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# An Extension of Mixed Sky-hook and ADD to Magneto-Rheological Dampers

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**Abstract:** The paper presents an extension of the mixed Sky-hook and ADD to Magneto-Rheological dampers. Firstly, a semi-active automotive suspension equipped with a nonlinear Magneto-Rheological damper is introduced. The interest of this nonlinear model is that the bi-viscous and hysteretic behaviors of MR dampers can be taken into account in the controller design. Hence, the designed controller is more adaptive with real MR dampers. Finally, the new mixed Sky-hook and ADD algorithm for MR dampers is proposed to enhance the passenger comfort. The performances of the proposed control method are then analyzed, based on simulations on a nonlinear vehicle model. The results show that the Mixed Sky-Hook and ADD can be successfully extended to Magneto-Rheological dampers.

*Keywords:* Application, control strategy

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## 1. INTRODUCTION

It is a well-known fact that semi-active suspensions have been more and more widely used in automobile industry. The controllable damping force makes them outperform the passive suspensions. Despite the passivity constraint (i.e it is impossible to deliver forces on the body having the same direction of the suspension elongation speed), the semi-active suspensions have considerable advantages compared with fully active suspensions. They can potentially achieve the majority of the performance criteria (see Ivers and Miller (1989), Patten et al. (1994)) while they are smaller in weight and volume, cheaper in price and they significantly consume less energy than active ones.

The Magneto-Rheological (MR) dampers are dissipative nonlinear components driven by electric current and belong to a class of semi-active suspensions. In recent years, MR dampers have become very attractive devices in automotive applications and in other fields like civil, aerospace... The highlights of MR dampers are the fail-safe characteristic (they operate as passive dampers once failures happen), low-power consumption, fast response... However, the main drawback of these devices is the high nonlinearity. This creates difficulties in the controller design. In fact, the modeling of the transient response, the Force-Velocity and Force-Deflection characteristics of an MR damper always raise difficult problems. There are different modeling approaches for example physical signification of coefficients by Jr et al. (1997), coefficients related to the hysteresis by Guo et al. (2006), non parametric by Savaresi et al. (2005a), coefficients related to

damping, stiffness and amplitude of force Choi and Sung (2008). Among these, the approach proposed by Guo et al. (2006) based on a tangent hyperbolic function to model the hysteresis characteristic is very interesting in terms of both modeling and controller design. With some modifications on the original model in Guo et al. (2006), Lozoya-Santos et al. (2009) proposed a new one where the electric current is considered as a control input which is suitable for controller synthesis (see also Do et al. (2010b)).

During the last few decades, semi-active suspension control has attracted many researchers. In general, there are three important issues: passenger comfort (ride quality), handling (road-holding) and suspension travel (stroke limits) control. Although multi-objective controller synthesis is a trend in semi-active suspension design (see in Lu and DePoyster (2002), Güvenc et al. (2006) and Poussot-Vassal et al. (2008)), it is true that the most important and basic issue to be studied is the passenger comfort. One of the first comfort-oriented control methods, which has been successfully applied in commercial vehicles, is the Sky-hook control proposed by Karnopp et al. (1974). In this linear-model based control design, the damping coefficient is adjusted continuously or switched between a maximum and a minimum value. Then numerous approaches have been also developed such as optimal control (Savaresi et al. (2005b)), clipped optimal control (Giorgetti et al. (2006), Canale et al. (2006)), or  $H_\infty$  control (Rossi and Lucente (2004), Sammier et al. (2003)). Recently, the mixed Sky-hook and ADD (SH-ADD) algorithm proposed by Savaresi and Spelta (2007) has been known to be one of the most

efficient comfort-oriented controllers. The previous studies have shown that the Sky-hook provides the best comfort at low frequency and the ADD is ideal for maximizing the passenger comfort at high frequency. The successful combination of Sky-hook and ADD in Savaresi and Spelta (2007) has resulted in an almost optimal comfort-oriented controller.

The mixed Sky-hook and ADD above has been proposed for linear dampers where the nonlinear characteristics (i.e. the bi-viscous and the hysteretic behaviors) have not been taken into account. The application by analogy of this algorithm for nonlinear MR dampers may deteriorate the performance of the closed-loop system. The contribution of the paper is to propose a new Mixed Sky-hook and ADD suitable for MR dampers where the nonlinearities are taken into consideration.

This paper is organized as follows. In Section 2, the quarter car model with a nonlinear MR damper is presented. This is a simple model for semi-active suspension control. In Section 3, the conventional Mixed Sky-hook and ADD is recalled. The extension of the conventional Mixed Sky-hook and ADD to MR dampers is then presented in Section 4. In Section 5, the results obtained in simulations with a nonlinear quarter car model are discussed. Finally, some conclusions and perspectives are drawn in Section 6.

## 2. QUARTER CAR MODEL

Consider a simple model of quarter vehicle (see Fig. 1) made up of a sprung mass ( $m_s$ ) and an unsprung mass ( $m_{us}$ ). A spring with the stiffness coefficient  $k_s$  and a semi-active damper connect these two masses. The wheel tire is modeled by a spring with the stiffness coefficient  $k_t$ . In this model,  $z_s$  (respectively  $z_{us}$ ) is the vertical position of  $m_s$  (respectively  $m_{us}$ ) and  $z_r$  is the road profile. It is assumed that the wheel-road contact is ensured.

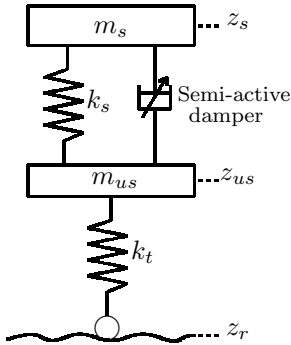


Fig. 1. Model of quarter vehicle with a semi-active damper.

The dynamical equations of a quarter vehicle are governed by

$$\begin{cases} m_s \ddot{z}_s = -k_s z_{def} - F_{damper} \\ m_{us} \ddot{z}_{us} = k_s z_{def} + F_{damper} - k_t (z_{us} - z_r) \end{cases} \quad (1)$$

where  $z_{def} = z_s - z_{us}$  is the damper deflection ( $m$ ) (assumed to be measured or estimated),  $\dot{z}_{def} = \dot{z}_s - \dot{z}_{us}$  is the deflection velocity ( $m/s$ ) (can be directly computed from  $z_{def}$ ).

$F_{damper}$  is the semi-active damper force. In this paper, to represent the behavior of an MR damper, the following nonlinear equation, as in Guo et al. (2006), is used

$$F_{damper} = c_0 \dot{z}_{def} + k_0 z_{def} + f_I \tanh(c_1 \dot{z}_{def} + k_1 z_{def}) \quad (2)$$

where  $c_0$  and  $k_0$  are the damping and stiffness coefficients of a passive damper (when  $f_I = 0$ ),  $c_1$  and  $k_1$  are the damping and stiffness coefficient beyond the limits of elasticity and  $f_I$  is a controllable force within the limits of elasticity. The passivity constraint of a semi-active damper is obtained by considering only the positivity constraint

$$0 \leq f_{min} \leq f_I \leq f_{max} \quad (3)$$

## 3. CONVENTIONAL MIXED SKY-HOOK AND ADD CONTROL

The conventional Sky-hook, ADD and mixed Sky-hook and ADD are linear-model based control designs where the damping coefficient is adjusted continuously or switched between a maximum and a minimum value and the semi-active damper force  $F_{damper} = c \dot{z}_{def}$  where  $\dot{z}_{def}$  is the deflection velocity and the damping coefficient  $c_{min} \leq c \leq c_{max}$ . The dynamical equations of a quarter car are the following

$$\begin{cases} m_s \ddot{z}_s = -k_s z_{def} - c \dot{z}_{def} \\ m_{us} \ddot{z}_{us} = k_s z_{def} + c \dot{z}_{def} - k_t (z_{us} - z_r) \end{cases} \quad (4)$$

In the system (4), the damping coefficient  $c$  ( $Nm/s$ ) is the control input.

### 3.1 Sky-hook Control

The well-known Sky-hook control is first developed for the linear semi-active suspension systems by Karnopp et al. (1974). The two-state Sky-hook algorithm for two-state damper is given by

$$c = \begin{cases} c_{max} & \text{if } \dot{z}_s \dot{z}_{def} > 0 \\ c_{min} & \text{if } \dot{z}_s \dot{z}_{def} \leq 0 \end{cases} \quad (5)$$

### 3.2 ADD Control

The ADD is developed for the linear suspension systems in Savaresi et al. (2005b) using optimal-control theory. It has been proven to be optimal in minimizing the body car acceleration without any road preview. The implementation of ADD control requires a two-state damper and is given by

$$c = \begin{cases} c_{max} & \text{if } \ddot{z}_s \dot{z}_{def} > 0 \\ c_{min} & \text{if } \ddot{z}_s \dot{z}_{def} \leq 0 \end{cases} \quad (6)$$

### 3.3 Mixed Sky-hook and ADD Control

The Sky-hook provides the best comfort at low frequency while the ADD improves considerably the comfort at high frequency. The Mixed Sky-hook and ADD algorithm guarantees the best behavior of both Sky-hook and ADD.

$$c = \begin{cases} c_{max} & \text{if } (\ddot{z}_s^2 - \alpha \dot{z}_s^2 \leq 0 \wedge \dot{z}_s \dot{z}_{def} \geq 0) \vee \\ & (\ddot{z}_s^2 - \alpha \dot{z}_s^2 > 0 \wedge \dot{z}_s \dot{z}_{def} \geq 0) \\ c_{min} & \text{if } (\ddot{z}_s^2 - \alpha \dot{z}_s^2 \leq 0 \wedge \dot{z}_s \dot{z}_{def} < 0) \vee \\ & (\ddot{z}_s^2 - \alpha \dot{z}_s^2 > 0 \wedge \dot{z}_s \dot{z}_{def} < 0) \end{cases} \quad (7)$$

In (7),  $\alpha$  is the SH-ADD crossover frequency (at which the frequency responses of closed-loop systems using Sky-hook

and ADD controllers intersect) and the amount  $(\ddot{z}_s^2 - \alpha \dot{z}_s^2)$  is the frequency-range selector. See in Savaresi and Spelta (2007) for more details on Mixed Sky-hook and ADD Control.

#### 4. MIXED SKY-HOOK AND ADD FOR MR DAMPERS

The frequency response analysis (see Appendix B) in this section is done using the quarter car vehicle model (parameters found in table 1) with a nonlinear spring whose force is presented in Fig. 5.a (i.e the stiffness coefficient  $k_s$  is not constant). The terms "Soft MRD" and "Hard MRD" represent the MR damper with  $f_I = f_{min}$  and the MR damper with  $f_I = f_{max}$ , respectively.

##### 4.1 Sky-hook Control for MR Dampers

*Sky-hook for MRD - A:* As seen in (1), (2) and (3) and from conventional Sky-hook control design in Section 3.1, by analogy, the classical Sky-hook Control for MR dampers can be given as

$$f_I = \begin{cases} f_{max} & \text{if } \dot{z}_s \dot{z}_{def} > 0 \\ f_{min} & \text{if } \dot{z}_s \dot{z}_{def} \leq 0 \end{cases} \quad (8)$$

*Sky-hook for MRD - B:* The main idea of the Skyhook for linear suspension system is that the damper exerts a force that reduces the velocity of the body mass  $\dot{z}_s$ . By using the same principle, the modified Sky-hook for MR damper will be as follows

$$f_I = \begin{cases} f_{max} & \text{if } \dot{z}_s \rho > 0 \\ f_{min} & \text{if } \dot{z}_s \rho \leq 0 \end{cases} \quad (9)$$

where  $\rho = \tanh(c_1 \dot{z}_{def} + k_1 z_{def})$ .

The comparison of performance between two strategies "Sky-hook for MRD - A" and "Sky-hook for MRD - B" is presented in Fig. 2.

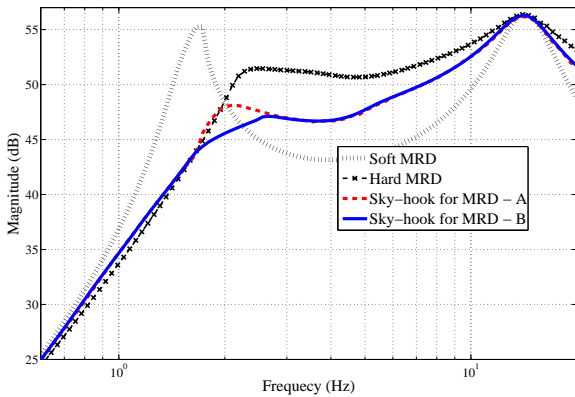


Fig. 2. Sky-hook for MR - Frequency Responses  $\ddot{z}_s/z_r$

##### 4.2 ADD Control for MR Dampers

*ADD for MRD - A:* As seen in (1), (2) and (3) and from conventional ADD control design in Section 3.2, by

analogy, the classical ADD Control for MR dampers can be given as

$$f_I = \begin{cases} f_{max} & \text{if } \ddot{z}_s \dot{z}_{def} > 0 \\ f_{min} & \text{if } \ddot{z}_s \dot{z}_{def} \leq 0 \end{cases} \quad (10)$$

*ADD for MRD - B:* The following modified ADD for MR dampers is inspired by the physical meaning of the existing ADD algorithm presented in (6). The proof is given in Appendix A. But it turns out very simple to explain. Looking at the (1) and (2), the only variable parameter is  $f_I$ , so when  $\ddot{z}_s \tanh(c_1 \dot{z}_{def} + k_1 z_{def}) > 0$ , for example  $\ddot{z}_s$  and  $\tanh(c_1 \dot{z}_{def} + k_1 z_{def})$  are positive,  $\ddot{z}_s$  will rapidly decrease to zero if  $f_I = f_{max}$ . On the contrary, when  $\ddot{z}_s \tanh(c_1 \dot{z}_{def} + k_1 z_{def}) \leq 0$ ,  $\ddot{z}_s$  will be kept not floating away from zero if  $f_I = f_{min}$ , and so on. Finally, the modified ADD for MR dampers is given as

$$f_I = \begin{cases} f_{max} & \text{if } \ddot{z}_s \rho > 0 \\ f_{min} & \text{if } \ddot{z}_s \rho \leq 0 \end{cases} \quad (11)$$

where  $\rho = \tanh(c_1 \dot{z}_{def} + k_1 z_{def})$ .

The comparison of performance between two strategies "ADD for MRD - A" and "ADD for MRD - B" is presented in Fig. 3.

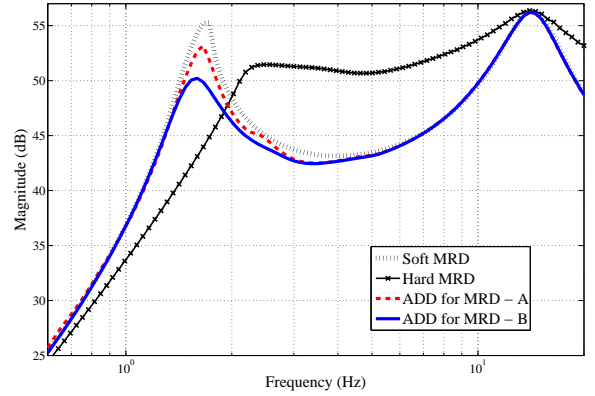


Fig. 3. ADD for MR - Frequency Responses  $\ddot{z}_s/z_r$

##### 4.3 A New Mixed Sky-hook and ADD Control for MR Dampers

As seen in Fig. 2 and Fig. 3, strategies B result in better performances than strategies A do. The reason for this improvement is that the nonlinearity of the MR damper is taken into account in strategies B while the same is not true in strategies A. The presence of nonlinearity  $\rho$  in strategies B creates a reduction in the first resonances of the closed-loop frequency responses.

Quite obviously, the modified Sky-hook (9) and the modified ADD (11) are used for the implementation of the Mixed Sky-hook and ADD Control for MR Dampers which is given as

$$f_I = \begin{cases} f_{max} & \text{if } (\ddot{z}_s^2 - \alpha \dot{z}_s^2 \leq 0 \wedge \dot{z}_s \rho > 0) \vee \\ & (\ddot{z}_s^2 - \alpha \dot{z}_s^2 > 0 \wedge \dot{z}_s \rho > 0) \\ f_{min} & \text{if } (\ddot{z}_s^2 - \alpha \dot{z}_s^2 \leq 0 \wedge \dot{z}_s \rho \leq 0) \vee \\ & (\ddot{z}_s^2 - \alpha \dot{z}_s^2 > 0 \wedge \dot{z}_s \rho \leq 0) \end{cases} \quad (12)$$

where  $\rho = \tanh(c_1 \dot{z}_{def} + k_1 z_{def})$ .

As in Section 3.3, the amount  $(\dot{z}_s^2 - \alpha \dot{z}_s^2)$  is the frequency-range selector and the SH-ADD crossover frequency  $\alpha = 2\pi f_{SHADD}$  rad/s where  $f_{SHADD} = 2.1$  Hz.

As seen in Fig. 4, the mixed SH-ADD for MR dampers dramatically improves the first resonance compared with the modified ADD (11) while keeping the best quality of each strategy (Sky-hook at low frequency and ADD at high frequency) in the whole range of frequency (except for a slight increase in the frequency response at low frequency).

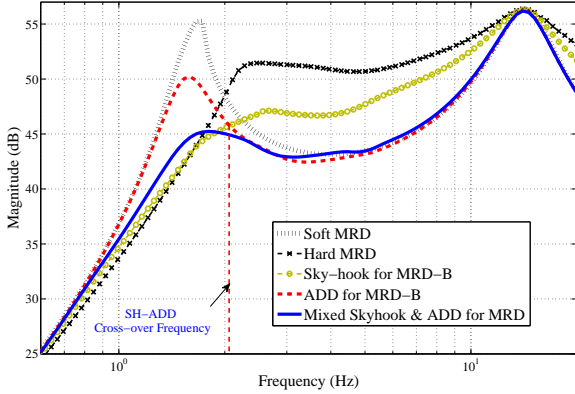


Fig. 4. Comparison between different control methods

## 5. RESULTS ANALYSIS

Table 1. Parameter values.

1/4 RMC	Value	MR damper	Value
$m_s$	315[kg]	$c_0$	737[Ns/m]
$m_{us}$	37.5[kg]	$k_0$	3884[N/m]
$k_t$	210000[N/m]	$f_{min}$	0[N]
—	—	$f_{max}$	870[N]
—	—	$c_1$	11.06[s/m]
—	—	$k_1$	58.3[1/s]
—	—	$C_{min}$	800[Nm/s]
—	—	$C_{max}$	8000[Nm/s]
—	—	$F_{min}$	-1440[N]
—	—	$F_{max}$	1440[N]

### 5.1 Model Parameters

The quarter vehicle using in this paper is the quarter car Renault Mégane Coupé (1/4 RMC) model (see Zin et al. (2004)) and the parameters are presented in table 1. The spring used in this simulation is a nonlinear one where the spring force is as in Fig. 5.a.

The MR damper model parameters are chosen according to the MR damper in Do et al. (2010a) and summarized in table 1. The Force-Velocity behavior of the MR damper is presented in Fig. 6.

To evaluate the efficiency of the controlled MR damper (the Sky-hook in (9), the ADD in (11) and the mixed Sky-hook and ADD in (12) for MR dampers), the simulation results are compared with those obtained with a nonlinear passive RMC damper (optimized for the Renault Mégane Coupé model to enhance the passenger comfort) whose force is a nonlinear function of  $\dot{z}_{def}$  (see Fig. 5.b).

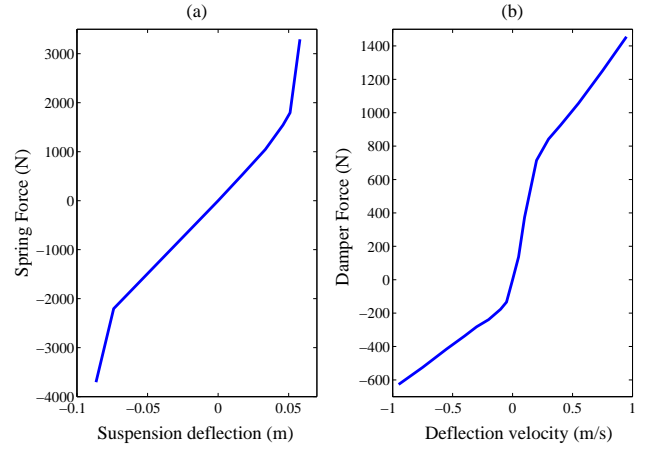


Fig. 5. Nonlinear RMC Spring (a) and Nonlinear Passive RMC damper (b) characteristics.

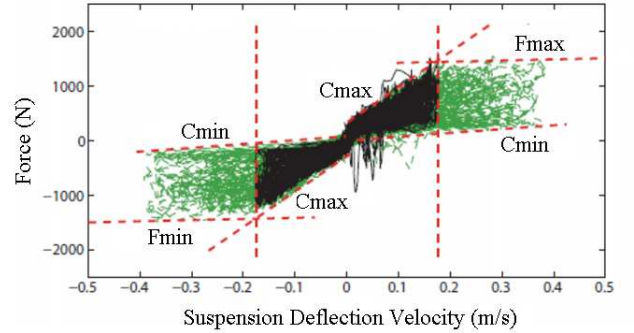


Fig. 6. Force vs. Suspension Deflection Velocity

### 5.2 Testing Scenarios and Results

The two testing scenarios for the evaluation of the proposed control method in time domain are presented as follows.

**Test 1** The road disturbance input is a 0.01 (m) step.

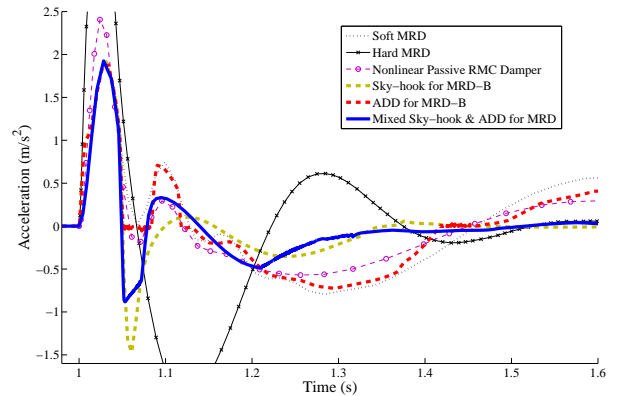


Fig. 7. Acceleration w.r.t 0.01 (m) step input

As seen in Fig. 7, the SH creates a high value peak (second peak) but it provides a good damping rate (as Hard MR damper). The ADD is good in reducing the peak values (as efficient as the Soft MR damper) but the damping rate is bad. The nonlinear passive RMC (which

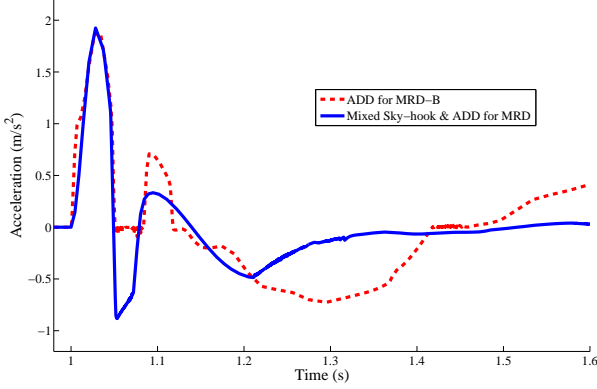


Fig. 8. Acceleration w.r.t 0.01 (m) step input: comparison between ADD for MRD and mixed SH-ADD for MRD

is optimized for the quarter car RMC model) provides an intermediate damping rate and peak value reduction. The mixed SH-ADD for MR damper is the most efficient in reducing the peak values and provide a very good damping rate. For better visibility, the comparison of two control methods, ADD and Mixed SH-ADD for MR dampers, is also presented in Fig. 8.

**Test 2** The road disturbance input is designed as an integrated white noise, band-limited within the frequency range [0-30] Hz (see Fig. 9).

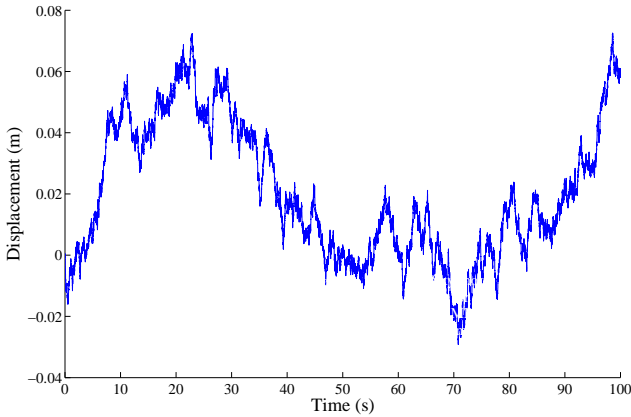


Fig. 9. Road profile  $z_r$

To evaluate the efficiency of the proposed method, the following criterion is used and compared between different strategies

$$J = \sqrt{\frac{1}{T} \int_0^T \dot{z}_s^2(t)} \quad (13)$$

where  $T$  is the simulation time. Here  $T=100$  s.

As in Fig. 10, compared with the nonlinear passive RMC damper (optimized for this RMC model), the MR damper with the proposed mixed Sky-hook and ADD improves 17 % in passenger comfort which cannot be achieved by Soft MR dampers, Hard MR dampers, modified Sky-hook MR dampers or modified ADD MR dampers. There is no doubt that the proposed modified mixed Sky-hook and ADD for MR dampers is the best in terms of passenger comfort.

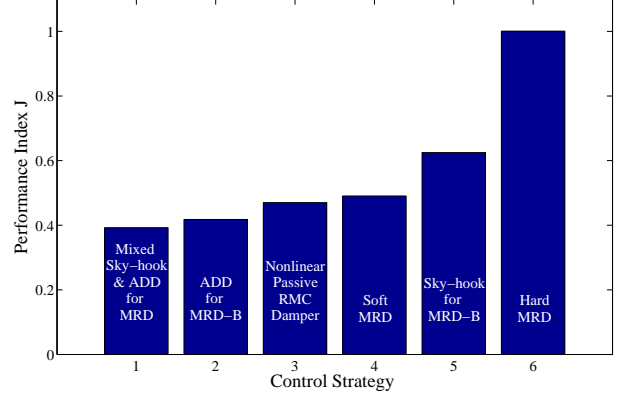


Fig. 10. Performance comparison

## 6. CONCLUSION

In this paper, a new mixed Sky-hook and ADD for MR dampers has been introduced. The simulation results in the frequency and time domains have shown that the Mixed Sky-Hook and ADD has been successfully extended to MR dampers. The MR damper model used in this paper is simple with only one current-dependent parameter (i.e the controllable force  $f_I$ ). It will be interesting to consider the control design problem with more general models for MR dampers (hence, more exact models) where both the damping coefficient and the stiffness coefficient beyond the limits of elasticity depend on input current (i.e  $c_1 = c_1(I)$  and  $k_1 = k_1(I)$ ). In addition, future works will concern also the multi-objective control to deal with the trade-off between comfort and road holding, between comfort and suspension travel or even between comfort, road holding and suspension travel.

### Appendix A. PROOF OF ADD FOR MR DAMPER

The proof presented here is based on that given in Savaresi et al. (2005b).

The dynamic equations of a quarter car model equipped with an MR damper are given by

$$\begin{cases} m_s \ddot{z}_s = -k_s z_{def} - c_0 \dot{z}_{def} - k_0 z_{def} \\ \quad - f_I \tanh(c_1 \dot{z}_{def} + k_1 z_{def}) \\ m_{us} \ddot{z}_{us} = k_s z_{def} + c_0 \dot{z}_{def} + k_0 z_{def} \\ \quad + f_I \tanh(c_1 \dot{z}_{def} + k_1 z_{def}) - k_t (z_{us} - z_r) \\ \dot{f}_I = -\beta f_I + \beta u \end{cases} \quad (A.1)$$

where  $\beta$  represents the bandwidth of a real MR damper and  $u$  is the control input which can take its values in  $[f_{Imin}, f_{Imax}]$ .

A state-space representation of (A.1) is given by

$$\dot{x} = f(x) + Bu + Pz_r \quad (A.2)$$

where  $x = (\dot{z}_s \ \dot{z}_{us} \ f_I \ z_s \ z_{us})^T$ ,

$$f(x) = \begin{pmatrix} \frac{-k}{m_s}(x_4 - x_5) - \frac{c_0}{m_s}(x_1 - x_2) - \frac{\rho(x)}{m_s}x_3 \\ \frac{k}{m_{us}}x_4 - \frac{k + k_t}{m_{us}}x_5 + \frac{c_0}{m_{us}}(x_1 - x_2) + \frac{\rho(x)}{m_{us}}x_3 \\ -\beta x_3 \\ x_1 \\ x_2 \end{pmatrix}$$

where  $k = k_s + k_0$  and  $\rho(x) = \tanh(c_1(x_1 - x_2) + k_1(x_4 - x_5))$ ,

$$B = (0 \ 0 \ \beta \ 0 \ 0)^T, \quad P = \begin{pmatrix} 0 & \frac{k_t}{m_{us}} & 0 & 0 & 0 \end{pmatrix}^T$$

The global optimization using the Minimum Principle of Pontryagin can be used for nonlinear system (A.2) with a constraint on control input  $u \in [f_{Imin}, f_{Imax}]$ . Consider only the known parts of (A.2) (without road disturbance  $Pz_r$ ). Let define the control problem

$$\text{minimize} \quad J = \int_{t_0}^{t_f} \ddot{z}_s^2 dt = \int_{t_0}^{t_f} l(x) dt \quad (\text{A.3})$$

where  $l(x) = (\frac{-k}{m_s}(x_4 - x_5) - \frac{c_0}{m_s}(x_1 - x_2) - \frac{\rho(x)}{m_s}x_3)^2$

subject to

$$\dot{x} = f(x) + Bu \quad (\text{A.4})$$

$$u \in [f_{Imin}, f_{Imax}] \quad (\text{A.5})$$

$$t_f = t_0 + \Delta T \quad (\text{A.6})$$

The Hamiltonian function is defined by

$$H(x, u, \lambda) = l(x) + \lambda^T [f(x) + Bu] \quad (\text{A.7})$$

Let  $(u^*(t), x^*(t))$  is the optimal solution of problem (A.3). The adjoint equation is

$$\dot{\lambda}(t) = -\nabla_x H(x^*(t), u^*(t), \lambda(t)) \quad (\text{A.8})$$

with  $\lambda(t_f) = 0$ .

Minimization of the Hamiltonian gives

$$u^*(t) = \arg \min_{u(t) \in [f_{Imin}, f_{Imax}]} H(x^*(t), u(t), \lambda(t)) \quad (\text{A.9})$$

The problems (A.8) and (A.9) are difficult to solve, however, it can be seen from Eq. (A.7) that the optimal control law of a semi-active suspension with comfort objective without preview is a genuine on-off strategy and is given by

$$u^*(t) = \begin{cases} f_{Imin} & \text{if } B^T \lambda(t) > 0 \\ f_{Imax} & \text{if } B^T \lambda(t) \leq 0 \end{cases} \quad (\text{A.10})$$

In order to find an explicit solution of  $u$ , consider the linear approximation of system (A.2) around the initial condition  $(x(t_0), u(t_0), z_r(t_0))$ . After some manipulations, one has

$$\dot{x} = Ax + Bu + Pz_r + E \quad (\text{A.11})$$

where

$$A = \begin{pmatrix} -\frac{a_0}{m_s} & \frac{a_0}{m_s} & \frac{\rho_0}{m_s} & -\frac{a_1}{m_s} & \frac{a_1}{m_s} \\ \frac{a_0}{m_{us}} & -\frac{a_0}{m_{us}} & \frac{\rho_0}{m_{us}} & \frac{a_1}{m_{us}} & -\frac{a_1}{m_{us}} \\ 0 & 0 & -\beta & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$a_0 = c_0 + x_{30}(1 - \rho_0^2)c_1$ ,  $a_1 = k + x_{30}(1 - \rho_0^2)k_1$ ,  $\rho_0 = \rho(x(t_0)) = \tanh(c_1(x_{10} - x_{20}) + k_1(x_{40} - x_{50}))$  and  $x_{i0}$  is the  $i^{th}$  component of the state vector  $x(t_0)$ .

$$E = \begin{pmatrix} \frac{((x_{10} - x_{20})c_1 + (x_{40} - x_{50})k_1)x_{30}(1 - \rho_0^2)}{m_s} \\ -\frac{((x_{10} - x_{20})c_1 + (x_{40} - x_{50})k_1)x_{30}(1 - \rho_0^2)}{m_{us}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By using the Lagrange Formula,  $x(t)$  at  $t_0 + \Delta T$  can be computed

$$x(t_0 + \Delta T) = e^{A(\Delta T)}x(t_0) + \int_{t_0}^{t_0 + \Delta T} e^{A(t_0 + \Delta T - \tau)}(Bu(\tau) + Pz_r(\tau) + E)d\tau \quad (\text{A.12})$$

Assume that, during the sampling interval  $\Delta T$ , the control input is constant  $u = \bar{u}$  and the body car acceleration  $\ddot{z}_s(t)$  or  $\dot{x}_1(t)$  does not change its sign. The optimal solution  $u$  is given as

$$\begin{aligned} \bar{u}_{opt}(t_0, t_0 + \Delta T) &= \arg \min_{\bar{u} \in [f_{Imin}, f_{Imax}]} (\dot{x}_1(t_0 + \Delta T))^2 \\ &= \begin{cases} \arg \min_{\bar{u} \in [f_{Imin}, f_{Imax}]} (\dot{x}_1(t_0 + \Delta T)) & \text{if } \ddot{z}_s(t_0) > 0 \\ \arg \min_{\bar{u} \in [f_{Imin}, f_{Imax}]} (\dot{x}_1(t_0 + \Delta T)) & \text{if } \ddot{z}_s(t_0) \leq 0 \end{cases} \\ &= \begin{cases} \arg \min_{\bar{u} \in [f_{Imin}, f_{Imax}]} (g_u(t_0 + \Delta T)) & \text{if } \ddot{z}_s(t_0) > 0 \\ \arg \min_{\bar{u} \in [f_{Imin}, f_{Imax}]} (g_u(t_0 + \Delta T)) & \text{if } \ddot{z}_s(t_0) \leq 0 \end{cases} \end{aligned} \quad (\text{A.13})$$

where  $g_{\bar{u}}(t_0 + \Delta T) = \frac{d(\bar{u} \int_{t_0}^{t_0 + \Delta T} e^{A(t-\tau)} Bu(\tau) d\tau)_1}{dt} \Big|_{t=t_0 + \Delta T}$  and

subscript "1" indicates the first element of the vector  $\bar{u} \int_{t_0}^{t_0 + \Delta T} e^{A(t-\tau)} Bu(\tau) d\tau$ . By using Taylor series expansion for  $e^{A(t-\tau)}$ , the following approximation is used for the calculation of  $g_{\bar{u}}(t_0 + \Delta T)$ :

$$e^{A(t-\tau)} = I + A(t-\tau) + \frac{1}{2}A^2(t-\tau)^2 + \frac{1}{6}A^3(t-\tau)^3 \quad (\text{A.14})$$

Finally one has

$$g_{\bar{u}}(t_0 + \Delta T) = -\bar{u}\rho_0\beta\gamma(\Delta T, \rho_0, x_{30}) \quad (\text{A.15})$$

Due to the length and the complexity, the explicit form of  $\gamma(\Delta T, \rho_0, x_{30})$  is not given here. But note that with the damper's bandwidth  $\beta = 40\pi$ ,  $\rho_0 \in [-1, 1]$  and the initial state  $x_{30} \in [0, 870]$ , one always has  $\gamma(\Delta T, \rho_0, x_{30}) > 0$  (see Fig. A.1). From Eq. (A.13) and Eq. (A.15), the optimal solution is finally given as:

$$\bar{u}_{opt}(t_0, t_0 + \Delta T) = \begin{cases} f_{Imin} & \text{if } \ddot{z}_s(t_0)\rho_0 > 0 \\ f_{Imax} & \text{if } \ddot{z}_s(t_0)\rho_0 \leq 0 \end{cases} \quad (\text{A.16})$$

The control law proposed in (11) has been proved.

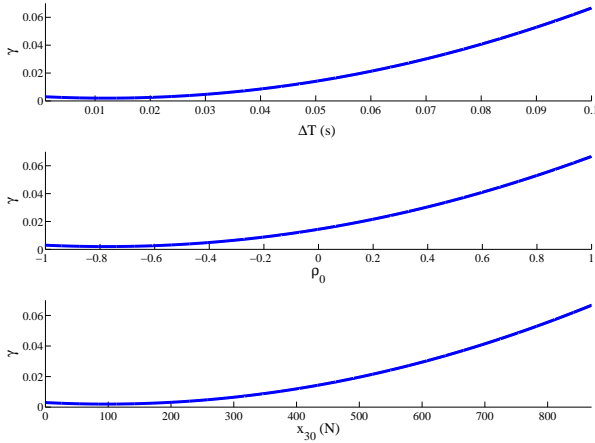


Fig. A.1.  $\gamma(\Delta T, \rho_0, x_{30})$

## Appendix B. NONLINEAR FREQUENCY RESPONSE - (PSEUDO-BODE)

The following nonlinear frequency response analysis is done by using the "variance gain" algorithm (see Savaresi et al. (2005b)) for nonlinear systems. The "variance gain" is simple and provides a good approximation to frequency response.

- Feed the system with a sinus signal  $z_{r_i} = A_r \sin(\omega_i t)$  ( $\omega_{min} \leq \omega_i \leq \omega_{max}$ ,  $i=1,2,3\dots N$  and  $t \in [0, T]$ ).
- For each input, measure output signals; for example, to evaluate the comfort, the body car acceleration  $\ddot{z}_{s_i}$  is measured.
- The approximate variance gain is computed and is defined as

$$F_{acc}(\omega_i) = \sqrt{\frac{\frac{1}{T} \int_0^T (\ddot{z}_{s_i})^2 dt}{\frac{1}{T} \int_0^T (z_{r_i})^2 dt}} \quad (i = 1, 2, 3 \dots N) \quad (B.1)$$

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