooted map are usually simply called *rooted maps* in the literature. A face-rooted map can be thought as a *plane map* (a connected graph embedded in the plane) by thinking of the root-face as the infinite face.

More generally, the k -fractional orientations of a face-rooted map can be identified with some orientations of the map $k \cdot G$ obtained from G by replacing each edge by k parallel edges. More precisely, the correspondence represented in Figure 4(a) is a bijection between the k -fractional orientations of G and the orientations of $k \cdot G$ respecting the *local minimality property*: no counterclockwise circuit of $k \cdot G$ is formed by two edges arising from the same edge of G . An edge of G is *saturated* if the flow is 1 in one direction (i.e., the k parallel edges of $k \cdot G$ all have the same direction). The *indegree* $\text{in}(v)$ of a vertex v of G is the sum of the flows toward v over all the edges incident to v . The *clockwise-degree* of a face f in G , denoted $\text{cw}(f)$, is the number of saturated edges incident to f having f on their right.

Let O be a k -fractional orientation. A *directed path* of O is a directed path of the underlying map such that each arc of the path has a positive flow. The orientation O is said *accessible* from a vertex v if any vertex is reachable from v by a directed path. A *clockwise circuit* of O is a simple cycle C of the underlying (face-rooted or corner-rooted) map G such that the flow of each edges is positive in the clockwise direction of C . A *counterclockwise circuit* is defined similarly. The face-rooted fractional orientation O is said *minimal* if it has no counterclockwise circuit (equivalently, the corresponding orientation of $k \cdot G$ is minimal). The following lemma is a straightforward extension (to the fractional setting) of a classical results for non-fractional orientations (see e.g. [Fel04]).

Lemma 2. *Let O be a k -fractional orientation of a face-rooted map G . Then G has a unique minimal k -fractional orientation O' such that all vertices have same indegrees in O as in O' .*

We now define some classes of orientations which are of importance in our bijections. A minimal orientation (fractional or not) is said *clockwise-minimal* if the root-face is a clockwise circuit. This orientation is said *accessible* if it is accessible from one of the outer vertices (in this case, O is in fact accessible from any outer vertex because the root-face is a circuit). We denote by \mathcal{O}_d (resp. \mathcal{O}_d^k) the set of non-fractional (resp. k -fractional) clockwise-minimal accessible orientations such that the root-face has degree d . We denote by $\tilde{\mathcal{O}}_d \subset \mathcal{O}_d$ (resp. $\tilde{\mathcal{O}}_d^k \subset \mathcal{O}_d^k$) the subset of these orientations such that every outer vertex has indegree 1. Observe that the outer face of an orientation in $\tilde{\mathcal{O}}_d^k$ is a simple cycle made of saturated edges oriented clockwise (since the outer vertices have indegree 1, this is the only way of avoiding a counterclockwise circuit) and that the inner edges incident to outer vertices are all saturated away from these vertices. Examples of orientations in $\mathcal{O}_5, \tilde{\mathcal{O}}_5, \mathcal{O}_4^3$ and $\tilde{\mathcal{O}}_4^3$ are represented in Figures 1 and 3.

3. MASTER BIJECTIONS BETWEEN ORIENTED MAPS AND MOBILES

In this section we describe the two “master bijections” Φ_- , Φ_+ . These bijections will be specialized respectively in Sections 4 and 5 in order to obtain bijections for d -angulations without and with boundary.

3.1. Master bijections Φ_+ , Φ_- in the non-fractional setting.

A *mobile* is a non-rooted plane tree with a proper coloring of its vertices in black and white (*proper* means that every edge join a white and a black vertex). A *blossoming mobile* is a mobile together with some *outgoing buds* (dangling outgoing half-edges)

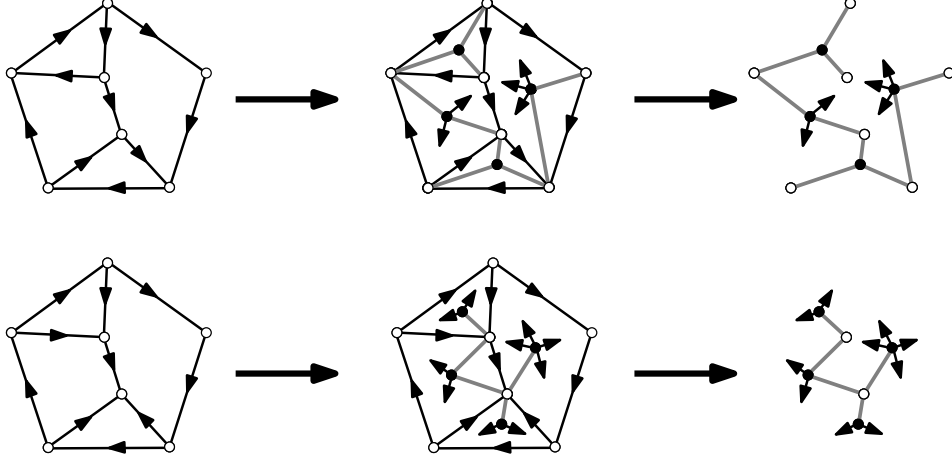


FIGURE 1. Top: Mapping Φ_+ applied to an orientation in \mathcal{O}_5 .
Bottom: Mapping Φ_- applied to an orientation in $\tilde{\mathcal{O}}_5$.

incident to black vertices. An example is shown in Figure 1(right). The *excess* of a blossoming mobile is the number of edges minus the number of outgoing buds. A mobile with excess δ is called a δ -mobile. The *degree* $\deg(v)$ of a vertex v of a blossoming mobile is the total number of incident edges and outgoing buds. The *indegree* $\text{in}(b)$ of a black vertex b is the number of incident edges.

We now define the mappings Φ_+ , Φ_- on orientations. Let O be a face-rooted orientation. A corner c of an inner face f is *clockwise* if the edge preceding c in clockwise order around f is oriented toward c . Observe that the number of clockwise corners of a face equals its clockwise degree. A clockwise corner c is called *inner-clockwise* if the edge preceding c is an inner edge.

Definition 3. Let O be a face-rooted orientation in \mathcal{O}_d (that is, O is clockwise-minimal and accessible). We consider the vertices of O as white and define the mappings Φ_+ , Φ_- illustrated in Figure 1 as follows.

- The embedded graph $\Phi_+(O)$ with black and white vertices is obtained by the following procedure. Insert a black vertex b in each inner face f of O . For each corner c of f , join the black vertex b to c by a non-oriented edge if c is a clockwise corner, and glue an outgoing bud to b otherwise (the order of edges and buds around b should match the order of clockwise and non-clockwise corners around f). Finally, delete the edges of O .
- The embedded graph $\Phi_-(O)$ is defined similarly except clockwise corner is replaced by inner-clockwise corner, and the outer vertices of O are erased at the end of the process (note that these vertices were isolated if $O \in \tilde{\mathcal{O}}_d$).

The following theorem is partly based on the results in [Ber07] and its proof is delayed until Section 7.

Theorem 4. Let d be a positive integer.

- The mapping Φ_+ is a bijection between the set \mathcal{O}_d of clockwise-minimal accessible orientations and the set of d -mobiles.

- The mapping Φ_- is a bijection between the subset $\tilde{\mathcal{O}}_d$ of clockwise-minimal accessible orientations and the set of $(-d)$ -mobiles.

Moreover, the parameter-correspondence for the bijections Φ_+ , Φ_- is shown in Figure 2.

orientation in \mathcal{O}_d	$\xleftrightarrow{\Phi_+}$	d -mobile
inner face f	$\xleftrightarrow[\text{cw}(f)=\text{in}(b)]{\text{deg}(f)=\text{deg}(b)}$	black vertex b
vertex v	$\xleftrightarrow{\text{in}(v)=\text{deg}(w)}$	white vertex w

orientation in \mathcal{O}_d	$\xleftrightarrow{\Phi_-}$	$(-d)$ -mobile
inner face f	$\xleftrightarrow[\text{cw}(f)-\epsilon(f)=\text{in}(b)]{\text{deg}(f)=\text{deg}(b)}$	black vertex b
inner vertex v	$\xleftrightarrow{\text{in}(v)=\text{deg}(w)}$	white vertex w

FIGURE 2. Parameter correspondence for the bijections Φ_+ , Φ_- , where $\epsilon(f)$ is the number of outer edges incident to the face f .

3.2. Master bijections Φ_+ , Φ_- in the fractional setting.

We now extend the master bijections Φ_+ , Φ_- to the fractional setting. A k -fractional mobile is a plane tree with vertices colored either black or white such that

- there is no black-black edge,
- the black-white edges are non-oriented,
- the white-white edges have a k -fractional orientation which is non-saturated (i.e., the flow in each direction is in $\{\frac{1}{k}, \dots, \frac{k-1}{k}\}$).

Observe that the notions of mobiles and 1-fractional mobiles coincide. A k -fractional mobile is *blossoming* if some outgoing buds are glued to black corners. The *excess* is the number of black-white edges, plus twice the number of white-white edges, minus the number of outgoing buds. A k -fractional blossoming mobile of excess δ is called a k -fractional δ -mobile. A 3-fractional 4-mobile and a 3-fractional (-4) -mobile are represented in Figure 3(right). As before, the *indegree* $\text{in}(b)$ of a black vertex b is the number of incident edges and its *degree* $\text{deg}(b)$ is this number plus the number of incident outgoing buds. The *degree* $\text{deg}(w)$ of a white vertex w is the number of incident black-white edges plus the sum of the flows (of the white-white edges) oriented toward w .

We now extend Theorem 4 to the k -fractional setting. A corner of an inner face f of O is said *clockwise* if the edge preceding c in clockwise order around f is saturated toward c . Observe that the number of clockwise corners of a face f equals its clockwise degree $\text{cw}(f)$. A clockwise corner c is called *inner-clockwise* if the edge preceding c is an inner edge, and *outer-clockwise* otherwise. For a k -fractional orientation O in \mathcal{O}_d^k the embedded graphs $\Phi_+(O)$ and $\Phi_-(O)$ are defined in the fractional setting exactly as in Definition 3 (non-fractional setting) except that at the last step, *only the saturated edges of O are deleted*. The mappings Φ_+ and Φ_- are illustrated in Figure 3.

Theorem 5. *Let d, k be positive integers.*

- The mapping Φ_+ is a bijection between the set \mathcal{O}_d^k of k -fractional clockwise-minimal accessible orientations and the set of k -fractional d -mobiles.
- The mapping Φ_- is a bijection between the subset $\tilde{\mathcal{O}}_d^k$ of k -fractional clockwise-minimal accessible orientations and the set of k -fractional $(-d)$ -mobiles.

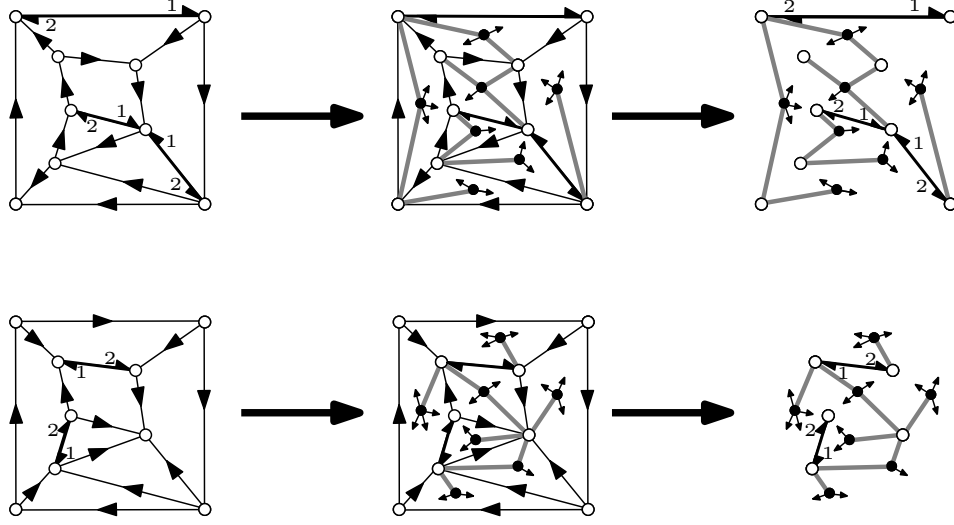


FIGURE 3. Top: Master bijection Φ_+ applied to a 3-fractional orientation in \mathcal{O}_4^3 . Bottom: Φ_- applied to an orientation in $\tilde{\mathcal{O}}_4^3$. The flows have been indicated only for non-saturated edges and multiplied by 3.

Moreover, the parameter-correspondence for the bijections Φ_+ , Φ_- is shown in Figure 2 (where \mathcal{O}_d and $\tilde{\mathcal{O}}_d$ should be replaced by \mathcal{O}_d^k and $\tilde{\mathcal{O}}_d^k$ respectively).

Before proving Theorem 5 we state an additional claim regarding corner-rooted orientations, which is useful for counting purposes. For any orientation O in $\tilde{\mathcal{O}}_d^k$, we call *exposed* the outgoing buds of the mobile $M = \Phi_-(O)$ corresponding to the outer-clockwise corners of O .

Claim 6. *Let d, k be positive integers, let O be an orientation in $\tilde{\mathcal{O}}_d^k$ and let $M = \Phi_-(O)$. There is a bijection between the set $\vec{\mathcal{O}}$ of corner-rooted maps inducing the face-rooted map O (i.e., the maps obtained by choosing a root-corner in the root-face of O), and the set \vec{M} of mobiles obtained from M by marking an exposed bud. Moreover, there is a bijection between the set M_{\rightarrow} of mobiles obtained from M by marking a non-exposed bud, and the set $M_{\rightarrow\circ}$ of mobiles obtained from M by marking a half-edge incident to a white vertex.*

We point out that there is a little subtlety in the Claim 6 related to the eventual symmetries (for instance, it could happen that the d corners of the root-face of O give less than d different corner-rooted maps).

Proof of Claim 6. There is an obvious bijection γ between the d corners in the root-face of O and the d exposed buds of M (since both sets are in natural bijection with the outer edges of O). The bijection γ induces a bijection between $\vec{\mathcal{O}}$ and \vec{M} since the definition of γ does not require any “symmetry breaking” convention (so that it respects the eventual symmetries of O and M). Let us now prove the second bijection. Let B be the set of non-exposed bud of M , and let H be the set of half-edges incident to a white vertex. We can define a bijection γ' between B

and H as follows: each saturated inner edge of O gives rise to a bud $b \in B$ (on the right side of e) and to a black-white edge e of M so we set $\gamma'(b) = h$ where h is the half-edge of e incident to the white vertex; each non-saturated inner edge e of O gives rise to two buds $b, b' \in B$ (one on each side) and we set $\gamma'(b) = h, \gamma'(b') = h'$, where h, h' are the two halves of e . The bijection γ' induces a bijection between M_{\rightarrow} and $M_{\rightarrow\circ}$ since the definition of γ' does not require any “symmetry breaking” convention (so that it respects the eventual symmetries of M). \square

Proof of Theorem 5. We call *marked orientation* a non-fractional face-rooted orientation with some *marked* faces of degree 2. We call *marked mobile* a non-fractional blossoming mobiles with some *marked* black vertices of degree 2. By Theorem 4 the marked orientations in \mathcal{O}_d (resp. $\tilde{\mathcal{O}}_d$) are in bijection by Φ_+ (resp. Φ_-) with marked d -mobiles (resp. $(-d)$ -mobiles). We will now prove Theorem 5 by identifying k -fractional orientations and k -fractional mobiles respectively with certain marked orientations and mobiles; see Figure 4.

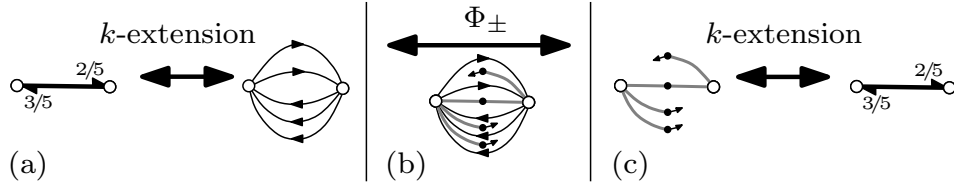


FIGURE 4. Reduction from the fractional to the non-fractional setting.

The k -extension of a face-rooted k -fractional orientation O is the marked orientation obtained by replacing each non-saturated edge e as indicated in Figure 4(a): if e has endpoint u, v with flow i/k toward u , then it is replaced by i edges oriented from v to u and $k - i$ edges oriented from u to v so as to respect the *local minimality condition* (no counterclockwise cycle), and the $k - 1$ created faces of degree 2 are marked. We call *marked block* of a marked orientation a maximal set of adjacent marked faces. We denote $\mathcal{O}^{(k)}$ the set of marked orientations such that every marked block has size $k - 1$ and respects the local minimality condition (no counterclockwise cycle inside the block). It is clear that the k -extension is a bijection between the set of k -fractional face-rooted orientations and $\mathcal{O}^{(k)}$. Moreover, a fractional orientation is in \mathcal{O}_d^k (resp. $\tilde{\mathcal{O}}_d^k$) if and only if its k -extension is in \mathcal{O}_d (resp. $\tilde{\mathcal{O}}_d$). Thus, the k -extension is a bijection between the sets \mathcal{O}_d^k and $\mathcal{O}_d \cap \mathcal{O}^{(k)}$ (resp. $\tilde{\mathcal{O}}_d^k$ and $\tilde{\mathcal{O}}_d \cap \mathcal{O}^{(k)}$).

We now consider a k -fractional blossoming mobile M . The k -extension of M is the marked mobile obtained by replacing each white-white edge e of M as indicated in Figure 4(c): if e has endpoints u, v and flow i/k toward u , then

- join u (resp. v) to $i - 1$ (resp. $k - i - 1$) new marked black vertices by edges placed just after the edge e in clockwise order around u (resp. v), and glue an outgoing bud to each of the new marked vertices,
- erase the orientation of e and insert a marked black vertex at its middle.

We call *1-vertex* or *2-vertex* respectively a marked vertex of a mobile incident to 1 or 2 edges. A *marked block* is a maximal set made of a 2-vertex b , some 1-vertices $b_1, \dots, b_r, b'_1, \dots, b'_s$ and their adjacent edges e_0, e'_0 incident to b and e_i (resp. e'_i)

incident to b_i (resp. b'_i), such that the edges e_0, e_1, \dots, e_r (resp. e'_0, e'_1, \dots, e'_r) are incident to the same white vertex and appear consecutively in this clockwise order around it. Observe that the marked blocks of a marked mobile are always disjoint. We denote by $\mathcal{M}^{(k)}$ the set of marked mobiles such that the every marked vertex belongs to a marked block, and these marked blocks contain $k - 1$ marked vertices. By definition, the k -extension of a mobile replaces each white-white edge by a marked-block containing $k - 1$ marked vertices. Therefore, the k -extension is a bijection between the set of k -fractional mobiles and the set $\mathcal{M}^{(k)}$. We call k -contraction the inverse bijection defined on $\mathcal{M}^{(k)}$ which replaces each marked block of the mobile by a non-saturated white-white edge.

It is clear from Figure 4(b) that the non-fractional version of the bijections Φ_+, Φ_- transform marked blocks (with $k - 1$ marked faces) of a marked orientation into marked blocks (with $k - 1$ marked vertices) of the corresponding marked mobile. Therefore, by Theorem 4, the mapping Φ_+ (resp. Φ_-) gives a bijections between the set $\mathcal{O}_d \cap \mathcal{O}^{(k)}$ (resp. $\tilde{\mathcal{O}}_d \cap \mathcal{O}^{(k)}$) of marked orientations and the set of d -mobiles (resp. $(-d)$ -mobiles) in $\mathcal{M}^{(k)}$. Moreover, it is clear from the definitions, that the image $\Phi_+(O)$ (resp. $\Phi_-(O)$) of a fractional orientation O in \mathcal{O}_d^k (resp. $\tilde{\mathcal{O}}_d^k$) by the fractional version of the mapping Φ_+ (resp. Φ_-) is obtained by first applying the k -extension to O , then applying the non-fractional version of the bijections Φ_+ (resp. Φ_-), and lastly applying the k -contraction to the resulting marked mobile. Therefore the mapping Φ_+ (resp. Φ_-) is a bijection between the set \mathcal{O}_d^k (resp. $\tilde{\mathcal{O}}_d^k$) and the set of k -fractional d -mobiles (resp. $(-d)$ -mobiles). Moreover, the parameter correspondence is clear from the definitions. This completes the proof of Theorem 5. \square

4. BIJECTIONS FOR d -ANGULATIONS OF GIRTH d

In this section, we prove the existence of a canonical class of fractional orientations for d -angulations of girth d . This allow us to identify the class \mathcal{C}_d of d -angulations of girth d with a class of orientations in $\tilde{\mathcal{O}}_d$. We then obtain a bijection between \mathcal{C}_d and a class of blossoming mobiles by specializing of the master bijection Φ_- to these orientations.

4.1. Fractional orientations for d -angulations of girth d .

Let D be a face-rooted d -angulation with e edges and v vertices. By Lemma 1, these numbers satisfy $e - d = \frac{d}{d-2}(v - d)$. Moreover, if the root-face of D is *simple* (its boundary is a simple cycle), then $e - d$ and $v - d$ are respectively the number of inner edges and inner vertices. Thus, it is tempting to look for a fractional orientation of the inner edges giving indegree $\frac{d}{d-2}$ at each inner vertex. More precisely, we define a $\frac{d}{d-2}$ -orientation of a face-rooted d -angulation D as a $(d - 2)$ -fractional orientation of D such that each inner vertex has indegree $\frac{d}{d-2}$ and each outer vertex has indegree 1. By Lemma 2, if D admits a $\frac{d}{d-2}$ -orientation, then there exists a unique minimal one. A minimal $\frac{d}{d-2}$ -orientation is represented in Figure 5(top-left).

Theorem 7. *A face-rooted d -angulation admits a $\frac{d}{d-2}$ -orientation if and only if it has girth d . In this case, the minimal $\frac{d}{d-2}$ -orientation is in $\tilde{\mathcal{O}}_d^{d-2}$.*

Before proving Theorem 7 we recall a classical results about the existence of (non-fractional) orientations with prescribed indegrees (see e.g. [Fel04]). Given a graph G and a function α from the vertices to the integers, an α -orientation is a (non-fractional) orientation with indegree $\alpha(v)$ at each vertex v .

Lemma 8. *Let $G = (V, E)$ and let α be a function from the vertices to the integers. There exists an α -orientation if and only if*

- (i) $\sum_{v \in V} \alpha(v) = |E|$
- (ii) $\sum_{v \in S} \alpha(v) \geq |E_S|$ for all subset S of vertices, where E_S is the set of edges with both ends in S .

Moreover, for a vertex v_0 , the following condition ensures that any α -orientation is accessible from v_0 .

- (iii) $\sum_{v \in S} \alpha(v) > |E_S|$ for all subset S of vertices not containing v_0 .

Proof of Theorem 7. We first prove the necessity of the girth D . Suppose that a d -angulation D has not girth D . The first possibility is that D is a tree (its girth is ∞), and in this case there can be no $\frac{d}{d-2}$ -orientation since the number of edges $e = v - 1$ is not sufficient to satisfy the indegree conditions. We now treat the other possibility: the girth is $c < d$. We consider a simple cycle C of length c and the submap D' of D enclosed by C which does not contain the root-face of D . Let V', E' be respectively the set of vertices and edges strictly inside C . By Lemma 1, $|E'| = \frac{d}{d-2}|V'| + \frac{c-d}{d-2} < \frac{d}{d-2}|V'|$. However, if D has a $\frac{d}{d-2}$ -orientation, then the sum of the indegree of the vertices in $|V'|$ is $\frac{d}{d-2}|V'|$. This is impossible by the previous inequality (since each edge in E' contributes at most 1), hence D has no $\frac{d}{d-2}$ -orientation.

We now consider a d -angulation D of girth d and want to prove that D has a $\frac{d}{d-2}$ -orientation. Let α be the function taking value $\frac{d}{d-2}$ on inner vertices and 1 on outer vertices. Clearly, there exists a $\frac{d}{d-2}$ -orientation of D if and only if there is a $(d-2)\alpha$ -orientation of $(d-2) \cdot D$. Thus, we only need to check the Properties (i), (ii) of Lemma 8 for $G = D = (V, E)$. Moreover, for a vertex v_0 , the accessibility of the $\frac{d}{d-2}$ -orientations from v_0 is granted if (iii) holds. Since D has girth d , its outer face is a simple cycle: there are d outer vertices. Hence $\sum_{v \in V} \alpha(v) = d + \frac{d}{d-2}(|V| - d)$, and this is equal to $|E|$ by Lemma 1. Thus, Property (i) holds. Let S be a subset of vertices. We first suppose that the induced subgraph $G_S = (S, E_S)$ is connected. If this is the case, either G_S is a tree and (ii), (iii) clearly hold (since $\sum_{v \in S} \alpha(v) \geq |S| > |E_S|$), or it is a map with faces of degree at least d (since the girth of G is d). In the second case, $|E_S| \leq d + \frac{d}{d-2}(|S| - d)$ by Lemma 1. Moreover, $\sum_{v \in S} \alpha(v) \geq d + \frac{d}{d-2}(|S| - d)$ with equality only if every outer vertex is in S . Thus (ii) holds and (iii) holds for any outer vertex v_0 . Now, if the induced subgraph $G_S = (S, E_S)$ is not connected, knowing that (ii), (iii) holds for every connected component of G_S ensures that it holds for G_S (since both sides of the inequalities are additive over components). This proves that D admits some $\frac{d}{d-2}$ -orientations and that they are all accessible from any outer vertex.

We now consider the minimal $\frac{d}{d-2}$ -orientations O of D . Since every outer vertex of O has indegree 1, its outer-face is a simple cycle made of saturated edges oriented clockwise (this is the only way of avoiding a counterclockwise circuit). Thus O is clockwise-minimal. The orientation O is also accessible from any outer vertex, hence O is in $\tilde{\mathcal{O}}_d^{d-2}$. \square

4.2. Bijection for d -angulations of girth d .

By Theorem 7, the class \mathcal{C}_d of d -angulations of girth d can be identified with the subset E_d of orientations in $\tilde{\mathcal{O}}_d^{d-2}$ such that every face has degree d and every inner vertex has indegree $\frac{d}{d-2}$. We now characterize the mobiles that are in bijection with the subset E_d .

We call *d -branching mobile* a blossoming mobile such that every black vertex has degree d and every white vertex has indegree $\frac{d}{d-2}$. A 5-branching mobile is shown in Figure 5(top-right). By Theorem 5 the master bijection Φ_- induces a bijection between the orientations in E_d with n inner faces and the d -branching mobiles of excess $-d$ with n black vertices. The additional condition of having excess $-d$ is in fact redundant (hence can be omitted) as claimed below.

Claim 9. *Any d -branching mobile has excess $-d$.*

Proof. Let e, e', o be respectively the number of edges, white-white edges and outgoing buds of a d -branching mobile. By definition, the excess is $\delta = e + e' - o$. Let b and w be respectively the number of black and white vertices. The condition on black vertices gives $e - e' + o = db$, so that $\delta = 2e - db$. The condition on white vertices gives $e = \frac{d}{d-2}w$ and combining this relation with $b + w = e + 1$ (so as to eliminate w) gives $b = \frac{2}{d}e + 1$, hence $\delta = -d$. \square

From Claim 9 and the preceding discussion we obtain:

Theorem 10. *For $d \geq 3$ and $n \geq 0$, face-rooted d -angulations of girth d with n inner faces are in bijection with d -branching mobiles with n black vertices.*

Theorem 10 is illustrated by Figure 5(top-part). Before closing this section we mention that a slight simplifications appears in the definition of d -branching mobiles when d is even.

Proposition 11. *If $d = 2b$ is even, the d -branching mobiles have only flows of the form $\frac{i}{b-1}$ with $i \in \{1, \dots, b-2\}$. In particular, the 4-branching mobiles are non-fractional.*

Similarly the minimal $\frac{d}{d-2}$ -orientation of any d -angulation has only flows of the form $\frac{i}{b-1}$ with $i \in \{0, \dots, b-1\}$ (i.e., it is a $(b-1)$ -fractional orientation).

Proof. Let d be even and let M be a d -branching mobile. Call *odd* a white-white edge of M having a flow of the form $\frac{i}{d-2}$ with i an odd integer. If the subset of odd edges is non-empty, it is a forest hence there is a vertex v incident to only 1 odd edge. Hence, the indegree of v has the form $\text{in}(v) = \frac{i}{d-2}$ with i odd. This contradicts the condition $\text{in}(v) = \frac{d}{d-2}$.

Similarly, for a minimal $\frac{d}{d-2}$ -orientation, we call *odd* an edge having a flow of the form $\frac{i}{d-2}$ with i odd. Odd edges are non-saturated hence form a forest (otherwise there would be a counterclockwise circuit). Hence, if there are odd edges, there is a vertex v incident to only 1 odd edge, contradicting the condition $\text{in}(v) = \frac{d}{d-2}$. \square

5. BIJECTIONS FOR d -ANGULATIONS OF GIRTH d WITH A p -GONAL BOUNDARY

In this section we deal with d -angulations with a boundary. For any integers $p \geq d \geq 3$, we call *p -gonal d -angulation* a map having one face of degree p whose contour is simple (its vertices are all distinct) and all the other faces of degree

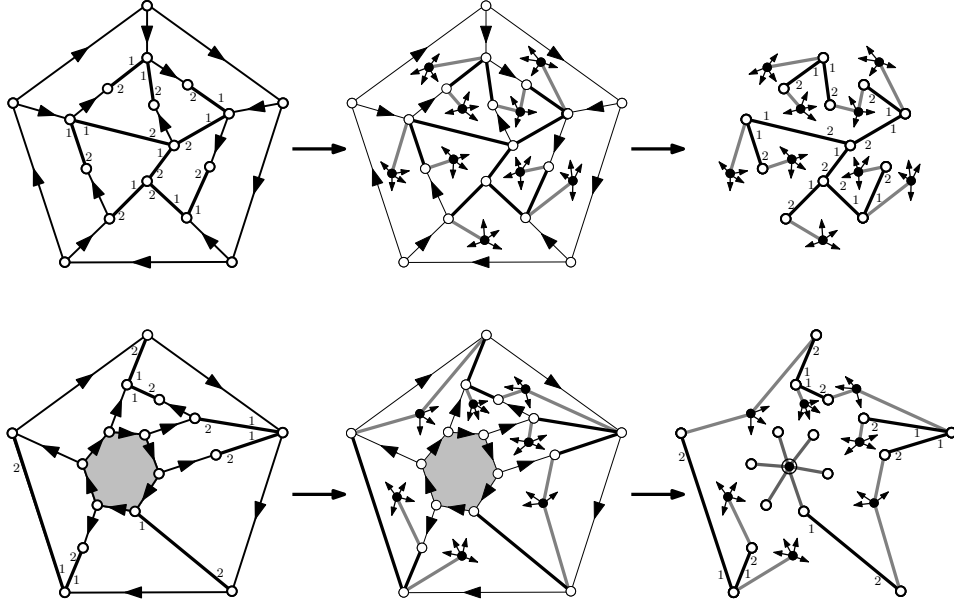


FIGURE 5. Top: a 5-angulation D of girth 5 and the 5-branching mobile $\Phi_{-}(D)$. Bottom: a non-separated 6-annular 5-angulation A and the $(6,5)$ -branching mobile $\Phi_{+}(A)$. The flows have only been indicated on non-saturated edges and multiplied by 3.

d . Our bijective strategy for p -gonal d -angulations is similar to the case without boundary: we first exhibit a canonical orientation for these maps and then characterize the bijection induced by a master bijection. However, an additional difficulty arises: one needs to factorize p -gonal d -angulations into two parts, one being a d -angulation without boundary and the other being a *non-separated* p -gonal d -angulations. The master bijections Φ_{-} and Φ_{+} then gives bijections for each part.

5.1. Orientations for p -gonal d -angulations.

Let A be a p -gonal d -angulation. The face of A of degree p is called *boundary-face* and the p incident vertices are called *boundary vertices*. A *pseudo $\frac{d}{d-2}$ -orientation* of A is a $(d-2)$ -fractional orientation such that non-boundary vertices have indegree $\frac{d}{d-2}$ and the boundary-face is a circuit of saturated edges. By Lemma 1, the number v of vertices and e of edges of A satisfy $e = \frac{d}{d-2}(v-p) + p + \frac{p-d}{d-2}$. Since $\frac{d}{d-2}(v-p)$ is the sum of the indegree of the non-boundary edges, this proves the following claim.

Claim 12. *The sum of indegree of the boundary vertices in a pseudo $\frac{d}{d-2}$ -orientation of a p -gonal d -angulation is $p + \frac{p-d}{d-2}$.*

We now prove that the existence of a pseudo $\frac{d}{d-2}$ -orientation characterizes the p -gonal d -angulations of girth d .

Lemma 13. *Let $p \geq d \geq 3$ be integers. A p -gonal d -angulation A admits a pseudo $\frac{d}{d-2}$ -orientation if and only if it has girth d . In this case, every pseudo $\frac{d}{d-2}$ -orientation is accessible from any boundary vertex.*

Proof. The proof of the necessity of girth d is the same as in the proof of Theorem 7. We now consider a p -gonal d -angulation A of girth d and prove that it admits a pseudo $\frac{d}{d-2}$ -orientation. From A we construct a d -angulation D of girth d as follows. Insert vertices and edges inside the boundary-face f_b so as to d -angulate the interior of f_b while keeping girth d . This yields a d -angulation D of girth d . We choose a root-face for D among the faces created inside f_d . By Theorem 7, D admits a minimal $\frac{d}{d-2}$ -orientation O . The orientation O induces an orientation O' of A in which every non-boundary vertex has indegree $\frac{d}{d-2}$. Let O'' be the orientation of A obtained from O' by reorienting the edges incident to the boundary-face f_b into a circuit of saturated edges. By definition, O'' is a pseudo $\frac{d}{d-2}$ -orientation.

Let us now show that the orientation O'' is accessible from any boundary vertex. Let v be a vertex of A . Recall that the $\frac{d}{d-2}$ -orientation O is accessible from any outer vertex of D so that there is a directed path of O from an outer vertex of D to v . Hence, there is a directed path of O'' from a boundary vertex to v . Since the boundary-face of O'' is a circuit, it means that there is a directed path of O'' from any boundary vertex to v . Thus, O'' is accessible from any boundary vertex.

Let \tilde{O} be another pseudo $\frac{d}{d-2}$ -orientation of A . We want to show that \tilde{O} is also accessible from the boundary vertices. Suppose the contrary and consider the set U of vertices u such that there is no directed path from a boundary vertex to u . Clearly any edge from a vertex u in U to a vertex not in U is saturated away from u . Thus, the sum of the indegrees of the vertices in U is equal to the number $|E_U|$ of edges with both ends in U . Since U contains no boundary vertex the sum of indegree is $\frac{d}{d-2}|U|$ both for \tilde{O} and O'' . But this shows that for O'' also the edges from a vertex in U to a vertex not in U are saturated away from U . Hence, the vertices in U are not reachable from the boundary vertices by a directed path of O'' . We reach a contradiction. \square

We call *p -annular d -angulation* a face-rooted p -gonal d -angulation of girth d whose root-face is not the boundary-face. For a p -annular d -angulation, a cycle of length d is said *separating* if one of the boundary vertex lies strictly on one side of C and one of the outer vertex lies strictly on the other side of C . A p -annular d -angulation is *non-separated* if it has no separating cycle.

Proposition 14. *Let $p > d \geq 3$. Any p -annular d -angulation A admits a unique minimal pseudo $\frac{d}{d-2}$ -orientation. This orientation is clockwise-minimal accessible (i.e., in \mathcal{O}_d^{d-2}) if and only if A is non-separated.*

Proof. Let A be a p -annular d -angulation. The map A admits some pseudo $\frac{d}{d-2}$ -orientations by Lemma 13. The existence and uniqueness of a minimal one is not directly granted by Lemma 2 because the definition of pseudo $\frac{d}{d-2}$ -orientations includes the requirement that the boundary-face is a saturated circuit. Instead, we consider the map \hat{A} obtained from A by contracting into a vertex, denoted v_b , the boundary-face of A (together with the p incident edges). The mapping γ which associates to a pseudo $\frac{d}{d-2}$ -orientations of A the induced orientation of \hat{A} is a bijection between the (non-empty) set S of pseudo $\frac{d}{d-2}$ -orientations of A with a boundary-face directed clockwise (there exists such orientations by Lemma 13) and the set \hat{S} of orientations of \hat{A} such that every vertex other than v_b has indegree $\frac{d}{d-2}$ (while v_b has indegree $\frac{p-d}{d-2}$). Moreover, an orientation $O \in S$ is minimal if and only if $\gamma(O)$ is minimal (indeed, the fact that the boundary-face of O is a circuit implies

that any counterclockwise circuit of $\gamma(O)$ is part of a counterclockwise circuit of O). Thus, applying Lemma 2 on the set \hat{S} of orientations implies that A admits a unique minimal pseudo $\frac{d}{d-2}$ -orientation.

We now prove second assertion of the proposition. Suppose first that A has a separating cycle C : a boundary vertex u_b lies on one side and an outer vertex u_o lies on the other side. We now prove that the minimal pseudo $\frac{d}{d-2}$ -orientation O is not accessible from u_o . Let D be the d -gonal d -angulation made of C and all the edges and vertices on the side of u_b (the side containing the boundary-face). By Lemma 1, the numbers e of edges and v of vertices in D satisfy $e = \frac{d}{d-2}(v - p) + p + \frac{p-d}{d-2}$. Moreover the right-hand-side is the sum of the indegree of the vertices in D (by claim 12). Thus every edge incident to C and lying in the region containing u_o are saturated away from C . Thus u_o cannot reach the vertices of D by a directed path. The orientation O is not accessible from the outer vertex u_o , hence O is not clockwise-minimal accessible.

Suppose conversely that A is not separating. We first prove that the minimal pseudo $\frac{d}{d-2}$ -orientation O is accessible from any outer vertex. We reason by contradiction and suppose that O is not accessible from one of the outer vertex u_o . In this case, u_o cannot reach the boundary vertices by a directed path (since Lemma 13 ensures that the boundary vertices can reach all the other vertices). We consider the set U of vertices of A that can reach the boundary vertices by a directed path. This set contains all the boundary vertices but not the outer vertex u_o . Let $A_U = (U, E_U)$ be the map induced by the vertices in U . Clearly, every edge between a vertex u in U and a vertex not in U is oriented away from U . Thus $|E_U|$ equals the sum of the indegree of the vertices in U . By claim 12 we get:

$$(1) \quad |E_U| = \frac{d}{d-2}(|U| - p) + p + \frac{p-d}{d-2}.$$

Now, since A has girth d , the non-boundary-faces of A_U have degree at least d . Therefore by Lemma 1, the equality (1) ensures that every face of A_U has degree d . In particular, the cycle C corresponding to the contour of the face of A_U in which the outer vertex u_o lies has length d . This cycle C is separating: a contradiction.

It remains to prove that the outer face of O is a clockwise cycle. Suppose the contrary. In this case, there is an outer edge e which is saturated with the root-face on its right. Let v, v' the end and origin of O . By accessibility from v there is a simple directed path P from v to a boundary vertex u . By accessibility from boundary vertices, there is a path simple directed P' from u to v' . The paths P and P' do not contain the saturated edge e . Thus, there is circuit C containing e in the concatenation of P , P' and e . Since e has the root-face on its right, the circuit C is counterclockwise, contradicting the minimality of O . \square

5.2. Bijection for p -gonal d -angulations of girth d .

We call (p, d) -branching mobile a $(d-2)$ -fractional blossoming mobile such that:

- every black vertex have degree d except for one, called the *special vertex* s , which has degree p and carries no bud,
- every white vertex that are not neighbor of s have indegree $\frac{d}{d-2}$, and the indegrees of the neighbors of s add up to $p + \frac{p-d}{d-2}$.

An example of $(6, 5)$ -branching mobile is represented in Figure 5(bottom-right).

By Proposition 14, the non-separated p -annular d -angulations can be identified with the class $\mathcal{O}_{p,d}$ of orientations in \mathcal{O}_d^{d-2} such that

- every face has degree d except for one non-root face of degree p , called boundary-face, whose contour is a clockwise circuit of saturated edges
- every non-boundary vertex has degree $\frac{d}{d-2}$ (and the sum of indegrees of the boundary vertices is $p + \frac{p-d}{d-2}$ by Claim 12).

By Theorem 5 (and the definition of Φ_+ which implies that an inner face of degree p whose contour is a clockwise circuit of saturated edges corresponds to a black vertex of the mobile incident to p edges and no bud), the class $\mathcal{O}_{p,d}$ of orientations are in bijection with the class of (p,d) -branching mobiles of excess d . As in the case of d -angulations without boundary, the additional requirement on the excess is redundant as claimed below.

Claim 15. *Any (p,d) -branching mobile has excess d .*

Proof. By definition, the excess is $\delta = e + e' - o$, where e, e', o are respectively the number of edges, white-white edges and outgoing buds. Let b and w be the number of black and white vertices. The condition on black vertices gives $e - e' + o = d(b - 1) + p$, so that $\delta = 2e - db + d - p$. The condition on white vertices gives $e = \frac{d}{d-2}(w - p) + p + \frac{p-d}{d-2}$ and combining this relation with $b + w = e + 1$ (so as to eliminate w) gives $b = \frac{2}{d}e + \frac{p}{d}$, hence $\delta = d$. \square

To summarize, by specialization of the master bijection Φ_+ in the $(d-2)$ -fractional setting, Proposition 14 gives the following result.

Theorem 16. *Let $p > d \geq 3$ and n be positive integers. The non-separated p -annular d -angulations (of girth d) with n inner faces are in bijection with (p,d) -branching mobiles with n black vertices.*

Theorem 16 is illustrated in Figure 5(bottom). As in the case without boundary, a slight simplifications appears in the definition of (p,d) -branching mobiles when d is even. First observe that when d is even the p -gonal d -angulations are bipartite (since the inner faces generate all cycles). Thus, there is no p -gonal d -angulations when p is odd and d even. Similarly there is no (p,d) -branching mobiles with p odd and d even. There is a further simplification:

Proposition 17. *If $d = 2b$ is even, then for any $p \geq b$, the $(2p,d)$ -branching mobiles have only flows of the form $\frac{i}{b-1}$ with $i \in \{1, \dots, b-2\}$. In particular, for $d = 4$, the $(2p,d)$ -branching mobiles are non-fractional.*

Similarly, the minimal pseudo $\frac{d}{d-2}$ -branching orientation of any $2p$ -annular d -angulation has only flows of the form $\frac{i}{b-1}$ with $i \in \{0, \dots, b-1\}$.

Proof. The proof is the same as the proof of Proposition 11. \square

We now explain how to deal bijectively with general (possibly separated) p -annular d -angulations. We call *pseudo-separating* a cycle of a p -annular d -angulations which is either separating or the contour of the root-face. A pseudo-separating cycle C of A defines:

- a p -annular d -angulation, denoted A_C , corresponding to the map on the side of C containing the boundary face (C is the contour of the root-face),

- a d -annular d -angulation, denoted A'_C , corresponding to the map on the side of C containing the root-face (C is the contour of the boundary face).

In order to make the decomposition $(A, C) \mapsto (A_C, A'_C)$ injective we consider *marked maps*. We call a p -annular d -angulation *marked* if a boundary vertex is marked. We now consider an arbitrary convention for each marked p -annular d -angulations by distinguishing one of the outer-vertices as *co-marked* (the co-marked vertex is entirely determined by the marked vertex). We then define for a marked p -annular d -angulation A^\bullet and a pseudo-separating cycle C , the marked annular d -angulations A_C^\bullet and $A_{C'}^\bullet$ obtained by marking A_C at the marked vertex of A^\bullet and marking $A_{C'}^\bullet$ at the co-marked vertex of A_C^\bullet .

Proposition 18. *Let $p > d \geq 3$. Any p -annular d -angulation A has a unique pseudo-separating cycle C such that A_C is non-separated. Moreover, the mapping Δ which associates to a marked p -annular d -angulation A^\bullet the pair $(A_C^\bullet, A_{C'}^\bullet)$ is a bijection between marked p -annular d -angulations and pairs made of a marked non-separated p -annular d -angulation and a marked d -annular d -angulation.*

Proposition 18 gives a bijective approach for general p -annular d -angulations: apply the master bijection Φ_+ (Theorem 16) to the non-separated p -annular d -angulation A_C , and the master bijection Φ_- (Theorem 10) to the d -angulation A'_C .

Proof. The set \mathcal{C} of pseudo-separating cycles of A is non-empty (it contains the cycle corresponding to the contour of the root-face) and is partially ordered by the relation \preceq defined by setting $C \preceq C'$ if $A_C \subseteq A_{C'}$ (here the inclusion is in terms of the vertex set and the edge set). Moreover, A_C is non-separated if and only if C is a minimal element of (\mathcal{C}, \preceq) . Observe also that for all $C, C' \in \mathcal{C}$, the intersection $A_C \cap A_{C'}$ is non-empty because it contains the boundary vertices and edges. Thus, if C, C' are not comparable for \preceq , then the cycles C, C' must intersect. Since they have both length d and A has no cycle of length less than d , the only possibility is that $A_C \cap A_{C'}$ is delimited by a separating cycle C'' of length d : $A_C \cap A_{C'} = A_{C''}$. This show that (\mathcal{C}, \preceq) has a unique minimal element, or equivalently, that A has a unique pseudo-separating cycle C such that A_C is non-separated.

We now prove the second statement. The injectivity of Δ is clear: the markings allow to glue back the maps $A_C^\bullet, A_{C'}^\bullet$ by identifying the root-face C of A_C^\bullet with the boundary-face of $A_{C'}^\bullet$, and identifying the co-marked vertex of A_C^\bullet with the marked vertex of $A_{C'}^\bullet$. To prove surjectivity we only need to observe that gluing a non-separated p -annular d -angulation and a marked d -annular d -angulation as described above preserves the girth d (hence the glued map A^\bullet is a p -annular d -angulation) and makes C a pseudo-separating cycle of A^\bullet . \square

6. COUNTING RESULTS

In this section, we establish equations for the generating functions of d -angulations without and with boundary (Subsections 6.2 and 6.1). We then obtain closed formulas in the cases $d = 3$ and $d = 4$. We call *generating function*, or *GF* for short, of a class \mathcal{C} counted according to a parameter P the formal power series $C(x) = \sum_{n \in \mathbb{N}} c_n x^n$, where c_n is the number of objects $c \in \mathcal{C}$ satisfying $P(c) = n$. We also denote $[x^n]G(x)$ the coefficient of x^n in a formal power series $G(x)$.

6.1. Counting rooted d -angulations of girth d .

By Theorem 10, counting d -angulations of girth d reduces to counting d -branching mobiles. We will characterize the generating function of d -branching mobiles by a system of equations (obtained by a straightforward recursive decomposition). We first need a few notations. For a positive integer j we define the polynomial h_j in the variables w_1, w_2, \dots by:

$$(2) \quad h_j(w_1, w_2, \dots) := [t^j] \frac{1}{1 - \sum_{i>0} t^i w_i} = \sum_{r>0} \sum_{\substack{i_1, \dots, i_r > 0 \\ i_1 + \dots + i_r = j}} w_{i_1} \cdots w_{i_r}.$$

In other words, h_j is the (polynomial) generating function of integer compositions of j where the variable w_i is conjugated to the number of parts of size i .

We call *planted d -branching mobile* a $(d-2)$ -fractional blossoming mobile with a marked leaf (vertex of degree 1) such that non-marked black vertices have degree d and non-marked white vertices have indegree $\frac{d}{d-2}$. For $i \in \{0, \dots, d-2\}$, we denote by \mathcal{W}_i the family of planted d -branching mobiles where the marked leaf has indegree $\frac{i}{d-2}$ (the case $i=0$ correspond to a black marked leaf). We also denote by $W_i \equiv W_i(x)$ the generating function of \mathcal{W}_i counted according to the number of non-marked black vertices.

For $i = d-2$, the marked leaf of a mobile in \mathcal{W}_i is a white vertex connected to a black vertex b . The $d-1$ other half-edges incident to b are either buds or belong to an edge e leading to white vertex w . In the second case the sub-mobile planted at b and containing w belongs to \mathcal{W}_0 . This decomposition gives the equation $W_{d-2} = x(1 + W_0)^{d-1}$.

For $i \in \{0, \dots, d-3\}$, the marked leaf of a mobile in \mathcal{W}_i is connected to a white vertex w . Let v_1, \dots, v_r the other neighbors of w . For $j = 1 \dots, r$ the sub-mobile planted at w and containing v_j belongs to one of the classes $\mathcal{W}_{\alpha(j)}$, and the sum of the indegrees $\sum_{j=1}^r \alpha(j)$ is $\frac{d}{d-2} - \frac{d-2-i}{d-2} = \frac{i+2}{d-2}$. This decomposition gives

$$W_i = \sum_{r>0} \sum_{\substack{i_1, \dots, i_r > 0 \\ i_1 + \dots + i_r = i+2}} W_{i_1} \cdots W_{i_r} = h_{i+2}(W_1, \dots, W_{d-2}).$$

Hence, the series W_0, W_1, \dots, W_{d-2} satisfy the following system of equations:

$$(3) \quad \begin{cases} W_{d-2} &= x(1 + W_0)^{d-1}, \\ W_i &= h_{i+2}(W_1, \dots, W_{d-2}) \quad \forall i \in \{0, \dots, d-3\}, \\ W_i &= 0 \quad \forall i > d-2. \end{cases}$$

Observe that the system (3) determines W_0, W_1, \dots, W_{d-2} uniquely as formal power series. Indeed, it is clear that any solutions W_0, W_1, \dots, W_{d-2} of this system have zero constant coefficient. And from this observation it is clear that the other coefficients are uniquely determined by induction.

The following table shows the system for the first values, $d = 3, 4, 5$:

$d = 3$	$d = 4$	$d = 5$
$W_1 = x(1 + W_0)^2$ $W_0 = W_1^2$	$W_2 = x(1 + W_0)^3$ $W_0 = W_1^2 + W_2$ $W_1 = 2W_1W_2$	$W_3 = x(1 + W_0)^4$ $W_0 = W_1^2 + W_2$ $W_1 = 2W_1W_2 + W_3$ $W_2 = W_1^4 + 3W_1^2W_2 + 2W_1W_3 + W_2^2$

In the case where d is even, $d = 2b$, one easily checks that $W_i = 0$ for i odd (this follows from the fact that, for odd i , all monomials in $h_{i+2}(W_1, \dots, W_{d-2})$ contain a W_j with odd j , and is related to Proposition 11). Hence the system can be simplified. The series $V_i := W_{2i}$ satisfy the system:

$$(4) \quad \begin{cases} V_{b-1} &= x(1 + V_0)^{2b-1}, \\ V_i &= h_{i+1}(V_1, \dots, V_{b-1}) \quad \forall i \in \{0, \dots, b-2\}, \\ V_i &= 0 \quad \forall i > b-1. \end{cases}$$

The following table shows the system for the first values, $b = 2, 3, 4$:

$b = 2$	$b = 3$	$b = 4$
$V_1 = x(1 + V_0)^3$ $V_0 = V_1$	$V_2 = x(1 + V_0)^5$ $V_0 = V_1$ $V_1 = V_1^2 + V_2$	$V_3 = x(1 + V_0)^7$ $V_0 = V_1$ $V_1 = V_1^2 + V_2$ $V_2 = V_1^3 + 2V_1V_2 + V_3$

Let \mathcal{M}_d be the family of d -branching mobiles rooted at a corner incident to a black vertex, and let $M_d(x)$ be the GF of \mathcal{M}_d counted according to the number of non-root black vertices. Since each of the d half-edges incident to the root-vertex is either a bud or is connected to a planted mobile in \mathcal{W}_0 we have

$$M_d(x) = (1 + W_0)^d.$$

Proposition 19 (Counting d -angulations of girth d). *For $d \geq 3$, the generating function $F_d(x)$ of corner-rooted d -angulations of girth d counted according to the number of inner faces has the following expression:*

$$F_d = W_{d-2} - \sum_{i=0}^{d-3} W_i W_{d-2-i},$$

where the series W_0, \dots, W_{d-2} are the unique power series solutions of the system (3) (which simplifies to (4) in the even case $d = 2b$). Therefore F_d is algebraic. Moreover it satisfies

$$F'_d(x) = (1 + W_0)^d.$$

Proof. Theorem 10 and Claim 6 (first part) imply that F_d is the series counting d -branching mobiles rooted at an exposed outgoing bud. Call B (resp. H) the GF of d -branching mobiles (counted according to the number of black vertex) with a marked outgoing bud (resp. marked half-edge incident to a white vertex). Claim 6 (second part) yields $F_d = B - H$. A mobile with a marked outgoing bud identifies to a planted mobile in \mathcal{W}_{d-2} (planted mobile where the vertex of attachment is black), hence $B = W_{d-2}$. A mobile with a marked half-edge incident to a white vertex identifies with an ordered pair of planted mobiles (w, w') in $\mathcal{W}_i \times \mathcal{W}_{d-2-i}$ for some i in $\{0, \dots, d-3\}$, hence $H = \sum_{i=0}^{d-3} W_i W_{d-2-i}$.

For the expression of $F'_d(x)$, observe that this is the GF of corner-rooted d -angulations of girth d with a secondary marked inner face. Equivalently, $F'_d(x)$ counts face-rooted d -angulations of girth d with a secondary marked corner not incident to the root-face. By the master bijection Φ_- , marking a corner not incident to the root-face in the d -angulation is equivalent to marking a corner incident to a black vertex in the corresponding mobile. Hence, Theorem 10 gives $F'_d(x) = M_d(x) = (1 + W_0)^d$. \square

The cases $d = 3$ and $d = 4$ correspond to simple triangulations and simple quadrangulations; we recover (see Section 6.3) exact-counting formulas due to Brown [Bro64, Bro65]. For $d \geq 5$ the counting results are new (to the best of our knowledge). For $d = 5$ and $d = 6$, one gets $F_5 = x + 5x^3 + 121x^5 + 4690x^7 + 228065x^9 + O(x^{11})$, and $F_6 = x + 3x^2 + 17x^3 + 128x^4 + 1131x^5 + 11070x^6 + O(x^7)$.

For any $d \geq 3$ a simple analysis based on the Drmota-Lalley-Wood theorem [FS09, VII.6] ensures that for odd d , the coefficient $[x^{2n}]F'_d(x)$ (from the Euler relation the odd coefficients of $F'_d(x)$ are zero) is asymptotically $c_d \gamma_d^n n^{-3/2}$ for some computable positive constants c_d and γ_d depending on d ; and for even d , the coefficient $[x^n]F'_d(x)$ is asymptotically $c_d \gamma_d^n n^{-3/2}$ again with c_d and γ_d computable constants. Since $[x^n]F_d(x) = \frac{1}{n}[x^{n-1}]F'_d(x)$, the number of corner-rooted d -angulations of girth d follows (up to the parity requirement for odd d) the asymptotic behavior $c \gamma^n n^{-5/2}$ which is universal for planar map families.

6.2. Counting rooted p -gonal d -angulations of girth d .

We start by characterizing the generating function of (p, d) -branching mobiles. A (p, d) -branching mobile is said *marked* if one of the p corners incident to the special vertex is marked. Let $\mathcal{M}_{p,d}$ be the class of marked (p, d) -branching mobiles, and let $M_{p,d}(x)$ be the GF of this class counted according to the number of non-special black vertices. Given a mobile in $\mathcal{M}_{p,d}$, we consider the (white) neighbors v_1, \dots, v_p of the special vertex s . For $i = 1 \dots p$ the sub-mobiles M_1, \dots, M_{r_i} planted at v_i (and not containing s) belong to some classes $\mathcal{W}_{\alpha(i,1)}, \dots, \mathcal{W}_{\alpha(i,r_i)}$, where $1 + \sum_{j=1}^{r_i} \frac{\alpha(i,j)}{d-2}$ is the indegree of v_i . Moreover, by definition of (p, d) -branching mobiles, the sum of the indegrees of v_1, \dots, v_p is $p + \frac{p-d}{d-2}$. This decomposition gives for all $p \geq d \geq 3$

$$(5) \quad M_{p,d}(x) = h_{p-d}^{(p)}(W_1, \dots, W_{d-2}),$$

where for $j \geq 0$ the polynomial $h_j^{(p)}$ is the generating function of p -tuples of compositions of integers i_1, \dots, i_p with $i_1 + \dots + i_p = j$, where w_i marks the number of parts of size i . It is clear that $h_j^{(1)} = h_j$ and that $h_j^{(p)}$ satisfies:

$$(6) \quad h_j^{(p)}(w_1, w_2, \dots) := [t^j] \frac{1}{(1 - \sum_{i>0} t^i w_i)^p}.$$

Observe that in the special case $p = d$, there is a unique (d, d) -branching mobile, hence $M_{d,d}(x) = 1$ (which is coherent with (5) since $h_0^{(p)} = 1$).

We now turn to annular d -angulations. For $p \geq d \geq 3$, we denote by $\mathcal{A}_{p,d}$ the family of *marked* p -annular d -angulations of girth d (a boundary vertex is marked, or equivalently a corner of the boundary face is marked), and we denote by $\mathcal{N}_{p,d}$ the subfamily of those that are non-separated. Let $A_{p,d}(x)$ and $N_{p,d}(x)$ be the GF of these classes counted according to the number of non-boundary inner faces. By Theorem 16, the marked non-separated p -annular d -angulations are in bijection by Φ_+ with the marked (p, d) -branching mobiles (since marking the mobile is equivalent to marking the p -annular d -angulation on a boundary vertex) for $p > d \geq 3$. Thus, $N_{p,d}(x) = M_{p,d}(x)$ for all $p > d \geq 3$. Moreover, Proposition 18 directly implies:

$$A_{p,d}(x) = M_{p,d}(x)F'_d(x),$$

since the series $F'_d(x)$ (defined in Proposition 19) counts marked d -annular d -angulations. This relations also holds for $p = d$ since $M_{d,d}(x) = 1$. We summarize:

Proposition 20 (Counting p -gonal d -angulations of girth d). *For $p \geq d \geq 3$, the generating function $F_{p,d}(x)$ of p -gonal d -angulations of girth d rooted at a corner incident to the boundary-face, counted according to the number of non-boundary faces satisfies*

$$F'_{p,d}(x) = h_{p-d}^{(p)}(W_1, \dots, W_{d-2}) \cdot (1 + W_0)^d,$$

where $h_j^{(p)}$ is defined in (6) and the series W_1, \dots, W_{d-2} are specified by (3).

In the bipartite case, $d = 2b$ the expression simplifies to

$$F'_{2p,2b}(x) = h_{p-b}^{(2p)}(V_1, \dots, V_{b-1}) \cdot (1 + V_0)^{2b},$$

where the series V_1, \dots, V_{b-1} are specified by (4).

Proof. By definition, a p -annular d -angulation is a p -gonal d -angulation of girth d with a secondary marked face of degree d . Hence, $F_{p,d}'(x) = A_{p,d}(x) = M_{p,d}(x)F_d'(x)$. Moreover $F_d'(x) = (1 + W_0)^d$ by Proposition 19 and $M_{p,d}(x) = h_{p-d}^{(p)}(W_1, \dots, W_{d-2})$ by (5). \square

6.3. Exact-counting formulas for triangulations and quadrangulations.

In the case of triangulations and quadrangulations ($d = 3, 4$) the system of equations given by Proposition 20 takes a form amenable to the Lagrange inversion formula. We thus recover bijectively the counting formulas established by Brown [Bro64, Bro65] for simple triangulations and quadrangulations with boundary.

Proposition 21 (Counting simple p -gonal triangulations). *For $p \geq 3$, $n \geq 0$, let $t_{p,n}$ be the number of corner-rooted p -gonal triangulations with $n + p$ vertices which are simple (i.e., have girth 3) and have the root-corner in the p -gonal face (no restriction on the root-corner for $p = 3$). The generating function $T_p(x) = \sum_{n \geq 0} (2n + p - 2) t_{p,n} x^n$ satisfies*

$$T_p(x) := \binom{2p-4}{p-3} u^{2p-3}, \quad \text{where } u = 1 + x u^4.$$

Consequently, the Lagrange inversion formula gives:

$$t_{p,n} = \frac{2(2p-3)!}{(p-1)!(p-3)!} \frac{(4n+2p-5)!}{n!(3n+2p-3)!}.$$

Proof. By the Euler relation, a p -gonal triangulation with $n + p$ vertices has $2n + p$ faces. Hence $T_p(x^2)x^{p-3} = F'_{p,3}(x)$. By proposition 20,

$$F'_{p,3}'(x) = \binom{2p-4}{p-3} W_1^{p-3} (1 + W_0)^3 = x^{p-3} \binom{2p-4}{p-3} (1 + W_0)^{2p-3},$$

where W_0, W_1 are specified by $\{W_1 = x(1 + W_0)^2, W_0 = W_1^2\}$. Thus, $T_p(x^2) = \binom{2p-4}{p-3} (1 + W_0)^{2p-3}$. Moreover, $1 + W_0 = 1 + W_1^2 = 1 + x^2(1 + W_0)^4$, hence $1 + W_0(x) = u(x^2)$, where $u \equiv u(x)$ is the series specified by $u = 1 + x u^4$. Thus, $T_p(x) = \binom{2p-4}{p-3} u(x)^{2p-3}$. \square

Proposition 22 (Counting simple $2p$ -gonal quadrangulations). *For $p \geq 2$, $n \geq 0$, let $q_{p,n}$ be the number of corner-rooted $2p$ -gonal quadrangulations with $n + 2p$ vertices which are simple (i.e., have girth 4) and have the root-corner incident to the $2p$ -gonal face (no restriction on the root-corner for $p = 2$). The generating function $Q_p(x) = \sum_{n \geq 0} (n + p - 1) q_{p,n} x^n$ satisfies*

$$Q_p(x) = \binom{3p-3}{p-2} u^{3p-2}, \quad \text{where } u = 1 + x u^3.$$

Consequently, the Lagrange inversion formula gives:

$$q_{p,n} = \frac{3(3p-2)!}{(p-2)!(2p-1)!} \frac{(3n+3p-4)!}{n!(2n+3p-2)!}.$$

Proof. By the Euler relation, a $2p$ -gonal quadrangulation with $n+2p$ vertices has $n+p$ faces. Hence $Q_p(x)x^{p-2} = F'_{2p,4}(x)$. By Proposition 20,

$$F'_{2p,4}(x) = \binom{3p-3}{p-2} V_1^{p-2} (1+V_0)^4 = x^{p-2} \binom{3p-3}{p-2} (1+V_0)^{3p-2},$$

where V_0, V_1 are specified by $\{V_1 = x(1+V_0)^3, V_0 = V_1\}$. Thus, $Q_p(x) = \binom{3p-3}{p-2} (1+V_0)^{3p-2}$. Moreover, $1+V_0 = 1+V_1 = 1+x(1+V_0)^3$, hence $1+V_0(x) = u(x)$, where $u \equiv u(x)$ is the series specified by $u = 1+xu^3$. Thus, $Q_p(x) = \binom{3p-3}{p-2} u(x)^{3p-2}$. \square

7. PROOF THAT THE MAPPINGS Φ_+, Φ_- ARE BIJECTIONS

In this section, we prove Theorem 4 thanks to a reduction to a bijection Ψ described in [Ber07]. All the orientations in this section are non-fractional. The relation between Ψ and the master bijections Φ_+, Φ_- involves *duality*.

Duality. The *dual* O^* of an orientation O is obtained by the process represented in Figure 6(a):

- place a vertex v_f of O^* in each face f of O ,
- for each edge e of O having a face f on its left and f' on its right, draw a *dual edge* e^* of O^* oriented from v_f to $v_{f'}$ across e .

Note that the duality (as defined above) is not an involution (applying duality twice inverse the direction of every edge). Duality maps vertex-rooted orientations to face-rooted orientations. It is easy to check that a face-rooted orientation O is minimal if and only if O^* is *accessible*, that is, accessible from the root-vertex. Similarly, a vertex-rooted orientation O is accessible if and only if O^* is *maximal*, that is, has no clockwise circuit. We denote by \mathcal{S}_d and $\tilde{\mathcal{S}}_d$ respectively the set of orientations which are the image by duality of the sets \mathcal{O}_d and $\tilde{\mathcal{O}}_d$. Thus, \mathcal{S}_d is the set of vertex-rooted orientations which are accessible, have a root-vertex of indegree 0, and which are maximal for one of the faces incident to the root-vertex (it is maximal for each of these faces in this case); see Figure 6(b). The set $\tilde{\mathcal{S}}_d$ is the subset of \mathcal{S}_d made of the orientations such that each of the faces incident to the root-vertex have *counterclockwise degree* 1 (only one edge is oriented in counterclockwise direction around these faces).

Partial closure and partial opening.

We now present the bijection Ψ from [Ber07] between corner-rooted maximal accessible orientations and mobile of excess 1. The presentation of Ψ below is in fact taken from [BC, Section 7].

Let M be a blossoming mobile with p edges and q outgoing buds (hence excess $\delta = p - q$). The corresponding *fully blossoming* mobile M' is obtained from M by inserting *ingoing bud* (dangling ingoing half-edge) in each of the p corners of M following an edge in counterclockwise direction around a black vertex. A fully

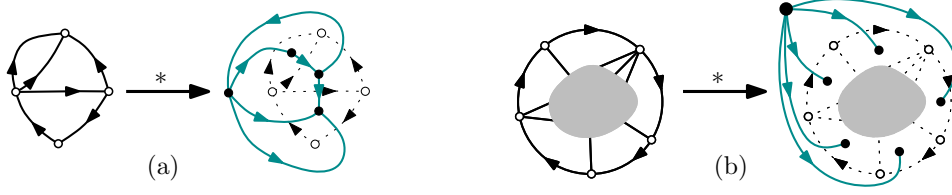
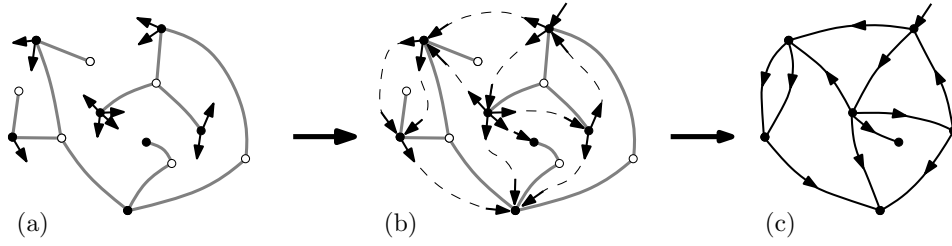


FIGURE 6. The dual of an orientation.

FIGURE 7. The rooted-closure of a mobile of excess $\delta = 1$.

blossoming mobile is represented in solid lines in Figure 7(b). A *counterclockwise walk* around M' (with the edges of the mobile M' on the left of the walker) sees a succession of outgoing buds and ingoing buds. Associating an opening parenthesis to outgoing buds and a closing parenthesis to ingoing buds, one obtains a cyclic binary word with q opening and p closing parentheses. This yields in turn a partial matching of the buds (an outgoing bud is matched with the next free ingoing bud counterclockwise around the mobile), leaving $|\delta|$ buds unmatched (the unmatched buds are ingoing if $\delta > 0$ and outgoing if $\delta < 0$).

The *partial closure* C of the blossoming mobile M is obtained by forming an oriented edge out of each matched pair of buds. Clearly the oriented edges can be formed in a planar way, and this process is uniquely defined (recall that all our maps are on the sphere). The partial closure is represented in Figure 7(a)-(b). We consider the partial closure C as a planar map with two types of edges (those of the mobile, which are non-oriented, and the new formed edges, which are oriented) and $|\delta|$ buds which are all incident to the same face, which we take to be the root-face of C . Note that, if $\delta \geq 0$, there are δ white corners incident to the root-face of C , because initially the number of such corners is equal to the number of edges of the mobile, and then each matched pair of buds decreases this number by 1. These corners, which stay incident to the root-face throughout the partial closure, are called *exposed white corners*.

Let O be an oriented map. The *partial opening* of O is the map C with vertices colored black or white and with two types of edges (oriented and non-oriented) obtained as follows.

- Color in black the vertices of O and insert a white vertex in each face of O .
- Around each vertex v of O draw a non-oriented edge from any corner c which follows an ingoing edge in clockwise order around v to the white vertex in the face containing c .

If O is corner-rooted, then the bud indicating the root-corner is interpreted as an ingoing half-edge and gives rise to an edge of C . For instance, the partial opening of the corner-rooted map in Figure 7(c) is the map in Figure 7(b).

The rooted-, positive- and negative- openings and closures

We now recall and extend the results given in [Ber07] about closures and openings. Observe that the partial closure C of a 1-mobile M has one dangling ingoing bud. The *rooted-closure* of M , denoted $\Psi(M)$, is obtained from the partial closure C by erasing every white vertex and every edge of the mobile; see Figure 7(b)-(c). The embedded graph $\Psi(M)$ is always connected (hence a map) as explained below and the dangling ingoing bud is considered as indicating the root-corner of $\Psi(M)$. The *rooted-opening* of a corner-rooted orientation O is obtained from its partial opening C by erasing all the ingoing half-edges of O (this leaves only the non-oriented edges of C and some outgoing buds incident to black vertices). The following result was proved in [Ber07] (see also [BC]).

Theorem 23. *The rooted-closure Ψ is a bijection between 1-mobiles and corner-rooted maximal accessible orientations. The rooted-opening is the inverse mapping.*

We now present the mappings Ψ_+ and Ψ_- defined respectively on blossoming mobiles of positive and negative excess; see Figure 8.

Definition 24. *Let M be a blossoming mobile of excess $\delta \neq 0$ and let C be its partial closure.*

- *If $\delta > 0$, then C has δ ingoing buds (incident to the root-face). The positive-closure of M , denoted $\Psi_+(M)$, is obtained from C by first creating a root-vertex v of $\Psi_+(M)$ in the root-face of C and connecting it to each ingoing bud (buds thus become part of an edge of O oriented away from v); second erasing the edges and white vertices of the mobile.*
- *If $\delta < 0$, then C has δ outgoing buds (incident to the root-face). The negative-closure of M , denoted $\Psi_-(M)$, is obtained from C by first creating a root-vertex v of $\Psi_-(M)$ in the root-face of C and connecting it to each outgoing bud and then reorienting these edges (buds thus become part of an edge of $\Psi_-(M)$ oriented away from v); second erasing the edges and white vertices of the mobile.*

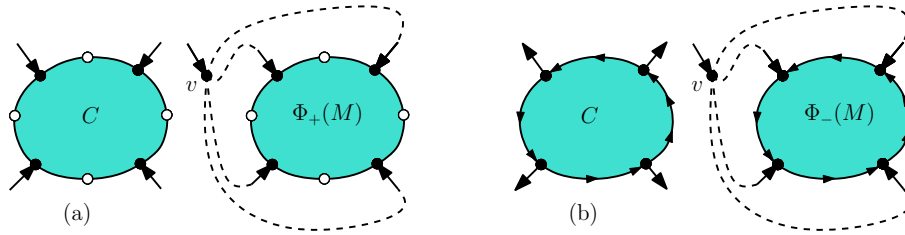


FIGURE 8. (a) Positive-closure Ψ_+ . (b) Negative-closure Ψ_- .

Theorem 25. *Let d be a positive integer.*

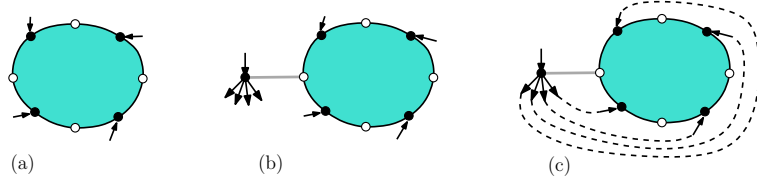


FIGURE 9. Formulation of the positive-closure Ψ_+ by reduction to the rooted-closure Ψ . Figure (a) shows generically the partial closure of a d -mobile for $d = 4$. In (b) one creates a black vertex b with d outgoing buds, and connects it to an exposed white corner. In (c) one performs the remaining matchings of buds to complete the positive-closure.

- The positive-closure Ψ_+ is a bijection between the set of d -mobiles and the set \mathcal{S}_d of orientations. Moreover, the mapping defined on \mathcal{O}_d by first applying duality (thereby obtaining an orientation in \mathcal{S}_d) and then applying the inverse mapping Ψ_+^{-1} is the mapping Φ_+ defined in Definition 3. Thus, Φ_+ is a bijection between \mathcal{O}_d and the set of d -mobiles.
- The negative-closure Ψ_- is a bijection between the set of $(-d)$ -mobiles and the subset $\tilde{\mathcal{S}}_d$ of orientations. Moreover, the mapping defined on $\tilde{\mathcal{O}}_d$ by first applying duality (thereby obtaining an orientation in $\tilde{\mathcal{S}}_d$) and then applying the inverse mapping Ψ_-^{-1} is the mapping Φ_- defined in Definition 3. Thus, Φ_- is a bijection between $\tilde{\mathcal{O}}_d$ and the set of $(-d)$ -mobiles.

Theorem 25 clearly implies Theorem 4 (indeed the correspondence of parameters mentioned in Theorem 4 is obvious from the definitions). It only remains to prove Theorem 25. Before doing so, we mention that a result can be stated for 0-mobiles: the 0-closure obtained by performing the partial closure and then erasing the edges of the mobile is a bijection between 0-mobiles and the set of $\bigcup_{d \in \mathbb{N}} \mathcal{S}_d$.

Proof. • We first treat the case of the positive-closure Ψ_+ . We begin by showing that the positive-closure of a d -mobile M is in \mathcal{S}_d . Let C be the partial closure of M and let $O = \Psi_+(M)$ be its positive-closure. As observed above, the mobile M has $d > 0$ exposed white corners. Let M' be the blossoming mobile obtained from M by creating a new black vertex b , joining b to an exposed white corner, and adding d outgoing buds to b ; see Figure 9. The excess of M' is 1, hence by Theorem 23 the rooted-closure $O' = \Psi(M')$ is maximal and accessible. Moreover, it is easily seen (Figure 9) that b is the root-vertex of O' (because the ingoing bud incident to b is not matched during the partial closure) and has indegree 0. Thus, the orientation O is in \mathcal{S}_d .

We also make the following observation (useful for the negative-closure):

Fact. the positive closure $O = \Psi_+(M)$ is in $\tilde{\mathcal{S}}_d$ if and only if each of the exposed white corners of M is incident to a (white) leaf of M .

Indeed, a white vertex w_f of M has an exposed white corner if and only if it corresponds to a face f of O incident to the root-vertex b . Moreover, the clockwise degree of f is (as always) the degree of w_f .

We now prove that the positive-closure is a bijection by defining the inverse mapping. Let O be a vertex-rooted orientation in \mathcal{S}_d . We define the *positive-opening*

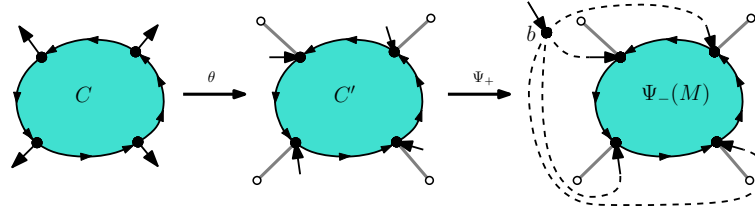


FIGURE 10. Formulation of the negative-closure Ψ_- , by reduction to the positive-closure Ψ_+ .

of O , as the embedded graph with buds M obtained by applying the partial opening of O , and then erasing every ingoing half-edge of O as well as the root-vertex b (and the incident outgoing half-edges). In order to prove that M is a blossoming mobile (i.e. a tree with buds), we consider a minimal accessible orientation O' obtained from O by choosing a root-corner for O among the corners incident to the root-vertex b . By Theorem 23, the rooted-opening of O' gives a blossoming mobile M' . It is clear from definitions that M is obtained from M' by erasing the root-vertex b . Moreover, b is a leaf of M' (since b is incident to no ingoing half-edge except the bud indicating the root-corner of O), hence M is a mobile. Moreover, since the rooted-closure and rooted-opening are inverse bijections, it is clear that positive-closure and positive-opening are inverse bijections.

Lastly, it is clear from the definitions that taking an orientation O in \mathcal{O}_d , and applying the positive-opening Ψ_+^{-1} to the dual orientation O^* gives the mobile $\Phi^+(O)$ as defined in Definition 3.

• We now treat the case of the negative-closure Ψ_- . Let M be a $(-d)$ -mobile. We denote by $\theta(M)$ the d -mobile obtained from M by transforming each of its d unmatched outgoing buds into an edge connected to a new white leaf. It is clear (Figure 10) that the positive-closure of $\theta(M)$ is equal to the negative-closure of M , hence $\Psi_- = \Psi_+ \circ \theta$. Moreover, θ is clearly a bijection between the set of $(-d)$ -mobiles and the set of d -mobiles such that every exposed white corner belongs to a leaf (the inverse mapping θ^{-1} replaces each edge incident to an exposed leaf by an outgoing bud). By the fact underlined above, this shows that $\Psi_- = \Psi_+ \circ \theta$ is a bijection between the set of $(-d)$ -mobiles and the set $\tilde{\mathcal{S}}_d$.

Lastly, by denoting $*$ the duality mapping, one gets from the above $\Psi_-^{-1} \circ * = \theta^{-1} \circ \Psi_+^{-1} \circ * = \theta^{-1} \circ \Phi_+$. In other words, taking an orientation O in $\tilde{\mathcal{O}}_d$, and applying the negative-opening Ψ_-^{-1} to O^* , is the same as applying the mapping Φ_+ and then replacing each edge of the mobile incident to an outer vertex by an outgoing bud. By definition, this is the same as applying the mapping Φ_- to O . This completes the proof of Theorem 25 and of Theorem 4. \square

8. ADDITIONAL REMARKS

We have presented a general bijective strategy for planar maps that relies on certain orientations with no counterclockwise circuit. We have applied the approach to an infinite collection $(\mathcal{C}_{d,p})_{d \geq 3, p \geq d}$ of families planar maps, where $\mathcal{C}_{d,p}$ denotes the set of d -angulations of girth d with a boundary of size p . For this

purpose we have introduced the $\frac{d}{d-2}$ -orientations on d -angulations of girth d . In future work we shall further explore and exploit the properties of these orientations:

$\frac{d}{d-2}$ -orientations and Schnyder decompositions. Schnyder showed in [Sch89] that the 3-orientations of triangulations (case $d = 3$) can be given more structure by adding a coloring of the edges. More precisely, for each 3-orientation, it is possible to color the edges with three colors in such a way that each color defines a spanning tree of the triangulation oriented away from the outer vertices. Moreover, there is a precise rule for the crossing of the 3 spanning-trees making the correspondence between the orientations and the colorings bijective. As detailed in an article to come [BFb], a similar property holds for any $d \geq 3$. More precisely, it is shown there that given an $\frac{d}{d-2}$ -orientation of a d -angulation G (seen as an orientation of the graph $(d-2) \cdot G$) it is possible to color the edges of $(d-2) \cdot G$ with d colors, in such a way that each color is a spanning tree oriented away from the outer vertices. Moreover, there is a precise rule for the crossing of the d spanning-trees making the correspondence between the fractional-orientations and the colorings bijective.

Extension of the bijections to planar maps of fixed girth. We have presented a bijection for the class of d -angulations of girth d . This class is the set of “extremal” maps in the larger family of maps of girth d (which have faces of degree at least d). In a future article [BFa], we will show that an extension of our bijection for the d -angulations of girth d allow to count bijectively all the maps of girth d (having a d -gonal outer face) according to the degree distribution of the faces. Indeed, the extended bijection associates to any map of girth d a mobile such that a face of degree i of the map corresponds to a black vertex of degree i in the mobile.

REFERENCES

- [BC] O. Bernardi and G. Chapuy. A bijection for covered maps, or a shortcut between Harer-Zagier’s and Jackson’s formulas. Submitted. ArXiv:1001.1592.
- [Ber07] O. Bernardi. Bijective counting of tree-rooted maps and shuffles of parenthesis systems. *Electron. J. Combin.*, 14(1):R9, 2007.
- [BFa] O. Bernardi and É. Fusy. Bijections for planar maps of fixed girth. In preparation (title subject to change).
- [BFb] O. Bernardi and É. Fusy. Schnyder decompositions for regular plane graphs and application to drawing. Submitted.
- [BFG04] J. Bouttier, P. Di Francesco, and E. Guitter. Planar maps as labeled mobiles. *Electron. J. Combin.*, 11(1):R69, 2004.
- [BFG07] J. Bouttier, P. Di Francesco, and E. Guitter. Blocked edges on eulerian maps and mobiles: Application to spanning trees, hard particles and the Ising model. *J. Phys. A*, 40:7411–7440, 2007.
- [BMJ06] M. Bousquet-Mélou and A. Jehanne. Polynomial equations with one catalytic variable, algebraic series and map enumeration. *J. Combin. Theory Ser. B*, 96(5):623 – 672, 2006.
- [BMS02] M. Bousquet-Mélou and G. Schaeffer. The degree distribution in bipartite planar maps: applications to the Ising model, 2002. arXiv:math.CO/0211070.
- [Bro64] W.G. Brown. Enumeration of triangulations of the disk. *Proc. London Math. Soc.*, 14(3):746–768, 1964.
- [Bro65] W.G. Brown. Enumeration of quadrangular dissections of the disk. *Canad. J. Math.*, 21:302–317, 1965.
- [CS04] Philippe Chassaing and Gilles Schaeffer. Random planar lattices and integrated superBrownian excursion. *Probab. Theory Related Fields*, 128(2):161–212, 2004.

- [CV81] R. Cori and B. Vauquelin. Planar maps are well labeled trees. *Canad. J. Math.*, 33(5):1023–1042, 1981.
- [dFOdM01] H. de Fraysseix and P. Ossona de Mendez. On topological aspects of orientations. *Discrete Math.*, 229:57–72, 2001.
- [dFPP90] H. de Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10(1):41–51, 1990.
- [Fel04] S. Felsner. Lattice structures from planar graphs. *Electron. J. Combin.*, 11(1), 2004.
- [FPS08] É. Fusy, D. Poulalhon, and G. Schaeffer. Dissections, orientations, and trees, with applications to optimal mesh encoding and to random sampling. *Transactions on Algorithms*, 4(2):Art. 19, 2008.
- [FS09] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, New York, NY, USA, 2009.
- [Fus07] É. Fusy. *Combinatoire des cartes planaires et applications algorithmiques*. PhD thesis, École Polytechnique, 2007.
- [GJ83] I. P. Goulden and D. M. Jackson. *Combinatorial Enumeration*. John Wiley, New York, 1983.
- [PS06] D. Poulalhon and G. Schaeffer. Optimal coding and sampling of triangulations. *Algorithmica*, 46(3-4):505–527, 2006.
- [Sch89] W. Schnyder. Planar graphs and poset dimension. *Order*, 5(4):323–343, 1989.
- [Sch98] G. Schaeffer. *Conjugaison d'arbres et cartes combinatoires aléatoires*. PhD thesis, Université Bordeaux I, 1998.
- [Tut63] W. T. Tutte. A census of planar maps. *Canad. J. Math.*, 15:249–271, 1963.