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Pricing and Hedging Basis Risk under No Good Deal Assumption

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Abstract

We consider the problem of pricing and hedging an option written on a non-exchangeable asset when trading in a correlated asset is possible. This is a typical case of incomplete market where it is well known that the super-replication concept provides generally too high prices. Here, following J.H. Cochrane and J. Saá-Requejo, we study valuation under No Good Deal (NGD) Assumption. First, we clarify the notion of NGD for dynamic strategies, compute a lower and an upper bound and prove that in fact NGD price can be strictly higher than the one previously computed in the literature. We also propose a hedging strategy by imposing a criterion on the variance of the replication's error. Finally, we provide various numerical illustrations showing the efficiency of NGD pricing and hedging.

1 Introduction

In this paper, we provide new elements for pricing and hedging Basis Risk. We consider the problem of an agent receiving (or paying) a derivative written on a risky asset V on which trading is not possible, not allowed or costly. For example, for liquidity reasons, an investor can sell an option on a stock and prefer to hedge with the associated index, or in the commodities market hedge with Fioul Oil 1% an option on Fioul Oil Straight Run 0,5%. In all these cases, one considers a more liquid asset S which is highly correlated to V and then prices and hedges investing in S and cash only.

This is a typical incomplete market and the natural extension of No Arbitrage pricing, i.e. replication, is the super-replication concept. But, in the Black Scholes diffusion world, it is well known that it leads to unreasonably high values. For example, the super-replication price of a call option on a non-tradable asset is equal to the initial value of this asset if it is possible to buy it at initiation.

Another method has been introduced by J.H. Cochrane and J. Saá-Requejo [CSR01]: the No Good Deal (NGD) pricing. The idea is to exclude from admissible strategies, portfolios which have too high "Sharp Ratio" because similarly to arbitrage opportunities, good deals will quickly disappear as investors would grab them up. How should we define Sharp ratio? In the economic theory, the Sharp ratio of a claim measures the degree to

which the expected return of the claim is in excess of the risk free rate, as a proportion of the standard deviation of this claim. For dynamic strategy, the definition of Sharp ratio is not so clear and there exists different versions in the literature. We refer to J.H. Cochrane and J. Saá-Requejo [CSR01], T. Björk and I. Slinko [BS06], E. Bayraktar and V. Young [BY08] or S. Klöppel and M. Schweizer [KS07] among others.

S. Klöppel and M. Schweizer [KS07] define Sharp Ratio globally and find that the NGD constraint, i.e. imposing a bound on the Sharp ratio of any portfolio based on exchangeable claims, is equivalent to a bound on the variance of the density of the pricing measures. Note that this definition of Sharp Ratio and No Good Deal price is linked to the notion of coherent risk measure and coherent NGD utility function [Che08].

J.H. Cochrane and J. Saá-Requejo [CSR01] and T. Björk and I. Slinko [BS06] use an instantaneous notion of Sharp Ratio and the authors assert that the NGD constraint lead to a bound on the market (coverable and uncoverable) risk premium. We remark that only a bound on the coverable risk premium naturally appears and that consequently their notion of NGD price is not directly relied to instantaneous definition of Sharp Ratio. We also show that it is also not relied to the global Sharp Ratio.

We choose to define No Good Deal using a global Sharp Ratio similar to the one of S. Klöppel and M. Schweizer [KS07]. Then we introduce NGD price as the minimum initial wealth such that there exists a strategy leading to a residual wealth (after delivering the claim) having a positive coherent NGD utility function (see (16) and (17)). As the super-replication price, the NGD price can be dually represented by the supremum over all pricing measure with bounded variance (by a constant linked to NGD constraint). The pricing measure (also called equivalent martingale measure or EMM) can be represented by their densities which depend on the coverable and uncoverable risk premium. This last quantity is a stochastic process and it is not possible to transfer our maximization constraint on it. In fact, if we set a bound on the market (coverable and uncoverable) risk premium, then the global Sharp Ratio is bounded but the reverse is not true. Thus, we were not able to solve the maximization problem induced by NGD pricing. We propose some upper and lower bounds for it and give analytical recipe to compute them. Then we show that our lower bound can be significantly higher than the prices computed by J.H. Cochrane and J. Saá-Requejo [CSR01] or T. Björk and I. Slinko [BS06] (which are equal).

One aspect not developed in our knowledge until now, is the hedging strategy associated to No Good Deal prices. We first show that in contrast to the super-replication notion, no natural strategy appears from No Good Deal concept. To overcome this drawback, we propose to impose a hedging risk criterium. Since the market is incomplete, it can not lead to pure replication strategies. Thus we propose to minimize the variance under the historical probability of the replication error. This notion has been firstly introduced by D. Duffie and H.R. Richardson [DR91] and M. Schweizer [Sch92] and extended by C. Gourieroux and *al.* (see [GLP98]) and lead to hedging and pricing refereed as Minimum Variance. It is a quadratic minimization type problem and the idea is thus to project our derivative on the set of all strategies induced by the tradable asset S and the cash. Since those assets are not martingale under the historical probability, this is not technically possible and therefore we will use the classical tool of change of numéraire. We obtain a closed form formula for the hedging strategies and the error associated to this strategy. This error can be divided in two parts. The first one is an initial wealth effect linked to the fact that we don't start from the Minimum Variance price. The second tends to zero

when the two assets are perfectly correlated and is the variance of the error linked to the non coverable risk.

In the last part, we perform numerical experiments. We consider a non exchangeable asset which is more risky but provides higher returns than the exchangeable one. We are typically in the case of a very liquid index and a less liquid constituent of this index. We first compute and compare NGD prices : the price of J.H. Cochrane and J. Saá-Requejo [CSR01] (or[BS06]) and our lower and upper bounds. We also analyze the sense of variation of NGD prices and their convergence. In a second time, we compute our minimum variance strategy and compare it to other possible hedging strategies such that buy and hold or Black Scholes strategies. The comparison is made through three points of view : probability of super-replication, expected loss and VaR. We find that our strategy leads to better result.

The rest of the paper is organized as follows: in section 2, we present the financial model. We defined the set of EMM, which must be in L^2 in our context. We rely the coverable and uncoverable risk premium to the variance of densities of EMM. In section 3, we review the various notions of “Sharp Ratio” in the literature and their implication for No Good Deal price definition. In section 4, we provide comparison between those No Good Deal prices and especially, we show that they can be strictly different depending on parameters of the model. Section 5 deals with computation of the minimal variance hedging strategy. Section 6 is devoted to numerical experiments. The technical proofs of the paper are group in Appendix.

2 The financial model

We consider the problem of pricing and hedging a derivative written on a risky asset V on which trading is not possible or not allowed. We assume that we can observe the price of V at each time. We will investigate the case where there exists a risky asset S , which is similar to V and is traded in the market. This similarity will be measured thanks to the correlation between the risk’s sources of both assets. The financial market contains also a non risky asset called S^0 .

Let $(W_t)_{0 \leq t \leq T}$ and $(W_t^*)_{0 \leq t \leq T}$ be two independent real-valued Brownian motion, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

$$dS_t^0 = S_t^0 r dt \tag{1}$$

$$dS_t = S_t(\mu_S dt + \sigma_S dW_t) \tag{2}$$

$$dV_t = V_t(\mu_V dt + \sigma_V(\rho dW_t + \sqrt{1 - \rho^2} dW_t^*)) \tag{3}$$

where

- r is the \mathbb{R} -valued instantaneous risk free rate,
- μ_S and σ_S are \mathbb{R} -valued drift and volatility of S ,
- μ_V and σ_V are \mathbb{R} -valued drift and volatility of V ,
- ρ is the correlation between risk sources of both assets (W and $\rho W + \sqrt{1 - \rho^2} W^*$), with $-1 < \rho < 1$.

We will use the notations $h_S = \frac{\mu_S - r}{\sigma_S}$ and $h_V = \frac{\mu_V - r}{\sigma_V}$ for Sharp ratio (in the classical sense) of the assets S et V respectively.

We shall denote by $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ the \mathbb{P} -augmentation of the filtration $\mathcal{F}_t^{W, W^*} = \sigma(W(s), W^*(s) / 0 \leq s \leq t)$ generated by $(W, W^*)^*$. \mathbb{F} represents the flow of total information on $[0, T]$ where $T > 0$ is a finite time horizon.

We also introduce the following notation : for any probability \mathbb{Q} on (Ω, \mathcal{F}) , $L^0(\mathbb{Q})$, $L^\infty(\mathbb{Q})$, $L^p(\mathbb{Q})$ for $p > 0$ will denote respectively the set of measurable, measurable and \mathbb{Q} -almost surely bounded, and measurable and such that the p -moment exists random variables. For any martingale M , $L_{loc}^2(M)$ will be the space of progressively measurable processes λ such that $\int_0^t \lambda_s^2 ds < M, M >_s < +\infty$.¹

The expectation and the variance computed under \mathbb{P} will be denote by \mathbb{E} and Var , the expectation and the variance computed under \mathbb{Q} will be denote by $\mathbb{E}^{\mathbb{Q}}$ and $\text{Var}^{\mathbb{Q}}$.

Let us introduce the set of pricing measure

$$\mathcal{M}^e(\mathbb{P}) = \{ \mathbb{Q} \sim \mathbb{P} : S/S^0 \text{ is a } \mathbb{Q} \text{ martingale} \}.$$

The Fundamental Theorem of Asset Pricing asserts that under some kind of no arbitrage condition $\mathcal{M}^e(\mathbb{P}) \neq \emptyset$, see F. Delbaen and W. Schachermayer [DS94]². In our simple setup it is easy to prove directly that $\mathcal{M}^e(\mathbb{P})$ is non empty (see below) and thus that no arbitrage condition holds.

In the context of pricing with the No Good Deal principle, we need to introduce the space

$$\mathcal{M}^2(\mathbb{P}) := L^2(\mathbb{P}) \cap \mathcal{M}^e(\mathbb{P}).^3$$

We will see in Lemma 1 that $\mathcal{M}^2(\mathbb{P})$ is also non empty. To this end we have to precise the set of pricing measure. Let $\lambda \in L_{loc}^2((W, W^*))$, we define Z_T^λ and Y_T^λ by

$$Z_T^\lambda = \exp \left(-h_S W_T - \frac{1}{2} h_S^2 T + \int_0^T \lambda_s dW_s^* - \frac{1}{2} \int_0^T \lambda_s^2 ds \right) \quad (4)$$

$$= Z_T^0 Y_T^\lambda \quad (5)$$

h_S is interpreted as the risk premium of the hedgeable risk W and $-\lambda$ as the risk premium of the non-hedgeable risk W^* . From now, we call \mathbb{Q}^λ the probability measure such that $Z_T^\lambda = d\mathbb{Q}^\lambda/d\mathbb{P}$, for $\lambda \in L_{loc}^2((W, W^*))$. We show in Lemma 1 below that \mathbb{Q}^λ is a so called pricing measure, i.e. $\mathbb{Q}^\lambda \in \mathcal{M}^e(\mathbb{P})$. Note that in a market where only the information on the tradeable asset S is available (i.e. $\mathbb{F} = \sigma(W(s) / 0 \leq s \leq T)$), $\mathcal{M}^2(\mathbb{P}) = \{\mathbb{Q}^0\}$.

Lemma 1. *We denote by Λ the set of $\lambda \in L_{loc}^2((W, W^*))$ such that Z_T^λ is a square integrable martingale. The space $\mathcal{M}^2(\mathbb{P})$ is explicitly given by*

$$\mathcal{M}^2(\mathbb{P}) = \left\{ \mathbb{Q} \mid \exists \lambda \in \Lambda \text{ s.t. } \frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T^\lambda \right\} \quad (6)$$

and is non empty.

¹ $\langle M, M \rangle$ is the bracket of M , see D. Revuz and M. Yor [RY94] p 118 for a precise Definition : $\langle M, M \rangle$ is the unique continuous process vanishing in 0 such that $M^2 - \langle M, M \rangle$ is a continuous local martingale. For example, for a Brownian motion W , $d \langle W, W \rangle_t = dt$.

² In general this holds true for local martingale.

³ by $\mathbb{Q} \in L^2(\mathbb{P})$, we mean that the density of \mathbb{Q} w.r.t. \mathbb{P} is in L^2 .

Proof. See Appendix 7.2.1 □

For the sequel, we need to express $\text{Var}(Z_T^\lambda)$ for any $\lambda \in \Lambda$.

Lemma 2. *Let $\lambda \in \Lambda$ then*

$$\text{Var}(Z_T^\lambda) = e^{h_S^2 T} \mathbb{E}^{\tilde{\mathbb{Q}}} \left(e^{\int_0^T \lambda_s^2 ds} \right) - 1, \quad (7)$$

where

$$d\tilde{\mathbb{Q}}/d\mathbb{P} = \exp \left(-2h_S W_t - 2h_S^2 t + 2 \int_0^t \lambda_s dW_s^* - 2 \int_0^t \lambda_s^2 ds \right).$$

If λ is a constant process we get that

$$\text{Var}(Z_T^\lambda) = e^{(h_S^2 + \lambda^2)T} - 1. \quad (8)$$

Note that a bound on the process λ implies a bound on the L^2 moment of the density Z_T^λ . This remark will be fundamental when defining the Sharp Ratio of a wealth process.

Proof. See Appendix 7.2.2 □

We now define the space of trading strategies in (S^0, S) denoted by \mathcal{S} . Two kinds of constraints need to be impose on a strategy (Φ^0, Φ^1) : (i) conditions such that the associated wealth $X_t := \Phi_t^0 S_t^0 + \Phi_t^1 S_t$ is in $L^2(\mathbb{P})$ (ii) conditions in order to avoid strategies leading to arbitrage.

Definition 1. *A strategy $(\Phi^0, \Phi^1) \in \mathcal{S}$ is a \mathbb{R}^2 -valued predictable process such that :*
(i) *the associated wealth process X defined by $X_t := \Phi_t^0 S_t^0 + \Phi_t^1 S_t \in L^2(\mathbb{P})$*
(ii) *$\frac{X_t}{S_t^0}$ is a \mathbb{Q} -martingale under all $\mathbb{Q} \in \mathcal{M}^2(\mathbb{P})$.*

The technic of numéraire change is a classical tool in Finance. The main idea is to express the financial assets in units of another asset called numéraire. In general, this asset is a bank account or some bonds but theoretically it could be any process U such that Assumption 1 below is satisfied.

Assumption 1. *A numéraire U is an \mathbb{F} -adapted, positive semi-martingale such that $1/U$ is also a semi-martingale and $U_0 = 1$.*

For example S^0 satisfied Assumption 1. We define the notion of self financing with respect to some numéraire U .

Definition 2. *A strategy $(\Phi^0, \Phi^1) \in \mathcal{S}$ is U -self financed in $\left(\frac{S^0}{U}, \frac{S}{U}\right)$ if and only if :*

$$\begin{aligned} (\Phi^0, \Phi^1) &\in L \left(\frac{S^0}{U}, \frac{S}{U} \right) \\ \Phi^0 d \left(\frac{S^0}{U} \right) + \Phi^1 d \left(\frac{S}{U} \right) &= d \left(\frac{\Phi^0 S^0 + \Phi^1 S}{U} \right) \end{aligned} \quad (9)$$

The set of such strategies is called \mathcal{A}_2^U .

Note that if X is a semi-martingale, $L(X)$ is the set of progressively measurable processes integrable with respect to X (see Protter [Pro90], p134).

Then if $(\Phi^0, \Phi^1) \in \mathcal{A}_2^U$, then we get

$$X_t = U_t \left(X_0 + \int_0^t \Phi_s^0 d\frac{S_s^0}{U_s} + \int_0^t \Phi_s^1 d\frac{S_s}{U_s} \right). \quad (10)$$

If $U_t = S_t^0$, we obtain the classical wealth representation

$$X_t = S_t^0 \left(X_0 + \int_0^t \Phi_s^1 d\frac{S_s}{S_s^0} \right). \quad (11)$$

A 1-self financed strategy is called shortly self financed and the set of such strategy is called \mathcal{A}_2 . The following Lemma gives the equivalence between notion of self-financing:

Lemma 3. *Under Assumption 1, $\mathcal{A}_2 = \mathcal{A}_2^U$*

Proof. See Appendix 7.2.3. □

Note that, under Assumption 1, if $(\Phi^0, \Phi^1) \in \mathcal{A}_2$ then equation (10) holds.

3 Good deal definition

Roughly speaking, a good deal is an asset or a strategy whose Sharp ratio is too high. Similarly to arbitrage opportunities, good deals will quickly disappears as investors would use them prioritically. The idea of J.H. Cochrane and J. Saá-Requejo [CSR01] is thus to exclude good deals as well as arbitrage opportunities. How should we define the Sharp ratio? In the economic theory, the Sharp ratio of a claim measures the degree to which the expected return of the claim is in excess of the risk free rate, as a proportion of the standard deviation of this claim. To formalize this in an abstract setup, there exists several definitions in the literature. We first analyze them in our context and conclude to the “right” definition to use.

The first definition, the so called conditional instantaneous Sharp Ratio can be found for example in T. Björk and I. Slinko [BS06] or E. Bayraktar and V. Young [BY08]. Let X_t be the value of a self financing strategy at time t , the Sharp ratio is defined by :

$$SR^1(X_t) = \frac{\frac{1}{dt} \mathbb{E} \left(\frac{dX_t}{X_t} / \mathcal{F}_t \right) - r}{\frac{1}{dt} \sqrt{\text{Var} \left(\frac{dX_t}{X_t} / \mathcal{F}_t \right)}} \quad (12)$$

Note first, that the Sharp ratio is not a number but a stochastic process.

Of course, the value of the Sharp ratio will depend upon the type of strategy which are allowed. In T. Björk and I. Slinko [BS06] , only the trading in the non risky asset S^0 and the exchangeable asset S are allowed. Let X_t be the value at time t of a self-financing strategy $(\Phi^0, \Phi^1) \in \mathcal{A}_2$: $X_t = \Phi_t^0 S_t^0 + \Phi_t^1 S_t$. Using the self-financing condition and Equations (1) and (2) :

$$\begin{aligned} dX_t &= \Phi_t^0 dS_t^0 + \Phi_t^1 dS_t = \Phi_t^0 r S_t^0 dt + \Phi_t^1 dS_t = r(X_t - \Phi_t^1 S_t) dt + \Phi_t^1 S_t (\mu_S dt + \sigma_S dW_t) \\ &= \left(X_t \left(r + \Phi_t^1 S_t \frac{\mu_S - r}{X_t} \right) \right) dt + S_t \Phi_t^1 \sigma_S dW_t \end{aligned}$$

Thus

$$\begin{aligned}\mathbb{E}\left(\frac{dX_t}{X_t}/\mathcal{F}_t\right) &= \left(r + \Phi_t^1 S_t \frac{\mu_S - r}{X_t}\right) dt \\ \frac{1}{dt} \text{Var}\left(\frac{dX_t}{X_t}/\mathcal{F}_t\right) &= \frac{S_t^2 (\Phi_t^1)^2 \sigma_S^2}{X_t^2} dt \\ SR^1(X_t) &= \frac{\mu_S - r}{\sigma_S} = h_S\end{aligned}$$

This last quantity is the Sharp ratio (in the classical sense) of the risky asset S and the stochastic process conditional instantaneous Sharp ratio reduces in fact to a number. Note that T. Björk and I. Slinko [BS06] point out that they consider the Sharp ratio of the entire economy (see Remark 3.4). But in fact, they only consider trading in the tradeable underlayings (S^0 and S) and in derivatives which can be attained thanks to strategy in those tradeable underlayings.

E. Bayraktar and V. Young [BY08] also use the notion defined in (12) but they consider the Sharp ratio of a portfolio consisting of the tradeable underlayings and the derivative H they want to price. The difference is major since the price of H depends on the non tradeable asset V . In a first time, they find the portfolio in the tradeable underlayings that minimizes the local variance of the global portfolio (including the derivative). The price of the derivative is then obtained by fixing the instantaneous Sharp ratio to some given value.

We now turn to a second kind of definition of Sharp ratio, the so called unconditional global Sharp Ratio which can be found in S. Klöppel and M. Schweizer [KS07]. The unconditional global Sharp ratio of a claim measures the degree to which the expected return of the claim is in excess of the expected return computed under a risk neutral pricing measure, as a proportion of the standard deviation of this claim. The definition is formally given for any claim X on $(\Omega, \mathcal{F}, \mathbb{P})$ and depends on a measure $\mathbb{Q} \in \mathcal{M}^2(\mathbb{P})$.

$$SR^2(X, \mathbb{Q}) = \frac{\mathbb{E}(X) - \mathbb{E}^{\mathbb{Q}}(X)}{\sqrt{\text{Var}(X)}} \quad (13)$$

If X is constant or $\text{Var}(X) = \infty$, $SR^2(X, \mathbb{Q}) = 0$. The Sharp ratio will be defined if $X \in L^2(\mathbb{P})$ (as $\mathbb{Q} \in L^2(\mathbb{P})$, the Cauchy Schwarz inequality implies that $X \in L^1(\mathbb{Q})$).

For $x \in \mathbb{R}$ and $\mathbb{Q} \in \mathcal{M}^e(\mathbb{P})$ let

$$\mathcal{C}(x, \mathbb{Q}) = \{X \in L^0(\mathbb{P}) : X^- \in L^\infty(\mathbb{P}) \text{ and } \mathbb{E}^{\mathbb{Q}}(X) \leq x\}.$$

This set can be interpreted as the set of claim, which are bounded from below (in order to avoid doubling strategies) and such that there price under the pricing measure \mathbb{Q} is less than x and thus affordable from x if we believe that the pricing measure is \mathbb{Q} . It is easy to see that if $X \in \mathcal{C}(x, \mathbb{Q}) \cap \{X : \mathbb{E}(X) < \infty\} \subset L^1(\mathbb{P})$ the Sharp ratio is also well-defined.

It is clear from the definition that this second notion of Sharp ratio is intimately linked to the choice of a pricing measure : if you believe the right pricing measure is \mathbb{Q} and if you consider a claim which is affordable from some initial wealth, then the Sharp ratio measures, in proportion of standard deviation, the excess between the expected value and the price. It is also a global measure of the performance of a claim X . Moreover it has the following remarkable property, which is report without proof in S. Klöppel and M. Schweizer [KS07] :

Proposition 1. *Let $\mathbb{Q} \in \mathcal{M}^2(\mathbb{P})$ and $x \in \mathbb{R}$ then*

$$\sup_{X \in \mathcal{C}(x, \mathbb{Q}) \cap \{X : \mathbb{E}(X) < \infty\}} SR^2(X, \mathbb{Q}) = \sqrt{\text{Var } Z_T}.$$

Proof. First we prove that the supremum is less or equal to $\sqrt{\text{Var } Z_T}$. If $X \in \mathcal{C}(x, \mathbb{Q}) \cap \{X : \mathbb{E}(X) < \infty\} \setminus L^2(\mathbb{P})$ then $SR^2(X, \mathbb{Q}) = 0$ else as $\mathbb{E}(X) - \mathbb{E}^{\mathbb{Q}}(X) = \mathbb{E}[(X - \mathbb{E}(X))(1 - Z_T)]$, the required inequality follows from Cauchy-Schwarz inequality. In order to prove that there is in fact equality, consider the sequence $X_n = x - Z_T \mathbf{1}_{Z_T \leq n}$, then $X_n \in \mathcal{C}(x, \mathbb{Q}) \cap \{X : \mathbb{E}(X) < \infty\}$. Moreover,

$$SR^2(X_n, \mathbb{Q}) = \frac{\mathbb{E}((Z_T \mathbf{1}_{Z_T \leq n}(Z_T - 1)))}{\sqrt{\text{Var}(Z_T \mathbf{1}_{Z_T \leq n})}},$$

which converges by Lebesgues Theorem to $\sqrt{\text{Var}(Z_T)}$. \square

So, for a given pricing measure and an initial wealth, imposing a bound on the Sharp ratio SR^2 of all affordable claim is equivalent to impose exactly the same bound on the variance of the density of the pricing measure. Note that the bound does not depend on initial wealth (which could be chosen equal to 0). As already mention in the introduction, pricing under the No Good Deal assumption requires to compute the supremum of the discounted claim under all the pricing measure when excluding the Good Deals, i.e. when putting a bound on the Sharp Ratio of all affordable claim. With SR^2 definition, it means to compute the supremum of the discount claim under all pricing measure with a bounded variance. The result of Proposition 1 is thus very important for the resolution of our problem of pricing, since with the definition of SR^1 , it is not possible to achieve the same conclusion; recall that $SR^1(X_t) = h_S$. The information obtains thanks to a bound on SR^2 is thus richer than the one using SR^1 . So, we will choose to define the Sharp Ratio by equation (13), i.e. with SR^2 .

Below we precise the restriction used for pricing under No Good Deal by J.H. Cochrane and J. Saá-Requejo ([CSR01]) and T. Björk and I. Slinko ([BS06]) and explain why in our opinion it is not directly relied to a restriction on the Sharp Ratio neither defined by SR^1 nor by SR^2 . First, we recall that J.H. Cochrane and J. Saá-Requejo ([CSR01]) defined their No Good Deal pricing rule by imposing a bound on $\frac{1}{dt} \mathbb{E} \left[\left(\frac{dZ_t^\lambda}{Z_t^\lambda} \right)^2 \right]$, which is equivalent to a bound on the process risk premium on the non coverable risk $(\lambda_t)_t$ (recall Equation (4) for definition of $(\lambda_t)_t$). T. Björk and I. Slinko ([BS06]) also defined their No Good Deal pricing rule by putting a bound on $(\lambda_t)_t$.

The first question is how to rely a restriction on $(\lambda_t)_t$ and a bound on SR^1 or SR^2 . The argument of T. Björk and I. Slinko ([BS06]), following L. Hansen and R. Jagannathan [HJ91], is to say that $|SR^1(X_t)| = |h_S| \leq |(-h_S, \lambda_t)|_{\mathbb{R}^2}$. Then instead of imposing a bound on the Sharp Ratio $SR^1(X_t)$ they rather put a bound on $|(-h_S, \lambda_t)|_{\mathbb{R}^2}$. This is of course mathematically correct but from our opinion it is not economic meaning full because from the first definition of Sharp ratio only a bound on h_S , the risk premium on the coverable risk W , naturally appears. So imposing a bound on $(\lambda_t)_t$ is not economically relied to the Sharp Ratio definition SR^1 ⁴.

⁴The argument of J.H. Cochrane and J. Saá-Requejo ([CSR01]) in order to link h_S and $\frac{1}{dt} \mathbb{E} \left[\left(\frac{dZ_t}{Z_t} \right)^2 \right]$ is less clear, specially in continuous time, but they also refer to L. Hansen and R. Jagannathan [HJ91].

The next question is then : is it mathematically equivalent to put a bound on SR^2 or on $(\lambda_t)_t$? The answer is no, except as if $(\lambda_t)_t$ is a constant process (see Equation (8)). In the general case, as we only know that $(\lambda_t)_t$ is progressively measurable, the story is completely different. From Equation (7) and Proposition 1, if you have a bound on λ_t then SR^2 is also bounded. But the reverse is not automatically true. In fact, we will present in section 4 a counter example which shows that a price with constraint on SR^2 can be significantly greater than a price with constraint on the risk premium on the non hedgeable risk $(\lambda_t)_t$. To conclude this paragraph, the No Good Deal Prices computed by J.H. Cochrane and J. Saá-Requejo [CSR01] and T. Björk and I. Slinko [BS06] by putting a bound on λ is smaller than the one computed by putting a bound on SR^2 and, in our opinion, not directly related to the No Good Deal principle.

We now define a good deal of level β for a pricing measure $\mathbb{Q} \in \mathcal{M}^e(\mathbb{P})$ named shortly as $GD(\beta, \mathbb{Q})$.

Definition 3. Let $\beta > 0$ and $\mathbb{Q} \in \mathcal{M}^e(\mathbb{P})$, X is a (β, \mathbb{Q}) -good deal $GD(\beta, \mathbb{Q})$ if $\exists x \in \mathbb{R}$ such that $X \in \mathcal{C}(x, \mathbb{Q}) \cap \{X : \mathbb{E}(X) < \infty\}$ and $SR^2(X, \mathbb{Q}) > \beta$.

Following the No Good Deal literature, we will assume that

Assumption 2. There exists $\mathbb{Q} \in \mathcal{M}^2(\mathbb{P})$ and $\beta > 0$, such there is no (β, \mathbb{Q}) -good deal ($NGD(\beta, \mathbb{Q})$), i.e. for all $x \in \mathbb{R}$ and $X \in \mathcal{C}(x, \mathbb{Q}) \cap \{X : \mathbb{E}(X) < \infty\}$, $SR^2(X, \mathbb{Q}) \leq \beta$.

From Proposition 1, it follows that:

Theorem 1. Assumption 2 is equivalent to

$$\mathcal{M}^{2,\beta}(\mathbb{P}) := \left\{ \mathbb{Q} \in \mathcal{M}^2(\mathbb{P}) : \|Z_T\|_{L^2(\mathbb{P})} \leq \sqrt{1 + \beta^2} \right\} \neq \emptyset.$$

Note again that the initial wealth does not influence No Good Deal notion and don't appear in the characterization above.

Proof. The first implication is a direct consequence of Proposition 1. Now assume that $\mathcal{M}^{2,\beta}(\mathbb{P}) \neq \emptyset$ and choose $\mathbb{Q} \in \mathcal{M}^{2,\beta}(\mathbb{P})$ then again by Proposition 1, for all $x \in \mathbb{R}$ and $X \in \mathcal{C}(x, \mathbb{Q}) \cap \{X : \mathbb{E}(X) < \infty\}$

$$SR^2(X, \mathbb{Q}) \leq \sup_{X \in \mathcal{C}(x, \mathbb{Q}) \cap \{X : \mathbb{E}(X) < \infty\}} SR^2(X, \mathbb{Q}) = \sqrt{\text{Var } Z_T} \leq \beta,$$

and $NGD(\beta, \mathbb{Q})$ holds. □

We end this section by remarking that there exists arbitrage opportunities which are not good deals. In a general context, it is thus necessary to have Assumption 2 together with a No Arbitrage Opportunity Assumption.

Consider a Cox-Ross-Rubinstein model of one period where the exchangeable asset is equal to S_0 at time 0 and S_0u with probability $0 < p < 1$ and S_0d with probability $0 < 1 - p < 1$, where $d < u$, at time 1. We also assume the existence of a non risky asset equal to 1 at time 0 and $1 + r$ at time 1. We assume that $d \geq 1 + r$. We consider the claim X_1 which is obtained at time 1 from the following strategy at time 0 : buy one unit of risky asset and finance this by getting short of S_0 units of non risky asset. The value of

this strategy is equal to 0 at time 0 and $X_1 = S_1 - S_0(1 + r)$ at time 1 and is clearly an arbitrage opportunity. Now consider a probability measure $(q, 1 - q)$, it is easy to see that

$$SR^2(X_1, q) = \frac{p - q}{\sqrt{p(1 - p)}}.$$

For example if $q > p$, for all β , X_1 is not a (β, q) -good deal. If $q < p$, it is sufficient to choose β such that $\beta > \frac{p - q}{\sqrt{p(1 - p)}}$ in order to show that X_1 is not a (β, q) -good deal.

4 No Good Deal Pricing

In this section, we will investigate the notion of pricing for a contingent claim H depending on the non-traded asset V . Since the market is imperfect, this notion must be clarified. A standard tool is the use of the super-replication price: intuitively, it is the minimal price which ensures in any situation the hedgeability of H . Mathematically, it is defined as the minimal initial wealth such that there exists a strategy leading to a terminal value almost surely over the claim. For example, for a call option, since H depends on the non traded asset, one can show that the super-replication price is $+\infty$ if the investor is not endowed with at least one unit of V (else it is equal to V_0).⁵

The super-replication price has a so called dual representation ; it is equal to the supremum over all pricing measure $\mathbb{Q} \in \mathcal{M}^e(\mathbb{P})$ of the expectation of the discounted payoff, i.e. $\mathbb{E}^{\mathbb{Q}}(\frac{H}{S_T^0})$.

The definition chosen by J.H. Cochrane and J. Saá-Requejo [CSR01] and T. Björk and I. Slinko [BS06] is the following

$$\tilde{p}_0(H) = \sup_{\lambda \in L_{loc}^2((W, W^*)), s.t. \lambda \in [-\lambda^{max}, \lambda^{max}]} \mathbb{E} \left[Z_T^\lambda \frac{H}{S_T^0} \right], \quad (14)$$

where

$$\lambda^{max} = \sqrt{\frac{1}{T} \ln(1 + \beta^2) - h_S^2}. \quad (15)$$

Note that from Equation (7), if $\lambda = (\lambda_t(\omega)) \in [-\lambda^{max}, \lambda^{max}]$ then $Z_T^\lambda \in \mathcal{M}^{2,\beta}(\mathbb{P})$ but the reverse is not true in general as already mentioned in section 3.

No rigorous justification is given by the authors for the choice of pricing rule (14) as a dual representation of some No Good Deal price. To do so, we need to use the notion of coherent risk measure as already notice in S. Klöppel and M.Schweizer [KS07] or A.S. Cherny [Che08]. We set u as the coherent utility function⁶ related to the No Good Deal valuation, i.e.

$$u(X) = \inf_{\mathbb{Q} \in \mathcal{M}^{2,\beta}(\mathbb{P})} \mathbb{E}^{\mathbb{Q}} \left[\frac{X}{S_T^0} \right]. \quad (16)$$

⁵In fact, if we start with a finite wealth X_0 , since H depends on W^* through the non traded V , we have that for any strategy (Φ^0, Φ) , $\mathbb{P}[\Phi_T^0 S_T^0 + \Phi_T S_T < H] \neq 0$. Now if the investor is endowed with one unit of V : $\mathbb{P}(V_T > (V_T - K)_+) = 1$.

⁶Since $\mathcal{M}^{2,\beta}(\mathbb{P})$ is non-empty, Theorem 2.2 of A.S. Cherny [Che08] ensures that u is a so called coherent utility function.

The notion of hedgeability used in the super-replication price is now replaced by the notion of having a positive coherent utility : the No Good Deal upper-bound price is the minimal initial wealth such that there exists a strategy leading to a residual wealth having a positive utility. More precisely,

$$p_0(H) = \inf \left\{ m \in \mathbb{R} \mid \exists \Phi \in \mathcal{A}_2 \text{ s.t. } u \left(m + \int_0^T \Phi_t^0 dS_t^0 + \int_0^T \Phi_t^1 dS_t - H \right) \geq 0 \right\}. \quad (17)$$

Note that if u is identity we are back to the super-replication price definition.

Theorem 2. *Under the Assumption 2, the dual representation of the No Good Deal price defined in (17) is*

$$p_0(H) = \sup_{\mathbb{Q} \in \mathcal{M}^{2,\beta}(\mathbb{P})} \mathbb{E}^{\mathbb{Q}} \left[\frac{H}{S_T^0} \right]. \quad (18)$$

Proof. Let $\Phi \in \mathcal{A}_2$, then from Lemma 3, $\Phi \in \mathcal{A}_2^{S^0}$, and using self financing Equation (9) and (11), m being the initial value of the strategy Φ , we get that

$$m + \int_0^T \Phi_t^0 dS_t^0 + \int_0^T \Phi_t^1 dS_t = \Phi_T^0 S_T^0 + \Phi_T^1 S_T = S_T^0 \left(m + \int_0^T \Phi_t^1 d \frac{S_t}{S_t^0} \right).$$

Let $\mathbb{Q} \in \mathcal{M}^{2,\beta}(\mathbb{P})$, as S_T^0 is deterministic, we get that

$$\mathbb{E}^{\mathbb{Q}} \left[m + \int_0^T \Phi_t^0 dS_t^0 + \int_0^T \Phi_t^1 dS_t \right] = S_T^0 m.$$

And from Definition of u , see (16),

$$u \left(m + \int_0^T \Phi_t^0 dS_t^0 + \int_0^T \Phi_t^1 dS_t - H \right) = m - \sup_{\mathbb{Q} \in \mathcal{M}^{2,\beta}(\mathbb{P})} \mathbb{E}^{\mathbb{Q}} \left[\frac{H}{S_T^0} \right].$$

Thus

$$\begin{aligned} p_0(H) &= \inf \left\{ m \in \mathbb{R} \mid \exists \Phi \in \mathcal{A}_2 \text{ s.t. } m \geq \sup_{\mathbb{Q} \in \mathcal{M}^{2,\beta}(\mathbb{P})} \mathbb{E}^{\mathbb{Q}} \left[\frac{H}{S_T^0} \right] \right\} \\ &= \sup_{\mathbb{Q} \in \mathcal{M}^{2,\beta}(\mathbb{P})} \mathbb{E}^{\mathbb{Q}} \left[\frac{H}{S_T^0} \right]. \end{aligned} \quad (19)$$

This concludes the proof. \square

Remark 1. *In the super-replication theory, there exists so called super-hedging strategies such that starting from the super-replication price and following some super-hedging strategy, H is fully hedge. But in the case of No Good Deal Pricing, no particular strategy appears : see Equation (19). For example, Buy and Hold strategy will do the job. In the next paragraph, we will introduce some hedging criterium starting with an initial wealth equal to the No Good Deal price.*

The computation of the supremum in (18) is not easy. From D. Revuz and M. Yor [RY94], the probability in the space $\mathcal{M}^{2,\beta}(\mathbb{P})$ can be represented by their densities, i.e. Z_T^λ (see Lemma 1). But since λ is a stochastic process, this optimization problem is difficult to handle.

In Theorem 3, we propose to analyze the No Good Deal Price $p_0(H)$ (simply denote by p_0 from now). First, we provide some upper and lower bounds for the No Good Deal Price p_0 . The upper bound will be obtained by removing the positivity assumption and relaxing the martingale condition on the pricing measure density. To define our lower bound, we assume that the risk premium λ of the non-hedgeable risk W^* is independent of the hedgeable risk W . This allows us to fully compute the optimization problem (18) when relaxing the positivity assumption on the pricing measure density. Then in order to obtain an equivalent martingale measure we just add the solution of (14).

Then we investigate the link between No Good Deal Price p_0 and the price \tilde{p}_0 proposed by J.H. Cochrane and J. Saá-Requejo [CSR01] and by Björk and I. Slinko [BS06] (see (14)). We show that the No Good Deal Price can be strictly greater than \tilde{p}_0 . Note that this is possible for the Call option but also for any claim H that has a closed form price in the Black and Scholes model.

We also introduce a “degenerated” version of p_0 , called \hat{p}_0 , defined as the supremum of the discounted payoff over particular pricing measure in $\mathcal{M}^{2,\beta}(\mathbb{P})$. In fact, we assume that the risk premium process on non coverable risk W^* , λ , is a constant number. We introduce this price because with this restriction, it is strictly equivalent to put a bound on λ or SR^2 , i.e. $\text{Var}(Z_T)$ (see Equation (8)).

$$\hat{p}_0(H) = \sup_{Z_T^\lambda \in \mathcal{M}^{2,\beta}(\mathbb{P}) \text{ s.t. } \lambda \in \mathbb{R}} \mathbb{E} \left[Z_T^\lambda \frac{H}{S_T^0} \right]$$

Theorem 3. *Assume that $\frac{1}{T} \ln(1 + \beta^2) \geq h_S^2$, $H = (V_T - K)_+$ and Assumption 2 holds. Let*

$$\bar{\beta} = \sqrt{(1 + \beta^2)e^{-h_S^2 T} - 1}. \quad (20)$$

Then

$$p_0^{LB} = \varepsilon \tilde{p}_0 + (1 - \varepsilon)e^{-rT} \mathbb{E}(Z_T^0 Y^{down} H) \leq p_0 \leq p_0^{UB} = e^{-rT} \mathbb{E}(Z^{UB} H),$$

where $\varepsilon \in (0, 1)$, Y^{down} is defined in Lemma 4 and

$$Z^{UB} = Z_T^0 + e^{h_S^2 T/2} \bar{\beta} \frac{H - \mathbb{E}(H | \sigma(W_t, t \leq T))}{\sqrt{\mathbb{E} \left[H^2 - \mathbb{E}(H | \sigma(W_t, t \leq T))^2 \right]}}. \quad (21)$$

Moreover,

$$p_0 \geq \tilde{p}_0 = \hat{p}_0 = e^{-rT} BS(V_0, T, K, \mu_V - \sigma_V \rho h_S + \sigma_V \lambda^{max} \sqrt{1 - \rho^2}, \sigma_V) \quad (22)$$

where the functional BS give a kind of Black-Scholes price as a function of the initial price of the stock, the maturity and the strike of the option, the drift and the volatility of

the stock : see Equation (49) in the Appendix for the precise definition and see (15) for the definition of λ^{max} . Finally

$$p_0^{UB} \geq p_0 \geq p_0^{LB} \geq \tilde{p}_0, \quad (23)$$

and there exists some situations where $p_0 \geq p_0^{LB} > \tilde{p}_0$.

Proof. Step 1: Computation of \hat{p}_0

We begin by choosing a $\lambda \in \mathbb{R}$, and by computing

$$p_0^\lambda = \mathbb{E} \left[e^{-rT} Z_T^\lambda (V_T - K)_+ \right]. \quad (24)$$

As λ is a constant process, we have seen (equation (8)) that $Z_T^\lambda \in \mathcal{M}^{2,\beta}(\mathbb{P})$ if and only if

$$\|Z_T\|_{L^2(\mathbb{P})} = e^{\frac{1}{2}(h_S^2 + \lambda^2)T} \leq \sqrt{1 + \beta^2} \Leftrightarrow \lambda \in [-\lambda^{max}, \lambda^{max}],$$

see (15) for the definition of λ^{max} . Thus,

$$\hat{p}_0 = \sup_{\lambda \in [-\lambda^{max}, \lambda^{max}]} p_0^\lambda.$$

From Girsanov Theorem (see for example D. Revuz and M. Yor [RY94]), for any process $(\lambda_t)_t$,

$$W_t^\lambda := W_t + h_S t \quad \text{and} \quad W_t^{\lambda,*} := W_t^* - \int_0^t \lambda_s ds \quad (25)$$

are standard brownian motion under \mathbb{Q}^λ defined by (4).

Thus, for all constant λ , the process V satisfy the stochastic differential equation:

$$dV_t = V_t((\mu_V - \sigma_V \rho h_S + \sigma_V \lambda \sqrt{1 - \rho^2})dt + \sigma_V(\rho dW_t^\lambda + \sqrt{1 - \rho^2} dW_t^{\lambda,*})). \quad (26)$$

We denote by η^λ the drift of this process, i.e.

$$\eta^\lambda = \mu_V - \sigma_V \rho h_S + \sigma_V \lambda \sqrt{1 - \rho^2}.$$

From Appendix (49), we are able now to state that the quantity p_0^λ is given by a Black-Scholes type formula:

$$p_0^\lambda = e^{-rT} BS(V_0, T, K, \eta^\lambda, \sigma_V). \quad (27)$$

Note that BS is an increasing function of η (see Appendix (51)) and consequently p_0^λ is increasing in λ . Back to our optimization problem \tilde{p}_0 , we get that

$$\begin{aligned} \hat{p}_0 &= \sup_{\lambda \in [-\lambda^{max}, \lambda^{max}]} p_0^\lambda = p_0^{\lambda^{max}} \\ &= e^{-rT} BS(V_0, T, K, \mu_V - \sigma_V \rho h_S + \sigma_V \lambda^{max} \sqrt{1 - \rho^2}, \sigma_V). \end{aligned}$$

Step 2: Computation of \tilde{p}_0

The proof is based on a comparison Theorem for solution of stochastic differential equations. In fact, for a progressively predictable process λ_t , following the proof of step 1, we know that the process V^λ follows the SDE (26) replacing λ by λ_t . As $\lambda_t(\omega) \leq \lambda^{max}$,

applying a comparison Theorem (see proposition 2.18 p.393 of I. Karatzas and S. Shreve [KS91]), we get that $V_t^\lambda \leq \bar{V}_t$, $\mathbb{P} - p.s$, where the process \bar{V}_t satisfies

$$d\bar{V}_t = \bar{V}_t \left((\mu_V - \sigma_V \rho h_S + \sigma_V \sqrt{1 - \rho^2} \lambda^{max}) dt + \sigma_V dU_t^* \right),$$

with $\bar{V}_0 = V_0$ and $U_t^* = \rho dW_t^\lambda + \sqrt{1 - \rho^2} dW_t^{*,\lambda}$ a brownian motion under the probability \mathbb{Q}^λ . Thus,

$$\mathbb{E}^{\mathbb{Q}^\lambda} \left[e^{-rT} (V_T^\lambda - K)_+ \right] \leq \mathbb{E}^{\mathbb{Q}^\lambda} \left[e^{-rT} (\bar{V}_T - K)_+ \right].$$

If we compute the right hand side of this inequality using equation (49) (see Appendix), we obtain that

$$\mathbb{E}^{\mathbb{Q}^\lambda} \left[e^{-rT} (\bar{V}_T - K)_+ \right] = e^{-rT} BS(V_0, T, K, \mu_V - \sigma_V \rho h_S + \sigma_V \lambda^{max} \sqrt{1 - \rho^2}, \sigma_V)$$

Thus as $\hat{p}_0 \leq \tilde{p}_0$, step 1 shows that

$$\tilde{p}_0 = \hat{p}_0 = e^{-rT} BS(V_0, T, K, \mu_V - \sigma_V \rho h_S + \sigma_V \lambda^{max} \sqrt{1 - \rho^2}, \sigma_V). \quad (28)$$

Step 3: Definition and computation of p_0^{UB}

Let $\mathcal{F}_T^W = \sigma(W_t, t \leq T)$ and define

$$p_0^{UB} = \sup_{\substack{Z, \mathbb{E}Z^2 \leq 1 + \beta^2 \\ \mathbb{E}(Z | \mathcal{F}_T^W) = Z_T^0}} \mathbb{E} \left[Z \frac{H}{S_T^0} \right] \quad (29)$$

Intuitively, this is an upper bound because we remove the positivity assumption and relax the martingale one on the pricing density Z . Note that the assumption $\mathbb{E}(Z | \mathcal{F}_T^W) = Z_T^0$ is equivalent to S/S^0 is martingale with respect to the hedgeable information only. We are going to prove that

$$p_0^{UB} = p_0^0 + e^{-rT} e^{h_S^2 T / 2} \bar{\beta} \sqrt{\mathbb{E} \left[H^2 - \mathbb{E}(H | \mathcal{F}_T^W)^2 \right]}, \quad (30)$$

see (27) with $\lambda = 0$ for a definition of p_0^0 .

p_0^{UB} is an upper bound: we show that any element $\mathbb{Q}^\lambda \in \mathcal{M}^{2,\beta}(\mathbb{P})$ satisfies the constraints of Problem 29. As $\mathbb{E}(Z_T^\lambda)^2 \leq 1 + \beta^2$, using (5), we get that

$$\mathbb{E} \left(Z_T^\lambda | \mathcal{F}_T^W \right) = Z_T^0 \mathbb{E} \left(Y_T^\lambda | \mathcal{F}_T^W \right) = Z_T^0 \mathbb{E} \left(1 + \int_0^T \lambda_t Y_t^\lambda dW_t^* | \mathcal{F}_T^W \right) = Z_T^0,$$

see, for the last equality, exercise 3.20 of [RY94].

Z^{UB} is the optimal solution of problem 29: First, we show that Z^{UB} (see (21)) satisfies constraints of Problem 29. It is straightforward that $\mathbb{E}(Z^{UB} | \mathcal{F}_T^W) = Z_T^0$. Furthermore, since $H - \mathbb{E}(H | \mathcal{F}_T^W)$ is orthogonal to \mathcal{F}_T^W and thus orthogonal to Z_T^0 :

$$\begin{aligned} \mathbb{E}(Z^{UB})^2 &= \mathbb{E}(Z_T^0)^2 + e^{h_S^2 T} \bar{\beta}^2 \frac{\mathbb{E} \left[H - \mathbb{E}(H | \mathcal{F}_T^W) \right]^2}{\mathbb{E} \left[H^2 - \mathbb{E}(H | \mathcal{F}_T^W)^2 \right]} = \mathbb{E}(Z_T^0)^2 + e^{h_S^2 T} \bar{\beta}^2 \\ &= e^{h_S^2 T} + (1 + \beta^2) - e^{h_S^2 T} = 1 + \beta^2, \end{aligned}$$

because $\mathbb{E}[H\mathbb{E}(H | \mathcal{F}_T^W)] = \mathbb{E}[\mathbb{E}(H | \mathcal{F}_T^W)]^2$ and using successively (8) and (20). Now, we prove that Z^{UB} reaches the maximal value of Problem 29.

$$\begin{aligned}\mathbb{E}(Z^{UB}H) &= \mathbb{E}(Z_T^0H) + e^{h_s^2 T/2} \bar{\beta} \frac{\mathbb{E}(H^2) - \mathbb{E}[H\mathbb{E}(H | \mathcal{F}_T^W)]}{\sqrt{\mathbb{E}[H^2 - \mathbb{E}(H | \mathcal{F}_T^W)]^2}} \\ &= \mathbb{E}(Z_T^0H) + e^{h_s^2 T/2} \bar{\beta} \sqrt{\mathbb{E}[H^2 - \mathbb{E}(H | \mathcal{F}_T^W)]^2}.\end{aligned}$$

Let Z such that $\mathbb{E}Z^2 \leq 1 + \beta^2$ and $\mathbb{E}(Z | \mathcal{F}_T^W) = Z_T^0$, we get that

$$\begin{aligned}\mathbb{E}(ZH) &= \mathbb{E}[Z(H - \mathbb{E}(H | \mathcal{F}_T^W))] + \mathbb{E}[Z\mathbb{E}(H | \mathcal{F}_T^W)] \\ &= \mathbb{E}[(Z - Z_T^0)(H - \mathbb{E}(H | \mathcal{F}_T^W))] + \mathbb{E}[\mathbb{E}(Z | \mathcal{F}_T^W)\mathbb{E}(H | \mathcal{F}_T^W)] \\ &\leq \sqrt{\mathbb{E}[Z - Z_T^0]^2} \sqrt{\mathbb{E}[H - \mathbb{E}(H | \mathcal{F}_T^W)]^2} + \mathbb{E}(Z_T^0H) \\ &\leq \mathbb{E}(Z_T^0H) + e^{h_s^2 T/2} \bar{\beta} \sqrt{\mathbb{E}[H - \mathbb{E}(H | \mathcal{F}_T^W)]^2} = \mathbb{E}(Z^{UB}H),\end{aligned}$$

where we have use successively that Z_T^0 is orthogonal to $H - \mathbb{E}(H | \mathcal{F}_T^W)$, Cauchy-Schwartz inequality and

$$\begin{aligned}\mathbb{E}[Z - Z_T^0]^2 &= \mathbb{E}(Z)^2 - 2\mathbb{E}(ZZ_T^0) + \mathbb{E}(Z_T^0)^2 \\ &\leq 1 + \beta^2 - 2\mathbb{E}(\mathbb{E}(Z | \mathcal{F}_T^W)Z_T^0) + \mathbb{E}(Z_T^0)^2 \\ &\leq 1 + \beta^2 - \mathbb{E}(Z_T^0)^2 = 1 + \beta^2 - e^{h_s^2 T} = e^{h_s^2 T} \bar{\beta}^2\end{aligned}$$

Step 4: Definition and computation of p_0^{LB}

The definition of the lower bound is a little more tricky. We first reformulate our problem using the probability \mathbb{Q}^{Z^0} . \mathbb{Q}^{Z^0} is defined by $d\mathbb{Q}^{Z^0}/d\mathbb{P} = (Z_T^0)^2/\mathbb{E}(Z_T^0)^2 = (Z_T^0)^2 e^{-h_s^2 T}$ (see (8)). Note that this kind of probability will be used in quadratic hedging part of the paper. Using Bayes formula and recalling definition of Y_T^λ (see (5)), we get that

$$\mathbb{E}\left[Z_T^\lambda \frac{H}{S_T^0}\right] = e^{-rT} \mathbb{E}\left[(Z_T^0)^2 Y_T^\lambda \frac{H}{Z_T^0}\right] = e^{(h_s^2 - r)T} \mathbb{E}^{\mathbb{Q}^{Z^0}}\left[Y_T^\lambda \frac{H}{Z_T^0}\right]$$

We now rewrite the constraints of problem (18) :

$$\mathbb{E}[(Z_T^\lambda)^2] = \mathbb{E}[(Z_T^0)^2 (Y_T^\lambda)^2] = e^{h_s^2 T} \mathbb{E}^{\mathbb{Q}^{Z^0}}[(Y_T^\lambda)^2]$$

Using the definition of $\bar{\beta}$ (20), we get that

$$\mathbb{E}[(Z_T^\lambda)^2] \leq 1 + \beta^2 \Leftrightarrow \mathbb{E}^{\mathbb{Q}^{Z^0}}[(Y_T^\lambda)^2] \leq 1 + \bar{\beta}^2. \quad (31)$$

We now assume that λ , the risk premium of the non-hedgeable risk W^* , is independent of the hedgeable risk W . Let $\mathcal{F}_t^{W^*} = \sigma(W_u^*, u \leq t)$. Then $Y_T^\lambda \in \mathcal{F}_T^{W^*}$ and thus

$$\mathbb{E}\left[Z_T^\lambda \frac{H}{S_T^0}\right] = e^{(h_s^2 - r)T} \mathbb{E}^{\mathbb{Q}^{Z^0}}\left[Y_T^\lambda \frac{H}{Z_T^0}\right] = e^{(h_s^2 - r)T} \mathbb{E}^{\mathbb{Q}^{Z^0}}\left[Y_T^\lambda \mathbb{E}^{\mathbb{Q}^{Z^0}}\left[\frac{H}{Z_T^0} \mid \mathcal{F}_T^{W^*}\right]\right].$$

We now compute $\mathbb{E}^{\mathbb{Q}^{Z^0}} \left[\frac{H}{Z_T^0} \mid \mathcal{F}_T^{W^*} \right]$ using \mathbb{Q}^0 (see (4) with $\lambda = 0$). As $d\mathbb{Q}^{Z^0}/d\mathbb{Q}^0 = Z_T^0 e^{-h_S^2 T}$,

$$\mathbb{E}^{\mathbb{Q}^{Z^0}} \left[\frac{H}{Z_T^0} \mid \mathcal{F}_T^{W^*} \right] = \frac{\mathbb{E}^{\mathbb{Q}^0} \left[e^{-h_S^2 T} Z_T^0 \frac{H}{Z_T^0} \mid \mathcal{F}_T^{W^*} \right]}{\mathbb{E}^{\mathbb{Q}^0} \left[e^{-h_S^2 T} Z_T^0 \mid \mathcal{F}_T^{W^*} \right]} = e^{-h_S^2 T} \mathbb{E}^{\mathbb{Q}^0} \left[H \mid \mathcal{F}_T^{W^*} \right],$$

because $\mathbb{E}^{\mathbb{Q}^0} \left[e^{-h_S^2 T} Z_T^0 \mid \mathcal{F}_T^{W^*} \right] = e^{-h_S^2 T} \mathbb{E}^{\mathbb{Q}^0} [Z_T^0] = e^{-h_S^2 T} \mathbb{E} \left[(Z_T^0)^2 \right] = 1$.
 $\mathbb{E}^{\mathbb{Q}^0} [H \mid \mathcal{F}_T^{W^*}]$ is fully calculable. If we rewrite V_t with \mathbb{Q}^0 (see (26)):

$$V_t = V_0 e^{\sigma_V \sqrt{1-\rho^2} W_T^*} \times e^{\left(\mu_V - \sigma_V \rho h_S - \frac{\sigma_V^2}{2} \right) T + \sigma_V \rho W_T^0}.$$

Then,

$$\mathbb{E}^{\mathbb{Q}^0} [H \mid \mathcal{F}_T^{W^*}] = \mathbb{E}^{\mathbb{Q}^0} [(V_T - K)_+ \mid \mathcal{F}_T^{W^*}] = \psi(W_T^*), \quad (32)$$

with

$$\begin{aligned} \psi(x) &= \mathbb{E}^{\mathbb{Q}^0} \left[\left(V_0 e^{\sigma_V \sqrt{1-\rho^2} x} \times e^{\left(\mu_V - \sigma_V \rho h_S - \frac{\sigma_V^2}{2} \right) T + \sigma_V \rho W_T^0} - K \right)_+ \right] \\ &= \mathbb{E}^{\mathbb{Q}^0} \left[\left(V_0 e^{\sigma_V \sqrt{1-\rho^2} x} e^{\left(\mu_V - \sigma_V \rho h_S - (1-\rho^2) \frac{\sigma_V^2}{2} - \rho^2 \frac{\sigma_V^2}{2} \right) T + \sigma_V \rho W_T^0} - K \right)_+ \right]. \end{aligned}$$

Therefore ψ can be expressed with a Black Scholes type formula (see (49) in Appendix):

$$\psi(x) = BS \left(V_0 e^{\sigma_V \sqrt{1-\rho^2} x}, T, K; \mu_V - \sigma_V \rho h_S - (1-\rho^2) \frac{\sigma_V^2}{2}, \sigma_V \rho \right).$$

Note that we get similarly, for all $Y \in \mathcal{F}_T^{W^*}$

$$\mathbb{E} \left[Z_T^0 Y \frac{H}{S_T^0} \right] = e^{-rT} \mathbb{E}^{\mathbb{Q}^{Z^0}} [Y \psi(W_T^*)] \quad (33)$$

So going back to our optimization problem

$$\begin{aligned} p_0 &\geq \sup_{\substack{\mathbb{Q}^\lambda \in \mathcal{M}^{2,\beta}(\mathbb{P}) \\ \lambda_t \in \mathcal{F}_t^{W^*}}} e^{-rT} \mathbb{E}^{\mathbb{Q}^\lambda} [(V_T - K)_+] \\ &= \sup_{\substack{\lambda_t \in \mathcal{F}_t^{W^*} \\ \mathbb{E}^{\mathbb{Q}^{Z^0}} (Y_T^\lambda)^2 \leq 1 + \bar{\beta}^2}} e^{-rT} \mathbb{E}^{\mathbb{Q}^{Z^0}} [Y_T^\lambda \psi(W_T^*)] \end{aligned} \quad (34)$$

$$\begin{aligned} &= \sup_{\substack{Y > 0, \mathbb{E}^{\mathbb{Q}^{Z^0}} Y = 1 \\ \mathbb{E}^{\mathbb{Q}^{Z^0}} Y^2 \leq 1 + \bar{\beta}^2}} e^{-rT} \mathbb{E}^{\mathbb{Q}^{Z^0}} [Y \psi(W_T^*)] \end{aligned} \quad (35)$$

We now able to state our new optimization problem used for the computation of the lower bound of Problem 18 (we relax the positivity Assumption on Y).

$$p_0^{down} = \sup_{\substack{Y \geq 0, \mathbb{E}^{\mathbb{Q}^{Z^0}} Y = 1 \\ \mathbb{E}^{\mathbb{Q}^{Z^0}} Y^2 \leq 1 + \bar{\beta}^2}} e^{-rT} \mathbb{E}^{\mathbb{Q}^{Z^0}} [Y \psi(W_T^*)] \quad (36)$$

Let Y^{down} be the solution of problem 36 (see Lemma 4 below) then $\varepsilon Y_T^{\lambda^{max}} + (1 - \varepsilon) Y^{down}$ satisfies conditions of problem 35 (the two first conditions are obviously satisfied and the third one comes directly from Cauchy-Schwartz inequality). From (33),

$$e^{-rT} \mathbb{E}^{\mathbb{Q}^{Z^0}} [Y^{down} \psi(W_T^*)] = e^{-rT} \mathbb{E} [Z_T^0 Y^{down} H]$$

and recalling (33), (24), (27) and (28), we get that

$$\begin{aligned} e^{-rT} \mathbb{E}^{\mathbb{Q}^{Z^0}} [Y_T^{\lambda^{max}} \psi(W_T^*)] &= e^{-rT} \mathbb{E} [Z_T^0 Y_T^{\lambda^{max}} H] = p_0^{\lambda^{max}} \\ &= e^{-rT} BS(V_0, T, K, \eta^{\lambda^{max}}, \sigma_V) = \tilde{p}_0 \end{aligned}$$

So we have found a lower bound for p_0 :

$$p_0 \geq \varepsilon \tilde{p}_0 + (1 - \varepsilon) e^{-rT} \mathbb{E} [Z_T^0 Y^{down} H]$$

It remains to find a solution for problem 36 : this is done in the lemma below which proof is postponed in Appendix 7.2.4.

Lemma 4. *The solution of Problem 36 is :*

if $1 - \bar{\beta} \frac{\mathbb{E}^{\mathbb{Q}^{Z^0}} (\psi(W_T^*))}{\sqrt{\text{Var}^{\mathbb{Q}^{Z^0}} \psi(W_T^*)}} \geq 0$ then

$$\begin{aligned} Y^{down} &= 1 + \bar{\beta} \frac{\psi(W_T^*) - \mathbb{E}^{\mathbb{Q}^{Z^0}} (\psi(W_T^*))}{\sqrt{\text{Var}^{\mathbb{Q}^{Z^0}} \psi(W_T^*)}} \\ p_0^{down} &= \mathbb{E} (\psi(W_T^*)) + \bar{\beta} \sqrt{\text{Var}^{\mathbb{Q}^{Z^0}} \psi(W_T^*)}. \end{aligned}$$

if $1 - \bar{\beta} \frac{\mathbb{E}^{\mathbb{Q}^{Z^0}} (\psi(W_T^*))}{\sqrt{\text{Var}^{\mathbb{Q}^{Z^0}} \psi(W_T^*)}} < 0$ then

$$\begin{aligned} Y^{down} &= \frac{(\psi(W_T^*) - \alpha)_+}{\mathbb{E}^{\mathbb{Q}^{Z^0}} (\psi(W_T^*) - \alpha)_+}, \\ p_0^{down} &= \alpha + (1 + \bar{\beta})^2 \mathbb{E}^{\mathbb{Q}^{Z^0}} (\psi(W_T^*) - \alpha)_+, \end{aligned}$$

⁷The equality between problems 34 and 35 comes from the following observations : let Y^{opt} be the solution of problem 35, and $Y_t^{opt} = \mathbb{E}^{\mathbb{Q}^{Z^0}} (Y^{opt} | \mathcal{F}_t^{W^*})$. As $(Y_t^{opt})_t$ is a L^2 ($\mathbb{Q}^{Z^0}, \mathcal{F}^{W^*}$)-martingale, from Theorem of martingale representation (see for example D. Revuz and M. Yor [RY94]) there exists $k_t \in L_{loc}^2(W^*)$ such that $dY_t^{opt} = k_t dW_t^*$. Let $\lambda_t^{opt} = k_t / Y_t^{opt}$ (note that $Y_t^{opt} > 0$), $\lambda_t^{opt} \in \mathcal{F}_t^{W^*}$. By Ito formula $Y_T^{opt} = Y_0^{opt} + \int_0^T \lambda_t^{opt} Y_t^{opt} dW_t^* = 1 + \int_0^T \lambda_t^{opt} Y_t^{opt} dW_t^* = Y_T^{\lambda^{opt}}$. Thus λ^{opt} satisfies condition of 34 and problem 35 is lower than problem 34. Let λ satisfying condition of 34 then Y_T^λ satisfies condition of 35 and thus the two problems are equal.

where there exists a positive number α such that

$$\frac{\mathbb{E}^{\mathbb{Q}^{Z^0}} (\psi(W_T^*) - \alpha)_+^2}{\left(\mathbb{E}^{\mathbb{Q}^{Z^0}} (\psi(W_T^*) - \alpha)_+\right)^2} = 1 + \bar{\beta}^2.$$

Step 4: Proof of $p_0^{LB} \geq \tilde{p}_0$ First we note that $Y_T^{\lambda^{max}}$ satisfies the constraints of Problem 36 which implies that $p_0^{down} \geq \tilde{p}_0$. Thus, using the definition of p_0^{LB}

$$p_0^{LB} - \tilde{p}_0 = (1 - \varepsilon)(p_0^{down} - \tilde{p}_0) \geq 0.$$

There exists situations where the inequality is strict :

Example 1. Computations have been made with $\mu_V = 0.04$, $\sigma_V = 0.32$, $\mu_S = 0.0272$, $\sigma_S = 0.256$, $V_0 = 15$, $S_0 = 100$, $K = 15$, $r = 0.02$, $T = 0.25$, $\beta = 2$ and $\rho = 0.8$. Those parameters are the one used in our numerical section and represent a meaningful economic situation (see details in section 6). With these parameters, $\tilde{p}_0 = 2.37$ while $p_0^{LB} = 2.59$. The lower bound of the No Good Deal price is 8.4% higher than \tilde{p}_0 . In section 6, we will provide other examples with higher gap. □

Remark 2. Using the same line of arguments as in step 1 above, it is easy to see that $\hat{p}_0(H)$ is also equal to the supremum of the discounted payoff over pricing measure in $\mathcal{M}^{2,\beta}(\mathbb{P})$ such that λ is a deterministic process. In fact, in this case

$$\mathbb{E} \left[Z_T^\lambda \frac{H}{S_T^0} \right] = e^{-rT} BS \left(V_0, T, K, \mu_V - \sigma_V \rho h_S + \sigma_V \frac{1}{T} \int_0^T \lambda_t dt \sqrt{1 - \rho^2}, \sigma_V \right).$$

As BS is increasing in his drift term, the maximum in λ of $\mathbb{E} \left[Z_T^\lambda \frac{H}{S_T^0} \right]$ will be attained for maximum value of $\frac{1}{T} \int_0^T \lambda_t dt$. Using Equation (7), the constraint

$$\|Z_T\|_{L^2(\mathbb{P})} = e^{\frac{1}{2}(h_S^2 T + \int_0^T \lambda_t^2 dt)} \leq \sqrt{1 + \beta^2} \Leftrightarrow \frac{1}{T} \sqrt{\int_0^T \lambda_t^2 dt} \leq \lambda^{max}$$

As from Cauchy Schwartz inequality, $\frac{1}{T} \int_0^T \lambda_t dt \leq \frac{1}{T} \sqrt{\int_0^T \lambda_t^2 dt}$, the maximum in

$$\sup_{Z_T^\lambda \in \mathcal{M}^{2,\beta}(\mathbb{P}) \text{ s.t. } \lambda_t \in \mathbb{R}} \mathbb{E} \left[Z_T^\lambda \frac{H}{S_T^0} \right]$$

is also attained by λ^{max} .

5 Minimal quadratic error hedging

The preceding section allows us to propose a price compatible with the No Good Deal criterium. But as mention in Remark 1, there is no natural hedging strategy associated to this criterium. In this section, we will consider the criterium of minimizing the quadratic

error. For a given initial wealth X_0 , we want to find the self-financed strategy in the *tradable* assets that minimizes the quadratic error (under the historical probability), i.e. the difference between the claim and the final value of the strategy. This concept has been introduced by H. Föllmer and D. Sonderman [FS86], in the martingale case. It is also study by D. Duffie and H. Richardson [DR91] and by M. Schweizer [Sch92]. The general proof was given by C. Gourieroux, J.P. Laurent and H. Pham [GLP98]. In a first time, we will consider the case of a general contingent claim H . Of course, when we study quadratic hedging we have to assume that:

Assumption 3. *The contingent claim H belongs to $L^2(\mathbb{P})$.*

Mathematically, we want to solve the optimization problem:

$$v(H) := \inf_{(\Phi^0, \Phi^1) \in \mathcal{A}_2} \mathbb{E} [H - (\Phi_T^0 S_T^0 + \Phi_T^1 S_T)]^2 \quad (37)$$

$$= \inf_{(\Phi^0, \Phi^1) \in \mathcal{A}_2} \mathbb{E} \left[H - \left(X_0 + \int_0^T (\Phi_t^0 dS_t^0 + \Phi_t^1 dS_t) \right) \right]^2, \quad (38)$$

for the second equality, we denote by $X_0 = \Phi_0^0 S_0^0 + \Phi_0^1 S_0$ and use the self-financing equation (9).

The first question is whether this problem admits a solution or not? The answer is yes and we will construct it explicitly. In fact, from the definition of \mathcal{A}_2 , we can see directly that the solution exists. It is well known that $L^2(\mathbb{P})$ is an Hilbert space under the inner product $(\cdot | \cdot)$ defined by $(X|Y) = \mathbb{E}(XY)$ and the associated norm $\|\cdot\|$. The set $\{\Phi_T^0 S_T^0 + \Phi_T^1 S_T \mid (\Phi^0, \Phi^1) \in \mathcal{A}_2\}$ is a linear closed subset of $L^2(\mathbb{P})$ (see F. Delbaen and W. Schachermayer [DS96] Thm. 2.2.) Thus Problem (37) admits a solution by an Hilbert space projection Theorem (see for example Luenberger [Lue69]). The natural ideal followed by D. Duffie and H. Richardson [DR91] and later by M. Schweizer [Sch92] is to use orthogonality and say that Φ^{0*} and Φ^{1*} are solutions of Problem 37 if and only for any $(\Phi^0, \Phi^1) \in \mathcal{A}_2$, $(H - (\Phi_T^{0*} S_T^0 + \Phi_T^{1*} S_T) | \Phi_T^0 S_T^0 + \Phi_T^1 S_T) = 0$. This leads to a PDE (see Equation 3.1 in [Sch92] for example) which is not straightforward to solve explicitly. The other natural idea is to use a projection argument and to get the explicit projection of H on S^0 and S . But as S^0 and S are not martingale this is not technically possible. So we follow the idea of C. Gourieroux, J.P. Laurent and H. Pham [GLP98] and transform the initial problem in order to get (local) martingales and achieve the projection argument. Let U be a numéraire, such that Assumption 1 and $U_T \in L^2(\mathbb{P})$, then from Self-financing Equation (10) and Lemma 3, we can rewrite our problem as follows :

$$\begin{aligned} v(H) &= \inf_{(\Phi^0, \Phi^1) \in \mathcal{A}_2^U} \mathbb{E} \left[U_T^2 \left(\frac{H}{U_T} - \left(X_0 + \int_0^T \left(\Phi_t^0 d \frac{S_t^0}{U_t} + \Phi_t^1 d \frac{S_t}{U_t} \right) \right) \right)^2 \right] \\ &= \inf_{(\Phi^0, \Phi^1) \in \mathcal{A}_2^U} \mathbb{E}(U_T^2) \mathbb{E}^{\mathbb{Q}^U} \left[\frac{H}{U_T} - \left(X_0 + \int_0^T \left(\Phi_t^0 d \frac{S_t^0}{U_t} + \Phi_t^1 d \frac{S_t}{U_t} \right) \right) \right]^2, \end{aligned}$$

where the probability \mathbb{Q}^U is defined by $d\mathbb{Q}^U/d\mathbb{P} = U_T^2/\mathbb{E}(U_T^2)$. The idea is to find the right U such that $\frac{S^0}{U}$ and $\frac{S}{U}$ are \mathbb{Q}^U (local) martingale and thus be able to do the projection of $\frac{H}{U_T}$ on $\frac{S^0}{U}$ and $\frac{S}{U}$ (by Galtchouk-Kunita-Watanabe Projection Theorem, see for example

J. Jacod [Jac79]). In contrary to C. Gourieroux, J.P. Laurent and H. Pham [GLP98], we do not introduce the so-called variance-optimal martingale measure in order to solve our problem but we show directly that the fact that $\frac{S^0}{U}$ and $\frac{S}{U}$ are \mathbb{Q}^U (local) martingale imposes a particular form on U (see Lemma 5 below). We then solve the problem using Galtchouk-Kunita-Watanabe Projection Theorem for a general H (see Theorem 4) and Ito calculus for a call option (see Theorem 5). The proofs of both Theorems are essentially technical and postponed to Appendix.

Lemma 5. *Let U such Assumption 1 holds and $U_T \in L^2(\mathbb{P})$. We further assume that $\ln(U)$ is an Ito process, i.e. there exist progressively measurable processes a, λ in $L^2_{loc}((W, W^*))$ and $c \in L^1([0, T])$ such that*

$$dU_t^\lambda = U_t^\lambda (a_t dW_t + \lambda_t dW_t^* + c_t dt).$$

Then, $\frac{S^0}{U^\lambda}$ and $\frac{S}{U^\lambda}$ are local martingale under the measure \mathbb{Q}^{U^λ} defined by $d\mathbb{Q}^{U^\lambda}/d\mathbb{P} = (U_T^\lambda)^2 / \mathbb{E}(U_T^\lambda)^2$ if and only if $a_t = -h_S$ and $c_t = r - \lambda_t^2 - h_S^2$, i.e.

$$U_t^\lambda = e^{-h_S W_t + \int_0^t \lambda_s dW_s^* - 3/2(h_S^2 t + \int_0^t \lambda_s^2 ds) + rt}. \quad (39)$$

If λ is deterministic, then $\frac{S^0}{U^\lambda}$ and $\frac{S}{U^\lambda}$ are \mathbb{Q}^{U^λ} -martingale.

Proof. See Appendix 7.2.5. □

Remark 3. *If we do not assume that U is an Ito process but only that $\frac{S^0}{U}$ and $\frac{S}{U}$ are \mathbb{Q}^U local martingale, then one can easily shows using Bayes formula that the probability $\mathbb{Q}^{0,U}$ defined by $d\mathbb{Q}^{0,U}/d\mathbb{P} = U_T/\mathbb{E}(U_T)$ ⁸, belongs to $\mathcal{M}^e(\mathbb{P})$ ⁹. Thus there exists some progressively measurable process λ such that $d\mathbb{Q}^{0,U}/d\mathbb{P} = Z_T^\lambda$ (recall Equation (4) for the Definition of Z_T^λ) and the final value of U comes from an Ito process.*

From Lemma 5, we get an explicit form for the numéraire U^λ but there are still a lot of possible choices. In a first time, we can restricted our attention to constant process λ : this allows us to compute $\mathbb{E}[U_T^\lambda]$ and $\mathbb{E}[U_T^{\lambda^2}]$. We choose to use the particular numéraire U^0 and thus solve

$$v(H) = \inf_{(\Phi^0, \Phi^1) \in \mathcal{A}_2^{U^0}} \mathbb{E}((U_T^0)^2) \mathbb{E}^{\mathbb{Q}^{U^0}} \left[\frac{H}{U_T^0} - \left(X_0 + \int_0^T (\Phi_t^0 d\frac{S_t^0}{U_t^0} + \Phi_t^1 d\frac{S_t}{U_t^0}) \right) \right]^2 \quad (40)$$

Two reasons motivate this choice. The first one is a financial argument : going back to Equation (39), the only process U^λ which is replicable from the tradeable assets (i.e. which does not depends on W^*), and thus can be called a numéraire from a financial point of view is U^0 . The second reason is the mathematical tractability, see Remark 4.

From now on, we will write U for U^0 , thus

$$U_t = e^{-h_S W_t + (r - 3/2 h_S^2) t}, \quad (41)$$

$$d\mathbb{Q}^U/d\mathbb{P} = e^{-2h_S W_T - 2h_S^2 T}. \quad (42)$$

⁸ $\mathbb{Q}^{0,U}$ is the variance optimal probability used by C. Gourieroux, J.P. Laurent and H. Pham [GLP98].

⁹This is true only if S^0 is deterministic, which is not the case of Lemma 5.

see (62) for the last equation.

We define the following two processes W^U and $W^{*,U}$ which, thanks to Girsanov Theorem will be brownian motions under the probability \mathbb{Q}^U :

$$W_t^U = W_t + 2h_S t, W_t^{*,U} = W_t^*. \quad (43)$$

The following Theorem gives the solution to Problem (38) for general H .

Theorem 4. *Assume that Assumption 3 holds. Consider the following Galtchouk-Kunita-Watanabe decomposition $(\Phi^{0,H}, \Phi^{1,H}, b)$ for $0 \leq t \leq T$*

$$\mathbb{E}^{\mathbb{Q}^U} \left(\frac{H}{U_T} | \mathcal{F}_t \right) = \mathbb{E}^{\mathbb{Q}^U} \left(\frac{H}{U_T} \right) + \int_0^t \Phi_l^{0,H} d\frac{S_l^0}{U_l} + \int_0^t \Phi_l^{1,H} d\frac{S_l}{U_l} + \int_0^t b_l dW_l^{*,U}. \quad (44)$$

Then Problem (37) is equivalent to

$$\begin{aligned} & \inf_{(\Phi^0, \Phi^1) \in \mathcal{A}_2^U} \mathbb{E}(U_T^2) \left[\left[\mathbb{E}^{\mathbb{Q}^U} \left(\frac{H}{U_T} \right) - X_0 \right]^2 + \mathbb{E}^{\mathbb{Q}^U} \left(\int_0^T b_t^2 dt \right) + \right. \\ & \left. \mathbb{E}^{\mathbb{Q}^U} \left(\int_0^T \left(h_S (\Phi_t^{0,H} - \Phi_t^0) \frac{S_t^0}{U_t} + (\sigma_S + h_S) (\Phi_t^{1,H} - \Phi_t^1) \frac{S_t}{U_t} \right)^2 dt \right) \right] \end{aligned} \quad (45)$$

If $(\Phi^{0,H}, \Phi^{1,H}) \in \mathcal{A}_2^U$, then $\Phi^0 = \Phi^{0,H}$ and $\Phi^1 = \Phi^{1,H}$ are solutions of Problem 37. The minimum is equal to

$$v(H) = e^{(2r - h_S^2)T} \left[\left(\mathbb{E}^{\mathbb{Q}^U} \left(\frac{H}{U_T} \right) - X_0 \right)^2 + \mathbb{E}^{\mathbb{Q}^U} \left(\int_0^T b_t^2 dt \right) \right].$$

Proof. See Appendix 7.2.6 □

If we want to hedge some practical examples of derivative H , we have to perform the Galtchouk-Kunita-Watanabe of $\mathbb{E}^{\mathbb{Q}^U} \left(\frac{H}{U_T} | \mathcal{F}_t \right)$ and find explicitly $(\Phi^{0,H}, \Phi^{1,H}, b)$. This will be done by Ito Formula. We will compute explicitly the solution for a call option on the non traded asset, i.e. $H = (V_T - K)_+$, in Theorem 5.

Remark 4. *If we choose to solve Problem 40 with $\lambda \neq 0$ instead of $\lambda = 0$, we are not able to find so easily a self-financing strategy, which achieves the minimum. In fact when $\lambda \neq 0$, in Problem 45 the strategy (Φ^0, Φ^1) also appears in the second term. So if for minimizing we put to zero both integrals, we get two equations and thus a unique strategy as a solution. Unfortunately, this strategy is not self-financed. Thus we have to introduce the self-financing constraints and then minimize the sum of the integrals (and not put each of them to zero). This problem is not mathematically tractable.*

Theorem 5. *If $H = (V_T - K)_+$, the solution of Problem 37 is given by*

$$\Phi_t^{0,H} = \frac{U_t}{S_t^0} \left[\frac{\sigma_S + h_S}{\sigma_S} \left(X_0 + \int_0^t \left(h_S K_l + \rho \frac{L_l}{U_l} \right) dW_l^U \right) - \frac{1}{\sigma_S} \left(h_S K_t + \rho \frac{L_t}{U_t} \right) \right], \quad (46)$$

and

$$\Phi_t^{1,H} = \frac{U_t}{\sigma_S S_t} \left[\left(h_S K_t + \rho \frac{L_t}{U_t} \right) - h_S \left(X_0 + \int_0^t \left(h_S K_l + \rho \frac{L_l}{U_l} \right) dW_l^U \right) \right]. \quad (47)$$

The minimum is equal to

$$v(H) = e^{(2r-h_S^2)T} \left[(e^{-rT} BS(V_0, T, K, \mu_V - \sigma_V h_{S\rho}, \sigma_V) - X_0)^2 + (1 - \rho^2) \mathbb{E}^{\mathbb{Q}^U} \left(\int_0^T \left(\frac{L_t}{U_t} \right)^2 dt \right) \right], \quad (48)$$

where

$$\begin{aligned} K_t &= \frac{e^{-r(T-t)}}{U_t} BS(V_t, T-t, K, \mu_V - \sigma_V h_{S\rho}, \sigma_V) \\ L_t &= \sigma_V e^{-r(T-t) + (\mu_V - \sigma_V h_{S\rho})(T-t)} V_t \mathcal{N}(d_1(V_t, T-t, K, \mu_V - \sigma_V h_{S\rho}, \sigma_V)), \end{aligned}$$

and BS and d_1 are defined in Equation (49).

Proof. See Appendix 7.2.7 □

Remark 5. Note that $K_t U_t = e^{-r(T-t)} BS(V_t, T-t, K, \mu_V - \sigma_V h_{S\rho}, \sigma_V)$ is the Black-Scholes price of a call on V with strike K and maturity T , if the pricing measure is \mathbb{Q}^0 . This can happen in two contexts : the first one is the minimal variance martingale criterium. This is also the case, if V is tradable (i.e. the market is complete) and if $e^{-rt} V_t$ is a \mathbb{Q}^0 martingale (which is implied by $\mu_V - \sigma_V h_{S\rho} = r$). In this case, the process L_t represents the “Delta” of this option.

If we want to find the minimal initial wealth p_{opt} needed to perform the quadratic hedging, it is clear from Equation (48) that $p_{opt} = e^{-rT} BS(V_0, T, K, \mu_V - \sigma_V h_{S\rho}, \sigma_V)$. It is the so called Minimum Variance price : p_0^0 in our notation (see (27)). But as our initial capital is the No Good Deal Price p_0 , the optimal quadratic error is greater than the one starting with capital p_{opt} : we have an extra term equal to $e^{(2r-h_S^2)T} (p_{opt} - p_0)^2$.

6 Numerical Results

This section will be divided into two parts. We first investigate the pricing issue applying the results of the section 4. We will compute and compare NGD prices (the price define by J.H. Cochrane and J. Saá-Requejo ([CSR01]) and our bounds : see Theorem 3) and also other notions of price as mean variance price and price derived from Black Scholes methodology (see below). Furthermore we will show numerically and theoretically convergence of NGD prices with respect to the correlation ρ between the risk sources and also the limit Good deal level β .

In a second time, we compute the strategy found in section 5 and compare it to other possible strategies as Buy and Hold or Black-Scholes (recall that no natural hedging strategy is linked to No Good Deal price). This comparison is made through three kind of risk measure : probability of super-replication, expected loss and Value at Risk.

6.1 Pricing

6.1.1 Framework

We will consider different prices in our numerical experiments. Section 4 tell us that we are not able, so far, to find a “tractable” closed formula for the No Good Deal price. Thus, we will compute the bounds p_0^{UB} and p_0^{LB} as defined in Theorem 3. We intend to exhibit various situations where p_0^{LB} (and thus the NGD price) is above the NGD price defined by J.H. Cochrane and J. Saá-Requejo ([CSR01]), see equation (14). In all graphics, the first one will be denoted by “NGD-UB” (resp. “NGD-LB”) for the upper (resp. lower) bound, and the second by “NGD-CSR”.

We will also be interested by the value of p_0^0 defined by equation (24) whose explicit expression is given in equation (27) (for $\lambda = 0$). This price is known as the minimal variance price and is defined as the derivative’s price compute with the minimal variance measure \mathbb{Q}^0 (see [GLP98]). It will be denoted in the graphics by “MV-Price”.

We also consider a price which is used some times in practice when dealing with Basis Risk. As we consider that the processes V and S are highly correlated, one can use the evolution property of S (i.e. the drift μ_S and the volatility σ_S) starting from V_0 to induce the evolution of V . Thus, we consider a new option, whose payoff is $\left(\frac{V_0}{S_0}S_T - K\right)_+$ and whose underlying is the tradable asset S . Therefore, we can compute the Black-Scholes price of this claim, denoted $S-BS_t$:

$$S-BS_t = e^{-r(T-t)} \frac{V_0}{S_0} BS\left(S_t, T-t, K \frac{S_0}{V_0}, r, \sigma_S\right).$$

In the sequel, it will be designed by the “S-BS Price”.

Finally, we look at the “real” Black-Scholes price of the contingent claim, denoted by $V-BS_t$. This is the price of an option on V in a market constituted by S^0 and V , when V is tradable. Of course, this price has no economic sense in case of Basis Risk. More precisely, it is defined by

$$V-BS_t = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}^V} [(V_T - K)_+ | \mathcal{F}_t] = e^{-r(T-t)} BS(V_t, T-t, K, r, \sigma_V),$$

where the probability \mathbb{Q}^V is the martingale probability for V , i.e.

$$\frac{d\mathbb{Q}^V}{d\mathbb{P}} = \exp\left(-h_V(\rho W_T + \sqrt{1-\rho^2}W_T^*) - \frac{1}{2}h_V^2 T\right).$$

As the preceeding, we will note it “V-BS Price”.

We will perform our computation for the set of parameters described in table 1. The idea

μ_V	σ_V	V_0	μ_S	σ_S	S_0	r	T
0.04	0.32	15	0.0272	0.256	100	2%	0.25

Table 1: *Set of parameters*

is that V is more risky than S (i.e. the volatility is higher) but provides with a higher return (the drift is also higher). We choose to start from a different initial stock value

as this for example the case for an action and an index. But experiments perform with similar initial stock value leads to the same kind of conclusion.

The two main parameters are β and the correlation ρ . The first one measures if a strategy is a good deal and thus has to be excluded from the market and the second one measures the similarity of the two assets V and S . Economic literature asserts that a reasonable value for β is 2 (see for example J.H. Cochrane and J. Saá-Requejo ([CSR01])) and as we are interested in hedging basis risk, we will choose assets which are well correlated, $\rho = 0.8$ at least.

In figure 1, we plot the different prices w.r.t ρ for three different values of β : 0.6, 2 and 3.4, and different value of K (at, in and out the money, i.e. $K = 15, 10$ or 20). The correlation ρ belongs to $[0.1, 0.95]$ with a step equal to 0.05. Figure 2 is the same, swapping the role of β and ρ . We choose β in $[0.6; 3.4]$ with a step equal to 0.2 and ρ equals successively to 0.2, 0.5 and 0.8.

6.1.2 Value of prices

We observe first that the NGD prices are considerably smaller than the initial value of stock, which is equal to 15 in our example. Note that, this bound is not always the super-replication price because the underlying is not tradable. But in the case where the investor is endowed with a unit of V , it is clearly the super-replication cost. The higher is the correlation between both assets, the smaller are the NGD prices. For example, if $\beta = 2$, $K = 15$, $\rho = 0.8$ then $p_0^{UB} = 3.1$ and $p_0^{LB} = 2.59$ while $\tilde{p}_0 = 2.37$

One of the main result of section 4 is that “NGD-CSR” is strictly below “NGD-LB” in some situations, which appears clearly in figure 1 and 2. For the economic meaningful following situation : highly correlated assets ($\rho = 0.8$) and $\beta = 2$ and an at the money option, the lower bound is 8.4% over “NGD-CSR”. Beside this economic classical case, we note that “NGD-CSR” is mostly strictly under “NGD-LB” especially when the option is at and out the money. The gap between both prices is 21% for $\beta = 3.4$, $K = 17.5$ and $\rho = 0.4$ ($\tilde{p}_0 = 2.136$ while $p_0^{LB} = 2.597$) and can even reach a value of 25%.

With our set of parameters, the “V-BS Price” is closed to the “MV-Price” but this is not true in general (for example if we put $\sigma_S = 0.02$ and $\rho = 0.8$, “MV-Price” = 0.08 while “V-BS Price” = 0.48). Similarly, “S-BS Price” is very low in our example. This comes from the choice of the volatility of V which is much higher than those of S (recall that BS function is increasing with volatility). Thus, in our example, “S-BS Price” clearly underestimated the price of the option.

6.1.3 Variation of prices with ρ and β

Note from figures 1 and 2 that NGD prices decrease with ρ and increase with β . This is an expected results for “NGD-CSR”: see formula (22) and observe that “BS” is an increasing function of the drift (see appendix (51)). As this drift is equal to $\mu_V - \sigma_V \rho h_S + \sigma_V \lambda^{max} \sqrt{1 - \rho^2}$, which is an increasing function of β and a decreasing function of ρ , the result is straightforward. It is also clear that p_0^{UB} increase with β : see equation (30). The growth of p_0^{UB} in ρ or the variation of p_0^{LB} are theoretically less clear.

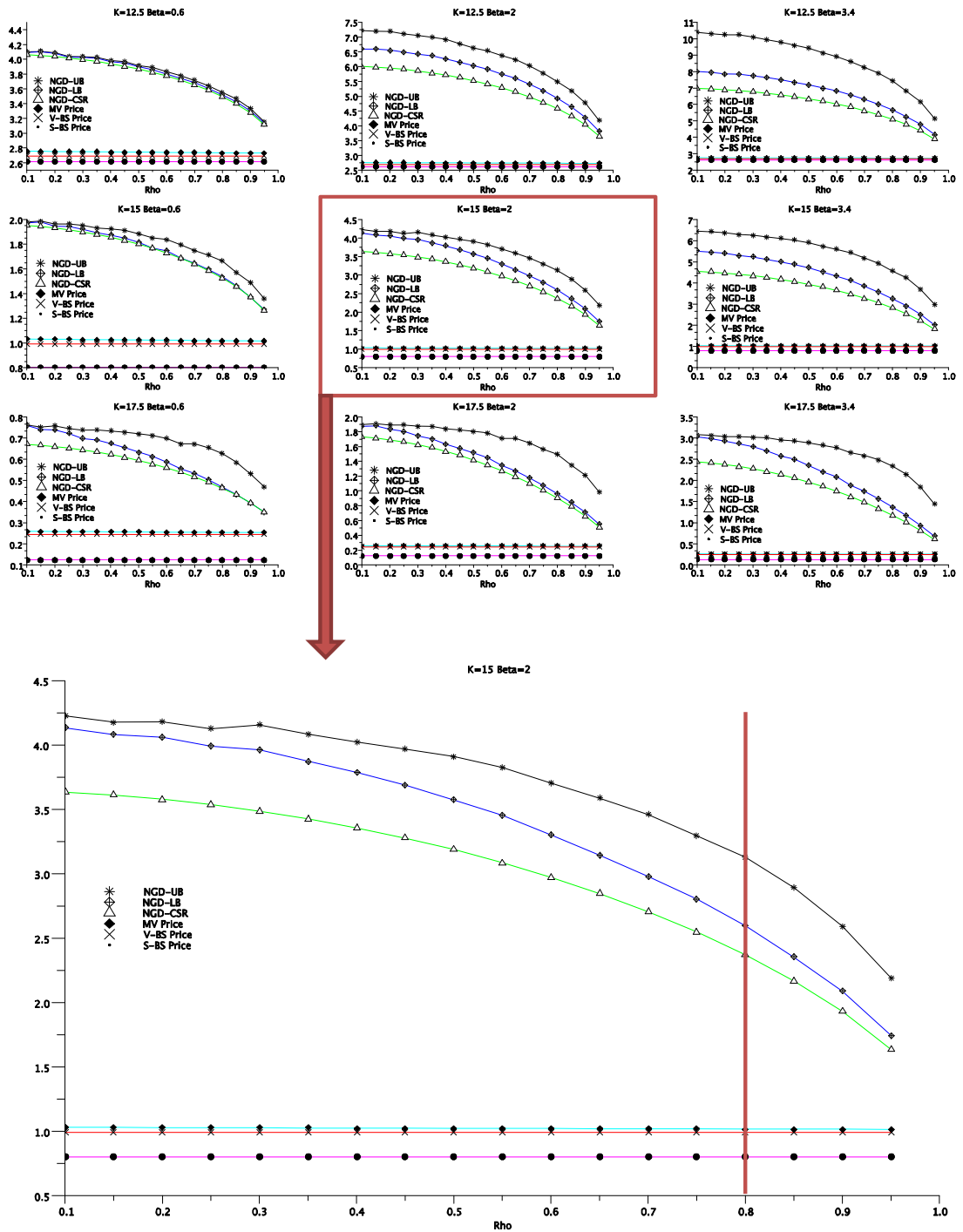


Figure 1: Evolution of prices w.r.t. ρ for different values of β and K .

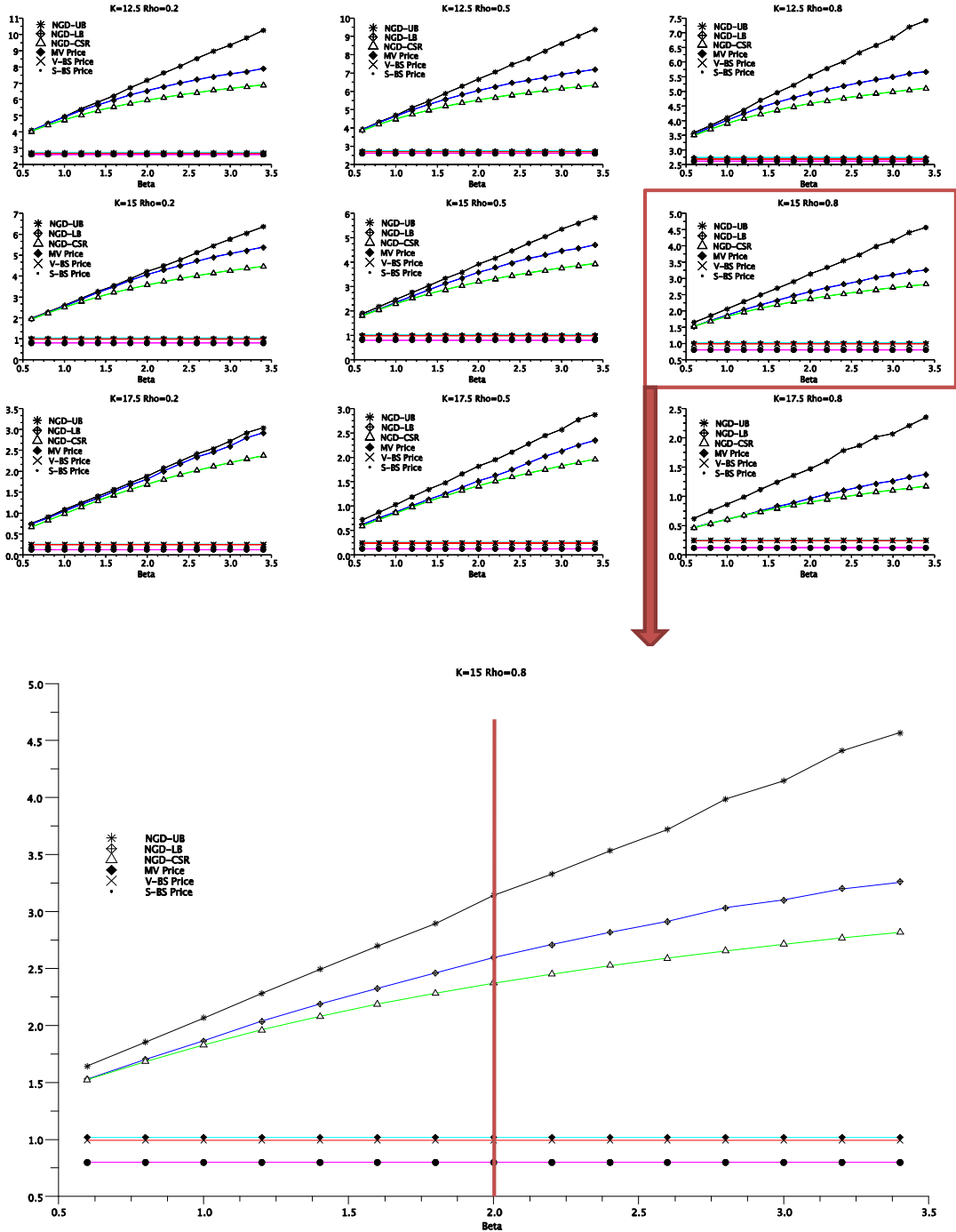


Figure 2: Evolution of prices w.r.t. β for different values of ρ and K .

Now we investigate the limit cases in β . Note that p_0 is clearly increasing in β as $\mathcal{M}^{2,\beta}(\mathbb{P})$ (see Theorem 1 and see (18)) : when β goes to infinity, $\mathcal{M}^{2,\beta}(\mathbb{P})$ tends to $\mathcal{M}^2(\mathbb{P})$ and thus p_0 converges to the super-replication price. At the opposite in Theorem 3, we assume that $\beta \geq \sqrt{e^{h_s^2 T} - 1} = 0.014$ with our parameters. In this limit case, $\bar{\beta} = 0$ (see (20)) and $\lambda^{max} = 0$ (see (15)) thus $p_0^{\lambda^{max}} = p_0^0 = p_0^{UB}$ (see (30)). It follows that all NGD prices : “NGD-UB”, “NGD-LB” and “NGD-CSR” but also p_0 converges to “MV Price” as observed in figure 2.

We observe that all No Good Deal prices (including p_0) converge to “MV Price” when $\rho \rightarrow 1$. We show below that this is theoretically correct. It is clear from (22) that “NGD-CSR” converges to p_0^0 , i.e. “MV-Price”. But without referring to our Theorem, when $\rho = 1$, the non exchangeable asset V depends only depends on W , which is now the single source of risk : the market is complete and the set $\mathcal{M}^2(\mathbb{P})$ is reduced to \mathbb{Q}^0 . Thus, the contingent claim $(V_T - K)_+$ is perfectly replicable and its price under the unique equivalent measure \mathbb{Q}^0 is “MV-Price”. For the upper bound, since H depends only on W , $\mathbb{E}[H|\mathcal{F}_T^W] = H$ and thus $p_0^{UB} = p_0^0$ (30). From (23), we deduce the convergence result for the lower bound “NGD-LB” and also for p_0 .

6.2 Hedging

It is essential when pricing to give an hedging strategy. As previously mention, with the notion of No Good Deal, no natural strategy is pointed out. Thus, this part presents three simple meaningful strategies to be implemented and compare to our strategy found in section 5. This will allow us to evaluate its quality. We denote by X_T^{Strat} the final value achieved using the strategy “Strat” and starting with an initial wealth X_0 . X_0 will be successively equal to “MV Price”, “NGD-CSR” and “(NGD-UB+NGD-LB)/2” (denoted by “NGD” in the following). We choose this value for NGD price because it can be interpreted as a mid price. The strategy “Strat” is one of the followings :

- Buy and Hold in cash (“BaHCash”): we put all the initial wealth X_0 in cash, thus $X_T^{BaHCash} = X_0 e^{rT}$.
- Buy and Hold in S (“BaHS”): we put all the initial wealth X_0 in the risky tradable asset S , thus $X_T^{BaHS} = X_0 S_T / S_0$.
- Black Scholes (“BS”): starting from “S-BS” price at time 0 and following a Black-Scholes strategy we replicate at time T the payoff

$$\left(\frac{V_0}{S_0} S_T - K \right)_+.$$

The difference between the initial wealth X_0 and the price “S-BS” is put in cash. Thus $X_T^{BS} = \left(\frac{V_0}{S_0} S_T - K \right)_+ + (X_0 - S-B S_0) e^{rT}$.

- No-Good-Deal (“NGD”): starting from X_0 , we follow the strategy obtained in section 5 (see Theorem 5).

To measure the hedging error we adopt three points of view : probability of super-replication, expected loss and Value at Risk. For the probability of super-replication,

we evaluate $\mathbb{P}[X_T^{Strat} \geq (V_T - K)_+]$. It is economically meaningful but has two drawbacks. The first one is theoretical : even if the probability is closed to one, the loss might be huge. Moreover, from a numerical point of view, the usual estimator of probability is very unstable because as it integrates a “one or nothing” function : two close trajectories could lead to significantly different results. For the expected loss, we compute $\mathbb{E}[(V_T - K)_+ - X_T^{Strat}]_+$. This is the classical expected shortfall risk measure. It allows to evaluate the size of the loss, but does not tell how often this loss occurs. From a numerical view point its estimation is more stable. Both preceding notions are deeply dependent on the level chosen for the initial wealth (“price effect”). If you start with a significantly higher initial wealth, you will do much better in terms of super-replication and expected losses. As the “NGD” might be five times higher than “MV-Price”, we expected that probability of super-replication and expected losses will perform better starting from “NGD” prices. To overcome this drawback, we introduce the Value At Risk risk indicator and compute the VaR of the loss at 99%, i.e. the value v such that

$$\mathbb{P}[X_T^{Strat} - (V_T - K)_+ \geq -v] = 99\%.$$

Formally, it is the value we have to add to our strategy to replicate the derivative with a probability equal to 99%. It is also the maximal loss, with probability 99% arising from following the strategy X^{Strat} and delivering the option $(V_T - K)_+$. VaR is a widely used measure of risk.

We plot the results of the simulation in figures 3 and 4 with $\beta = 2$ and $K = 15$. We choose these values because it is reasonable from an economical point of view for β and because the results of simulations does not change a lot for in or out the money derivatives. We only study the variation in ρ since β , in contrary to ρ , influences our strategy “NGD” only through the initial wealth X_0 (see (46) and (47)).

In order to interpret our numerical results note that better situations are characterized by probability of super-replication closed to 1, small expected loss and low VaR.

We first remark that our strategy has slightly better results, especially when the correlation is high, which is satisfying : starting from “NGD-CSR” or from “NGD” the probability of super-replication are closed from one, the expected loss and the VaR are small.

We can classify our strategies in two categories : the first one contains the “naive” Buy and Hold strategies (“BaHCash” and “BaHS”) and the second the more elaborated one : the mean variance strategy “NGD” and the Black Scholes strategy “BS”. We see in figures 3 and 4 that each category have a similar behavior.

Next, as expected, the results obtained starting from initial prices “NGD-CSR” and “NGD” are very similar : only the level varies. For a correlation of 0.8, the ratio between “NGD” divided by “NGD-CSR” is equal to 1.2 and in average (on the strategy) the probability of super-replication increases of 4%, the expected loss decreases of 35% and the VaR decreases of 19% when starting from “NGD” instead of “NGD-CSR”.

We now observe the dependence in ρ . First when ρ is small all strategies seem to perform similarly : note that this is not true in general choosing another set of parameters. When ρ increases, the prices “NGD-CSR”, “NGD” and “MV” should decrease (see section

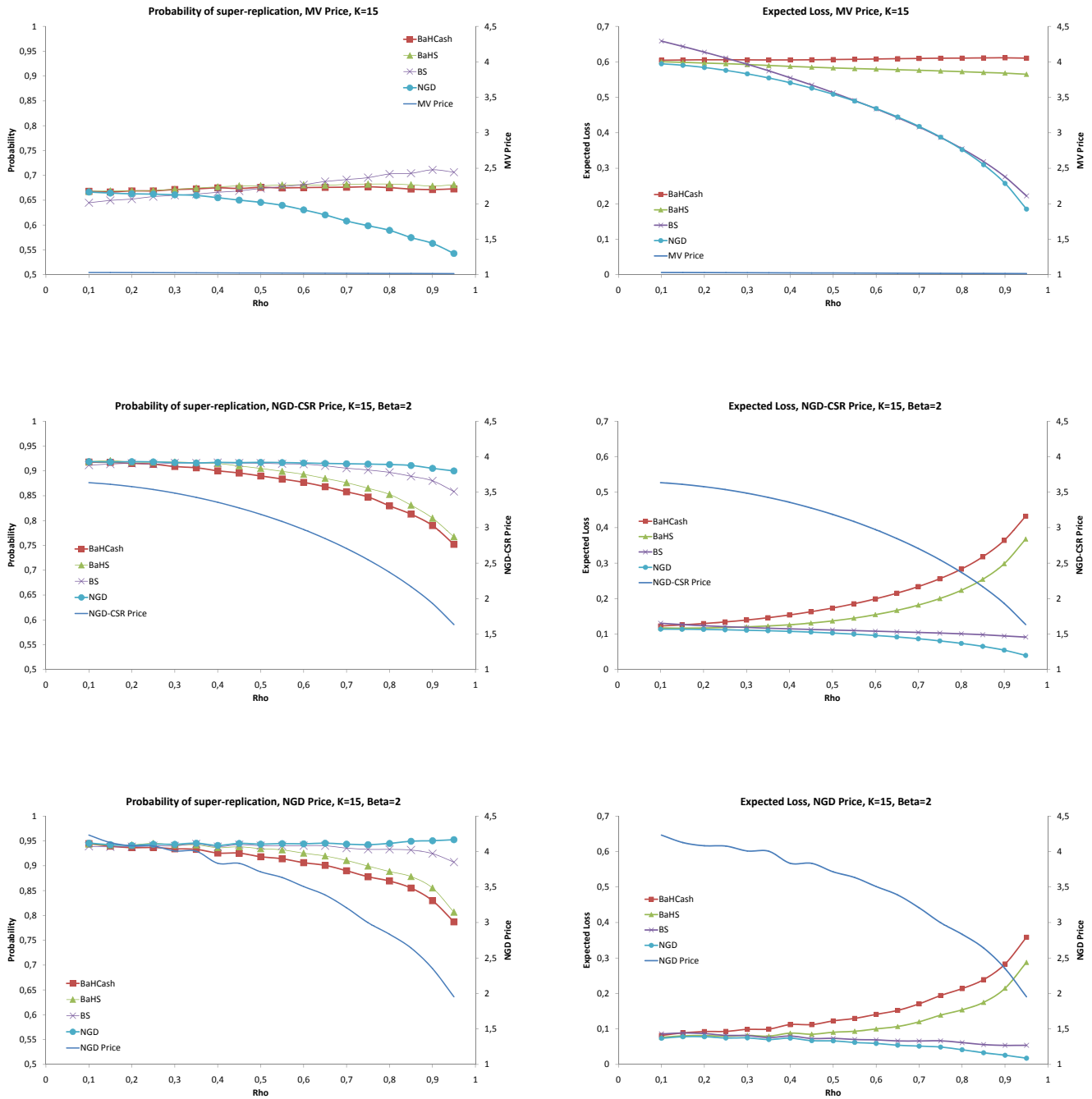


Figure 3: Comparison of the probability of super-replication and expected loss for the different strategies starting from “MV-Price”, “NGD-CSR” and the middle of “NGD-LB” and “NGD-UB”.

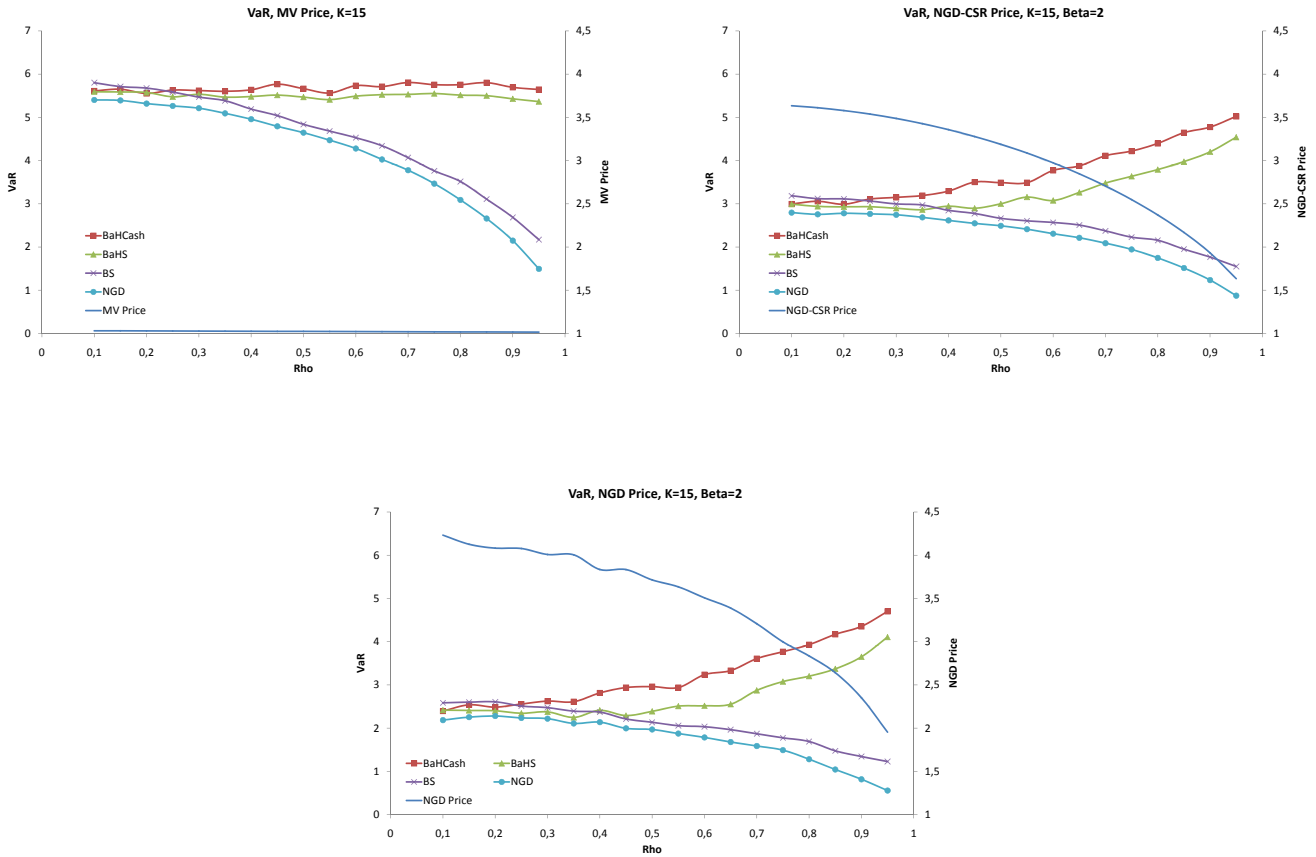


Figure 4: Comparison of the value at risk for the different strategies starting from “MV-Price”, “NGD-CSR” and the middle of “NGD-LB” and “NGD-UB”.

6.1.3). We see that for our set of parameters, “MV” remains almost constant. Now, in the strategies “BaHCash” and “BaHS”, the correlation appears only in the initial wealth. Thus starting from X_0 equal to “NGD-CSR” and “NGD” and recalling the definition of our risk measures, it is clear that the probability of super-replication should decrease, the expected loss and the VaR should increase with the ρ . Starting from “MV” price, the three risk measures should not vary a lot. This is what we observe in figures 3 and 4.

In contrary to buy and hold strategies, “NGD” and “BS” intend to approach (*p.s.* for “BS” and L^2 for “NGD”) the optional call payoff. When ρ increases, both risky assets S and V become similar in term of risk, thus it seems natural that the risk of loss arising from hedging a call written on V with a strategy in S should decrease. Thus we should observe an increase of the probability of super-replication and a decrease of the expected loss and the VaR. Looking to figures 3 and 4, we see that this is true for expected loss and VaR. For the probability of super-replication, this is definitively not true for “NGD” strategy starting from “MV” price. Recall that the minimum variance principle implies to minimize the variance of loss, thus we expect to get a loss which is similar to a Dirac mass

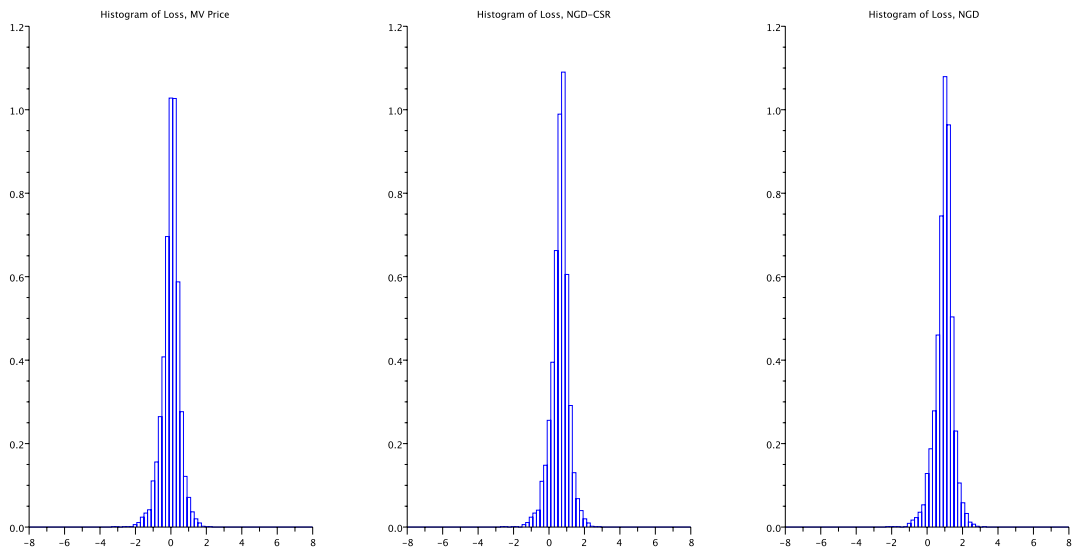


Figure 5: *Histogram of the loss for the three different prices and $K = 15$, $\beta = 2$ and $\rho = 0.8$*

in zero : this is confirm by numerical experiment (see left of figure 5). When evaluating numerically a loss which is similar to a Dirac mass, it is intuitive that the associated probability should be around 1/2 (and the expected loss around 0). Note that starting from an other price than “MV” the distribution of loss is not centered around 0 anymore (see 5) and we don’t have the same numerical problem.

For the probability of super-replication starting from NGD prices, the results are less clear. Following the “NGD” strategy, the probability seems to be more or less constant and following “BS”, it seems to be decreasing. Note that they are still numerical issues associated to the evaluation of a probability which are combined with the “price effect”. Finally, we remark that our two sophisticated approaches “NGD” and “BS” allow to overcome the fact when ρ increases the prices decreases : even if we start with less cash, we perform a better hedging.

7 Appendices

7.1 On Black-Scholes formula

We recall the following formula which is analogous to the Black Scholes formula. All proofs are omitted since they are completely similar to the one of Black Scholes model which can be found for example in M. Musiela and M. Rutkowski [MR07] (p.94 and followings).

Let Y be a geometric brownian motion, with drift η and volatility φ , i.e.

$$Y_t = Y_0 \exp \left(\left(\eta - \frac{\varphi^2}{2} \right) t + \varphi W_t \right).$$

Then, the function $BS(Y_t, T-t, K, \eta, \varphi)$ defined by

$$BS(Y_t, T-t, K, \eta, \varphi) = \mathbb{E}[(Y_T - K)_+ | \mathcal{F}_t],$$

can be explicitly expressed as

$$BS(Y_t, T-t, K, \eta, \varphi) = Y_t e^{\eta(T-t)} \mathcal{N}(d_1) - K \mathcal{N}(d_0) \quad (49)$$

where

$$\begin{aligned} d_1 = d_1(Y_t, T-t, K, \eta, \varphi) &= \frac{\ln\left(\frac{Y_t}{K}\right) + \left(\eta + \frac{\varphi^2}{2}\right)(T-t)}{\varphi\sqrt{T-t}} \\ d_0 = d_0(Y_t, T-t, K, \eta, \varphi) &= d_1 - \varphi\sqrt{T-t} \\ \mathcal{N}(d) &= \int_{-\infty}^d \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \end{aligned}$$

By Ito Formula one can show that

$$dBS(Y_t, T-t, K, \eta, \varphi) = \varphi e^{\eta(T-t)} \mathcal{N}(d_1) Y_t dW_t. \quad (50)$$

Moreover, we get that

$$\frac{\partial BS}{\partial \eta}(Y_t, T-t, K, \eta, \varphi) = (T-t) K \mathcal{N}(d_0) > 0, \quad (51)$$

which implies that the Black-Scholes formula is an increasing function of η .

7.2 Proofs

7.2.1 Proof of Lemma 1

Proof. We begin with the inclusion \subset in (6).

Let $\mathbb{Q} \in \mathcal{M}^2(\mathbb{P})$, as \mathbb{Q} is equivalent to \mathbb{P} , there exists two processes λ and γ in $L_{loc}^2((W, W^*))$ such that (see M. Musiela and M. Rutkowski [MR07] p577 Prop B.2.1) :

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp\left(\int_0^t \gamma_s dW_s - \frac{1}{2} \int_0^t \gamma_s^2 ds + \int_0^t \lambda_s dW_s^* - \frac{1}{2} \int_0^t \lambda_s^2 ds\right).$$

Under the probability \mathbb{Q} , we can define the brownian motion W_t^γ (see Girsanov Theorem) by

$$W_t^\gamma := W_t - \int_0^t \gamma_s ds$$

From Equation (2), we get that

$$\frac{dS_t}{S_t} = (\mu_S + \sigma_S \gamma_t) dt + \sigma_S dW_t^\gamma.$$

Using Ito formula it is easy to see that

$$d\frac{S_t}{S_t^0} = \frac{S_t}{S_t^0} ((\mu_S + \sigma_S \gamma_t - r) dt + \sigma_S dW_t^\gamma).$$

As S/S^0 is a \mathbb{Q} -martingale, we get that the drift term is equal to zero and thus $\gamma_t = -h_S$. So we have proved that $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T^\lambda$. To achieve the proof of the first inclusion, remark that as $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^2(\mathbb{P})$, we get that $\lambda \in \Lambda$ by definition of Λ . For the reverse inequality, let $\lambda \in \Lambda$ and \mathbb{Q}^λ such that $\frac{d\mathbb{Q}^\lambda}{d\mathbb{P}} = Z_T^\lambda$. Following the same line of arguments as above, we get that

$$d\frac{S_t}{S_t^0} = \sigma_S \frac{S_t}{S_t^0} dW_t^\lambda,$$

where W^λ is a brownian motion defined in (25).

Using Ito formula again and Fubini-Tonelli Theorem, one get that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^\lambda} \left(\int_0^T \left(\frac{S_t}{S_t^0} \right)^2 dt \right) &= S_0^2 \mathbb{E}^{\mathbb{Q}^\lambda} \left(\int_0^T e^{2\sigma_S W_t^\lambda - \sigma_S^2 t} dt \right) = S_0^2 \int_0^T \mathbb{E}^{\mathbb{Q}^\lambda} \left(e^{2\sigma_S W_t^\lambda - \sigma_S^2 t} \right) dt \\ &= S_0^2 \int_0^T e^{\sigma_S^2 t} dt = S_0^2 \frac{e^{\sigma_S^2 T} - 1}{\sigma_S^2} < \infty, \end{aligned}$$

recall that $\mathbb{E}(e^{iW_t}) = e^{-\frac{1}{2}t}$. Thus $\frac{S_t}{S_t^0} \in L^2(\mathbb{Q}^\lambda)$ and S/S^0 is not only a local martingale but a real \mathbb{Q}^λ -martingale (see M. Musiela and M. Rutkowski [MR07] p571).

To finish the proof of the reverse inequality, we note that from the Definition of Λ , $\frac{d\mathbb{Q}^\lambda}{d\mathbb{P}} \in L^2(\mathbb{P})$. It remains to prove that Λ , and thus $\mathcal{M}^2(\mathbb{P})$, is non-empty. Consider some constant process λ , using (8), we get that $\mathbb{E}((Z_T^\lambda)^2) < \infty$ and thus any constant λ belongs to Λ . \square

7.2.2 Proof of Lemma 2

Proof. Let $\lambda \in \Lambda$, since Z_T^λ is a martingale, $\mathbb{E}Z_T^\lambda = Z_0^\lambda = 1$. We define the following process

$$\bar{Z}_t^\lambda = \exp \left(-2h_S W_t - 2h_S^2 t + 2 \int_0^t \lambda_s dW_s^* - 2 \int_0^t \lambda_s^2 ds \right).$$

\bar{Z}^λ is a Doléans-Dade process and thus a continuous local martingale (see [KS91], p.191). We are going to show that \bar{Z}^λ is a martingale.

We clearly have that $\bar{Z}_t^\lambda \leq (Z_t^\lambda)^2$, $\forall t \in [0, T]$. Since Z^λ is a square integrable martingale, the Doob maximal inequality (see [KS91], Theorem 1.3.8 p.14) implies that

$$\mathbb{E} \left(\sup_{t \in [0, T]} Z_t^\lambda \right)^2 \leq 4\mathbb{E}[(Z_T^\lambda)^2] < +\infty.$$

Let τ be a stopping time such that $\mathbb{P}(0 \leq \tau \leq T) = 1$, then $\bar{Z}_\tau^\lambda \leq \sup_{t \in [0, T]} \bar{Z}_t^\lambda$. So we deduce that

$$\mathbb{E} \left[\sup_{\tau \in [0, T]} \bar{Z}_\tau^\lambda \right] \leq \mathbb{E} \left[\sup_{t \in [0, T]} \bar{Z}_t^\lambda \right] \leq \mathbb{E} \left[\sup_{t \in [0, T]} (Z_t^\lambda)^2 \right] = \mathbb{E} \left[\left(\sup_{t \in [0, T]} Z_t^\lambda \right)^2 \right] < +\infty.$$

Thus \bar{Z}^λ is a continuous local martingale of class (DL) (see [KS91], definition 1.4.8 p.24). This shows that \bar{Z}^λ is a martingale (see [KS91], problem 1.5.19 (i) p.36) and thus $\mathbb{E}(\bar{Z}_T^\lambda) = 1$. Then, we can define the following probability measure

$$d\tilde{\mathbb{Q}}/d\mathbb{P} = \bar{Z}_T^\lambda.$$

Using Bayes Formula

$$\mathbb{E}((Z_T^\lambda)^2) = \mathbb{E}^{\tilde{\mathbb{Q}}}\left(e^{h_s^2 T + \int_0^T \lambda_s^2 ds}\right)$$

Thus

$$\text{Var}(Z_T^\lambda) = e^{h_s^2 T} \mathbb{E}^{\tilde{\mathbb{Q}}}\left(e^{\int_0^T \lambda_s^2 ds}\right) - 1.$$

□

7.2.3 Proof of Lemma 3

The proof is inspired from C. Gourieroux, J.P. Laurent and H. Pham [GLP98] Proposition 3.2.

Proof. We first prove that $\mathcal{A}_2 \subset \mathcal{A}_2^U$. Let $(\Phi^0, \Phi^1) \in \mathcal{A}_2$, we set $\Phi_t^{i,n} = \Phi_t^i \mathbf{1}_{|\Phi_t^0| + |\Phi_t^1| \leq n}$ for $i = 0, 1$ and $X_t^n = \Phi_t^{0,n} S_t^0 + \Phi_t^{1,n} S_t$. Then $dX_t^n = \Phi_t^{0,n} dS_t^0 + \Phi_t^{1,n} dS_t$ and from definition of integrability with respect to (S^0, S) (see Protter [Pro90], p.134), we get that

$$\int_0^t \left(\Phi_l^{0,n} dS_l^0 + \Phi_l^{1,n} dS_l \right) \xrightarrow[n \rightarrow +\infty]{\text{semi. mart.}} \int_0^t \left(\Phi_l^0 dS_l^0 + \Phi_l^1 dS_l \right).$$

Noting that $X_0^n = X_0$ for n big enough, we deduce that

$$X_t^n = X_0^n + \int_0^t \left(\Phi_l^{0,n} dS_l^0 + \Phi_l^{1,n} dS_l \right) \xrightarrow[n \rightarrow +\infty]{\text{semi. mart.}} X_0 + \int_0^t \left(\Phi_l^0 dS_l^0 + \Phi_l^1 dS_l \right) \quad (52)$$

Using the integration by part formula

$$d\left(\frac{X_t^n}{U_t}\right) = \frac{1}{U_t} dX_t^n + X_t^n d\left(\frac{1}{U_t}\right) + d\left\langle \frac{1}{U_t}, X_t^n \right\rangle.$$

As $dX_t^n = \Phi_t^{0,n} dS_t^0 + \Phi_t^{1,n} dS_t$, we get that $d\left\langle \frac{1}{U_t}, X_t^n \right\rangle = \Phi_t^{0,n} d\left\langle \frac{1}{U_t}, S_t^0 \right\rangle + \Phi_t^{1,n} d\left\langle \frac{1}{U_t}, S_t \right\rangle$. Using again the integration by part formula, we get that : $d\left\langle \frac{1}{U_t}, S_t^0 \right\rangle = d\left(\frac{S_t^0}{U_t}\right) - \left(\frac{1}{U_t} dS_t^0 + S_t^0 d\left(\frac{1}{U_t}\right)\right)$ and a similar expression for $d\left\langle \frac{1}{U_t}, S_t \right\rangle$. Thus, we deduce that : $d\left(\frac{X_t^n}{U_t}\right) = \Phi_t^{0,n} d\left(\frac{S_t^0}{U_t}\right) + \Phi_t^{1,n} d\left(\frac{S_t}{U_t}\right)$. Thus, recalling (52),

$$X_t^n = U_t \left(X_0^n + \int_0^t \left(\Phi_l^{0,n} d\left(\frac{S_l^0}{U_l}\right) + \Phi_l^{1,n} d\left(\frac{S_l}{U_l}\right) \right) \right) \xrightarrow[n \rightarrow +\infty]{\text{semi. mart.}} X_0 + \int_0^t \left(\Phi_l^0 dS_l^0 + \Phi_l^1 dS_l \right)$$

and

$$\int_0^t \left(\Phi_l^{0,n} d\left(\frac{S_l^0}{U_l}\right) + \Phi_l^{1,n} d\left(\frac{S_l}{U_l}\right) \right) \xrightarrow[n \rightarrow +\infty]{\text{semi. mart.}} \frac{1}{U_t} \left(X_0 + \int_0^t \left(\Phi_l^0 dS_l^0 + \Phi_l^1 dS_l \right) \right) - X_0.$$

Thus, as the right hand side of the last equation is a semi-martingale, $(\Phi^0, \Phi^1) \in L\left(\left(\frac{S^0}{U}, \frac{S}{U}\right)\right)$ and

$$\begin{aligned} \int_0^t \left(\Phi_l^0 d\left(\frac{S_l^0}{U_l}\right) + \Phi_l^1 d\left(\frac{S_l}{U_l}\right) \right) &= \frac{1}{U_t} \left(X_0 + \int_0^t (\Phi_l^0 dS_l^0 + \Phi_l^1 dS_l) \right) - X_0 \\ &= \frac{1}{U_t} (\Phi_t^0 S_t^0 + \Phi_t^1 S_t) - X_0, \end{aligned}$$

using that $(\Phi^0, \Phi^1) \in \mathcal{A}_2$. Thus we get that $(\Phi^0, \Phi^1) \in \mathcal{A}_2^U$. For the reverse inequality, the proof is similar using the following integration by part formula :

$$d\left(U_t \frac{X_t^n}{U_t}\right) = \frac{X_t^n}{U_t} dU_t + U_t d\left(\frac{X_t^n}{U_t}\right) + d < \frac{X_t^n}{U_t}, U_t > .$$

□

7.2.4 Proof of Lemma 4

We will prove the following lemma for ease of exposure. Let $X \in L^2$, $X \geq 0$ such that $X\mathbf{1}_{X>0}$ has a density with respect to Lebesgue measure and γ a positive number.

$$p_0^{opt} = \sup_{\substack{Y \geq 0, \mathbb{E}Y = 1 \\ \mathbb{E}Y^2 \leq 1 + \gamma^2}} \mathbb{E}[YX] \quad (53)$$

Lemma 6. *The solution of Problem 53 is :*

if $1 - \gamma \frac{\mathbb{E}(X)}{\sqrt{\text{Var } X}} \geq 0$ then

$$\begin{aligned} Y^{opt} &= 1 + \gamma \frac{X - \mathbb{E}(X)}{\sqrt{\text{Var } X}} \\ p_0^{opt} &= \mathbb{E}(X) + \gamma \sqrt{\text{Var } X}. \end{aligned}$$

if $1 - \gamma \frac{\mathbb{E}(X)}{\sqrt{\text{Var } X}} < 0$ then

$$\begin{aligned} Y^{opt} &= \frac{(X - \alpha)_+}{\mathbb{E}(X - \alpha)_+}, \\ p_0^{opt} &= \alpha + (1 + \gamma)^2 \mathbb{E}(X - \alpha)_+, \end{aligned}$$

where there exists α , a positive number, such that¹⁰

$$\frac{\mathbb{E}(X - \alpha)_+^2}{\mathbb{E}^2(X - \alpha)_+} = 1 + \gamma^2. \quad (54)$$

Proof. If $1 - \gamma \frac{\mathbb{E}(X)}{\sqrt{\text{Var } X}} \geq 0$, then it is straightforward that $\mathbb{E}Y^{opt}X = \mathbb{E}(X) + \gamma\sqrt{\text{Var } X}$. Let Y such that $Y \geq 0$, $\mathbb{E}Y = 1$ and $\mathbb{E}Y^2 \leq 1 + \gamma^2$ then $\text{Var } Y \leq \gamma^2$ and by Cauchy-Schwartz inequality

$$\begin{aligned} \mathbb{E}(YX) &= \mathbb{E}((Y - \mathbb{E}Y)(X - \mathbb{E}X)) + \mathbb{E}X \leq \sqrt{\text{Var } X} \sqrt{\text{Var } Y} + \mathbb{E}X \\ &\leq \gamma \sqrt{\text{Var } X} + \mathbb{E}X = \mathbb{E}[Y^{opt}X]. \end{aligned}$$

¹⁰The term $\mathbb{E}^2[A]$ denotes $(\mathbb{E}[A])^2$

To prove that Y^{opt} is the optimal solution of (53), it remains to check that it satisfies the constraints. $Y^{opt} = 1 - \gamma \frac{\mathbb{E}X}{\sqrt{\text{Var} X}} + \gamma \frac{X}{\sqrt{\text{Var} X}} \geq 0$ by assumption (recall that $X \geq 0$). The two others constraints are straightforward.

If $1 - \gamma \frac{\mathbb{E}(X)}{\sqrt{\text{Var} X}} < 0$, assume that there exists α such that condition (54) is satisfied. Then it is straightforward that

$$\mathbb{E}[Y^{opt} X] = \frac{\mathbb{E}[(X - \alpha)_+ (X - \alpha + \alpha)]}{\mathbb{E}(X - \alpha)_+} = \alpha + \frac{\mathbb{E}(X - \alpha)_+^2}{\mathbb{E}(X - \alpha)_+} = \alpha + (1 + \gamma^2)\mathbb{E}(X - \alpha)_+$$

using condition 54. Let Y such that $Y \geq 0$, $\mathbb{E}Y = 1$ and $\mathbb{E}Y^2 \leq 1 + \gamma^2$ then by Cauchy-Schwartz inequality and condition (54)

$$\begin{aligned} \mathbb{E}[YX] &= \mathbb{E}(Y(X - \alpha)_+) + \alpha + \mathbb{E}(Y(X - \alpha)\mathbf{1}_{X < \alpha}) \\ &\leq \mathbb{E}(Y(X - \alpha)_+) + \alpha \leq \sqrt{\mathbb{E}(X - \alpha)_+^2} \sqrt{\mathbb{E}Y^2} + \alpha \\ &\leq \sqrt{1 + \gamma^2} \sqrt{1 + \gamma^2} \mathbb{E}(X - \alpha)_+ + \alpha = \mathbb{E}[Y^{opt} X]. \end{aligned}$$

Y^{opt} is thus optimal solution for 53 because that it satisfies the constraints (see condition 54).

It remains to prove that there exists some α such that condition 54 is satisfied. Let $f(x) = \frac{\mathbb{E}(X-x)_+^2}{\mathbb{E}^2(X-x)_+}$ then $f(0) = \frac{\text{Var} X}{\mathbb{E}^2 X} + 1 < 1 + \gamma^2$ by assumption. Below we show that there exists $\alpha_0 > 0$ such that $f(\alpha_0) \geq 1 + \gamma^2$, thus by continuity of f there will exist some $\alpha > 0$ such that $f(\alpha) = 1 + \gamma^2$. We prove first that there exist α_0 such that $\mathbb{P}(X > \alpha_0) = \frac{1}{1 + \gamma^2}$. Such an α_0 exists because

$$\begin{aligned} 1 - \gamma \frac{\mathbb{E}(X)}{\sqrt{\text{Var} X}} < 0 &\Leftrightarrow \text{Var}[X\mathbf{1}_{X > 0}] < \gamma^2 \mathbb{E}^2[X\mathbf{1}_{X > 0}] \\ &\Leftrightarrow \mathbb{E}[X^2\mathbf{1}_{X > 0}] < (\gamma^2 + 1) \mathbb{E} \left[X \left(\sqrt{\mathbf{1}_{X > 0}} \right)^2 \right] \\ &\Rightarrow \mathbb{E}[X^2\mathbf{1}_{X > 0}] < (\gamma^2 + 1) \mathbb{E} [X^2\mathbf{1}_{X > 0}] \mathbb{P}(X > 0), \end{aligned}$$

by Cauchy Schwartz inequality. Thus $\mathbb{P}(X = 0) \leq \frac{\gamma^2}{1 + \gamma^2}$ and by continuity of $x \rightarrow \mathbb{P}(X \leq x)$, for $x > 0$ there exists α_0 such that $\mathbb{P}(X \leq \alpha_0) = \frac{\gamma^2}{1 + \gamma^2}$. Then by Cauchy Schwartz inequality,

$$\begin{aligned} \mathbb{E}^2(X - \alpha_0)_+ &= \mathbb{E}^2(X\mathbf{1}_{X > \alpha_0}) - 2\alpha_0\mathbb{P}(X > \alpha_0)\mathbb{E}(X\mathbf{1}_{X > \alpha_0}) + \alpha_0^2\mathbb{P}^2(X > \alpha_0) \\ &\leq \mathbb{P}(X > \alpha_0)\mathbb{E}(X^2\mathbf{1}_{X > \alpha_0}) - 2\alpha_0\mathbb{P}(X > \alpha_0)\mathbb{E}(X\mathbf{1}_{X > \alpha_0}) + \alpha_0^2\mathbb{P}^2(X > \alpha_0) \\ &\leq \frac{1}{1 + \gamma^2} (\mathbb{E}X^2\mathbf{1}_{X > \alpha_0} - 2\alpha_0\mathbb{E}X\mathbf{1}_{X > \alpha_0} + \alpha_0^2\mathbb{P}(X > \alpha_0)) \\ &\leq \frac{1}{1 + \gamma^2} \mathbb{E}(X - \alpha_0)_+^2. \end{aligned}$$

Thus $f(\alpha_0) \geq 1 + \gamma^2$ which concludes the proof. \square

7.2.5 Proof of Lemma 5

Proof. To show that $\frac{S^0}{U^\lambda}$ and $\frac{S}{U^\lambda}$ are local martingales under the measure \mathbb{Q}^{U^λ} , we are going to compute the stochastic differential equation satisfied by these processes and see under which conditions they have no drift term. We set two processes W^{U^λ} and W^{*,U^λ} which, thanks to Girsanov Theorem will be brownian motions under the probability \mathbb{Q}^{U^λ} :

$$W_t^{U^\lambda} = W_t - 2a_t t, \quad W_t^{U^\lambda,*} = W_t^* - 2\lambda_t t. \quad (55)$$

Then, the processes U^λ , S , and S^0 satisfies:

$$\begin{aligned} dU_t^\lambda &= U_t^\lambda \left(a_t dW_t^{U^\lambda} + \lambda_t dW_t^{U^\lambda,*} + (c_t + 2a_t^2 + 2\lambda_t^2) dt \right) \\ dS_t &= S_t \left((r + \sigma_S h_S + 2a_t \sigma_S) dt + \sigma_S dW_t^{U^\lambda} \right) \\ dS_t^0 &= r S_t^0 dt \end{aligned}$$

Thus, by Ito formula applied to $f(x, y) = \frac{x}{y}$, we have

$$\begin{aligned} d \frac{S_t^0}{U_t^\lambda} &= r \frac{S_t^0}{U_t^\lambda} dt - \frac{S_t^0}{U_t^{\lambda^2}} dU_t^\lambda + \frac{S_t^0}{U_t^{\lambda^3}} d \langle U^\lambda \rangle_t \\ &= \frac{S_t^0}{U_t^\lambda} \left[(r - c_t - a_t^2 - \lambda_t^2) dt - a_t dW_t^{U^\lambda} - \lambda_t dW_t^{U^\lambda,*} \right], \end{aligned}$$

and

$$\begin{aligned} d \frac{S_t}{U_t^\lambda} &= \frac{dS_t}{U_t^\lambda} - \frac{S_t}{U_t^{\lambda^2}} dU_t^\lambda + \frac{S_t}{U_t^{\lambda^3}} d \langle U^\lambda \rangle_t - \frac{1}{U_t^{\lambda^2}} d \langle S, U^\lambda \rangle_t \\ &= \frac{S_t}{U_t} \left[(r + \sigma_S h_S + a_t \sigma_S - c_t - a_t^2 - \lambda_t^2) dt + (\sigma_S - a_t) dW_t^U - \lambda_t dW_t^{U,*} \right], \end{aligned}$$

Thus, these processes are local martingale if and only if

$$r + \sigma_S h_S + a_t \sigma_S - c_t - a_t^2 - \lambda_t^2 = 0 \quad \text{and} \quad r - c_t - a_t^2 - \lambda_t^2 = 0 \quad (56)$$

$$\int_0^T \left(\frac{S_t^0}{U_t^\lambda} \right)^2 dt < \infty \quad \text{and} \quad \int_0^T \left(\frac{S_t}{U_t^\lambda} \right)^2 dt < \infty. \quad (57)$$

The inequalities in (57) hold true because $\frac{S^0}{U^\lambda}$ and $\frac{S}{U^\lambda}$ are continuous. The unique solution of this system (56) is $c_t = r - \lambda_t^2 - h_S^2$ and $a_t = -h_S$. With these parameters, the process U_t^λ is the same as those described by (39). We also get that

$$d \left(\frac{S_t^0}{U_t^\lambda} \right) = \frac{S_t^0}{U_t^\lambda} \left[h_S dW_t^{U^\lambda} - \lambda_t dW_t^{U^\lambda,*} \right], \quad (58)$$

$$d \left(\frac{S_t}{U_t^\lambda} \right) = \frac{S_t}{U_t} \left[(\sigma_S + h_S) dW_t^{U^\lambda} - \lambda_t dW_t^{U^\lambda,*} \right]. \quad (59)$$

Note that if λ is such that

$$\mathbb{E}^{\mathbb{Q}^{U^\lambda}} \int_0^T \left(\frac{S_t^0}{U_t^\lambda} \right)^2 dt < \infty \quad \text{and} \quad \mathbb{E}^{\mathbb{Q}^{U^\lambda}} \int_0^T \left(\frac{S_t}{U_t^\lambda} \right)^2 dt < \infty,$$

$\frac{S^0}{U^\lambda}$ and $\frac{S}{U^\lambda}$ are \mathbb{Q}^{U^λ} martingale (see M. Musiela and M. Rutkowski [MR07] p571). This is for example the case with deterministic λ . \square

7.2.6 Proof of Theorem 4

Proof. The result of section 5 shows that we have to solve (see (40))

$$v(H) = \inf_{(\Phi^0, \Phi^1) \in \mathcal{A}_2^U} \mathbb{E}(U_T^2) \mathbb{E}^{\mathbb{Q}^U} \left[\frac{H}{U_T} - \left(X_0 + \int_0^T \Phi_t^0 d\frac{S_t^0}{U_t} + \int_0^T \Phi_t^1 d\frac{S_t}{U_t} \right) \right]^2 \quad (60)$$

Let $K_t = \mathbb{E}^{\mathbb{Q}^U} \left(\frac{H}{U_T} \middle| \mathcal{F}_t \right)$, using Galtchouk-Kunita-Watanabe decomposition on the \mathbb{Q}^U martingale K under Assumption 3, we get that

$$K_t = \mathbb{E}^{\mathbb{Q}^U} \left(\frac{H}{U_T} \right) + \int_0^t \Phi_l^{0,H} d\frac{S_l^0}{U_l} + \int_0^t \Phi_l^{1,H} d\frac{S_l}{U_l} + R_t^H, \quad 0 \leq t \leq T,$$

where R^H is a L^2 -martingale orthogonal to $\frac{S^0}{U}$ and $\frac{S}{U}$, i.e. $\langle R_t^H, \frac{S_t^0}{U_t} \rangle = 0$ and $\langle R_t^H, \frac{S_t}{U_t} \rangle = 0$. Thus as $K_T = \frac{H}{U_T}$ problem (60) can be rewrite as

$$\begin{aligned} v(H) &= \inf_{(\Phi^0, \Phi) \in \mathcal{A}_2^U} \mathbb{E}(U_T^2) \mathbb{E}^{\mathbb{Q}^U} \left[\mathbb{E}^{\mathbb{Q}^U} \left(\frac{H}{U_T} \right) - X_0 + \int_0^T (\Phi_t^{0,H} - \Phi_t^0) d\frac{S_t^0}{U_t} \right. \\ &\quad \left. + \int_0^T (\Phi_t^{1,H} - \Phi_t^1) d\frac{S_t}{U_t} + R_T^H \right]^2 \\ &= \inf_{(\Phi^0, \Phi) \in \mathcal{A}_2^U} \mathbb{E}(U_T^2) \left[\left(\mathbb{E}^{\mathbb{Q}^U} \left(\frac{H}{U_T} \right) - X_0 \right)^2 + (R_T^H)^2 \right. \\ &\quad \left. + \mathbb{E}^{\mathbb{Q}^U} \left[\int_0^T (\Phi_t^{0,H} - \Phi_t^0) d\frac{S_t^0}{U_t} + \int_0^T (\Phi_t^{1,H} - \Phi_t^1) d\frac{S_t}{U_t} \right]^2 \right]. \end{aligned} \quad (61)$$

As $\frac{S^0}{U}$ and $\frac{S}{U}$ are not orthogonal, we can not continue directly the computation. We have to decompose this two processes on W^U and $W^{*,U}$ which are orthogonal (see equation (55) for definition of those processes). Since R^H is a square integrable martingale, the Theorem of Martingale representation (see for example D. Revuz and M. Yor) asserts that there exists some progressively measurable processes a and b such that $\mathbb{E} \int_0^T a_t^2 dt < +\infty$ and $\int_0^T |b_t| dt < +\infty$:

$$R_t^H = \int_0^t a_l dW_l^U + \int_0^t b_l dW_l^{*,U}$$

Recalling equation (58) and (59) with $\lambda = 0$, the orthogonality conditions lead to

$$a_t \frac{S_t^0}{U_t} h_S = 0 \quad \text{and} \quad a_t \frac{S_t}{U_t} (h_S + \sigma_S) = 0.$$

Thus $a_t = 0$ and $R_t^H = \int_0^t b_t dW_t^{*,U}$. Replacing R^H in equation (61) and using again (58) and (59) with $\lambda = 0$

$$\begin{aligned} v(H) &= \inf_{(\Phi^0, \Phi^1) \in \mathcal{A}_2^U} \mathbb{E}(U_T^2) \left[\left[\mathbb{E}^{\mathbb{Q}^U} \left(\frac{H}{U_T} \right) - X_0 \right]^2 + \mathbb{E}^{\mathbb{Q}^U} \left(\int_0^T b_t dW_t^{*,U} \right)^2 + \right. \\ &\quad \left. \mathbb{E}^{\mathbb{Q}^U} \left(\int_0^T \left(h_S(\Phi_t^{0,H} - \Phi_t^0) \frac{S_t^0}{U_t} + (\sigma_S + h_S)(\Phi_t^{1,H} - \Phi_t^1) \frac{S_t}{U_t} \right) dW_t^U \right)^2 \right] \\ &= \inf_{(\Phi^0, \Phi^1) \in \mathcal{A}_2^U} \mathbb{E}(U_T^2) \left[\left[\mathbb{E}^{\mathbb{Q}^U} \left(\frac{H}{U_T} \right) - X_0 \right]^2 + \mathbb{E}^{\mathbb{Q}^U} \left(\int_0^T b_t^2 dt \right) + \right. \\ &\quad \left. \mathbb{E}^{\mathbb{Q}^U} \left(\int_0^T \left(h_S(\Phi_t^{0,H} - \Phi_t^0) \frac{S_t^0}{U_t} + (\sigma_S + h_S)(\Phi_t^{1,H} - \Phi_t^1) \frac{S_t}{U_t} \right)^2 dt \right) \right] \end{aligned}$$

The minimum is clearly obtain for $(\Phi^0, \Phi^1) \in \mathcal{A}_2^U$ such that $h_S(\Phi_t^{0,H} - \Phi_t^0) \frac{S_t^0}{U_t} + (\sigma_S + h_S)(\Phi_t^{1,H} - \Phi_t^1) \frac{S_t}{U_t} = 0$ $\mathbb{Q}^U - p.s.$ If $(\Phi^{0,H}, \Phi^{1,H}) \in \mathcal{A}_2^U$, then $\Phi^0 = \Phi^{0,H}$ and $\Phi^1 = \Phi^{1,H}$ are solutions of Problem 37. \square

7.2.7 Proof of Theorem 5

Proof. Using the results of Theorem 4, it is sufficient to compute the Galtchouk- Kunita-Watanabe decomposition of the process K_t . We first remark that K can be rewritten using Bayes Formula as

$$K_t = \mathbb{E}^{\mathbb{Q}^U} \left(\frac{(V_T - K)_+}{U_T} \middle| \mathcal{F}_t \right) = \frac{\mathbb{E}(U_T(V_T - K)_+ | \mathcal{F}_t)}{\mathbb{E}(U_T^2 | \mathcal{F}_t)}$$

As from (41), $U_t = e^{-h_S W_t + (r - \frac{3}{2} h_S^2) t} = Z_t^0 e^{(r - h_S^2) t}$ and \mathbb{Q}^0 is defined in (4) by $d\mathbb{Q}^0/d\mathbb{P} = Z_T^0$. We obtain using Bayes Formula again that

$$\mathbb{E}(U_T(V_T - K)_+ | \mathcal{F}_t) = e^{(r - h_S^2)(T-t)} U_t \mathbb{E}^{\mathbb{Q}^0}((V_T - K)_+ | \mathcal{F}_t).$$

As

$$\mathbb{E}(U_T^2 | \mathcal{F}_t) = e^{-2h_S W_t + (2r - 3h_S^2)T + 2h_S^2(T-t)} = U_t^2 e^{(2r - h_S^2)(T-t)}, \quad (62)$$

we get that

$$K_t = e^{-r(T-t)} \frac{\mathbb{E}^{\mathbb{Q}^0}((V_T - K)_+ | \mathcal{F}_t)}{U_t}.$$

But, the process V under the probability \mathbb{Q}^0 , is a geometric Brownian motion, and we can achieve these decomposition using the Black-Scholes formula. In fact

$$\begin{aligned} \frac{dV_t}{V_t} &= \mu_V V_t dt + \sigma_V V_t (\rho dW_t + \sqrt{1 - \rho^2} dW_t^*) \\ &= (\mu_V - \sigma_V h_S \rho) V_t dt + \sigma_V V_t (\rho dW_t^0 + \sqrt{1 - \rho^2} dW_t^{*,0}). \end{aligned}$$

The processes W^0 and $W^{0,*}$ are Brownian motion under \mathbb{Q}^0 (see Equation (25) for definition). It follows from Black Scholes formula (49) that

$$K_t = \frac{e^{-r(T-t)}}{U_t} BS(V_t, T-t, K, \mu_V - \sigma_V h_S \rho, \sigma_V)$$

and by formula (50)

$$\begin{aligned} dBS(V_t, T-t, K, \mu_V - \sigma_V h_S \rho, \sigma_V) \\ = \sigma_V e^{(\mu_V - \sigma_V h_S \rho)(T-t)} V_t \mathcal{N}(d_1) (\rho dW_t^0 + \sqrt{1 - \rho^2} dW_t^{*,0}), \end{aligned}$$

where we used the short notation d_1 for $d_1(V_t, T-t, K, \mu_V - \sigma_V h_S \rho, \sigma_V)$. Using Integration by part formula,

$$\begin{aligned} dK_t &= \frac{e^{-r(T-t)}}{U_t} d(BS(V_t, T-t, K, \mu_V - \sigma_V h_S \rho, \sigma_V)) + \\ &BS(V_t, T-t, K, \mu_V - \sigma_V h_S \rho, \sigma_V) d \frac{e^{-r(T-t)}}{U_t} + \\ &d < \frac{e^{-r(T-t)}}{U_t}, BS(V_t, T-t, K, \mu_V - \sigma_V h_S \rho, \sigma_V) >. \end{aligned}$$

Using Ito formula,

$$d \frac{e^{-r(T-t)}}{U_t} = \frac{e^{-r(T-t)}}{U_t} (h_S^2 dt + h_S dW_t^0)$$

Thus

$$d < \frac{e^{-r(T-t)}}{U_t}, BS > = e^{(\mu_V - \sigma_V h_S \rho)(T-t)} \sigma_V \rho h_S \frac{e^{-r(T-t)}}{U_t} V_t \mathcal{N}(d_1) dt$$

And

$$\begin{aligned} dK_t &= \left[h_S^2 K_t + e^{(\mu_V - \sigma_V h_S \rho)(T-t)} \sigma_V \rho h_S \frac{e^{-r(T-t)}}{U_t} V_t \mathcal{N}(d_1) \right] dt + h_S K_t dW_t^0 + \\ &\sigma_V e^{(\mu_V - \sigma_V h_S \rho)(T-t)} \frac{e^{-r(T-t)}}{U_t} V_t \mathcal{N}(d_1) (\rho dW_t^0 + \sqrt{1 - \rho^2} dW_t^{*,0}) \\ &= \left(h_S K_t + \rho \sigma_V e^{(\mu_V - \sigma_V h_S \rho)(T-t)} \frac{e^{-r(T-t)}}{U_t} V_t \mathcal{N}(d_1) \right) dW_t^U + \\ &\sqrt{1 - \rho^2} \sigma_V e^{(\mu_V - \sigma_V h_S \rho)(T-t)} \frac{e^{-r(T-t)}}{U_t} V_t \mathcal{N}(d_1) dW_t^{*,U} \end{aligned}$$

See Equation (43) for definition of W^U and $W^{*,U}$: $W_t^U = W_t^0 + h_S t$ and $W_t^{*,U} = W_t^0$. So we get that

$$dK_t = \left(h_S K_t + \rho \frac{L_t}{U_t} \right) dW_t^U + \sqrt{1 - \rho^2} \frac{L_t}{U_t} dW_t^{*,U} \quad (63)$$

with

$$L_t = \sigma_V e^{-r(T-t) + (\mu_V - \sigma_V h_S \rho)(T-t)} V_t \mathcal{N}(d_1). \quad (64)$$

Going back to the Galtchouk-Kunita-Watanabe (44) of K_t , we are looking for $\Phi^{0,H}$, $\Phi^{1,H}$ and b such that

$$\begin{aligned} dK_t &= \Phi_t^{0,H} d\frac{S_t^0}{U_t} + \Phi_t^{1,H} d\frac{S_t}{U_t} + b_t dW_t^{*,U} \\ &= \left(h_S \frac{S_t^0}{U_t} \Phi_t^{0,H} + (h_S + \sigma_S) \frac{S_t}{U_t} \Phi_t^{1,H} \right) dW_t^U + b_t dW_t^{*,U}, \end{aligned}$$

recall Equations (58) and (59) with $\lambda = 0$. Comparing with equation (63), we obtain that

$$\begin{aligned} h_S \frac{S_t^0}{U_t} \Phi_t^{0,H} + (h_S + \sigma_S) \frac{S_t}{U_t} \Phi_t^{1,H} &= h_S K_t + \rho \frac{L_t}{U_t} \\ b_t &= \sqrt{1 - \rho^2} \frac{L_t}{U_t} \end{aligned} \quad (65)$$

Recall from Theorem 4 that we are looking for $(\Phi^{0,H}, \Phi^{1,H}) \in \mathcal{A}_2^U$. So we impose the self financing condition

$$\begin{aligned} \Phi_t^{0,H} \frac{S_t^0}{U_t} + \Phi_t^{1,H} \frac{S_t}{U_t} &= X_0 + \int_0^t \Phi_l^{0,H} d\frac{S_l^0}{U_l} + \int_0^t \Phi_l^{1,H} d\frac{S_l}{U_l} \\ &= X_0 + \int_0^t \left(h_S \frac{S_l^0}{U_l} \Phi_l^{0,H} + (h_S + \sigma_S) \frac{S_l}{U_l} \Phi_l^{1,H} \right) dW_l^U \\ &= X_0 + \int_0^t \left(h_S K_l + \rho \frac{L_l}{U_l} \right) dW_l^U, \end{aligned}$$

where we have use Equation (65) to get the last equality. Using equation (65) again, we get that

$$\begin{aligned} \Phi_t^{0,H} &= \frac{U_t}{S_t^0} \left[\frac{\sigma_S + h_S}{\sigma_S} \left(X_0 + \int_0^t \left(h_S K_l + \rho \frac{L_l}{U_l} \right) dW_l^U \right) - \frac{1}{\sigma_S} \left(h_S K_t + \rho \frac{L_t}{U_t} \right) \right] \\ \Phi_t^{1,H} &= \frac{U_t}{\sigma_S S_t} \left[\left(h_S K_t + \rho \frac{L_t}{U_t} \right) - h_S \left(X_0 + \int_0^t \left(h_S K_l + \rho \frac{L_l}{U_l} \right) dW_l^U \right) \right] \end{aligned}$$

In order to prove that $(\Phi^{0,H}, \Phi^{1,H}) \in \mathcal{A}_2^U$, it remains to prove that $(\Phi^{0,H}, \Phi^{1,H}) \in L\left(\left(\frac{S^0}{U}, \frac{S}{U}\right)\right)$, i.e.

$$\begin{aligned} \int_0^T \left(\Phi_t^{0,H} \right)^2 d \langle \frac{S^0}{U} \rangle_t &= \int_0^T \left(\Phi_t^{0,H} \right)^2 h_S^2 \left(\frac{S_t^0}{U_t} \right)^2 dt < \infty \\ \int_0^T \left(\Phi_t^{1,H} \right)^2 d \langle \frac{S}{U} \rangle_t &= \int_0^T \left(\Phi_t^{1,H} \right)^2 (h_S + \sigma_S)^2 \left(\frac{S_t}{U_t} \right)^2 dt < \infty \end{aligned}$$

This holds true because $\Phi_t^{0,H}$, $\Phi_t^{1,H}$, S_t^0 , S_t and U_t are continuous on $[0, T]$. \square

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