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EXPLICIT ASSOCIATOR RELATIONS FOR MULTIPLE ZETA VALUES

ISMAEL SOUDÈRES

ABSTRACT. Associators were introduced by Drinfel'd in [Dri91] as a monodromy representation of a Knizhnik-Zamolodchikov equation. Associators can be briefly described as formal series in two non-commutative variables satisfying three equations. These three equations yield a large number of algebraic relations between the coefficients of the series, a situation which is particularly interesting in the case of the original Drinfel'd associator, whose coefficients are multiple zetas values. In the first part of this paper, we work out these algebraic relations among multiple zeta values by direct use of the defining relations of associators. While well-known for the first two relations, the algebraic relations we obtain for the third (pentagonal) relation, which are algorithmically explicit although we do not have a closed formula, do not seem to have been previously written down. The second part of the paper shows that if one has an explicit basis for the bar-construction of the moduli space $\mathcal{M}_{0,5}$ of genus zero Riemann surfaces with 5 marked points at one's disposal, then the task of writing down the algebraic relations corresponding to the pentagon relation becomes significantly easier and more economical compared to the direct calculation above. We discuss the explicit basis described by Brown and Gangl, which is dual to the basis of the enveloping algebra of the braids Lie algebra, $U\mathfrak{B}_5$.

In order to write down the relation between multiple zeta values, we then remark that it is enough to write down the relations associated to elements that generate the bar construction as an algebra. This corresponds to looking at the bar construction modulo shuffle, which is dual to the Lie algebra of 5-strand braids. We write down, in the appendix, the associated algebraic relations between multiple zeta values in weights 2 and 3.

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1. INTRODUCTION

In the first part of this introduction we recall the necessary definitions concerning associators, and in the second part, we recall the definitions and main results concerning multiple zeta values. In the third part, we give the outline of the paper and state the main results.

1.1. Associators. Let k be a field of characteristic 0. Let $U\mathfrak{F}_2 = k\langle\langle X_0, X_1 \rangle\rangle$ be the ring of formal series over k in two non-commutative variables. The coproduct Δ on $U\mathfrak{F}_2$ is defined by

$$\Delta(X_0) = X_0 \otimes 1 + 1 \otimes X_0 \quad \Delta(X_1) = X_1 \otimes 1 + 1 \otimes X_1.$$

An element $\Phi = \Phi(X_0, X_1) \in U\mathfrak{F}_2$ is *group-like* if it satisfies $\Delta(\Phi) = \Phi \hat{\otimes} \Phi$ where $\hat{\otimes}$ denotes the complete tensor product.

Remark 1.1. We remark that the constant term of a group-like element is 1.

Definition 1.2. If S is a finite set, let S^* denote the set of words with letters in S , that is the dictionary over S . If $S = \{s_1, \dots, s_n\}$ we may write, $\{s_1, \dots, s_n\}^*$.

Let $\mathcal{W}_{0,1}$ be the dictionary over $\{X_0, X_1\}$.

We remark that the monomial in $U\mathfrak{F}_2$ are words in $\mathcal{W}_{0,1}$; the empty word \emptyset in $\mathcal{W}_{0,1}$ will be 1 by convention when considered in $U\mathfrak{F}_2$. The following definition allows us to define a filtration on $U\mathfrak{F}_2$.

Definition 1.3. The depth $dp(W)$ of a monomial $W \in U\mathfrak{F}_2$, that is an element of $\mathcal{W}_{0,1}$, is the number of X_1 , and its weight (or length) $wt(W) = |W|$ is the number of letters.

The algebra $U\mathfrak{F}_2$ is filtered by the weight, and its graded pieces of weight d are the subspaces generated by the monomial of length d ; $U\mathfrak{F}_2$ is thus a graded algebra.

Let $U\mathfrak{B}_5$ be the enveloping algebra of \mathfrak{B}_5 , the completion (with respect to the natural grading) of the pure sphere braid Lie algebra; that is, $U\mathfrak{B}_5$ is the quotient of $C\langle\langle X_{ij} \rangle\rangle$ with $1 \leq i \leq 5$ and $1 \leq j \leq 5$ by the relations

- $X_{ii} = 0$ for $1 \leq i \leq 5$,
- $X_{ij} = X_{ji}$ for $1 \leq i, j \leq 5$,
- $\sum_{j=1}^5 X_{ij} = 0$ for $1 \leq i \leq 5$,
- $[X_{ij}, X_{kl}] = 0$ if $\{i, j\} \cap \{k, l\} = \emptyset$.

Definition 1.4 (Drinfel'd [Dri91]). A group-like element Φ in $U\mathfrak{F}_2$, with zero coefficient in degree 1, together with a $\mu \in k^*$ is an associator if it satisfies the following equations

$$(I) \quad \Phi(X_0, X_1)\Phi(X_1, X_0) = 1$$

$$(II) \quad e^{\frac{\mu}{2}X_0}\Phi(X_\infty, X_0)e^{\frac{\mu}{2}X_\infty}\Phi(X_1, X_\infty)e^{\frac{\mu}{2}X_1}\Phi(X_0, X_1) = 1 \quad \text{with } X_0 + X_1 + X_\infty = 0$$

$$(III) \quad \Phi(X_{12}, X_{23})\Phi(X_{34}, X_{45})\Phi(X_{51}, X_{12})\Phi(X_{23}, X_{34})\Phi(X_{45}, X_{51}) = 1 \quad \text{in } U\mathfrak{B}_5$$

We will write an associator as

$$\Phi(X_0, X_1) = \sum_{W \in \mathcal{W}_{0,1}} Z_W W = 1 + \sum_{\substack{W \in \mathcal{W}_{0,1} \\ W \neq \emptyset}} Z_W W.$$

We have $Z_\emptyset = 1$ because Φ is group-like.

In [Dri91], Drinfel'd gives an explicit associator Φ_{KZ} over \mathbb{C} , known as the Drinfel'd associator and associated to a Knizhnik-Zamolodchikov equation (KZ equation). More precisely, consider the KZ equation (see also [Fur03][§3]).

$$(KZ) \quad \frac{\partial g}{\partial u} = \left(\frac{X_0}{u} + \frac{X_1}{1-u} \right) \cdot g(u)$$

where g is an analytic function in one complex variable u with values in $\mathbb{C}\langle\langle X_0, X_1 \rangle\rangle$ (analytic means that each coefficient is an analytic function). This equation has singularities only at 0, 1 and ∞ . The equation (KZ) has a unique solution on $C = \mathbb{C} \setminus (]-\infty, 0] \cup [1, \infty[)$ having a specified value at a given point in C , because C is simply connected. Moreover, at 0 (resp. 1), there exists a unique solution $g_0(u)$ (resp. $g_1(u)$) such that

$$g_0(u) \sim u^{X_0} \quad (u \rightarrow 0) \quad (\text{resp. } g_1(u) \sim (1-u)^{X_1} \quad (u \rightarrow 1)).$$

As g_0 and g_1 are invertible with specified asymptotic behavior, they must coincide up to multiplication on the right by an invertible element in $\mathbb{C}\langle\langle X_0, X_1 \rangle\rangle$.

Definition 1.5. The *Drinfel'd associator*¹, Φ_{KZ} , is the element in $\mathbb{C}\langle\langle X_0, X_1 \rangle\rangle$ defined by

$$g_0(u) = g_1(u)\Phi_{KZ}(X_0, X_1).$$

In [Dri91], Drinfel'd proved the following result.

Proposition 1.6. *The element Φ_{KZ} is a group-like element and it satisfies (I), (II) with $\mu = 2i\pi$, and (III) of definition 1.4. That is,*

$$(IKZ) \quad \Phi_{KZ}(X_0, X_1)\Phi_{KZ}(X_1, X_0) = 1$$

$$(IIKZ) \quad e^{i\pi X_0}\Phi_{KZ}(X_\infty, X_0)e^{i\pi X_\infty}\Phi_{KZ}(X_1, X_\infty)e^{i\pi X_1}\Phi_{KZ}(X_0, X_1) = 1 \\ \text{with } X_0 + X_1 + X_\infty = 0$$

$$(IIIKZ) \quad \Phi_{KZ}(X_{12}, X_{23})\Phi_{KZ}(X_{34}, X_{45})\Phi_{KZ}(X_{51}, X_{12})\Phi_{KZ}(X_{23}, X_{34}) \\ \Phi_{KZ}(X_{45}, X_{51}) = 1 \quad \text{in } U\mathfrak{B}_5.$$

1.2. Multiple zeta values. For a p -tuple $\mathbf{k} = (k_1, \dots, k_p)$ of strictly positive integers with $k_1 \geq 2$, the multiple zeta value $\zeta(\mathbf{k})$ is defined as

$$\zeta(\mathbf{k}) = \sum_{n_1 > \dots > n_p > 0} \frac{1}{n_1^{k_1} \dots n_p^{k_p}}.$$

Definition 1.7. The depth of a p -tuple of integers $\mathbf{k} = (k_1, \dots, k_p)$ is $dp(\mathbf{k}) = p$, and its weight $wt(\mathbf{k})$ is $wt(\mathbf{k}) = k_1 + \dots + k_p$.

To the tuple \mathbf{k} , with $n = wt(\mathbf{k})$, we associate the n -tuple:

$$\bar{\mathbf{k}} = (\underbrace{0, \dots, 0}_{k_1-1 \text{ times}}, 1, \dots, \underbrace{0, \dots, 0}_{k_p-1 \text{ times}}, 1) = (\varepsilon_n, \dots, \varepsilon_1)$$

¹In [Dri91], Drinfel'd actually defined ϕ_{KZ} rather than Φ_{KZ} , where $\phi_{KZ}(X_0, X_1) = \Phi_{KZ}(\frac{1}{2i\pi}X_0, \frac{1}{2i\pi}X_1)$ and is defined via the KZ equation $\frac{\partial g}{\partial u} = \frac{1}{2i\pi} \left(\frac{X_0}{u} + \frac{X_1}{1-u} \right) \cdot g(u)$.

and the word in $\{X_0, X_1\}^*$

$$X_{\varepsilon_n} \cdots X_{\varepsilon_1}.$$

This makes it possible to associate a multiple zeta value $\zeta(W)$ to each word W in $X_0\{X_0, X_1\}X_1$ (where W begins with X_0 and ends with X_1).

Following Kontsevich and Drinfel'd, one can write the multiple zeta values as a Chen iterated integral [Che73]

$$\zeta(\mathbf{k}) = \int_0^1 (-1)^p \frac{du}{u - \varepsilon_n} \circ \cdots \circ \frac{du}{u - \varepsilon_1}.$$

Note that, as $k_1 \geq 2$, we have $\varepsilon_n = 0$. This expression as an iterated integral leads directly to an expression of the multiple zeta values as an integral over a simplex

$$\zeta(\mathbf{k}) = \int_{\Delta_n} (-1)^p \frac{dt_1}{t_1 - \varepsilon_1} \wedge \cdots \wedge \frac{dt_n}{t_n - \varepsilon_n}$$

where $\Delta_n = \{0 < t_1 < \cdots < t_n < 1\}$.

Thanks to the work of Ecalle, Zagier, Ihara, Kaneko or Furusho we can extend the definition of multiple zeta values to tuples without the condition $k_1 \geq 2$ (see [Rac02], [IKZ06] or [Fur03]). These extended multiple zeta values are called regularized multiple zeta values and we speak of regularizations. We will be interested in a specific regularization, the *shuffle regularization*.

Definition 1.8 (Shuffle product). A shuffle of $\{1, 2, \dots, n\}$ and $\{1, \dots, m\}$ is a permutation σ of $\{1, 2, \dots, n+m\}$ such that :

$$\sigma(1) < \sigma(2) < \cdots < \sigma(n) \quad \text{and} \quad \sigma(n+1) < \sigma(n+2) < \cdots < \sigma(n+m).$$

The set of all the shuffles of $\{1, 2, \dots, n\}$ and $\{1, \dots, m\}$ is denoted by $\text{sh}(n, m)$

Let $V = X_{i_1} \cdots X_{i_n}$ and $W = X_{i_{n+1}} \cdots X_{i_{n+m}}$ be two words in $\mathcal{W}_{0,1}$. The shuffle of V and W is the collection of words

$$\text{sh}(V, W) = (X_{i_{\sigma^{-1}(1)}} X_{i_{\sigma^{-1}(2)}} \cdots X_{i_{\sigma^{-1}(n+m)}})_{\sigma \in \text{sh}(n, m)}.$$

Working in $\mathbb{C}\langle\langle X_0, X_1 \rangle\rangle$, we will also consider the sum

$$V \text{ III } W = \sum_{U \in \text{sh}(V, W)} U = \sum_{\sigma \in \text{sh}(n, m)} X_{i_{\sigma^{-1}(1)}} X_{i_{\sigma^{-1}(2)}} \cdots X_{i_{\sigma^{-1}(n+m)}}$$

and extend the shuffle product III by linearity.

Definition 1.9. The *shuffle regularization* of the multiple zeta values is the collection of real numbers $(\zeta^{\text{III}}(W))_{W \in \mathcal{W}_{0,1}}$ such that:

- (1) $\zeta^{\text{III}}(X_0) = \zeta^{\text{III}}(X_1) = 0$,
- (2) $\zeta^{\text{III}}(W) = \zeta(W)$ for all $W \in X_0 \mathcal{W}_{0,1} X_1$,
- (3) $\zeta^{\text{III}}(V) \zeta^{\text{III}}(W) = \sum_{U \in \text{sh}(V, W)} \zeta^{\text{III}}(U)$ for all $V, W \in \mathcal{W}_{0,1}$

These regularized multiple zeta values $\zeta^{\text{III}}(W)$, for W not in $X_0 \mathcal{W}_{0,1} X_1$, are in fact linear combinations of the usual multiple zeta values, as it has been shown explicitly by Furusho in [Fur03].

The coefficients of the Drinfel'd associator can be written in an explicit way using convergent multiple zeta values [Fur03].

Proposition 1.10. *Using the shuffle regularization we can write ([LM96], [Fur03])*

$$\Phi_{KZ}(X_0, X_1) = \sum_{W \in \mathcal{W}_{0,1}} (-1)^{\text{wt}(W)} \zeta^{\text{III}}(W) W.$$

1.3. Main results. In Theorem 2.2 and Theorem 2.8 we will give explicit relations between the coefficients of the series defining an associator Φ equivalent to the relation (I) and (II) satisfied by Φ . Both were well-known as it is easy to expand the product of the associators in $U\widehat{\mathfrak{F}}_2$ even if the author does not know whether the relations of Theorem 2.8 have been written in the literature. In the case of the pentagon relation (III), writing down relations between the coefficients implies fixing a basis B of $U\mathfrak{B}_5$. Even if fixing such a basis breaks the natural symmetry of the pentagon relation (III), it is then possible to write a new family of relations between the coefficients of Φ equivalent to (III_{KZ}). More precisely, decomposing a word W in the letters $X_{3,4}, X_{4,5}, X_{2,4}, X_{1,2}, X_{2,3}$ in the basis B we have

$$W = \sum_{b \in B} l_{b,W} b,$$

and we obtain the following theorem.

Theorem (Theorem 2.12). *The relation (III) is equivalent to the family of relations*

$$\forall b \in B \ (b \neq 1) \quad \sum_{W \in \{X_{3,4}, X_{4,5}, X_{2,4}, X_{1,2}, X_{2,3}\}^*} l_{b,W} C_{5,W} = 0$$

where $C_{5,W}$ are explicitly given by :

$$C_{5,W} = \sum_{\substack{U_1, \dots, U_5 \in W \\ U_1 \cdots U_5 = W}} Z_{\rho_1(U_1)} Z_{\rho_2(U_2)} Z_{\rho_3(U_3)} Z_{\rho_4(U_4)} Z_{\rho_5(U_5)}.$$

Applying this theorem to the particular basis B_4 coming from the identification

$$U\mathfrak{B}_5 \simeq k\langle\langle X_{3,4}, X_{4,5}, X_{2,4} \rangle\rangle \rtimes k\langle\langle X_{1,2}, X_{2,3} \rangle\rangle,$$

one can compute the coefficients $l_{b,W}$ using the equation defining $U\mathfrak{B}_5$. In particular it is easy to see that $l_{b,W}$ is in \mathbb{Z} in that case.

After each family of relations between the coefficients, we apply our results to the particular case of the Drinfel'd associator and give the corresponding family between multiple zeta values in equations (3), (8) and (11).

In Section 3 of the article, we explain how these families of relations between multiple zeta values are induced by iterated integrals on $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$ using the bar construction studied by Brown in [Bro09]. The geometry of $\mathcal{M}_{0,5}$ allows us in Proposition 3.16 to interpret the coefficients $C_{5,W}$ using iterated integrals.

Proposition (Proposition 3.16). *For any bar symbol ω_W dual to a word W in the letters $X_{34}, X_{45}, X_{24}, X_{12}, X_{23}$, we have*

$$C_{5,W} = \int_{\gamma} \text{Reg}(\omega_W, \gamma)$$

where $\text{Reg}(\omega, D)$ is the regularisation of a bar symbol in $\oplus \mathbb{H}^1(\mathcal{M}_{0,5})^{\otimes n}$ along boundary components $D \subset \partial\mathcal{M}_{0,5}$.

This is a consequence of Theorem 3.14 which links the family of relations (11) to the bar construction.

Theorem (Theorem 3.14). *The relation (III_{KZ}) is equivalent to the family of relations*

$$\forall b_4 \in B_4 \quad \int_{\gamma} \text{Reg}(b_4^*, \gamma) = 0$$

which is exactly the family of relations (11). Here $(b_4^*)_{b_4 \in B_4}$ denotes the basis in $V(\mathcal{M}_{0,5})$, the bar construction on $\mathcal{M}_{0,5}$, dual to the basis B_4 of $U\mathfrak{B}_5$ described earlier.

More generally, we then have that, for any basis $F = (f)_{f \in F}$ on $V(\mathcal{M}_{0,5})$, the pentagon relation (III_{KZ}) is equivalent to

$$\forall f \in F \quad \int_{\gamma} \text{Reg}(f, \gamma) = 0.$$

Brown, Gangl and Levin in [BGL10] obtain, using different methods and for another purpose, the same basis B_4^* of $V(\mathcal{M}_{0,5})$. In their work, the basis B_4^* is described using combinatorial objects. More precisely, they use maximal triangulations of rooted decorated polygons.

Instead of looking all the elements of a basis F of $V(\mathcal{M}_{0,5})$, it is enough to consider only a subbasis that generates $V(\mathcal{M}_{0,5})$ as a shuffle algebra. Indeed, if ω in $V(\mathcal{M}_{0,5})$ is equal to $f_1 \sharp f_2$, the iterated integral $\int_{\gamma} \omega$ is equal to $\int_{\gamma} f_1 \int_{\gamma} f_2$. Thus it does not give a new relation between multiple zeta values. Considering a set of generators of the shuffle algebra leads to computing many less relations. In degree 2 and 3 we have respectively 4 and 10 generators instead of 19 and 65 elements in the vector space basis. In the appendix, we will give these relations in degree 2 and 3 using the basis B_4 .

The multiplicative generators that we have found don't have a particularly simple expression in terms of symbols ω_W dual to words W in the letters $X_{34}, X_{45}, X_{24}, X_{12}, X_{23}$. But it seems to be linked with our particular choice of identification. The identification

$$U\mathfrak{B}_5 \simeq k\langle\langle X_{3,4}, X_{1,4}, X_{2,4} \rangle\rangle \rtimes k\langle\langle X_{1,2}, X_{2,3} \rangle\rangle$$

leads to another basis \tilde{B}_4 of $U\mathfrak{B}_5$. Then, multiplicative generators can be found with a particularly simple expression in terms of symbols ω_W dual to words W in the letters $X_{34}, X_{14}, X_{24}, X_{12}, X_{23}$. More precisely, writing such a word W as

$$W = \sum_{\tilde{b}_4 \in \tilde{B}_4} l_{\tilde{b}_4, W} \tilde{b}_4,$$

we can write $\tilde{b}_4^* = \sum_W l_{\tilde{b}_4, W} \omega_W$. The generators are elements \tilde{b}_4^* such that the number of $l_{\tilde{b}_4, W}$ is as minimal as possible. This seems to be a general fact.

2. COMBINATORIAL DESCRIPTION OF ASSOCIATORS RELATIONS

The goal of this section is, for any associator and for the particular case of Φ_{KZ} , to give an explicit expression for the relations between the coefficients derived from the associator relations (I), (II) and (III). For each of these relations, we will first study the case of a general associator and then deduce, for the Drinfel'd associator, relations between the regularized multiple zeta values. Let

$$\Phi = \sum_{W \in \mathcal{W}_{0,1}} Z_W W$$

be an associator. The idea will be to expand the product in the right hand side of the equations (I), (II) and (III) in a suitable basis of the space $U\mathfrak{F}_2$ or $U\mathfrak{B}_5$.

2.1. The symmetry, (I) and (I_{KZ}). Let P_2 be the product $\Phi(X_0, X_1)\Phi(X_1, X_0)$. As the monomials in $U\mathfrak{F}_2$, that is words in $\mathcal{W}_{0,1}$, form a basis of $U\mathfrak{F}_2$, we can write P_2 as

$$P_2 = \sum_{W \in \mathcal{W}_{0,1}} C_{2,W} W = 1 + \sum_{W \in \mathcal{W}_{0,1}, W \neq \emptyset} C_{2,W} W.$$

The relation (I) tells us that for each $W \in \mathcal{W}_{0,1}$, W being nonempty, we have

$$(1) \quad C_{2,W} = 0.$$

Example 2.1. In low degree we have the following relations:

- In degree one, there are just 2 words : X_0 and X_1 and (1) gives:

$$\begin{aligned} C_{2,X_0} &= Z_{X_0} + Z_{X_1} = 0 \\ C_{2,X_1} &= Z_{X_1} + Z_{X_0} = 0 \end{aligned}$$

- In degree two there are 4 words X_0X_0 , X_0X_1 , X_1X_0 and X_1X_1 and (1) gives:

$$\begin{aligned} C_{2,X_0X_0} &= Z_{X_0X_0} + Z_{X_0}Z_{X_1} + Z_{X_1}X_1 = 0 \\ C_{2,X_0X_1} &= Z_{X_0X_1} + Z_{X_0}Z_{X_0} + Z_{X_1}X_0 = 0 \\ C_{2,X_1X_0} &= Z_{X_1X_0} + Z_{X_1}Z_{X_1} + Z_{X_0}X_1 = 0 \\ C_{2,X_1X_1} &= Z_{X_1X_1} + Z_{X_1}Z_{X_0} + Z_{X_0}X_0 = 0 \end{aligned}$$

- In degree three there are 8 words. Looking at the coefficients of the words $X_0X_0X_1$ in P_2 , equation (1) gives:

$$Z_{X_0X_0X_1} + Z_{X_0X_0}Z_{X_0} + Z_{X_0}Z_{X_1X_0} + Z_{X_1X_1X_0} = 0$$

Let θ be the automorphism of $U\mathfrak{F}_2$ that sends X_0 to X_1 and X_1 to X_0 . Then we have:

Theorem 2.2. *The relation (I) is equivalent to the family of relations*

$$(2) \quad \forall W \in \mathcal{W}_{0,1} \setminus \{\emptyset\}, \quad \sum_{\substack{U_1, U_2 \in \mathcal{W}_{0,1} \\ U_1 U_2 = W}} Z_{U_1} Z_{\theta(U_2)} = 0.$$

Proof. As $\Phi(X_1, X_0) = \theta(\Phi(X_0, X_1))$, we have

$$\begin{aligned} \Phi(X_1, X_0) &= \theta \left(1 + \sum_{\substack{W \in \mathcal{W}_{0,1} \\ W \neq \emptyset}} Z_W W \right) = 1 + \sum_{\substack{W \in \mathcal{W}_{0,1} \\ W \neq \emptyset}} Z_W \theta(W) \\ &= 1 + \sum_{\substack{W \in \mathcal{W}_{0,1} \\ W \neq \emptyset}} Z_{\theta(W)} W. \end{aligned}$$

Then, expanding the product P_2 and reorganizing, we have

$$\begin{aligned} \Phi(X_0, X_1) \Phi(X_1, X_0) &= \left(1 + \sum_{\substack{U_1 \in \mathcal{W}_{0,1} \\ U_1 \neq \emptyset}} Z_{U_1} U_1 \right) \left(1 + \sum_{\substack{U_2 \in \mathcal{W}_{0,1} \\ U_2 \neq \emptyset}} Z_{\theta(U_2)} U_2 \right) \\ &= \sum_{\substack{W \in \mathcal{W}_{0,1} \\ W \neq \emptyset}} \left(\sum_{\substack{U_1, U_2 \in \mathcal{W}_{0,1} \\ U_1 U_2 = W}} Z_{U_1} Z_{\theta(U_2)} \right) W. \end{aligned}$$

□

Corollary 2.3. *The relation (I_{KZ}) is equivalent to the family of relations*

$$(3) \quad \forall W \in \mathcal{W}_{0,1}, \quad \sum_{\substack{U_1, U_2 \in \mathcal{W}_{0,1} \\ U_1 U_2 = W}} (-1)^{dp(U_1)} \zeta^{\mathfrak{m}}(U_1) (-1)^{dp(\theta(U_2))} \zeta^{\mathfrak{m}}(\theta(U_2)) = 0,$$

that family being equivalent to the following

$$(4) \quad \forall W \in \mathcal{W}_{0,1}, \quad \sum_{\substack{U_1, U_2 \in \mathcal{W}_{0,1} \\ U_1 U_2 = W}} (-1)^{|U_2|} \zeta^{\mathfrak{m}}(U_1) \zeta^{\mathfrak{m}}(\theta(U_2)) = 0.$$

2.2. The 3-cycle or the hexagon relation, (II) and (II_{KZ}). For any element $P = \sum_{W \in \mathcal{W}_{0,1}} a_W W$ in $U\mathfrak{F}_2$, let $C_{0,1}(P|W)$ be the coefficient a_W of the monomial W .

Let P_3 be the product

$$P_3 = e^{\frac{\mu}{2}X_0}\Phi(X_\infty, X_0)e^{\frac{\mu}{2}X_\infty}\Phi(X_1, X_\infty)e^{\frac{\mu}{2}X_1}\Phi(X_0, X_1).$$

We can write P_3 as

$$P_3 = \sum_{W \in \mathcal{W}_{0,1}} C_{0,1}(P_3|W)W = \sum_{W \in \mathcal{W}_{0,1}} C_{3,W}W.$$

The relation (II) tells us that for each $W \in \mathcal{W}_{0,1}$, $W \neq \emptyset$, we have

$$(5) \quad C_{3,W} = 0.$$

In order to make these coefficients explicit, we will need some definitions.

Definition 2.4. Let α_0 (resp. α_1 and α_∞) be the endomorphism of $U\mathfrak{F}_2$ defined on X_0 and X_1 by :

$$\alpha_0(X_0) = X_0 \quad \text{and} \quad \alpha_0(X_1) = 0,$$

respectively

$$\alpha_1(X_0) = 0 \quad \text{and} \quad \alpha_1(X_1) = X_1$$

and

$$\alpha_\infty = -(\alpha_0 + \alpha_1).$$

Let $\tilde{\alpha}_i$ be the composition of α_i with $X_0, X_1 \mapsto 1$.

The following proposition is a consequence of the expression of the exponential

$$\forall P \in U\mathfrak{F}_2 \quad \exp(P) = \sum_{n \geq 0} \frac{P^n}{n!}$$

and of the equality

$$(6) \quad (-X_0 - X_1)^n = \sum_{\substack{W \in \mathcal{W}_{0,1} \\ |W|=n}} (-1)^{|W|} W.$$

Proposition 2.5. Let W be a word in $\mathcal{W}_{0,1}$, we have

$$\begin{aligned} C_{0,1}(e^{\frac{\mu}{2}X_0}|W) &= \frac{\mu^{|W|}}{2^{|W|}|W|!} \tilde{\alpha}_0(W), \\ C_{0,1}(e^{\frac{\mu}{2}X_1}|W) &= \frac{\mu^{|W|}}{2^{|W|}|W|!} \tilde{\alpha}_1(W) \quad \text{and} \\ C_{0,1}(e^{\frac{\mu}{2}X_\infty}|W) &= (-1)^{|W|} \frac{\mu^{|W|}}{2^{|W|}|W|!}. \end{aligned}$$

In order to describe the coefficient of $\Phi(X_i, X_j)$, with either one of the variables being X_∞ , we introduce a set of different decompositions of W into sub-words.

Definition 2.6. Let W be a word in $\mathcal{W}_{0,1}$. For $i \in \{0, 1\}$, let $\text{dec}_{0,1}(W, X_i)$ be the set of tuples $(V_1, X_i^{k_1}, V_2, X_i^{k_2}, \dots, V_p, X_i^{k_p})$ with $1 \leq p < \infty$, $V_j \in \mathcal{W}_{0,1}$ and $V_2, \dots, V_p \neq \emptyset$ such that

$$W = V_1 X_i^{k_1} V_2 X_i^{k_2} \dots V_p X_i^{k_p}.$$

We will write $(\mathbf{V}, \mathbf{k}) \in \text{dec}_{0,1}(W, X_i)$ instead of $(V_1, X_i^{k_1}, V_2, X_i^{k_2}, \dots, V_p, X_i^{k_p}) \in \text{dec}_{0,1}(W, X_i)$ and $|\mathbf{V}|$ (resp. $|\mathbf{k}|$) will denote $|V_1| + \dots + |V_p|$ (resp. $k_1 + \dots + k_p$).

The following proposition describes the coefficient of W in the series $\Phi(X_0, X_1)$, $\Phi(X_\infty, X_0)$ and $\Phi(X_1, X_\infty)$.

Proposition 2.7. *Let W be a word in $\mathcal{W}_{0,1}$. We have*

$$C_{0,1}(\Phi(X_0, X_1)|W) = Z_W.$$

The coefficients $C_{0,1}(\Phi(X_\infty, X_0)|W)$ and $C_{0,1}(\Phi(X_1, X_\infty)|W)$ can be written as

$$C_{0,1}(\Phi(X_\infty, X_0)|W) = \sum_{(\mathbf{V}, \mathbf{k}) \in \text{deco}_{0,1}(W, X_0)} (-1)^{|\mathbf{V}|} Z_{X_0^{|\mathbf{V}_1|} X_1^{k_1} X_0^{|\mathbf{V}_2|} X_1^{k_2} \dots X_0^{|\mathbf{V}_p|} X_1^{k_p}}$$

and

$$C_{0,1}(\Phi(X_1, X_\infty)|W) = \sum_{(\mathbf{V}, \mathbf{k}) \in \text{deco}_{0,1}(W, X_1)} (-1)^{|\mathbf{V}|} Z_{X_1^{|\mathbf{V}_1|} X_0^{k_1} X_1^{|\mathbf{V}_2|} X_0^{k_2} \dots X_1^{|\mathbf{V}_p|} X_0^{k_p}}.$$

Proof. The first statement is immediate. Let $\mathcal{L}_{2,c}(\mathbb{N})$ denote the set of double p -tuples ($0 \leq p < \infty$) of integers $((l_1, \dots, l_p), (k_1, \dots, k_p))$ with $k_i, l_i \in \mathbb{N}$ and let (\mathbf{l}, \mathbf{k}) denote an element of $\mathcal{L}_{2,c}(\mathbb{N})$. We can write $\Phi(X_\infty, X_0)$ as

$$\Phi(X_\infty, X_0) = \sum_{(\mathbf{l}, \mathbf{k}) \in \mathcal{L}_{2,c}(\mathbb{N})} Z_{X_0^{l_1} X_1^{k_1} \dots X_0^{l_p} X_1^{k_p}} X_\infty^{l_1} X_0^{k_1} \dots X_\infty^{l_p} X_0^{k_p}$$

which equals

$$\sum_{(\mathbf{l}, \mathbf{k}) \in \mathcal{L}_{2,c}(\mathbb{N})} Z_{X_0^{l_1} X_1^{k_1} \dots X_0^{l_p} X_1^{k_p}} (-1)^{|\mathbf{l}|} (X_0 + X_1)^{l_1} X_0^{k_1} \dots (X_0 + X_1)^{l_p} X_0^{k_p}.$$

Reorganizing, we see that the expression of $C_{0,1}(\Phi(X_\infty, X_0)|W)$ follows from (6); the case of $C_{0,1}(\Phi(X_1, X_\infty)|W)$ is identical. \square

Theorem 2.8. *The relation (II) is equivalent to the family of relations*

$$(7) \quad \forall W \in \mathcal{W}_{0,1} \setminus \{\emptyset\},$$

$$\begin{aligned} & \sum_{\substack{W_1, \dots, W_6 \in \mathcal{W}_{0,1} \\ W_1 \dots W_6 = W}} \frac{\mu^{|W_1|}}{2^{|W_1|} |W_1|!} \tilde{\alpha}_0(W_1) \times \\ & \left(\sum_{\substack{(\mathbf{U}, \mathbf{k}) \in \\ \text{deco}_{0,1}(W_2, X_0)}} (-1)^{|\mathbf{U}|} Z_{X_0^{|\mathbf{U}_1|} X_1^{k_1} \dots X_0^{|\mathbf{U}_p|} X_1^{k_p}} \right) (-1)^{|W_3|} \frac{\mu^{|W_3|}}{2^{|W_3|} |W_3|!} \times \\ & \left(\sum_{\substack{(\mathbf{V}, \mathbf{l}) \in \\ \text{deco}_{0,1}(W_4, X_1)}} (-1)^{|\mathbf{V}|} Z_{X_1^{|\mathbf{V}_1|} X_0^{l_1} \dots X_1^{|\mathbf{V}_p|} X_0^{l_p}} \right) \frac{\mu^{|W_5|}}{2^{|W_5|} |W_5|!} \tilde{\alpha}_1(W_5) Z_{W_6} = 0. \end{aligned}$$

Proof. The relation (II) is equivalent to the family of relations

$$\forall W \in \mathcal{W}_{0,1} \setminus \{\emptyset\} \quad C_{0,1}(P_3, W) = 0.$$

As P_3 is a product of six factors, this is equivalent to

$$\forall W \in \mathcal{W}_{0,1} \setminus \{\emptyset\}$$

$$C_{0,1}(e^{\frac{\mu}{2} X_0}, W) C_{0,1}(\Phi(X_\infty, X_0), W) \cdot$$

$$C_{0,1}(e^{\frac{\mu}{2} X_\infty}, W) C_{0,1}(\Phi(X_1, X_\infty), W) \cdot$$

$$C_{0,1}(e^{\frac{\mu}{2} X_1}, W) C_{0,1}(\Phi(X_0, X_1), W) = 0.$$

The proposition follows, then, from Proposition 2.5 and 2.7. \square

Corollary 2.9. *The relation (Π_{KZ}) is equivalent to the family of relations*

$$(8) \quad \forall W \in \mathcal{W}_{0,1} \setminus \{\emptyset\},$$

$$\sum_{\substack{W_1, \dots, W_6 \in \mathcal{W}_{0,1} \\ W_1 \cdots W_6 = W}} \frac{(i\pi)^{|W_1|}}{|W_1|!} \tilde{\alpha}_0(W_1) \times$$

$$\left(\sum_{\substack{(\mathbf{U}, \mathbf{k}) \\ \in \text{dec}_{0,1}(W_2, X_0)}} (-1)^{|W_2|} \zeta^{\mathfrak{m}}(X_0^{|\mathbf{U}_1|} X_1^{k_1} \cdots X_0^{|\mathbf{U}_p|} X_1^{k_p}) \right) (-1)^{|W_3|} \frac{(i\pi)^{|W_3|}}{|W_3|!} \times$$

$$\left(\sum_{\substack{(\mathbf{V}, \mathbf{l}) \\ \in \text{dec}_{0,1}(W_4, X_1)}} \zeta^{\mathfrak{m}}(X_1^{|\mathbf{V}_1|} X_0^{l_1} \cdots X_1^{|\mathbf{V}_p|} X_0^{l_p}) \right) \frac{(i\pi)^{|W_5|}}{|W_5|!} \tilde{\alpha}_1(W_5) (-1)^{dp(W_6)} \zeta^{\mathfrak{m}}(W_6) = 0.$$

2.3. The 5-cycle or the pentagon relation, (III) and (III_{KZ}) . In order to find families of relations between the coefficient equivalent to (I) and (II) we have decomposed the product P_2 and P_3 on a basis of $U\mathfrak{F}_2$, the words in X_0 and X_1 . We will do the same thing here but the monomials in the variable X_{ij} are not a linear basis of $U\mathfrak{B}_5$ because there are relations between the X_{ij} . Using the relation defining $U\mathfrak{B}_5$ we can see that $X_{51} = -X_{12} - X_{13} - X_{14}$ and that

$$\begin{aligned} X_{51} &= -X_{54} - X_{53} - X_{52} \\ &= 2X_{23} + 2X_{24} + 2X_{34} + X_{12} + X_{13} + X_{14}. \end{aligned}$$

Then, as the characteristic of k is zero, we have $X_{51} = X_{23} + X_{24} + X_{34}$. In this section, we will expand the product in the R.H.S of III using this relation and then decompose this product on a linear basis of $U\mathfrak{B}_5$. Let B denote a linear basis of $U\mathfrak{B}_5$ (with 1 being an element of B) and B_4 the basis of $U\mathfrak{B}_5$ coming from the identification

$$U\mathfrak{B}_5 \simeq k\langle\langle X_{24}, X_{34}, X_{45} \rangle\rangle \rtimes k\langle\langle X_{12}, X_{23} \rangle\rangle.$$

This identification is induced by the morphism $U\mathfrak{B}_5 \rightarrow U\mathfrak{F}_2$ that maps X_{i4} to 0 ($1 \leq i \leq 5$), X_{12} to X_0 , X_{23} to X_1 , the images of the other generators being easily deduced from those. The basis B_4 is formed by 1 and the monomials, that is words, $U_{245}V_{123}$ where U_{245} is a word in ${}_{24}\mathcal{W}_{34,45} = \{X_{24}, X_{34}, X_{45}\}^*$ and V_{123} is in $\mathcal{W}^{12,23} = \{X_{12}, X_{23}\}^*$. Speaking of the empty word \emptyset in B_4 , we will mean 1 when seen in $U\mathfrak{B}_5$ and \emptyset when seen as the word.

Let \mathcal{W} be the dictionary $\{X_{24}, X_{34}, X_{45}, X_{12}, X_{23}\}^*$, and let ${}_{24}\mathcal{W}_{34}^{23}$ and ${}_{24}\mathcal{W}_{34}^{12,23}$ be respectively the sub-dictionary $\{\tilde{X}_{23}, \tilde{X}_{24}, \tilde{X}_{34}\}^*$ and $\{\tilde{X}_{12}, \tilde{X}_{23}, \tilde{X}_{24}, \tilde{X}_{34}\}^*$.

Let P_5 be the product in $U\mathfrak{B}_5$.

$$\Phi(X_{12}, X_{23})\Phi(X_{34}, X_{45})\Phi(X_{51}, X_{12})\Phi(X_{23}, X_{34})\Phi(X_{45}, X_{51}).$$

As $X_{51} = X_{23} + X_{24} + X_{34}$, we can write P_5 without using X_{51}

$$\begin{aligned} P_5 &= \Phi(X_{12}, X_{23})\Phi(X_{34}, X_{45})\Phi(X_{23} + X_{24} + X_{34}, X_{12})\Phi(X_{23}, X_{34}) \\ &\quad \Phi(X_{45}, X_{23} + X_{24} + X_{34}). \end{aligned}$$

Expanding the terms $(X_{23} + X_{24} + X_{34})^n$ as

$$\sum_{\substack{W \in {}_{24}\mathcal{W}_{34}^{23} \\ |W|=n}} W,$$

we have

$$(9) \quad P_5 = \sum_{W \in \mathcal{W}} C_{5,W} W.$$

Despite that this writing is not unique, as a decomposition of P_5 in \mathcal{W} , those $C_{5,W}$ are the coefficients of a word W just after expanding the product P_5 without X_{51} (that is replacing X_{51} by $X_{23} + X_{24} + X_{34}$) and as such are unique and well defined.

Lemma 2.10. *Let $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$ be the morphisms from $U\mathfrak{B}_5$ to $U\mathfrak{F}_2$ defined respectively on the monomial X_{24} by, $X_{34}, X_{45}, X_{12}, X_{23}$ as :*

$$\begin{aligned} \rho_1(X_{12}) &= X_0, & \rho_1(X_{23}) &= X_1, & \rho_1(X_{34}) &= 0, & \rho_1(X_{45}) &= 0, & \rho_1(X_{24}) &= 0, \\ \rho_2(X_{12}) &= 0, & \rho_2(X_{23}) &= 0, & \rho_2(X_{34}) &= X_0, & \rho_2(X_{45}) &= X_1, & \rho_2(X_{24}) &= 0, \\ \rho_3(X_{12}) &= X_1, & \rho_3(X_{23}) &= X_0, & \rho_3(X_{34}) &= X_0, & \rho_3(X_{45}) &= 0, & \rho_3(X_{24}) &= X_0, \\ \rho_4(X_{12}) &= 0, & \rho_4(X_{23}) &= X_0, & \rho_4(X_{34}) &= X_1, & \rho_4(X_{45}) &= 0, & \rho_4(X_{24}) &= 0, \\ \rho_5(X_{12}) &= 0, & \rho_5(X_{23}) &= X_1, & \rho_5(X_{34}) &= X_1, & \rho_5(X_{45}) &= X_0, & \rho_5(X_{24}) &= X_1. \end{aligned}$$

Proposition 2.11. *For all words $W \in \mathcal{W}$ ($W \neq \emptyset$), the coefficient $C_{5,W}$ is given by*

$$(10) \quad C_{5,W} = \sum_{\substack{U_1, \dots, U_5 \in \mathcal{W} \\ U_1 \dots U_5 = W}} Z_{\rho_1(U_1)} Z_{\rho_2(U_2)} Z_{\rho_3(U_3)} Z_{\rho_4(U_4)} Z_{\rho_5(U_5)},$$

where by convention $Z_0 = 0$.

Proof. It is enough to show that the i -Th factor of P_5 without using X_{51} can be written as

$$\sum_{U_i \in \mathcal{W}} Z_{\rho_i(U_i)} U_i.$$

The first, second and fourth factor being similar, we will discuss only the first one. It is clear in the case of $\Phi(X_{12}, X_{23})$ that either U_1 is in $\mathcal{W}^{12,23}$ and its coefficient is then $Z_{\rho_1(U_1)}$; or U_1 is not in $\mathcal{W}^{12,23}$ and it does not appear in $\Phi(X_{12}, X_{23})$ which means that its coefficient is 0.

The third and fifth factor are similar and thus we will treat only the former. We can write $\Phi(X_{23} + X_{24} + X_{34}, X_{12})$ as

$$\sum_{(l, \mathbf{k}) \in \mathcal{L}_{2,c}(\mathbb{N})} Z_{X_0^{l_1} X_1^{k_1} \dots X_0^{l_p} X_1^{k_p}} (X_{23} + X_{24} + X_{34})^{l_1} X_{12}^{k_1} \dots (X_{23} + X_{24} + X_{34})^{l_p} X_{12}^{k_p}.$$

We can rewrite the previous sum as running through all the words in the letters X_{12}, X_{23}, X_{24} and X_{34} because $(X_{23} + X_{24} + X_{34})^l$ is equal to

$$\sum_{\substack{W \in {}_{24}\mathcal{W}_{34}^{12,23} \\ |W|=l}} W.$$

Using the unique decomposition (as word) of $U_4 \in {}_{24}\mathcal{W}_{34}^{12,23}$ as

$$U_4 = V_1 X_{12}^{k_1} \dots V_p X_{12}^{k_p} \quad \text{with } V_i \in {}_{24}\mathcal{W}_{34}^{23},$$

we see that each word U_4 in ${}_{24}\mathcal{W}_{34}^{12,23}$ appears one and only one time in $\Phi(X_{23} + X_{24} + X_{34}, X_{12})$ with the coefficient $Z_{X_0^{|V_1|} X_1^{k_1} \dots X_0^{|V_p|} X_1^{k_p}}$. We finally have

$$\Phi(X_{23} + X_{24} + X_{34}, X_{12}) = \sum_{U_3 \in \mathcal{W}} Z_{\rho_3(U_3)} U_3.$$

□

Fixing a linear basis B of $U\mathfrak{B}_5$ (1 being an element of the basis) we have for every W in \mathcal{W} a decomposition (in $U\mathfrak{B}_5$)

$$W = \sum_{b \in B} l_{b,W} B \quad l_{b,W} \in k$$

and then a family of relations equivalent to (III).

Theorem 2.12. *The relation (III) is equivalent to the family of relations*

$$\forall b \in B (b \neq 1) \quad \sum_{W \in \mathcal{W}} l_{b,W} C_{5,W} = 0$$

where $C_{5,W}$ are given by Proposition 2.11.

Proof. As B is a basis, for a given W in \mathcal{W} there are only finitely many $l_{b,W}$ that are non zero. Moreover, as $U\mathfrak{B}_5$ is a graduated vector space (the grading is given by the weight, that is the length, of the monomial) with finite dimensional graduated part, for any b in B there are only finitely many $l_{b,W}$ that are non zero. The product P_5 is equal to

$$\begin{aligned} P_5 &= \sum_{W \in \mathcal{W}} C_{5,W} W \\ &= \sum_{W \in \mathcal{W}} C_{5,W} \left(\sum_{b \in B} l_{b,W} B \right) \\ &= \sum_{b \in B} \left(\sum_{W \in \mathcal{W}} l_{b,W} C_{5,W} \right) B. \end{aligned}$$

The relation (III) tells us that

$$P_5 = 1$$

which, because 1 is in B , means that $C_{5,\emptyset} = 1$ and

$$\forall b \in B (b \neq 1) \quad \sum_{W \in \mathcal{W}} l_{b,W} C_{5,W} = 0.$$

□

Using the more common basis B_4 we have:

Corollary 2.13. *The relation (III) is equivalent to the family of relations*

$$\forall b_4 \in B_4 (b_4 \neq 1) \quad \sum_{W \in \mathcal{W}} l_{b_4,W} C_{5,W} = 0$$

where the $C_{5,W}$ are given by Proposition 2.11.

Remark 2.14. In the case of the basis B_4 one can check that the coefficients $l_{b_4,W}$ are in \mathbb{Z} .

The previous corollary, applied to the particular case of the Drinfel'd associator and making explicit the $C_{5,W}$ in terms of multiple zeta values, gives:

Theorem 2.15. *With the convention that $\zeta^{\text{III}}(0) = 0$, the relation (III_{KZ}) is equivalent to the family of relations*

$$(11) \quad \forall b_4 \in B_4 (b_4 \neq 1) \quad \sum_W l_{b_4,W} \left(\sum_{U_1 \dots U_5 = W} (-1)^{dp_1(U_1) + dp_2(U_2) + dp_3(U_3) + dp_4(U_4) + dp_5(U_5)} \zeta^{\text{III}}(\rho_1(U_1)) \zeta^{\text{III}}(\rho_2(U_2)) \zeta^{\text{III}}(\rho_3(U_3)) \zeta^{\text{III}}(\rho_4(U_4)) \zeta^{\text{III}}(\rho_5(U_5)) \right) = 0$$

where $dp_i(U)$ is the depth of $\rho_i(U)$ and the words W, U_i are in \mathcal{W} .

3. BAR CONSTRUCTION AND ASSOCIATOR RELATIONS

In this section, we suppose that k is \mathbb{C} . We review the notion of bar construction and its links with multiple zeta values. Those results have been shown in greater generality in [Che73] and [Bro09]. We will recall the Brown variant of Chen's reduced bar construction in the case of the moduli spaces of curves of genus 0 with 4 and 5 marked points, $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$.

3.1. Bar Construction. The moduli space of curves of genus 0 with 4 marked points, $\mathcal{M}_{0,4}$, is

$$\mathcal{M}_{0,4} = \{(z_1, \dots, z_4) \in (\mathbb{P}^1)^4 \mid z_i \neq z_j \text{ if } i \neq j\} / \text{PGL}_2(k)$$

and is identified as

$$\mathcal{M}_{0,4} \simeq \{t \in (\mathbb{P}^1) \mid t \neq 0, 1, \infty\}$$

by sending the point $[(0, t, 1, \infty)] \in \mathcal{M}_{0,4}$ to t .

The moduli space of curves of genus 0 with 5 marked points, $\mathcal{M}_{0,5}$, is

$$\mathcal{M}_{0,5} = \{(z_1, \dots, z_5) \in (\mathbb{P}^1)^5 \mid z_i \neq z_j \text{ if } i \neq j\} / \text{PGL}_2(k)$$

and is identified as

$$\mathcal{M}_{0,5} \simeq \{(x, y) \in (\mathbb{P}^1)^2 \mid x, y \neq 0, 1, \infty \text{ and } x \neq y\}$$

by sending the point $[(0, xy, y, 1, \infty)] \in \mathcal{M}_{0,5}$ to (x, y) . This identification can be interpreted as the composition of

$$\mathcal{M}_{0,5} \longrightarrow \mathcal{M}_{0,4} \times \mathcal{M}_{0,4}$$

$$[(z_1, \dots, z_5)] \longmapsto [(z_1, z_2, z_3, z_5)] \times [(z_1, z_3, z_4, z_5)]$$

with the previous identification of $\mathcal{M}_{0,4}$ using the fact that

$$[(0, xy, y, \infty)] = [(0, x, 1, \infty)].$$

For $\mathcal{M} = \mathcal{M}_{0,4}$ or $\mathcal{M} = \mathcal{M}_{0,5}$, Brown has defined in [Bro09] a graded Hopf k -algebra

$$(12) \quad V(\mathcal{M}) = \bigoplus_{m=0}^{\infty} V_m(\mathcal{M}) \subset \bigoplus_{m=0}^{\infty} \mathbf{H}_{\text{DR}}^1(\mathcal{M})^{\otimes m}.$$

Here $V_0(\mathcal{M}) = k$, $V_1(\mathcal{M}) = \mathbf{H}_{\text{DR}}^1(\mathcal{M})$ and $V_m(\mathcal{M})$ is the intersection of the kernel \wedge_i for $1 \leq i \leq m-1$:

$$\wedge_i : \mathbf{H}_{\text{DR}}^1(\mathcal{M})^{\otimes m} \longrightarrow \mathbf{H}_{\text{DR}}^1(\mathcal{M})^{\otimes m-i-1} \otimes \mathbf{H}_{\text{DR}}^2(\mathcal{M}) \otimes \mathbf{H}_{\text{DR}}^1(\mathcal{M})^{\otimes i-1}$$

$$\nu_m \otimes \cdots \otimes \nu_1 \longmapsto \nu_m \otimes \cdots \otimes (\nu_{i+1} \wedge \nu_i) \otimes \cdots \otimes \nu_1.$$

Suppose that $\omega_1, \dots, \omega_k$ form a basis of $\mathbf{H}_{\text{DR}}^1(\mathcal{M})$, then the elements of $V_m(\mathcal{M})$ can be written as linear combinations of symbols

$$\sum_{I=(i_1, \dots, i_m)} c_I [\omega_{i_m} \mid \cdots \mid \omega_{i_1}],$$

with $c_I \in k$, which satisfies the integrability condition

$$(13) \quad \sum_{I=(i_1, \dots, i_m)} c_I \omega_{i_m} \otimes \cdots \otimes \omega_{i_{j+2}} \otimes (\omega_{i_{j+1}} \wedge \omega_{i_j}) \otimes \omega_{i_{j-1}} \otimes \cdots \otimes \omega_{i_m} = 0$$

for all $1 \leq j \leq m-1$.

Definition 3.1. The Brown's bar construction over \mathcal{M} is the tensor product

$$B(\mathcal{M}) = \mathcal{O}_{\mathcal{M}} \otimes V(\mathcal{M}).$$

Theorem 3.2 ([Bro09]). *The bar construction $B(\mathcal{M})$ is a commutative graded Hopf algebra isomorphic to the zeroTh cohomology group of Chen's reduced bar complex on $\mathcal{O}_{\mathcal{M}}$:*

$$B(\mathcal{M}) \simeq H^0(B(\Omega^\bullet \mathcal{O}_{\mathcal{M}})).$$

Let ν_m, \dots, ν_1 be m holomorphic 1-forms in $\Omega^1(\mathcal{M})$. The iterate integral of the word $\nu_m \cdots \nu_1$, denoted by

$$\int \nu_m \circ \cdots \circ \nu_1,$$

is the application that sends any path $\gamma : [0, 1] \rightarrow \mathcal{M}$ to

$$\int_{\gamma} \nu_m \circ \cdots \circ \nu_1 = \int_{0 < t_1 < \dots < t_m} \gamma^* \nu_1(t_1) \wedge \cdots \wedge \gamma^* \nu_m(t_m).$$

This value is called the iterated integral of $\nu_m \cdots \nu_1$ along γ . We extend these definitions by linearity to linear combinations of forms $\sum_I c_I \nu_{i_m} \cdots \nu_{i_1}$.

When, for any γ , the iterated integral

$$\int_{\gamma} \sum_I c_I \nu_m \circ \cdots \circ \nu_1$$

depends only of the homotopy class of γ , we say it is an homotopy invariant iterate integral and note it $\int \sum_I c_I \nu_m \circ \cdots \circ \nu_1$. Let $L(\mathcal{M})$ denote the set of all the homotopy invariant iterated integral.

Proposition 3.3 ([Bro09]). *The morphism ρ defined as*

$$\rho : B(\mathcal{M}) \longrightarrow L(\mathcal{M})$$

$$\sum_I c_I [\omega_{i_m} | \cdots | \omega_{i_1}] \longmapsto \int \sum_I c_I \omega_{i_m} \circ \cdots \circ \omega_{i_1}$$

is an isomorphism.

Remark 3.4. In particular for any such γ homotopically equivalent to zero we have for all $\sum_I c_I [\omega_{i_m} | \cdots | \omega_{i_1}]$ in $V(\mathcal{M})$:

$$\sum_I c_I \int_{\gamma} \omega_{i_m} \circ \cdots \circ \omega_{i_1} = 0$$

3.2. Bar Construction on $\mathcal{M}_{0,4}$, symmetry and hexagon relations. Here, we will show how the symmetry relations (I_{KZ}) and the hexagon (Π_{KZ}) relations are related to the bar construction on $\mathcal{M}_{0,4}$.

First of all we should remark that $B(\mathcal{M}_{0,4})$ is extremely simple.

Proposition 3.5. *Let ω_0 and ω_1 denote respectively the differential 1-form, in $\Omega^1(\mathcal{M}_{0,4})$, $\frac{dt}{t}$ and $\frac{dt}{t-1}$.*

Then, any element $[\omega_{\varepsilon_n} | \cdots | \omega_{\varepsilon_1}]$ with ε_i in $\{0, 1\}$ is an element of $V(\mathcal{M}_{0,4})$. Moreover, the family of these elements is a basis of $V(\mathcal{M}_{0,4})$.

Proof. As $\omega_0 \wedge \omega_1 = 0$, the integrability condition (13) is automatically satisfied so that any element $[\omega_{\varepsilon_n} | \cdots | \omega_{\varepsilon_1}]$ ($\varepsilon_i = 0, 1$) is an element of $V(\mathcal{M}_{0,4})$. Moreover, as (ω_0, ω_1) is a basis of $H_{DR}^1(\mathcal{M}_{0,4})$, the elements $[\omega_{\varepsilon_n} | \cdots | \omega_{\varepsilon_1}]$ form a basis of $V(\mathcal{M}_{0,4})$. \square

Sending X_0 on ω_0 and X_1 on ω_1 gives a one to one correspondence between words $W = X_{\varepsilon_n} \cdots X_{\varepsilon_1}$ in $\mathcal{W}_{0,1}$ and the elements $[\omega_{\varepsilon_n} | \cdots | \omega_{\varepsilon_1}]$ of the previous basis of $V(\mathcal{M}_{0,4})$. This correspondence allows us to identify $V(\mathcal{M}_{0,4})$ with the graded dual of $U\mathfrak{F}_2$,

$$V(\mathcal{M}_{0,4}) \simeq (U\mathfrak{F}_2)^*.$$

The word, $W = X_{\varepsilon_n} \cdots X_{\varepsilon_1}$ is sent to its dual $W^* = \omega_W = [\omega_{\varepsilon_n} | \cdots | \omega_{\varepsilon_1}]$.

Remark 3.6. Let α and β be two paths in a variety with $\alpha(1) = \beta(0)$. We will denote by $\beta \circ \alpha$ the composed path beginning with α and ending with β .

The iterated integral of $\omega = \omega_n \cdots \omega_1$ along $\beta \circ \alpha$ is then equal to

$$(14) \quad \sum_{k=0}^n \left(\int_{\beta} \omega_n \circ \cdots \circ \omega_{n-k+1} \right) \left(\int_{\alpha} \omega_{n-k} \circ \cdots \circ \omega_1 \right).$$

Following [Bro09] and considering the three dihedral structures on $\mathcal{M}_{0,4}$, one can define 6 tangential based points : $\vec{01}$, $\vec{10}$, $\vec{1\infty}$, $\vec{\infty 1, \infty 0}$ and $\vec{0\infty}$. Let p denote the path beginning at the tangential based point $\vec{01}$ and ending at $\vec{10}$ defined by $t \mapsto t$ and let p^{-1} denote its inverse $t \mapsto 1 - t$.

If γ is a path, starting at a tangential based point \vec{P} (and/or ending at a tangential based point \vec{P}') an iterated integral $\int \omega$ may be divergent. However, one can define (as in [Bro09]) a value to that divergent integral ; we speak of regularised iterated integral.

If W is a word in $X_0 \mathcal{W}_{0,1} X_1$, the iterated integral $\int_p \omega_W$ is convergent and is equal to $(-1)^{dp(W)} \zeta(W)$. If W is a word beginning by X_1 and/or ending by X_0 (that is in $\mathcal{W}_{0,1} \setminus X_0 \mathcal{W}_{0,1} X_1$), then the regularised iterated integral $\int_p \omega_W$ is equal to $(-1)^{dp(W)} \zeta^{\text{reg}}(W)$.

We may, thereafter, omit the term *regularised* in the expressions “regularised iterated integral ” or “regularised homotopy invariant iterated integral”.

Theorem 3.7. *The relation (IKZ) is equivalent to the family of relations*

$$\forall W \in \mathcal{W}_{0,1} \quad \int_{p \circ p^{-1}} \omega_W = 0$$

which is exactly the family (3).

Proof. Considering the KZ equation (KZ)

$$\frac{\partial g}{\partial u} = \left(\frac{X_0}{u} + \frac{X_1}{1-u} \right) \cdot g(u)$$

and the two normalized solutions at 0 and 1, g_0 and g_1 , $\Phi_{KZ}(X_0, X_1)$ is the unique element in $U\mathfrak{F}_2$ such that

$$g_0(u) = g_1(u) \Phi_{KZ}(X_0, X_1).$$

Using the symmetry of the situation we also have

$$g_1(u) = g_0(u) \Phi_{KZ}(X_1, X_0).$$

The equation (IKZ) comes from the unicity of such a solution normalized at 1

$$(15) \quad g_1(u) = g_1(u) \Phi_{KZ}(X_0, X_1) \Phi_{KZ}(X_1, X_0).$$

The elements $\Phi_{KZ}(X_0, X_1)$ and $\Phi_{KZ}(X_1, X_0)$ can be expressed using regularised iterated integrals as

$$\Phi_{KZ}(X_0, X_1) = \sum_{W \in \mathcal{W}_{0,1}} \left(\int_p \omega_W \right) W$$

and

$$\Phi_{KZ}(X_1, X_0) = \sum_{W \in \mathcal{W}_{0,1}} \left(\int_{p^{-1}} \omega_W \right) W.$$

Equation (15) corresponds to the comparison of the normalized solution g_1 with the solution given by analytic continuation of g_1 along $p \circ p^{-1}$. The product

$$\Phi_{KZ}(X_0, X_1) \Phi_{KZ}(X_1, X_0)$$

is then the series

$$\sum_{W \in \mathcal{W}_{0,1}} \left(\int_{p \circ p^{-1}} \omega_W \right) W.$$

As the path $p \circ p^{-1}$ is homotopically equivalent to 0, all the previous iterated integrals (for $W \neq \emptyset$) are 0. We deduce that (IKZ) is equivalent to

$$\forall W \in \mathcal{W}_{0,1} \quad \int_{p \circ p^{-1}} \omega_W = 0.$$

Now, fix any $W = X_{\varepsilon_n} \cdots X_{\varepsilon_1}$ in $\mathcal{W}_{0,1}$ and compute the regularised iterated integral $\int_{p \circ p^{-1}} \omega_W$. Using (14), we have

$$\int_{p \circ p^{-1}} \omega_W = \sum_{k=0}^n \left(\int_p \omega_{\varepsilon_n} \circ \cdots \circ \omega_{\varepsilon_{n-k+1}} \right) \left(\int_{p^{-1}} \omega_{\varepsilon_{n-k}} \circ \cdots \circ \omega_{\varepsilon_1} \right).$$

Setting $U_1 = X_{\varepsilon_n} \cdots X_{\varepsilon_{n-k+1}}$ and $U_2 = X_{\varepsilon_{n-k}} \cdots X_{\varepsilon_1}$, we have

$$\int_p \omega_{\varepsilon_n} \circ \cdots \circ \omega_{\varepsilon_{n-k+1}} = (-1)^{dp(U_1)} \zeta^{\text{m}}(U_1).$$

As p^{-1} is given by $t \mapsto 1 - t$, we have for ε in $\{0, 1\}$

$$(p^{-1})^*(\omega_\varepsilon) = \omega_{1-\varepsilon}.$$

Moreover, as $p^*(\omega_\varepsilon) = \omega_\varepsilon$, one computes

$$\begin{aligned} \int_{p^{-1}} \omega_{\varepsilon_{n-k}} \circ \cdots \circ \omega_{\varepsilon_1} &= \int_{0 < t_1 < \cdots < t_{n-k}} (p^{-1})^*(\omega_{\varepsilon_1}(t_1)) \wedge \cdots \wedge (p^{-1})^*(\omega_{\varepsilon_{n-k}}(t_{n-k})) \\ &= \int_{0 < t_1 < \cdots < t_{n-k}} \omega_{1-\varepsilon_1}(t_1) \wedge \cdots \wedge \omega_{1-\varepsilon_{n-k}}(t_{n-k}) \\ &= \int_p \omega_{1-\varepsilon_{n-k}} \circ \cdots \circ \omega_{1-\varepsilon_1} = \int_p \omega_{\theta(U_2)}, \end{aligned}$$

where θ exchange X_0 and X_1 . Finally, we obtain

$$\int_{p^{-1}} \omega_{U_2} = \int_p \omega_{\theta(U_2)} = (-1)^{dp(\theta(U_2))} \zeta^{\text{m}}(\theta(U_2))$$

and

$$0 = \int_{p \circ p^{-1}} \omega_W = \sum_{U_1 U_2 = W} (-1)^{dp(U_1)} \zeta^{\text{m}}(U_1) (-1)^{dp(\theta(U_2))} \zeta^{\text{m}}(\theta(U_2))$$

which is exactly the relation (3) for the word W . \square

Now, let c be the infinitesimal half circle around 0 in the lower half plan, connecting the tangential based point $0\overrightarrow{\infty}$ and $0\overrightarrow{1}$. The path c can be seen as the limit when ε tends to 0 of $c_\varepsilon : t \mapsto \varepsilon e^{i(\pi+t\pi)}$.

We have a natural 3 cycle on $\mathcal{M}_{0,4}$ given by $\tau : t \mapsto \frac{1}{1-t}$. Let γ be the path $c \circ \tau^2(p) \circ \tau^2(c) \circ \tau(p) \circ \tau(c) \circ p$.

Theorem 3.8. *The relation (IKZ) is equivalent to the family of relations*

$$\forall W \in \mathcal{W}_{0,1} \quad \int_\gamma \omega_W = 0$$

which is exactly the family (8).

Proof. Comparing the six different normalized solutions of (KZ) at the six different based points leads to six equations. Combining these equations, one obtains (IKZ) via the relation

$$(16) \quad g_0(u) = g_0(u) e^{i\pi X_0} \Phi_{KZ}(X_\infty, X_0) e^{i\pi X_\infty} \Phi_{KZ}(X_1, X_\infty) e^{i\pi X_1} \Phi_{KZ}(X_0, X_1)$$

where exponentials are coming from the relation between the solutions at the based points $0\vec{\infty}$ and $0\vec{1}$ (resp. $1\vec{0}$ and $1\vec{\infty}$, $\infty\vec{1}$ and $\infty\vec{0}$); that is from the monodromy around 0, 1 and ∞ .

Putting the six different relations in order to get the previous equation is the same as comparing the solution g_0 with the analytic continuation of g_0 along any path starting at $0\vec{1}$, joining the other tangential based points $1\vec{0}$, $1\vec{\infty}$, $\infty\vec{1}$, $\infty\vec{0}$, $0\vec{\infty}$ in that order and ending at $0\vec{1}$; staying all the time in the lower half plan. Such a path is homotopically equivalent to γ .

So, equation (16) gives a relation between g_0 and the solution obtained from g_0 by analytic continuation along γ . Then, the product in $U\mathfrak{F}_2$ in the R.H.S of (16) can be expressed using homotopy invariant iterated integrals

$$e^{i\pi X_0} \Phi_{KZ}(X_\infty, X_0) e^{i\pi X_\infty} \Phi_{KZ}(X_1, X_\infty) e^{i\pi X_1} \Phi_{KZ}(X_0, X_1) = \sum_{W \in \mathcal{W}_{0,1}} \left(\int_\gamma \omega_W \right) W.$$

As γ is homotopically equivalent to 0, for any word W in $\mathcal{W}_{0,1}$, one has

$$\int_\gamma \omega_W = 0.$$

The first part of the theorem is proved.

Using the decomposition of iterated integrals on a composed path (Equation (14)), we have

$$\forall W \in \mathcal{W}_{0,1} \quad \int_\gamma \omega_W = \sum_{\substack{U_1, \dots, U_6 \\ U_1 \cdots U_6 = W}} \int_c \omega_{U_1} \int_{\tau^2(p)} \omega_{U_2} \int_{\tau^2(c)} \omega_{U_3} \\ \int_{\tau(p)} \omega_{U_4} \int_{\tau(c)} \omega_{U_5} \int_p \omega_{U_6}.$$

Thus, in order to show that the family of relation

$$\forall W \in \mathcal{W}_{0,1} \quad \int_\gamma \omega_W = 0$$

gives exactly the family of relations (8), it is enough, to show that for any U in $\mathcal{W}_{0,1}$

$$\int_c \omega_U = C_{0,1}(e^{i\pi X_0} | U), \quad \int_{\tau^2(p)} \omega_U = C_{0,1}(\Phi_{KZ}(X_\infty, X_0) | U), \\ \int_{\tau^2(c)} \omega_U = C_{0,1}(e^{i\pi X_\infty} | U), \quad \int_{\tau(p)} \omega_U = C_{0,1}(\Phi_{KZ}(X_1, X_\infty) | U), \\ \int_{\tau(c)} \omega_U = C_{0,1}(e^{i\pi X_1} | U), \quad \int_p \omega_U = C_{0,1}(\Phi_{KZ}(X_0, X_1) | U).$$

In order to compute the iterated integral along c , $\tau(c)$ and $\tau^2(c)$, it is enough to compute the limit when ε tends to 0 of the iterate integral along c_ε , $\tau(c_\varepsilon)$ and $\tau^2(c_\varepsilon)$. As

$$c_\varepsilon^*(\omega_0) = i\pi dt \quad \text{and} \quad c_\varepsilon^*(\omega_1) = \varepsilon \frac{-i\pi e^{i(\pi+\pi t)} dt}{1 - \varepsilon e^{i(\pi+\pi t)}},$$

the iterate integral $\int_{c_\varepsilon} \omega_U$ tends to 0 except if $U = X_0^n$ and then $\int_{c_\varepsilon} \omega_{X_0^n} = \frac{(i\pi)^n}{n!}$ for all ε . Thus, we have

$$\int_c \omega_U = C_{0,1}(e^{i\pi X_0} | U).$$

Similarly we have

$$\tau(c_\varepsilon)^*(\omega_1) = i\pi \frac{dt}{1 - \varepsilon e^{i(\pi+\pi t)}} \quad \text{and} \quad \tau(c_\varepsilon)^*(\omega_0) = \frac{\varepsilon i\pi e^{i(\pi+\pi t)} dt}{1 - \varepsilon e^{i(\pi+\pi t)}}.$$

The iterate integral $\int_{\tau(c_\varepsilon)} \omega_U$ tends to 0 except if $U = X_1^n$ and then $\int_{c_\varepsilon} \omega_{X_0^n}$ tends to $\frac{(i\pi)^n}{n!}$ when ε tends to 0. Thus, we have

$$\int_{\tau(c)} \omega_U = C_{0,1}(e^{i\pi X_1} |U).$$

Computing $\tau^2(c_\varepsilon)^*$, we have

$$\tau^2(c_\varepsilon)^*(\omega_0) = -i\pi \frac{dt}{1 - \varepsilon e^{i(\pi+\pi t)}} \quad \text{and} \quad \tau^2(c_\varepsilon)^*(\omega_1) = -i\pi dt.$$

Then, we find that the limit when ε tends to 0 of $\int_{\tau^2(c_\varepsilon)} \omega_U$ is $\frac{(-i\pi)^{|U|}}{|U|!}$ which gives

$$\int_{\tau^2(c)} \omega_U = C_{0,1}(e^{i\pi X_\infty} |U).$$

The equality

$$\int_p \omega_U = C_{0,1}(\Phi_{KZ}(X_0, X_1) |U)$$

is obvious.

Cases of

$$\int_{\tau(p)} \omega_U \quad \text{and} \quad \int_{\tau^2(p)} \omega_U$$

are extremely similar and we will discuss only the last one. First, we should remark that

$$\tau^2(c)^*(\omega_0) = -\omega_0 + \omega_1 \quad \text{and} \quad \tau^2(c)^*(\omega_1) = -\omega_0.$$

For $U = X_{\varepsilon_n} \cdots X_{\varepsilon_1}$ ($\varepsilon_i = 0, 1$) we can rewrite the iterate integral $\int_{\tau^2(c)} \omega_U$ as

$$\int_p \tau^2(p)^*(\omega_{\varepsilon_n}) \circ \cdots \circ \tau^2(p)^*(\omega_{\varepsilon_1}).$$

We will now prove by induction on $n = |U|$ that in $V(\mathcal{M}_{0,4})$

$$(17) \quad [\tau^2(p)^*(\omega_{\varepsilon_n}) | \cdots | \tau^2(p)^*(\omega_{\varepsilon_1})] = \sum_{(\mathbf{V}, \mathbf{k}) \in \text{dec}_{0,1}(U, X_0)} (-1)^{|\mathbf{V}|} \omega_{X_0^{|\mathbf{V}_1|} X_1^{k_1} X_0^{|\mathbf{V}_2|} X_1^{k_2} \cdots X_0^{|\mathbf{V}_p|} X_1^{k_p}}$$

which will give using Proposition 2.7 the equality

$$\int_p \omega_U = C_{0,1}(\Phi_{KZ}(X_0, X_1) |U).$$

If $U = X_0$, the set $\text{dec}_{0,1}(U, X_0)$ has 2 elements, $((X_0), (0))$ and $((\emptyset), (1))$. Similarly, if $U = X_1$ then $\text{dec}_{0,1}(U, X_0)$ has only one element which is $((X_1), (0))$. In both cases (17) is satisfied.

Let $W = X_{j_n} \cdots X_{j_1}$ be a word in $\mathcal{W}_{0,1}$, and let ε be in $\{0, 1\}$. For the simplicity of notation, we shall write $[\omega_{j_n} | \cdots | \omega_{j_1} | \omega_\varepsilon]$ as

$$[\omega_W | \omega_\varepsilon] := [\omega_{j_n} | \cdots | \omega_{j_1} | \omega_\varepsilon].$$

We suppose now that

$$U = U_1 X_0 \quad \text{with} \quad |U_1| \geq 1.$$

We have a map from $\text{dec}_{0,1}(U, X_0)$ to $\text{dec}_{0,1}(U_1, X_0)$ that sends a decomposition

$$(\mathbf{V}, \mathbf{k}) = ((V_1, \dots, V_p), (k_1, \dots, k_p))$$

to

$$\begin{cases} (\mathbf{V}, (k_1, \dots, k_p - 1)) & \text{if } k_p \neq 0 \\ ((V_1, \dots, V_p'), \mathbf{k}) & \text{if } k_p = 0 \text{ and } V_p = V_p' X_0. \end{cases}$$

Any decomposition $(\mathbf{V}', \mathbf{k}')$ in $\text{dec}_{0,1}(U_1, X_0)$ has exactly two preimages by this map. If one writes

$$U_1 = V'_1 X_0^{k'_1} \cdots V'_1 X_0^{k'_p}$$

then it leads to two decompositions of U

$$((V'_1, \dots, V'_p), (k'_1, \dots, k'_p + 1)) \quad \text{and} \quad ((V'_1, \dots, V'_p, X_0), (k'_1, \dots, k'_p, 0)).$$

By induction we have in $V_{n-1}(\mathcal{M}_{0,4})$

$$[\tau^2(p)^*(\omega_{\varepsilon_n}) | \cdots | \tau^2(p)^*(\omega_{\varepsilon_2})] = \sum_{\substack{(\mathbf{V}', \mathbf{k}') \in \\ \text{dec}_{0,1}(U_1, X_0)}} (-1)^{|\mathbf{V}'|} \omega_{X_0^{|\mathbf{V}'_1|} X_1^{k'_1} \cdots X_0^{|\mathbf{V}'_p|} X_1^{k'_p}}.$$

We deduce from the previous equality and using the linearity of the tensor product that $[\tau^2(p)^*(\omega_{\varepsilon_n}) | \cdots | \tau^2(p)^*(\omega_{\varepsilon_1})]$ is equal to

$$\sum_{\substack{(\mathbf{V}', \mathbf{k}') \in \\ \text{dec}_{0,1}(U_1, X_0)}} (-1)^{|\mathbf{V}'|} [\omega_{X_0^{|\mathbf{V}'_1|} X_1^{k'_1} \cdots X_0^{|\mathbf{V}'_p|} X_1^{k'_p}} | - \omega_0 + \omega_1].$$

This sum can be decomposed as

$$\begin{aligned} & \sum_{\substack{(\mathbf{V}', \mathbf{k}') \in \\ \text{dec}_{0,1}(U_1, X_0)}} (-1)^{|\mathbf{V}'|+1} [\omega_{X_0^{|\mathbf{V}'_1|} X_1^{k'_1} \cdots X_0^{|\mathbf{V}'_p|} X_1^{k'_p}} | \omega_0] + \\ & \sum_{\substack{(\mathbf{V}', \mathbf{k}') \in \\ \text{dec}_{0,1}(U_1, X_0)}} (-1)^{|\mathbf{V}'|} [\omega_{X_0^{|\mathbf{V}'_1|} X_1^{k'_1} \cdots X_0^{|\mathbf{V}'_p|} X_1^{k'_p}} | \omega_1]. \end{aligned}$$

The first term of the sum is equal to

$$\sum_{\substack{(\mathbf{V}', \mathbf{k}') \in \\ \text{dec}_{0,1}(U_1, X_0)}} (-1)^{|\mathbf{V}'|+1} \omega_{X_0^{|\mathbf{V}'_1|} X_1^{k'_1} \cdots X_0^{|\mathbf{V}'_p|} X_1^{k'_p} X_0}$$

and the second term is equal to

$$\sum_{\substack{(\mathbf{V}', \mathbf{k}') \in \\ \text{dec}_{0,1}(U_1, X_0)}} (-1)^{|\mathbf{V}'|} \omega_{X_0^{|\mathbf{V}'_1|} X_1^{k'_1} \cdots X_0^{|\mathbf{V}'_p|} X_1^{k'_p+1}}.$$

The previous discussion on $\text{dec}_{0,1}(U, X_0)$ tells us that adding the two sums above, gives

$$\sum_{(\mathbf{V}, \mathbf{k}) \in \text{dec}_{0,1}(U, X_0)} (-1)^{|\mathbf{V}|} \omega_{X_0^{|\mathbf{V}_1|} X_1^{k_1} X_0^{|\mathbf{V}_2|} X_1^{k_2} \cdots X_0^{|\mathbf{V}_p|} X_1^{k_p}}.$$

This gives (17) when $U = U_1 X_0$.

If $U = U_1 X_1$ with $|U_1| \geq 1$, we have a one to one correspondence between $\text{dec}_{0,1}(U_1, X_0)$ and $\text{dec}_{0,1}(U, X_0)$ defined by

$$((V'_1, \dots, V'_p), (k'_1, \dots, k'_p)) \mapsto \begin{cases} ((V'_1, \dots, V'_p, X_1), (k'_1, \dots, k'_p, 0)) & \text{if } k'_p = 0 \\ ((V'_1, \dots, V'_p, X_1), (k'_1, \dots, k'_p)) & \text{otherwise.} \end{cases}$$

Then, (17) follows by induction using the linearity of the tensor product. \square

3.3. Bar Construction on $\mathcal{M}_{0,5}$ and the pentagon relations. Here, we will show how the pentagon relations (III_{KZ}) are related to the bar construction on $\mathcal{M}_{0,5}$.

The shuffle algebra $B(\mathcal{M}_{0,4})$ being much more complicated than $B(\mathcal{M}_{0,4})$ we will first review some facts explained in [Bro09]. We now fix a dihedral structure δ , as described in [Bro09], on $\mathcal{M}_{0,5}$. We will use the “standard” dihedral structure given by “cyclic” order on the marked points

$$z_1 < z_2 < z_3 < z_4 < z_5 (< z_1)$$

or with our normalization

$$0 < xy < y < 1 < \infty.$$

This corresponds to a good choice of coordinates to study the connected component of $\mathcal{M}_{0,5}(\mathbb{R})$ such that the marked points are in the order given by δ . We will refer to that component as the standard cell.

More precisely, let i, j, k, l denote distinct elements of $\{1, 2, 3, 4, 5\}$. The cross-ratio $[i j | k l]$ is defined by the formula:

$$[i j | k l] = \frac{z_i - z_k}{z_i - z_l} \frac{z_j - z_l}{z_j - z_k}.$$

Brown, in [Bro09, Sections 2.1 and 2.2], has defined coordinates on $\mathcal{M}_{0,5}$ (and more generally on $\mathcal{M}_{0,n}$) that respect the natural dihedral symmetry of the moduli spaces of curves. Following his work, for i and j in $\{1, 2, 3, 4, 5\}$ such that $i, i+1, j$ and $j+1$ are distinct, we set

$$u_{ij} = [i i+1 | j+1 j].$$

Let $\omega_{12}, \omega_{23}, \omega_{34}, \omega_{45}, \omega_{24}$ be the differential forms

$$\omega_{12} = d \log(u_{25}) = \frac{dx}{x}, \quad \omega_{23} = d \log(u_{31}u_{41}) = \frac{dx}{x-1},$$

$$\omega_{34} = d \log(u_{24}u_{41}) = \frac{dy}{y-1}, \quad \omega_{45} = d \log(u_{35}) = \frac{dy}{y}$$

$$\text{and } \omega_{24} = d \log(u_{41}) = \frac{d(xy)}{xy-1}.$$

If W is a word in $\mathcal{W} = \{X_{34}, X_{45}, X_{24}, X_{12}, X_{23}\}^*$ with $|W| = n$, we will write $\omega_W \in \mathbb{H}_{\text{DR}}(\mathcal{M}_{0,5})^{\otimes n}$ the bar symbol $[\omega_{i_n j_n} | \cdots | \omega_{i_1 j_1}]$. Note that the elements ω_W for W in \mathcal{W} are not all in $V(\mathcal{M}_{0,5})$ usually, only linear combination of such symbols are in $V(\mathcal{M}_{0,5})$.

Example 3.9. The elements $[\omega_{12}]$, $[\omega_{23}]$ and $[\omega_{12}|\omega_{23}]$ are in $V(\mathcal{M}_{0,5})$ even if $[\omega_{12}|\omega_{45}]$ is not. However $[\omega_{12}|\omega_{45}] + [\omega_{45}|\omega_{12}]$ is in $V(\mathcal{M}_{0,5})$.

Example 3.43 in [Bro09] (using [Bro09, Thm. 3.38 and Coro. 3.41]) tells us that the exact sequence

$$0 \longrightarrow \mathbb{C}\langle\langle X_{24}, X_{34}, X_{45} \rangle\rangle \longrightarrow U\mathfrak{B}_5 \longrightarrow \mathbb{C}\langle\langle X_{12}, X_{23} \rangle\rangle \longrightarrow 0$$

is dual to the exact sequence

$$0 \longrightarrow V(\mathcal{M}_{0,4}) \longrightarrow V(\mathcal{M}_{0,5}) \longrightarrow \mathbb{C}\left\langle \frac{dy}{y}, \frac{dy}{y-1}, \frac{x dy}{xy-1} \right\rangle \longrightarrow 0$$

which comes from the expression, in cubical coordinates, of the map $\mathcal{M}_{0,5} \longrightarrow \mathcal{M}_{0,4}$ which forgets the 4th point. Thus, the identification

$$U\mathfrak{B}_5 \simeq \mathbb{C}\langle\langle X_{24}, X_{34}, X_{45} \rangle\rangle \rtimes \mathbb{C}\langle\langle X_{12}, X_{23} \rangle\rangle$$

is dual (as graded algebra) to

$$V(\mathcal{M}_{0,5}) \simeq V(\mathcal{M}_{0,4}) \otimes \mathbb{C}\left\langle \frac{dy}{y}, \frac{dy}{y-1}, \frac{x dy}{xy-1} \right\rangle$$

and $V(\mathcal{M}_{0,5})$ is the graded dual $U\mathfrak{B}_5^*$ of $U\mathfrak{B}_5$.

The graded dual of the free algebra of formal series

$$R = \mathbb{C}\langle\langle X_{34}, X_{45}, X_{24}, X_{12}, X_{23} \rangle\rangle$$

is the shuffle algebra

$$T := \bigoplus_n (\mathbb{C}\omega_{34} \oplus \mathbb{C}\omega_{45} \oplus \mathbb{C}\omega_{24} \oplus \mathbb{C}\omega_{12} \oplus \mathbb{C}\omega_{23})^{\otimes n}.$$

Let Ω be the element in $R \otimes H_{\text{DR}}^1(\mathcal{M}_{0,5})$ defined by

$$\Omega = X_{12} \otimes \omega_{12} + X_{23} \otimes \omega_{23} + X_{34} \otimes \omega_{34} + X_{45} \otimes \omega_{45} + X_{24} \otimes \omega_{24}.$$

and

$$\text{Exp}(\Omega) := \sum_{W \in \mathcal{W}} W \otimes \omega_W \in R \otimes T.$$

The element $\text{Exp}(\Omega)$ corresponds to the identity of R and encodes the fact that the dual of a word W is ω_W . A word W (seen in $U\mathfrak{B}_5$) is written in the basis B_4 as

$$W = \sum_{b_4 \in B_4} l_{b_4, W} b_4.$$

Duality between R and T and between $U\mathfrak{B}_5$ and $V(\mathcal{M}_{0,5})$ tells us that, the basis $B_4^* = (b_4^*)_{b_4 \in B_4}$ of $V(\mathcal{M}_{0,5})$ dual to B_4 is given by

$$\forall b_4 \in B_4 \quad b_4^* = \sum_{W \in \mathcal{W}} l_{b_4, W} \omega_W.$$

Using the projection $R \rightarrow U\mathfrak{B}_5$ one can see $\text{Exp}(\Omega)$ in $U\mathfrak{B}_5 \otimes T$. Actually, by duality $\text{Exp}(\Omega)$ lies in $U\mathfrak{B}_5 \otimes V(\mathcal{M}_{0,5})$. So, writing each W in the basis B_4 leads to the following expression of $\text{Exp}(\Omega)$ in $U\mathfrak{B}_5 \otimes V(\mathcal{M}_{0,5})$

$$\text{Exp}(\Omega) = \sum_{b_4 \in B_4} b_4 \otimes b_4^* \in U\mathfrak{B}_5 \otimes V(\mathcal{M}_{0,5}).$$

Thus, $\text{Exp}(\Omega)$ realized the identification between the graded dual of $U\mathfrak{B}_5$ and $V(\mathcal{M}_{0,5})$ as it was remarked by Furusho in [Fur08]. This discussion can be resumed by the following proposition.

Proposition 3.10. *We have a natural identification*

$$U\mathfrak{B}_5^* \simeq V(\mathcal{M}_{0,5}),$$

$U\mathfrak{B}_5^*$ being the graded dual of $U\mathfrak{B}_5$.

This identification gives a basis B_4^ of $V(\mathcal{M}_{0,5})$ dual to B_4 the basis of $U\mathfrak{B}_5$ which comes from the identification*

$$U\mathfrak{B}_5 \simeq \mathbb{C}\langle\langle X_{24}, X_{34}, X_{45} \rangle\rangle \rtimes \mathbb{C}\langle\langle X_{12}, X_{23} \rangle\rangle.$$

The basis $B_4^ = (b_4^*)_{b_4 \in B_4}$ is explicitly given for all b_4 in the basis B_4 by*

$$(18) \quad b_4^* = \sum_{W \in \mathcal{W}} l_{b_4, W} \omega_W.$$

Let $\widehat{\mathcal{M}}_{0,5}$ be the universal covering of $\mathcal{M}_{0,5}$. A multi-valued function on $\widehat{\mathcal{M}}_{0,5}$ is an analytic function on $\widehat{\mathcal{M}}_{0,5}$. Consider the formal differential equation on $\widehat{\mathcal{M}}_{0,5}$

$$dL = \Omega L$$

where L takes values in $U\mathfrak{B}_5$, its coefficients being multi-valued functions on $\mathcal{M}_{0,5}$. As in the case of the equation (KZ), if we fix the value of L at some point of $\mathcal{M}_{0,5}$ or its asymptotic behavior at a tangential based points, the solution is unique.

The irreducible components of $\mathcal{M}_{0,5}$ are in one to one correspondence with the 2-partitions of $\{z_1, z_2, z_3, z_4, z_5\}$ and will be denoted as $z_{i_1}z_{i_2}|z_{i_3}z_{i_4}z_{i_5}$. We will here only consider the following components $D_{52} = z_1z_2|z_3z_4z_5$, $D_{13} = z_2z_3|z_4z_5z_1$, $D_{24} = z_3z_4|z_5z_1z_2$, $D_{35} = z_4z_5|z_1z_2z_3$, $D_{41} = z_5z_1|z_2z_3z_4$ (we may use the convention $D_{ij} = D_{ji}$). One remarks that those components are given by a partition that respect the dihedral structure δ and the numbering D_{ij} is coherent with the notation of [Bro09]. The divisors D_{ij} are given in the dihedral coordinates by $u_{ij} = 0$. Following Brown, we have 5 tangential based points (corresponding to the intersection of 2 irreducible components) given by the triangulation of the polygon corresponding to δ ; as we are working in $\mathcal{M}_{0,5}$ the polygon is a pentagon and a triangulation is given by two chords going out from a single vertex so one can number the triangulation by the number of its vertex, precisely one has

$$\begin{aligned} P_3 &= D_{35} \cap D_{13}, & P_1 &= D_{13} \cap D_{41}, & P_4 &= D_{41} \cap D_{24}, \\ P_2 &= D_{24} \cap D_{52}, & \text{and} & & P_5 &= D_{52} \cap D_{35}. \end{aligned}$$

Let L_i be the normalized solution at P_i (see [Bro09] Theorem 6.12).

Now, let's fix a basis $B = (b)_{b \in B}$ of $U\mathfrak{B}_5$ and its dual basis $B^* = (b^*)_{b \in B}$ in $V(\mathcal{M}_{m_0,5})$. The description of the situation in dimension 1 and section 5.2 in [Bro09] shows that Theorem 6.27 of Brown's article that [Bro09] can be rewritten as follow.

Proposition 3.11. *For any tangential based point, P_i , one can write $L_i(z)$ as*

$$\forall z \in \widehat{\mathcal{M}}_{0,5} \quad L_i(z) = \sum_{b \in B} \left(\int_{\gamma} b^* \right) b$$

where γ is a path from P_i to z and where iterated integrals are regularised iterated integrals.

Comparison of two different normalized solutions at two different based points P_i and P_j is then given by

$$\forall z \in \widehat{\mathcal{M}}_{0,5} \quad L_i(z) = L_j(z) \left(\sum_{b \in B} \left(\int_{\gamma} b^* \right) b \right)$$

where γ is any path going from P_i to P_j homotopically equivalent to a path γ' going from P_i to P_j in the standard cell of $\mathcal{M}_{0,5}(\mathbb{R})$.

Brown shows how to restrict any element ω in $B(\mathcal{M}_{0,5})$ to any boundary components D introducing a regularisation map $\text{Reg}(\omega, D)$ that send each $\frac{du_{ij}}{u_{ij}}$ to 0 if $u_{ij} = 0$ or if $u_{ij} = 1$ on D .

Proposition 3.12. *For any two consecutive tangential based points P_i and P_j with $j \cong i - 2 \pmod{5}$, one has*

$$\forall z \in \widehat{\mathcal{M}}_{0,5} \quad L_i(z) = L_j(z) \left(\sum_{b \in B} \left(\int_{p_{ji}} \text{Reg}(b^*, D_{ji}) \right) b \right)$$

where p_{ji} is the real segment going in D_{ji} from P_i to P_j .

Proof. The symmetry of the situation allows us to prove it only in the case where $i = 5, j = 3$ and B is the basis B_4 .

Let p_{35} be the path in D_{35} going from P_5 to P_3 , we need to show that

$$(19) \quad L_3(z)^{-1} L_5(z) = \sum_{b_4 \in B_4} \left(\int_{p_{35}} \text{Reg}(b_4^*, D_{35}) \right) b_4.$$

Brown, in [Bro09, Definition 6.18], defined Z^{35} to be the quotient $L_3(z)^{-1}L_5(z)$. Using the proof of Theorem 6.20 in [Bro09], we have

$$Z^{35} = L_3(z)^{-1}L_5(z) = \sum_{\substack{W=X_{i_n j_n} \cdots X_{i_1 j_1} \\ \in \{X_{12}, X_{23}\}}} \left(\int_p \frac{dt}{t - \varepsilon_n} \wedge \cdots \wedge \frac{dt}{t - \varepsilon_1} \right) W$$

with $\varepsilon_k = 0$ if $i_k = 1$ (and $j_k = 2$) and $\varepsilon_k = 1$ otherwise (that is, $i_k = 2$ and $j_k = 3$). Using the morphism $p_4 : U\mathfrak{B}_5 \rightarrow U\mathfrak{F}_2$ that send X_{i_4} to 0, X_{12} to X_0 and X_{23} to X_1 , we have:

$$Z^{35} = L_3(z)^{-1}L_5(z) = \sum_{\substack{W=X_{i_n j_n} \cdots X_{i_1 j_1} \\ \in \{X_{12}, X_{23}\}}} \left(\int_p \omega_{p_4(W)} \right) W.$$

We recall that an element b_4 of the basis B_4 is either 1 or a monomial of the form

$$(20) \quad b_4 = U_{245}V_{123} \quad U_{245} \in \{X_{24}, X_{34}, X_{45}\}^*, \quad V_{123} \in \{X_{12}, X_{23}\}^*.$$

So, in order to prove (19), it is enough to prove that:

- All the iterated integrals $\int_{p_{35}} \text{Reg}(b_4^*, D_{35})$ for $b_4 = U_{245}V_{123}$ with U_{245} not empty vanish :

$$\int_{p_{35}} \text{Reg}(b_4^*, D_{35}) = 0 \quad \Leftrightarrow \quad b_4 = U_{245}V_{123} \\ \text{with } U_{245} \in \{X_{24}, X_{34}, X_{45}\}^*, \quad U_{245} \neq \emptyset.$$

- All the iterated integrals $\int_{p_{35}} \text{Reg}(b_4^*, D_{35})$ for $b_4 = V_{123}$ are equals to

$$\int_p \omega_{p_4(V_{123})} = \int_p \omega_{p_4(b_4)}.$$

That is:

$$\int_{p_{35}} \text{Reg}(b_4^*, D_{35}) = \int_p \omega_{p_4(b_4)} \quad \Leftrightarrow \quad b_4 = V_{123} \in \{X_{12}, X_{23}\}^*$$

One computes that

$$\text{Reg}(u_{25}, D_{35}) = t, \quad \text{Reg}(u_{31}, D_{35}) = 1 - t$$

and $\text{Reg}(u_{ij}, D_{35}) = 0$ or 1 otherwise.

Thus, we have

$$\text{Reg}(\omega_{12}, D_{35}) = \frac{dt}{t}, \quad \text{Reg}(\omega_{23}, D_{35}) = \frac{dt}{t-1}$$

and $\text{Reg}(\omega_{ij}, D_{35}) = 0$ otherwise.

It is now enough to show that for b_4 in B_4

- b_4 is a word in the letter X_{12} and X_{23} (that is $b_4 \in \{X_{12}, X_{23}\}^*$) if and only if

$$b_4^* = \omega_{b_4} \quad \text{with } b_4 \in \{X_{12}, X_{23}\}^* \\ (= [\omega_{i_n j_n} | \cdots | \omega_{i_1 j_1}] \quad \text{with } X_{i_k j_k} \in \{X_{12}, X_{23}\})$$

- b_4 contains some X_{ij} with $i = 4$ or $j = 4$ if and only if

$$b_4^* = \sum \lambda_{W'} \omega_{W'} \quad (\lambda_{W'} \neq 0) \\ \text{such that } \forall W' \quad W' \notin \{X_{12}, X_{23}\}^*.$$

That is, if and only if b_4^* is a linear combination of bar symbol $\sum \lambda_{W'} \omega_{W'}$ ($\lambda_{W'} \neq 0$) with W' containing at least one of the letters X_{34}, X_{45}, X_{24} .

Using equations (20) and (18) that describe respectively b_4 and b_4^* , one sees that Equation (19) (and thus the proposition) follows directly from the relation defining $U\mathfrak{B}_5$. \square

From the previous proposition we deduce immediately the following corollary.

Corollary 3.13. *For any path γ in the standard cell homotopically equivalent to p_{ji} $j = i - 2 \pmod{5}$ ($1 \leq i, j \leq 5$) we have*

$$\forall \omega \in V(\mathcal{M}_{0,5}) \quad \int_{\gamma} \omega = \int_{p_{ji}} \text{Reg}(\omega, D_{ji}).$$

Let $\gamma = p_{35} \circ p_{52} \circ p_{24} \circ p_{41} \circ p_{13}$ denote the composed path beginning and ending at P_3 and extending the map $\text{Reg}(\omega, \gamma)$ to paths that are piecewise in some on the divisor D_{ij} .

Theorem 3.14. *The relation (III_{KZ}) is equivalent to the family of relations*

$$\forall b_4 \in B_4, b_4 \neq 1 \quad \int_{\gamma} \text{Reg}(b_4^*, \gamma) = 0$$

which is exactly the family (11).

Proof. For i in $\{1, 2, 3, 4, 5\}$ and $j = i - 2 \pmod{5}$, we define Z^{ji} by the formula

$$Z^{ji} = \left(\sum_{b_4 \in B_4} \left(\int_{p_{ji}} \text{Reg}(b_4^*, D_{ji}) \right) b_4 \right).$$

By Proposition 3.12, one has

$$\forall z \in \widehat{\mathcal{M}}_{0,5} \quad L_i(z) = L_j(z) Z^{ji}.$$

Comparison between the 5 normalized solutions L_i at the 5 tangential based points P_i gives

$$(21) \quad \forall z \in \widehat{\mathcal{M}}_{0,5} \quad L_3(z) = L_3(z) Z^{35} Z^{52} Z^{24} Z^{41} Z^{13}.$$

In the proof of Theorem 6.20 [Bro09] and in the example which follows it, Brown proves that the product of the Z^{ji} is equal to the L.H.S (that is the product of the Φ_{KZ}) of (III_{KZ}). So, Equation (III_{KZ}) can be written as

$$Z^{35} Z^{52} Z^{24} Z^{41} Z^{13} = 1.$$

It can also be proved directly using Proposition 3.12.

Equation (21) is given by the analytic continuation of the solution L_3 along any path in the standard cell beginning and ending at P_3 and going through P_1, P_4, P_2 and P_5 (in that order). Such a path is homotopically equivalent to γ (and to 0) and the product of the Z^{ji} can be written as

$$Z^{35} Z^{52} Z^{24} Z^{41} Z^{13} = \sum_{b_4 \in B_4} \left(\int_{\gamma} b_4^* \right) b_4.$$

As γ is homotopically equivalent to 0, each of the homotopy invariant regularised iterated integrals above are 0 (except for $b_4 = 1$). Thus, the product

$$Z^{35} Z^{52} Z^{24} Z^{41} Z^{13}$$

is equal to 1. We deduce from the previous discussion that the family of relation

$$\forall b_4 \in B_4, b_4 \neq 1 \quad \int_{\gamma} \text{Reg}(b_4^*, \gamma) = 0$$

implies relation (III_{KZ}). Moreover, one deduces from the equation

$$Z^{35} Z^{52} Z^{24} Z^{41} Z^{13} = \sum_{b_4 \in B_4} \left(\int_{\gamma} b_4^* \right) b_4.$$

that relation (III_{KZ}) (which says that the product of the Z^{j_i} is 1) implies

$$\forall b_4 \in B_4, b_4 \neq 1 \quad \int_{\gamma} \text{Reg}(b_4^*, \gamma) = 0.$$

The first part of the theorem is then proved.

Using the expression of b_4^* in terms of ω_W , the end of the theorem follows from Proposition 3.16. \square

From the previous theorem, one deduces the following corollary.

Corollary 3.15. *For any basis B of $U\mathfrak{B}_5$, the pentagon relation (III_{KZ}) is equivalent to*

$$\forall b \in B \quad \int_{\gamma} \text{Reg}(b^*, \gamma) = 0$$

where γ , as previously, is the path $= p_{35} \circ p_{52} \circ p_{24} \circ p_{41} \circ p_{13}$.

Following the proof of 3.12, one proves Proposition 3.16 that end the proof of Theorem 3.14.

Proposition 3.16. *For any bar symbol ω_W dual to a word W in the letters $X_{34}, X_{45}, X_{24}, X_{12}, X_{23}$, we have*

$$C_{5,W,KZ} = \int_{\gamma} \text{Reg}(\omega_W, \gamma)$$

where $C_{5,W,KZ}$ is the coefficient $C_{5,W}$ defined in (9) in the particular case of the Drinfel'd associator Φ_{KZ} .

Proof. To show the proposition, it is enough, using the decomposition of $\gamma = p_{35} \circ p_{52} \circ p_{24} \circ p_{41} \circ p_{13}$, to show that, for any U in $\{X_{34}, X_{45}, X_{24}, X_{12}, X_{23}\}^*$ and any i , one has

$$(-1)^{dp_i(U)} \zeta^{\text{tr}}(\rho_i(U)) = \int_{I_i} \text{Reg}(\omega_U, I_i)$$

where $I_5 = p_{13}, I_4 = p_{41}, I_3 = p_{24}, I_2 = p_{52}$ and $I_1 = p_{35}$.

As $\text{Reg}(\omega_{kl}, I_i) = \omega_{\rho_i(X_{kl})}$, the corollary follows. \square

4. APPENDIX

Here one can find the explicit relations given by the pentagon equation III_{KZ} in low degree. Writing the product

$$\begin{aligned} \Phi_{KZ}(X_{12}, X_{23}) \Phi_{KZ}(X_{34}, X_{45}) \Phi_{KZ}(X_{51}, X_{12}) \Phi_{KZ}(X_{23}, X_{34}) \\ \Phi_{KZ}(X_{45}, X_{51}) = \sum_{b_4} C_{b_4} b_4 \end{aligned}$$

in the basis B_4 the following tables give the relation $C_{b_4} = 0$ in terms of regularised multiple zeta values.

4.1. Degree 1, 2 and 3. In degree 1 the basis is given by the letters $X_{34}, X_{45}, X_{24}, X_{12}$ and X_{23} . The corresponding relations (equivalent to (III_{KZ})) are given in Table 1 below.

In degree 2 the basis B_4 is given by 19 monomials but we have only 4 multiplicative generators and the corresponding relations are given in Table 2. In degree 3 there are 10 multiplicative generators and the corresponding relations are given in Table 3 and Table 4. We have written in bold the particular monomial whose dual are multiplicative generator of $V(\mathcal{M}_{0,5})$.

In degree 4 and 5 the family of relation equivalent to III_{KZ} can be found on the web page of the author together with the sources of the Mathematica code.

Monomial	Dual bar symbol	Relations
$X_{1,2}$	$[\omega_{1,2}]$	$\zeta^{\text{III}}(X_0) - \zeta^{\text{III}}(X_1) = 0$
$X_{2,3}$	$[\omega_{2,3}]$	$2(\zeta^{\text{III}}(X_0) - \zeta^{\text{III}}(X_1)) = 0$
$X_{2,4}$	$[\omega_{2,4}]$	$\zeta^{\text{III}}(X_0) - \zeta^{\text{III}}(X_1) = 0$
$X_{3,4}$	$[\omega_{3,4}]$	$2(\zeta^{\text{III}}(X_0) - \zeta^{\text{III}}(X_1)) = 0$
$X_{4,5}$	$[\omega_{4,5}]$	$\zeta^{\text{III}}(X_0) - \zeta^{\text{III}}(X_1) = 0$

TABLE 1. Explicit set of relations equivalent to III_{KZ} in degree 1

Monomial	Dual bar symbol	Relations
$X_{2,4}X_{4,5}$	$-[\omega_{1,2} \omega_{2,4}] + [\omega_{2,4} \omega_{4,5}]$	$\zeta^{\text{III}}(X_0)\zeta^{\text{III}}(X_1) - \zeta^{\text{III}}(X_1)^2 = 0$
$X_{2,4}X_{3,4}$	$-[\omega_{1,2} \omega_{2,4}] + [\omega_{2,3} \omega_{2,4}]$ $-[\omega_{2,3} \omega_{3,4}] + [\omega_{2,4} \omega_{3,4}]$	$-\zeta^{\text{III}}(X_0)^2 + \zeta^{\text{III}}(X_1)\zeta^{\text{III}}(X_0) -$ $2\zeta^{\text{III}}(X_1)^2 + \zeta^{\text{III}}(X_0X_0) +$ $\zeta^{\text{III}}(X_0X_1) + \zeta^{\text{III}}(X_1X_0) +$ $\zeta^{\text{III}}(X_1X_1) = 0$
$X_{3,4}X_{4,5}$	$[\omega_{3,4} \omega_{4,5}]$	$2\zeta^{\text{III}}(X_0)^2 - \zeta^{\text{III}}(X_1)\zeta^{\text{III}}(X_0) -$ $\zeta^{\text{III}}(X_0X_1) - \zeta^{\text{III}}(X_1X_0) = 0$
$X_{1,2}X_{2,3}$	$[\omega_{1,2} \omega_{2,3}]$	$2\zeta^{\text{III}}(X_0)^2 - 2\zeta^{\text{III}}(X_1)\zeta^{\text{III}}(X_0) +$ $\zeta^{\text{III}}(X_1)^2 - \zeta^{\text{III}}(X_0X_1) -$ $\zeta^{\text{III}}(X_1X_0) = 0$

TABLE 2. Explicit set of relations equivalent to III_{KZ} in degree 2

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Monomial	Relations
$X_{2,4}X_{2,3}X_{1,2}$	$-2\zeta^{\text{III}}(X_0X_0X_1) - \zeta^{\text{III}}(X_0X_1X_0) = 0$
$X_{3,4}X_{2,3}X_{1,2}$	$-2\zeta^{\text{III}}(X_0X_0X_1) - \zeta^{\text{III}}(X_0X_1X_0) = 0$
$X_{2,4}X_{3,4}X_{2,4}$	$2\zeta^{\text{III}}(X_0X_0X_1) + \zeta^{\text{III}}(X_0X_1X_0) = 0$
$X_{3,4}X_{2,4}X_{2,3}$	$-2\zeta^{\text{III}}(X_0X_0X_1) - 2\zeta^{\text{III}}(X_0X_1X_0) - 2\zeta^{\text{III}}(X_1X_0X_0) = 0$
$X_{2,4}X_{1,2}X_{2,3}$	$-\zeta^{\text{III}}(X_0X_1X_0) - 2\zeta^{\text{III}}(X_1X_0X_0) = 0$
$X_{3,4}X_{1,2}X_{2,3}$	$-\zeta^{\text{III}}(X_0X_1X_0) - 2\zeta^{\text{III}}(X_1X_0X_0) = 0$
$X_{2,4}X_{2,4}X_{1,2}$	$-\zeta^{\text{III}}(X_0X_0X_1) - \zeta^{\text{III}}(X_0X_1X_0) - \zeta^{\text{III}}(X_1X_0X_0) = 0$
$X_{3,4}X_{2,3}X_{2,3}$	$-\zeta^{\text{III}}(X_0X_0X_1) - \zeta^{\text{III}}(X_0X_1X_0) - \zeta^{\text{III}}(X_1X_0X_0) = 0$
$\mathbf{X}_{3,4}\mathbf{X}_{2,4}\mathbf{X}_{2,4}$	$-\zeta^{\text{III}}(\mathbf{X}_0\mathbf{X}_0\mathbf{X}_1) - \zeta^{\text{III}}(\mathbf{X}_0\mathbf{X}_1\mathbf{X}_0) - \zeta^{\text{III}}(\mathbf{X}_1\mathbf{X}_0\mathbf{X}_0) = 0$
$X_{3,4}X_{3,4}X_{1,2}$	$-\zeta^{\text{III}}(X_0X_0X_1) - \zeta^{\text{III}}(X_0X_1X_0) - \zeta^{\text{III}}(X_1X_0X_0) = 0$
$X_{4,5}X_{4,5}X_{2,3}$	$-\zeta^{\text{III}}(X_0X_0X_1) - \zeta^{\text{III}}(X_0X_1X_0) - \zeta^{\text{III}}(X_1X_0X_0) = 0$
$\mathbf{X}_{1,2}\mathbf{X}_{2,3}\mathbf{X}_{2,3}$	$\zeta^{\text{III}}(\mathbf{X}_0\mathbf{X}_1\mathbf{X}_1) - \zeta^{\text{III}}(\mathbf{X}_1\mathbf{X}_0\mathbf{X}_0) = 0$
$X_{2,3}X_{1,2}X_{1,2}$	$\zeta^{\text{III}}(X_0X_1X_1) - \zeta^{\text{III}}(X_1X_0X_0) = 0$
$\mathbf{X}_{3,4}\mathbf{X}_{4,5}\mathbf{X}_{4,5}$	$\zeta^{\text{III}}(\mathbf{X}_0\mathbf{X}_1\mathbf{X}_1) - \zeta^{\text{III}}(\mathbf{X}_1\mathbf{X}_0\mathbf{X}_0) = 0$
$\mathbf{X}_{4,5}\mathbf{X}_{2,4}\mathbf{X}_{2,4}$	$\zeta^{\text{III}}(\mathbf{X}_0\mathbf{X}_1\mathbf{X}_1) - \zeta^{\text{III}}(\mathbf{X}_1\mathbf{X}_0\mathbf{X}_0) = 0$
$X_{4,5}X_{3,4}X_{3,4}$	$\zeta^{\text{III}}(X_0X_1X_1) - \zeta^{\text{III}}(X_1X_0X_0) = 0$
$X_{2,4}X_{2,4}X_{3,4}$	$\zeta^{\text{III}}(X_1X_0X_0) - \zeta^{\text{III}}(X_0X_0X_1) = 0$
$X_{2,4}X_{3,4}X_{2,3}$	$2\zeta^{\text{III}}(X_0X_0X_1) + 2\zeta^{\text{III}}(X_0X_1X_0) + 2\zeta^{\text{III}}(X_1X_0X_0) = 0$
$X_{1,2}X_{2,3}X_{1,2}$	$\zeta^{\text{III}}(X_1X_0X_1) - \zeta^{\text{III}}(X_0X_1X_0) = 0$
$X_{2,3}X_{1,2}X_{2,3}$	$\zeta^{\text{III}}(X_1X_0X_1) - \zeta^{\text{III}}(X_0X_1X_0) = 0$
$X_{2,4}X_{4,5}X_{2,4}$	$\zeta^{\text{III}}(X_1X_0X_1) - \zeta^{\text{III}}(X_0X_1X_0) = 0$
$X_{3,4}X_{4,5}X_{3,4}$	$\zeta^{\text{III}}(X_1X_0X_1) - \zeta^{\text{III}}(X_0X_1X_0) = 0$
$X_{4,5}X_{3,4}X_{4,5}$	$\zeta^{\text{III}}(X_1X_0X_1) - \zeta^{\text{III}}(X_0X_1X_0) = 0$
$X_{4,5}X_{3,4}X_{2,3}$	$2\zeta^{\text{III}}(X_0X_1X_1) + \zeta^{\text{III}}(X_1X_0X_1) = 0$
$X_{4,5}X_{2,4}X_{2,3}$	$-\zeta^{\text{III}}(X_0X_1X_0) + 2\zeta^{\text{III}}(X_0X_1X_1) - 2\zeta^{\text{III}}(X_1X_0X_0) + \zeta^{\text{III}}(X_1X_0X_1) = 0$
$X_{4,5}X_{2,4}X_{4,5}$	$-\zeta^{\text{III}}(X_0X_1X_0) - 2\zeta^{\text{III}}(X_1X_1X_0) = 0$
$X_{4,5}X_{2,4}X_{3,4}$	$-\zeta^{\text{III}}(X_0X_1X_0) + \zeta^{\text{III}}(X_0X_1X_1) - \zeta^{\text{III}}(X_1X_0X_0) - 2\zeta^{\text{III}}(X_1X_1X_0) = 0$
$X_{2,4}X_{4,5}X_{1,2}$	$-\zeta^{\text{III}}(X_1X_0X_1) - 2\zeta^{\text{III}}(X_1X_1X_0) = 0$
$X_{3,4}X_{2,4}X_{3,4}$	$-2\zeta^{\text{III}}(X_0X_0X_1) - \zeta^{\text{III}}(X_0X_1X_0) - \zeta^{\text{III}}(X_1X_0X_1) - 2\zeta^{\text{III}}(X_1X_1X_0) = 0$

TABLE 3. Explicit set of relations equivalent to III_{KZ} in degree 3 where we already have used the relations $\zeta^{\text{III}}(X_0^k) = \zeta^{\text{III}}(X_1^k) = 0$.

Monomial	Relations
$X_{2,4}X_{3,4}X_{1,2}$	$-\zeta^{\text{III}}(X_0X_0X_1) - \zeta^{\text{III}}(X_0X_1X_0) - \zeta^{\text{III}}(X_1X_0X_0)$ $-\zeta^{\text{III}}(X_1X_0X_1) - 2\zeta^{\text{III}}(X_1X_1X_0) = 0$
$X_{3,4}X_{2,4}X_{4,5}$	$\zeta^{\text{III}}(X_1X_0X_0) - \zeta^{\text{III}}(X_1X_1X_0) = 0$
$\mathbf{X}_{1,2}\mathbf{X}_{1,2}\mathbf{X}_{2,3}$	$\zeta^{\text{III}}(\mathbf{X}_1\mathbf{X}_1\mathbf{X}_0) - \zeta^{\text{III}}(\mathbf{X}_0\mathbf{X}_0\mathbf{X}_1) = 0$
$X_{2,3}X_{2,3}X_{1,2}$	$\zeta^{\text{III}}(X_1X_1X_0) - \zeta^{\text{III}}(X_0X_0X_1) = 0$
$\mathbf{X}_{3,4}\mathbf{X}_{3,4}\mathbf{X}_{4,5}$	$\zeta^{\text{III}}(\mathbf{X}_1\mathbf{X}_1\mathbf{X}_0) - \zeta^{\text{III}}(\mathbf{X}_0\mathbf{X}_0\mathbf{X}_1) = 0$
$X_{4,5}X_{4,5}X_{2,4}$	$\zeta^{\text{III}}(X_1X_1X_0) - \zeta^{\text{III}}(X_0X_0X_1) = 0$
$X_{4,5}X_{4,5}X_{3,4}$	$\zeta^{\text{III}}(X_1X_1X_0) - \zeta^{\text{III}}(X_0X_0X_1) = 0$
$X_{4,5}X_{3,4}X_{2,4}$	$\zeta^{\text{III}}(X_0X_1X_0) + \zeta^{\text{III}}(X_0X_1X_1) + \zeta^{\text{III}}(X_1X_1X_0) = 0$
$\mathbf{X}_{2,4}\mathbf{X}_{4,5}\mathbf{X}_{4,5}$	$\zeta^{\text{III}}(\mathbf{X}_1\mathbf{X}_1\mathbf{X}_0) - \zeta^{\text{III}}(\mathbf{X}_1\mathbf{X}_0\mathbf{X}_0) = 0$
$X_{2,4}X_{2,4}X_{4,5}$	$\zeta^{\text{III}}(X_0X_1X_0) + \zeta^{\text{III}}(X_1X_0X_0) + \zeta^{\text{III}}(X_1X_1X_0) = 0$
$\mathbf{X}_{2,4}\mathbf{X}_{3,4}\mathbf{X}_{3,4}$	$\zeta^{\text{III}}(\mathbf{X}_0\mathbf{X}_0\mathbf{X}_1) + \zeta^{\text{III}}(\mathbf{X}_0\mathbf{X}_1\mathbf{X}_0) - \zeta^{\text{III}}(\mathbf{X}_0\mathbf{X}_1\mathbf{X}_1)$ $+ \zeta^{\text{III}}(\mathbf{X}_1\mathbf{X}_0\mathbf{X}_0) + \zeta^{\text{III}}(\mathbf{X}_1\mathbf{X}_1\mathbf{X}_0) = 0$
$X_{2,4}X_{1,2}X_{1,2}$	$\zeta^{\text{III}}(X_0X_1X_1) + \zeta^{\text{III}}(X_1X_0X_1) + \zeta^{\text{III}}(X_1X_1X_0) = 0$
$X_{3,4}X_{1,2}X_{1,2}$	$\zeta^{\text{III}}(X_0X_1X_1) + \zeta^{\text{III}}(X_1X_0X_1) + \zeta^{\text{III}}(X_1X_1X_0) = 0$
$X_{3,4}X_{3,4}X_{2,3}$	$\zeta^{\text{III}}(X_0X_1X_1) + \zeta^{\text{III}}(X_1X_0X_1) + \zeta^{\text{III}}(X_1X_1X_0) = 0$
$X_{4,5}X_{2,3}X_{2,3}$	$\zeta^{\text{III}}(X_0X_1X_1) + \zeta^{\text{III}}(X_1X_0X_1) + \zeta^{\text{III}}(X_1X_1X_0) = 0$
$X_{3,4}X_{4,5}X_{2,4}$	$-\zeta^{\text{III}}(X_0X_1X_0) - \zeta^{\text{III}}(X_1X_0X_0) + \zeta^{\text{III}}(X_1X_0X_1)$ $+ \zeta^{\text{III}}(X_1X_1X_0) = 0$
$X_{3,4}X_{3,4}X_{2,4}$	$\zeta^{\text{III}}(X_0X_0X_1) + \zeta^{\text{III}}(X_0X_1X_1) - \zeta^{\text{III}}(X_1X_0X_0)$ $+ \zeta^{\text{III}}(X_1X_0X_1) + \zeta^{\text{III}}(X_1X_1X_0) = 0$
$\mathbf{X}_{2,4}\mathbf{X}_{4,5}\mathbf{X}_{3,4}$	$\zeta^{\text{III}}(\mathbf{X}_1\mathbf{X}_0\mathbf{X}_0) + \zeta^{\text{III}}(\mathbf{X}_1\mathbf{X}_0\mathbf{X}_1) + \zeta^{\text{III}}(\mathbf{X}_1\mathbf{X}_1\mathbf{X}_0) = 0$
$\mathbf{X}_{2,4}\mathbf{X}_{3,4}\mathbf{X}_{4,5}$	$\zeta^{\text{III}}(\mathbf{X}_0\mathbf{X}_1\mathbf{X}_0) + 2\zeta^{\text{III}}(\mathbf{X}_1\mathbf{X}_1\mathbf{X}_0) = 0$
$X_{3,4}X_{4,5}X_{2,3}$	$\zeta^{\text{III}}(X_1X_0X_1) + 2\zeta^{\text{III}}(X_1X_1X_0) = 0$
$X_{4,5}X_{2,4}X_{1,2}$	$\zeta^{\text{III}}(X_1X_0X_1) + 2\zeta^{\text{III}}(X_1X_1X_0) = 0$
$X_{3,4}X_{2,4}X_{1,2}$	$-\zeta^{\text{III}}(X_0X_0X_1) - \zeta^{\text{III}}(X_0X_1X_0) - \zeta^{\text{III}}(X_1X_0X_0)$ $+ \zeta^{\text{III}}(X_1X_0X_1) + 2\zeta^{\text{III}}(X_1X_1X_0) = 0$
$X_{2,4}X_{4,5}X_{2,3}$	$\zeta^{\text{III}}(X_0X_1X_0) + 2\zeta^{\text{III}}(X_1X_0X_0) + \zeta^{\text{III}}(X_1X_0X_1)$ $+ 2\zeta^{\text{III}}(X_1X_1X_0) = 0$

TABLE 4. Explicit set of relations equivalent to III_{KZ} in degree 3 where we already have used the relations $\zeta^{\text{III}}(X_0^k) = \zeta^{\text{III}}(X_1^k) = 0$.