Central Limit Theorems and Quadratic Variation in terms of Spectral Density

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CENTRAL LIMIT THEOREMS AND QUADRATIC VARIATION IN TERMS OF SPECTRAL DENSITY

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Abstract. We give a new proof and provide new bounds for the speed of convergence in the Central Limit Theorems of Breuer Major on stationary Gaussian time series. Our assumptions are given in terms of the spectral density of the time series. We then consider generalized quadratic variations of Gaussian fields with stationary increments under the assumption that their spectral density is asymptotically self-similar and prove Central Limit Theorems in this context.

1. Introduction

In this paper we essentially develop Central Limit Theorems that are well adapted to obtain asymptotic properties of quadratic variations of Gaussian fields with stationary increments. Moreover, we give bounds for the speed of convergence, which partially improve the bounds given by Nourdin and Peccati in [22]. We rely heavily on their methods but adopt a spectral point of view, which improves the final computations.

Before describing our theoretical results, let us describe the scope of applications that we have in mind. The finite distributional properties of the increments of a real valued Gaussian field $\{Y(t); t \in \mathbb{R}^\nu\}$ indexed by $\mathbb{R}^\nu$ ($\nu \geq 1$) may be described from its variogram, that is, the function

$$v_Y(t) := \mathbb{E}(((Y(s+t) - Y(s))^2) \tag{1}$$

or from its spectral measure, which is such that

$$v_Y(t) \;=\; 2\int_{\mathbb{R}^d} |e^{-it.x} - 1|^2 \; d\tau(x) \;,\; \forall t \in \mathbb{R}^\nu. \tag{2}$$

Here $t.x$ stands for the scalar product of the two vectors in $\mathbb{R}^\nu$ and $|x|$ denotes the Euclidean norm of the vector x. The spectral measure τ is a non negative even measure on $\mathbb{R}^\nu$. We will only consider absolutely continuous spectral measures, that is, measures that can be written as $d\tau(x) := F(x)d\mu^\nu(x)$. Here μ^ν denotes the Lebesgue measure on $\mathbb{R}^\nu$ (we skip the exponent for $\nu = 1$). The function F, called the *spectral density* of Y, is assumed to be a non-negative even function of $L^1\left(\mathbb{R}^\nu, \min\left(1, |x|^2\right) d\mu^\nu(x)\right)$. A typical example of such random fields is given by

$$Y(t) = \int_{\mathbb{R}^\nu} \left(e^{-it\cdot x} - 1\right) F(x)^{1/2} d\widetilde{W}^\nu(x), t \in \mathbb{R}^\nu, \tag{3}$$

where $\widetilde{W}^\nu$ is a complex centered Gaussian measure on $\mathbb{R}^\nu$ with Lebesgue control measure μ^ν, such that $\widetilde{W}^\nu(-A) = \overline{\widetilde{W}^\nu(A)}$ a.s. for any Borel set A of $\mathbb{R}^\nu$. In fact, if we are only interested by finite distributions of the random field Y, we can always assume that Y is given by such a spectral representation (3).

Centered Gaussian fields with stationary increments are widely used as models for real data, for example to describe rough surfaces or porous media that possess some homogeneity properties. In particular the fractional Brownian field (fBf), first defined in dimension $\nu = 1$ through a stochastic

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integral of moving-average type by Mandelbrot and Van Ness [21], admits such a representation with spectral density given by

$$F_H(x) = \frac{1}{|x|^{2H+\nu}}, \text{ with } H \in (0,1) \text{ called the Hurst parameter.}$$

The homogeneity of F_H implies a self-similarity property of the corresponding random field Y_H, namely

$$\forall \lambda > 0,\ \{Y_H(\lambda t)\,;\, t \in \mathbb{R}^\nu\} \overset{fdd}{=} \lambda^H \{Y_H(t)\,;\, t \in \mathbb{R}^\nu\}.$$

When $\nu \geq 2$, the choice of this power of the Euclidean norm for the spectral density is equivalent to the fact that the variogram is $v_{Y_H}(t) = c_H|t|^{2H}$, for some positive constant c_H. It induces the isotropy of the field Y_H (its law is invariant under vectorial rotations). Such a model is not adapted when anisotropic features are observed. Anisotropic but still self-similar generalizations are simply obtained by considering a spectral density given by $F(x) = \Omega(x)F_H(x)$ with Ω an homogeneous function of degree 0 satisfying $\Omega(x) = \Omega(x/|x|)$. Then the corresponding variogram is given in a similar form

$$v(t) = \omega(t/|t|)|t|^{2H}, \text{ with } \omega(t) = c_{H,\nu} \int_{|x|=1} |t \cdot x|^{2H} \Omega(x) dx,$$

where dx denotes the Lebesgue measure on $S^{\nu-1} = \{x \in \mathbb{R}^\nu; |x| = 1\}$. When using such a model, a typical question is the identification of the Hurst parameter H from real data. Many estimators for the Hurst parameter of a one-dimensional fBf (called fractional Brownian motion) have been proposed, based for example on time domain methods or spectral methods (see [11] and [3] and references therein). Quadratic variations are relevant estimators when considering H as the critical index of Hölder regularity for the sample paths. Moreover in [19] the authors give precise bounds of the bias of the variance and show that minimax rates are achieved for this kind of estimators. Generalized quadratic variations also apply to more general Gaussian processes and fields with stationary increments with the same Hölder regularity (see [17, 18] or [9, 10] for instance), for which the variogram satisfies

$$v(t) = \omega(t/|t|)|t|^{2H} + \underset{|t|\to 0}{O}\left(|t|^{2H+s}\right)$$

for $H \in (0,1)$ and $s \in (0, 2-2H)$ with ω a positive function on the sphere $S^{\nu-1}$ (and additional assumptions of regularity). This kind of assumption can be replaced by an assumption on the spectral density F (which is a priori stronger but does not require any extra assumption of regularity). More precisely, we will be interested in random fields for which

$$F(x) = \frac{\Omega(x)}{|x|^{2H+\nu}} + \underset{|x|\to+\infty}{O}\left(\frac{1}{|x|^{2H+\nu+\gamma}}\right), \tag{4}$$

with Ω an even function on the sphere $S^{\nu-1}$ (or a constant when $\nu = 1$). Our particular interest in this situation, where the self-similar spectral density is perturbed by a rest that decreases more rapidly at infinity, may be understood from previous work [7, 8]. This arises in particular when one considers a weighted projection of a self-similar random field. We develop here methods that reveal to be stable when adding such a perturbation to the spectral density. A source of inspiration for us has also been the paper of Chan and Wood [9], which deals with stationary random Gaussian fields with asymptotic self-similar properties.

The estimation of Ω or H goes through the consideration of quadratic variations of Y, observed on finer and finer grids. Typically, we assume to have observed values of the random field on a grid with uniform mesh, that is, $\{Y(k/n); k = (k_1, \ldots, k_\nu) \in \mathbb{Z}^\nu \text{ with } 0 \leq k_1, \ldots, k_\nu \leq n-1\}$. We want to have Central Limit Theorems for the quadratic variation of this sequence when n tends to ∞. One central idea used in this paper consists in a change of scale, so that we can as well consider a fixed mesh, but for a different random field at each scale, because of the fact that F is asymptotically homogeneous. The rest does not appear in the limit, and acts only on the speed of convergence, which is in $n^{-\alpha}$, for some $\alpha > 0$ depending in particular on γ. Once we have Central Limit Theorems for finite distributions through this scaling argument, we can also recover asymptotic properties for

continuous time quadratic variations, which may be used when dealing with increments of non linear functionals of Y instead of increments of Y.

Let us come back to the theoretical part of this paper, which constitutes its core. We revisit Breuer Major's Theorem, which is our main tool to obtain Central Limit Theorems, and use the powerful theory developed by Nourdin, Nualart, Ortiz-Latorre, Peccati and others to do so. This is described in the next section and we refer to it for more details. We would like to attract attention to a remark, which has its own interest: under appropriate additional assumptions, the Malliavin derivative of $\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H(X(k))$, where $(X(k))_{k\in\mathbb{Z}}$ is a stationary Gaussian time series and where the Hermite expansion of H starts with terms of order 2, can be written in terms of the integrated periodogram of the sequence $H'(X(k))$. Recall that the periodogram of this time series is defined as

$$\frac{1}{n}\left|\sum_{k=0}^{n-1} H'(X)(k)e^{ikx}\right|^2.$$

Up to our knowledge, this link between two different theories had not been given before. As a consequence, the techniques that we use for having the speed of convergence in Central Limit Theorems may be used for consistency of estimators given in terms of integrated periodograms.

Section 2 is devoted to the theoretical aspects (Central Limit Theorems, integrated periodograms, speed of convergence) in dimension one. We chose to give the proofs in this context, so that the reader can easily follow them. Once this done, we hope that it is not difficult to see how to adapt them in higher dimension, which we do more rapidly in Section 3. We then apply this to generalized quadratic variations in Section 4.

Acknowledgement. This work was done independently of the paper of Nourdin, Peccati and Podolskij [23], which has been posted on the web while we were finishing to write this one. Compared to the results of [23], we deliberately restricted to simple cases, but have found better bounds for the speeds of convergence. It would certainly be helpful to make a synthesis between the two papers. We chose not to do it here, but to stick to our initial project and to the applications we had in view, with assumptions given on spectral densities and not on variograms.

2. Breuer-Major Theorem revisited

In this section we will be interested in stationary centered Gaussian time series $X = (X(k))_{k\in\mathbb{Z}}$ as well as approximate ones. We will start from Breuer-Major Theorem and give a proof of it which is based on the Malliavin Calculus, as exploited by Nourdin, Nualart, Ortiz-Latorre, Peccati, among others, to develop Central Limit Theorems in the context of Wiener Chaos (see [26, 22] for instance). This kind of proof is implicit in the work of these authors, and explicit in the last paper of Nourdin, Peccati and Podolskij [23], where speeds of convergence are given in a very general context. Our interest, here, is to see that assumptions are particularly simple and meaningful when they are given on the spectral density of the time series. Meanwhile, we improve the estimates for the speed of convergence to the best possible through this method under the assumption that the spectral density is in some Sobolev space. Also, this study will lead us to asymptotic estimates on integrated periodograms, which have their own interest and are particularly relevant when one interests to spectral densities.

Let us first state the theorem of Breuer Major in the simplest one dimensional case. For $l \geq 1$, we consider the stationary centered time series $H_l(X) = (H_l(X)(k))_{k\in\mathbb{Z}}$ where $H_l(X)(k) = H_l(X(k))$ with H_l the l-th Hermite polynomial.

Theorem 2.1 (Breuer-Major)**.** *Let $(X(k))_{k\in\mathbb{Z}}$ be a centered stationary Gaussian time series. Assume that for $l \geq 1$, the sequence $r(k) = \mathbb{E}(X(j)X(j+k))$ satisfies the condition*

$$\sum_{k\in\mathbb{Z}} |r(k)|^l < \infty. \tag{5}$$

Then we have the following asymptotic properties for n tending to infinity:

(i)
$$\mathrm{Var}\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_l(X)(k)\right) \longrightarrow \sigma_l^2,$$

(ii)
$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_l(X)(k) \xrightarrow{d} \mathcal{N}(0,\sigma_l^2),$$

with

$$\sigma_l^2 = l!\sum_{k\in\mathbb{Z}} r(k)^l. \tag{6}$$

For $p \geq 1$, we introduce the Banach space $\ell^p(\mathbb{Z})$ of p-summable sequences equipped with the norm $\|u\|_{\ell^p(\mathbb{Z})} = \left(\sum_{k\in\mathbb{Z}} |u(k)|^p\right)^{1/p}$ for $u = (u(k))_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$. Then, Assumption (5) can be written as $r = (r(k))_{k\in\mathbb{Z}} \in \ell^l(\mathbb{Z})$. We recall that the sequence $r(k)$ can be seen as the Fourier coefficients of a positive even periodic finite measure, called the spectral measure of the time series (see [13] or [29] for instance). We will restrict to time series for which the spectral measure is absolutely continuous with respect to the Lebesgue measure μ.

We identify 2π-periodic functions both with functions on the torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and functions on $[-\pi,+\pi)$. The spaces $L^p(\mathbb{T})$ are spaces of measurable functions f on $[-\pi,+\pi)$ such that

$$\|f\|_p^p := \|f\|_{L^p(\mathbb{T})}^p := \frac{1}{2\pi}\int_{\mathbb{T}} |g(x)|^p d\mu(x) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} |f(x)|^p d\mu(x). \tag{7}$$

Let us come back to our assumption on the spectral measure. We call f_X its density with respect to the measure μ, and speak of *spectral density of the time series* as it is classical. Moreover we pose $f_X = |g|^2$. We could of course choose g non negative, but want to have some flexibility later on.

So, in the following we assume that there exists some function $g \in L^2(\mathbb{T})$ which satisfies the assumption $\overline{g(x)} = g(-x)$, such that

$$r(k) := \frac{1}{2\pi}\int_{\mathbb{T}} e^{-ikx}|g(x)|^2 d\mu(x) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-ikx}|g(x)|^2 d\mu(x). \tag{8}$$

Remark that the absolute continuity is only an additional property under (5) when $l > 2$. When the sequence $r(k)$ is square summable, one can find $f_X = |g|^2 \in L^2(\mathbb{T})$ by Plancherel's Theorem.

Remark 2.2. *Recall that, for U and V two centered Gaussian variables such that $\mathbb{E}(UV) = \rho$, we have $\mathbb{E}(H_l(U)H_l(V)) = l!\rho^l$. So, whenever the time series X has the spectral density f_X, the time series $H_l(X)$ has the density $f_{H_l(X)} = l!f_X^{*l}$, where the notation f_X^{*l} stands for l times the convolution of f_X by itself on the torus. Assumption* (5) *implies the absolute convergence of the Fourier series of the spectral density of $H_l(X)$. It also means that f_X^{*l} is continuous on $\mathbb{T}$ and that $\sigma_l^2 = l!f_X^{*l}(0)$. For all $l \geq 2$, Assumption* (5) *is in particular implied by the stronger assumption*

$$f_X \in L^{\frac{l}{l-1}}(\mathbb{T}), \tag{9}$$

since $\|r\|_{\ell^l(\mathbb{Z})} \leq \|f_X\|_{\frac{l}{l-1}}$ by Hausdorff-Young Inequality (see [20] *for instance).*

Remark 2.3. *Note also that the assumption that the Gaussian time series X has a spectral density f_X implies in particular that the time series $H_l(X)$ is a strictly stationary ergodic one, for any $l \geq 1$, (see* [12] *for instance).*

We will give a new proof of the theorem of Breuer Major under Assumption (8). We will use for instance Theorem 4 in [26], which asserts that the Central Limit Theorem is a consequence of the convergence of the variance given in (i) on one side, then of a quantity related to the Malliavin derivative on another side, so that one does not need to consider all moments as in the original proof of Breuer and Major. In a first subsection we recall the main tools in our framework.

2.1. **Complex Wiener chaos and Malliavin calculus.** Let W be a complex centered Gaussian measure on $[-\pi,+\pi)$ with Lebesgue control measure $\frac{1}{2\pi}\mu$ such that, for any Borel set A of $[-\pi,\pi)$ $W(-A) = \overline{W(A)}$ almost surely. We consider complex-valued functions ψ defined on $[-\pi,+\pi)$, considered as periodic functions of the torus that satisfy, for almost every $x \in \mathbb{T}$

$$\overline{\psi(x)} = \psi(-x).$$

We write $L^2_e(\mathbb{T})$ the real vector space of such functions that are square integrable with respect to the Lebesgue measure on $\mathbb{T}$. Endowed with the scalar product of $L^2(\mathbb{T})$, which we also note

$$\langle \psi, \varphi \rangle_\mu = \frac{1}{2\pi}\int_{\mathbb{T}} \psi(x)\overline{\varphi(x)}d\mu(x),$$

$L^2_e(\mathbb{T})$ is a real separable Hilbert space. Moreover, for any $\psi \in L^2_e(\mathbb{T})$, one can define its stochastic integral with respect to W as

$$I_1(\psi) = \int_{-\pi}^{+\pi} \psi(x)dW(x).$$

Then $I_1(\psi)$ is a real centered Gaussian variable with variance given by $\|\psi\|_2^2$, where $\|\cdot\|_2$ is the norm induced by the scalar product $\langle\cdot,\cdot\rangle_\mu$. For introducing the k-th Itô-Wiener integral, with $k \geq 1$, we consider the complex functions belonging to

$$L^2_e(\mathbb{T}^k) = \{\psi \in L^2(\mathbb{T}^k) : \psi(-x) = \overline{\psi(x)}\}.$$

The inner product in the real Hilbert space of complex functions of $L^2_e(\mathbb{T}^k)$ is given by

$$\langle \psi, \varphi \rangle_{\mu^k} = \frac{1}{(2\pi)^k}\int_{\mathbb{T}^k} \psi(x)\overline{\varphi(x)}d\mu^k(x).$$

The space $L^2_s(\mathbb{T}^k)$ denotes the subspace of functions of $L^2_e(\mathbb{T}^k)$ a.e. invariant under permutations of their arguments. By convention $L^2_s(\mathbb{T}^k) = \mathbb{R}$ for $k = 0$. Let us define $H(W)$ the subspace of random variables in $L^2(\Omega,\mathbb{P})$ measurable with respect to W. The k-Itô-Wiener integral I_k is defined in such a way that $(k!)^{-1/2}I_k$ is an isometry between $L^2_s(\mathbb{T}^k)$ and its range $\mathcal{H}_k \subset H(W)$, so that we have the orthogonal decomposition

$$H(W) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k,$$

where $\mathcal{H}_0$ is the space of real constants. Each $Y \in H(W)$ has an $L^2(\Omega,\mathbb{P})$ convergent decomposition

$$Y = \sum_{k=0}^{\infty} I_k(\psi_k), \quad \psi_k \in L^2_s(\mathbb{T}^k).$$

When moreover $\sum_{k=1}^{+\infty}(k+1)!\|\psi_k\|_2^2 < +\infty$, with $\|\psi_k\|_2^2 = \langle\psi_k,\psi_k\rangle_{\mu^k}$, the Malliavin derivative of Y, denoted by DY, is defined as the complex valued random process given on $\mathbb{T}$ by

$$D_tY = \sum_{k=1}^{+\infty} kI_{k-1}\left(\psi_k(\cdot,t)\right), t \in \mathbb{T}.$$

Furthermore if H_k is the k-th Hermite polynomial for the standard Gaussian measure and denoting by $\psi^{\odot_k}$ the k-tensor product of the function $\psi \in L^2_e(\mathbb{T})$ we have

$$H_k(I_1(\psi)) = I_k(\psi^{\odot_k}) := \int_{[-\pi,+\pi)^k} \psi(x_1)\dots\psi(x_k)dW(x_1)\dots dW(x_k), \tag{10}$$

so that its Malliavin derivative is given by $DH_k(I_1(\psi)) = kH_{k-1}(I_1(\psi))\psi$.

2.2. **Proof of Breuer Major Theorem under Assumption** (8)**.** As far as finite distributions are concerned, when the time series X admits a covariance function given by (8) for some function $g \in L^2_e(\mathbb{T})$, we can assume, without loss of generality, that

$$X(k) := \int_{-\pi}^{+\pi} e^{-ikx} g(x) dW(x). \tag{11}$$

Up to normalization, we can also assume that $r(0) = 1$, or equivalently that $\|g\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{T}} |g(x)|^2 d\mu(x) = 1$. For any $k \in \mathbb{Z}$, we write $g_k(x) = e^{-ikx} g(x) \in L^2_e(\mathbb{T})$ so that $X(k)$ may be written as the Itô-Wiener integral $I_1(g_k)$. Moreover $H_l(X)$ is in the Wiener chaos of order l with

$$H_l(X)(k) = I_l\left(g_k^{\odot_l}\right),\ k \in \mathbb{Z}.$$

Let us now proceed to the proof. The computation of the variance in (i) is direct. Let us write

$$Y_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_l(X)(k).$$

Then,

$$\begin{aligned}
\mathrm{Var}(Y_n) &= \frac{1}{n}\sum_{k=0}^{n-1}\sum_{k'=0}^{n-1} \mathrm{Cov}(H_l(X)(k), H_l(X)(k')) \\
&= \frac{l!}{n}\sum_{k=0}^{n-1}\sum_{k'=0}^{n-1} r(k-k')^l \\
&= l! \sum_{k=-(n-1)}^{n-1} \left(1 - \frac{|k|}{n}\right) r(k)^l,
\end{aligned}$$

which tends to $l!\sum_{k\in\mathbb{Z}} r(k)^l = l!\sigma_l^2$. Recall that this last sum is absolutely convergent because of the assumption on r. This concludes the proof when $l = 1$ since Y_n is a Gaussian variable in this case. When $l \geq 2$ we write $Y_n = I_l(F_n)$, with $F_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} g_k^{\odot_l}$. By Theorem 4 of [26], to prove Part (ii) it is sufficient to prove that

$$\|DY_n\|_2^2 \underset{n\to+\infty}{\longrightarrow} l\sigma_l^2 \text{ in } L^2(\Omega, \mathbb{P}),$$

with DY_n the Malliavin's Derivative of Y_n, which is given by

$$DY_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} l H_{l-1}(I_1(g_k)) g_k.$$

We first remark that

$$\begin{aligned}
\|DY_n\|_2^2 &= \frac{l^2}{n}\sum_{k,k'=0}^{n-1} H_{l-1}(I_1(g_k)) H_{l-1}(I_1(g_{k'}))\langle g_k, g_{k'}\rangle_\mu \\
&= \frac{l^2}{n}\sum_{k,k'=0}^{n-1} H_{l-1}(X)(k) H_{l-1}(X)(k') r(k-k') \\
&= \frac{l^2}{2\pi}\int_{-\pi}^{\pi} I_n^{(l-1)}(x) f_X(x) d\mu(x),
\end{aligned}$$

where

$$I_n^{(l-1)}(x) = \frac{1}{n}\sum_{k,k'=0}^{n-1} H_{l-1}(X)(k) H_{l-1}(X)(k') e^{i(k'-k)x} = \frac{1}{n}\left|\sum_{k=0}^{n-1} H_{l-1}(X)(k) e^{ikx}\right|^2,$$

is the periodogram of order n of the stationary sequence $\left(H_{l-1}(X)(k)\right)_{k\in\mathbb{Z}}$ (see [15] for instance). The end of the proof is a direct consequence of the next subsection, which is devoted to the limit of integrated periodograms. The fact that the Malliavin derivative may be written in terms of the periodogram is an unexpected phenomenon.

2.3. **Integrated periodograms.** We keep the notations of the last subsection, so that for $l \geq 1$,

$$I_n^{(l)}(x) := \frac{1}{n}\left|\sum_{k=0}^{n-1} H_l(X)(k)e^{ikx}\right|^2. \tag{12}$$

The periodogram $I_n^{(l)}$ is used as an estimator of the spectral density of the stationary sequence $(H_l(X)(k))_{k\in\mathbb{Z}}$, that is $l!f_X^{*l}$ since its Fourier coefficients are equal to $l!r(k)^l$. It is well known that $I_n^{(l)}(x)$ is not a consistent estimate of $l!f_X^{*l}(x)$, even when well defined because of continuity (see [15] for instance). However we can hope consistency results for

$$I_{\phi,n}^{(l)} := \frac{1}{2\pi}\int_{-\pi}^{+\pi} I_n^{(l)}(x)\phi(x)d\mu(x). \tag{13}$$

Here ϕ is a test function, which is real, even, integrable and has some smoothness properties to be stated later on. Such quantities $I_{\phi,n}^{(l)}$ are called *integrated periodograms*.

We have the following proposition, which gives in particular the asymptotic properties that are required for the proof of Breuer Major Theorem. We introduce $c_k(\phi) = \frac{1}{2\pi}\int_{\mathbb{T}}\phi(x)e^{-ikx}d\mu(x)$ the k-th Fourier coefficient of ϕ.

Proposition 2.4. *Assume that* $(r(k))_{k\in\mathbb{Z}} \in \ell^{l+1}(\mathbb{Z})$ *and* $\sum_{k\in\mathbb{Z}}|c_k(\phi)|^{l+1} < \infty$. *Then, as* n *tends to* ∞,

(i) $\mathbb{E}(I_{\phi,n}^{(l)})$ *tends to* $\frac{l!}{2\pi}\int_{\mathbb{T}} f_X^{*l}(x)\phi(x)d\mu(x)$.

(ii) $I_{\phi,n}^{(l)} - \mathbb{E}(I_{\phi,n}^{(l)})$ *tends to* 0 *in* $L^2(\Omega,\mathbb{P})$.

Remark that conditions on r and ϕ imply that the integral $\frac{1}{2\pi}\int f_X^{*l}(x)\phi(x)d\mu(x)$ may be given meaning as $\sum_{k\in\mathbb{Z}} r(k)^l\, c_k(\phi)$, using the absolute convergence of the series. When $\phi = f_X$, as in the proof of Breuer Major Theorem, the limit of the expectation is $l!\sum_{k\in\mathbb{Z}} r(k)^{l+1} = \sigma_{l+1}^2/(l+1)$.

Proof. The first assertion follows from the fact that

$$\mathbb{E}(I_{\phi,n}^{(l)}) = l!\sum_{-n+1}^{n-1}\left(1-\frac{|k|}{n}\right)r(k)^l\, c_k(\phi).$$

Next, in view of the second assertion, we consider the components of $\|DY_n\|_2^2$ in the Wiener chaos and use for this the multiplication formula (see [16] for instance), which we recall now:

$$H_l(X)(k)H_l(X)(k') = \sum_{p=0}^{l} p!(2(l-p))!\begin{pmatrix} l \\ p\end{pmatrix}^2 I_{2l-2p}(g_k^{\odot l}\otimes_p g_{k'}^{\odot l}),$$

with

$$g_k^{\odot l}\otimes_p g_{k'}^{\odot l} = \langle g_k, g_{k'}\rangle_\mu^p \left(g_k^{\odot l-p}\otimes g_{k'}^{\odot l-p}\right)_s.$$

Here, as usual, if $\psi \in L_e^2(\mathbb{T}^k)$, we note $(\psi)_s$ its symmetrization in $L_s^2(\mathbb{T}^k)$, for $k \geq 2$. For simplification, we note $s(k) := c_k(\phi)$. Then we have

$$I_{\phi,n} - \mathbb{E}(I_{\phi,n}) = \sum_{p=0}^{l-1} p!(2(l-p))!\begin{pmatrix} l \\ p\end{pmatrix}^2 U_{p,n},$$

with

$$\begin{aligned} U_{p,n} &= \frac{1}{n}\sum_{k,k'=0}^{n-1} s(k-k') I_{2l-2p}(g_k^{\odot l} \otimes_p g_{k'}^{\odot l}) \\ &= \frac{1}{n}\sum_{k,k'=0}^{n-1} s(k-k') r(k-k')^p I_{2l-2p}((g_k^{\odot l-p} \otimes g_{k'}^{\odot l-p})_s). \end{aligned}$$

Using orthogonality between components, it is sufficient to consider each of them separately. The next lemma gives the convergence in $L^2(\Omega,\mathbb{P})$ of each term.

Lemma 2.5. *Assume that* $\sum_{k\in\mathbb{Z}} |r(k)|^{l+1} < \infty$ *and* $\sum_{k\in\mathbb{Z}} |s(k)|^{l+1} < \infty$. *Let* $p < l$. *Then* $\mathbb{E}(|U_{p,n}|^2)$ *tends to* 0 *for* n *tending to* ∞.

Proof. We can write $U_{p,n}$ as $I_{2(l-p)}(F_{p,n})$, with

$$F_{p,n} := \frac{1}{n}\sum_{k,k'=0}^{n-1} s(k-k') r(k-k')^p (g_k^{\odot l-p} \otimes g_{k'}^{\odot l-p})_s.$$

By isometry, the $L^2(\Omega,\mathbb{P})$ norm of $U_{p,n}$ is equal, up to the constant $(2(l-p))!^{1/2}$, to the $L^2_s\left(\mathbb{T}^{2(l-p)},\right)$ norm of $F_{p,n}$. Now $F_{p,n}$ may be written as the mean of $(2(l-p))!$ terms, corresponding to permutations of the $2(l-p)$ variables. It is easy to see that all terms have the same norm, so that the $L^2_s\left(\mathbb{T}^{2(l-p)}\right)$ norm of $F_{p,n}$ is bounded by the $L^2_e\left(\mathbb{T}^{2(l-p)}\right)$ norm of one of the terms, that is

$$\left\| \frac{1}{n}\sum_{k,k'=0}^{n-1} s(k-k') r(k-k')^p g_k^{\odot l-p} \otimes g_{k'}^{\odot l-p} \right\|_2 .$$

Finally, $\mathbb{E}(|U_{p,n}|^2)$ is bounded by

$$\frac{(2(l-p))!}{(2\pi)^{2l-2p}} \int_{(-\pi,+\pi)^{l-p}\times(-\pi,+\pi)^{l-p}} K_{p,n}(x,y) f_X(x_1)\cdots f_X(x_{l-p}) f_X(y_1)\cdots f_X(y_{l-p}) d\mu^{l-p}(x) d\mu^{l-p}(y),$$

with

$$\begin{aligned} K_{p,n}(x,y) &:= \left| \frac{1}{n}\sum_{k,k'=0}^{n-1} r(k-k')^p s(k-k') e^{-ik(x_1+\cdots+x_{l-p})} e^{ik'(y_1+\cdots+y_{l-p})} \right|^2 \\ &= \frac{1}{n^2}\sum_{j,j',k,k'=0}^{n-1} r(k-k')^p s(k-k') r(j-j')^p s(j-j') e^{-i(k-j)(x_1+\cdots+x_{l-p})} e^{i(k'-j')(y_1+\cdots+y_{l-p})}. \end{aligned}$$

We pose $\rho_1(k) := |r(k)|^p |s(k)|$ and $\rho_2(k) = |r(k)|^{l-p}$, for $k \in \mathbb{Z}$ and denote by $\rho_{1,n}(k)$, respectively $\rho_{2,n}(k)$, the truncated series $\rho_{1,n}(k) = \rho_1(k)$, respectively $\rho_{2,n}(k) = \rho_2(k)$, when $|k| \le n-1$ and 0 otherwise. Assumption on r implies that $\rho_2 \in \ell^{\frac{l+1}{l-p}}(\mathbb{Z})$, while together with assumption on s and Hölder inequality imply that $\rho_1 \in \ell^{\frac{l+1}{p+1}}(\mathbb{Z})$, with $\|\rho_1\|_{\ell^{\frac{l+1}{p+1}}(\mathbb{Z})} \le \|r\|^p_{\ell^{l+1}(\mathbb{Z})} \|s\|_{\ell^{l+1}(\mathbb{Z})}$. Therefore the convolution product of ρ_1 and ρ_2 is well defined and is uniformly bounded:

$$\rho_1 * \rho_2(k) = \sum_{k'\in\mathbb{Z}} \rho_1(k-k')\rho_2(k') \le \|\rho_1\|_{\ell^{\frac{l+1}{p+1}}(\mathbb{Z})} \|\rho_2\|_{\ell^{\frac{l+1}{l-p}}(\mathbb{Z})}.$$

Then, the same bound holds for $\rho_{1,n} * \rho_{2,n}(k)$ and

$$\begin{aligned}\mathbb{E}\left(|U_{p,n}|^2\right) &\leq \frac{(2(l-p))!}{n^2} \sum_{j,j',k,k'=0}^{n-1} \rho_{1,n}(k-k')\rho_{1,n}(j-j')\rho_{2,n}(k-j)\rho_{2,n}(j'-k') \\ &\leq \frac{(2(l-p))!}{n^2} \sum_{k,j'=0}^{n-1} \left(\rho_{1,n} * \rho_{2,n}(k-j')\right)^2.\end{aligned}$$

It follows that

$$\mathbb{E}\left(|U_{p,n}|^2\right) \leq \frac{(2(l-p))!}{n} \sum_{|j|\leq n-1} \left(1 - \frac{|j|}{n}\right) \left(\rho_{1,n} * \rho_{2,n}(j)\right)^2,$$

such that $\mathbb{E}\left(|U_{p,n}|^2\right)$ is uniformly bounded with

$$\mathbb{E}(|U_{p,n}|^2) \leq (2(l-p))! \|r\|^{2l}_{\ell^{l+1}(\mathbb{Z})} \|s\|^2_{\ell^{l+1}(\mathbb{Z})}.$$

Let us now prove that $\mathbb{E}(|U_{p,n}|^2)$ tends to 0. We will use a density argument. For $\widetilde{s}$ a sequence with finite support, the quantity

$$\frac{1}{n^2}\mathbb{E}\left| \sum_{k,k'=0}^{n-1} \widetilde{s}(k-k') I_{2l-2p}(g_k^{\odot_{l-1}} \otimes_p g_{k'}^{\odot_{l-1}})(k') \right|^2$$

tends to 0. To prove the same with s in place of $\widetilde{s}$, for a given $\varepsilon > 0$ we write s as the sum of some $\widetilde{s}$ with finite support such that $\sum_{k\in\mathbb{Z}} |\widetilde{s}(k) - s(k)|^l < \varepsilon$. We conclude by a standard argument. □

We have completed the proof of Proposition 2.4, and in the same time the proof of Breuer Major Theorem under the assumption that the spectral measure has a density. □

This proposition on periodograms seems new. Actually Part (i) proves the asymptotic unbiasedness of the estimator $I^{(l)}_{\phi,n}$, while Part (ii) implies its consistency.

Remark 2.6. *If we are only interested in asymptotic unbiasedness, continuity of the function $f^{*l} * \phi$ at 0 is sufficient, see* [14]. *Remark that the assumptions given here imply that its Fourier series is absolutely convergent. Note also that consistency is proved through asymptotic normality under stronger assumptions of integrability in* [14]. *This proposition may also be compared with* [2], *where Central Limit Theorems are developed for integrated periodograms when the test functions are in the Sobolev space $\mathcal{H}^\alpha$ for $\alpha > 1/2$.*

Recall that $\mathcal{H}^\alpha := \mathcal{H}^\alpha(\mathbb{T})$ is the space of functions $\psi \in L^2(\mathbb{T})$ such that

$$\sum_{k\in\mathbb{Z}} |c_k(\psi)|^2 (1+|k|)^{2\alpha} < \infty.$$

We now give a bound for the speed of convergence in Proposition 2.4 when ϕ is a test function that satisfies a condition of Sobolev type. More precisely, we have the following proposition.

Proposition 2.7. *Assume that $r \in \ell^{l+1}(\mathbb{Z})$ and $\sum_{k\in\mathbb{Z}} |c_k(\phi)|^{l+1}(1+|k|)^{\alpha(l+1)} < \infty$, for some $\alpha > 0$. Then, for some constant C_α and for all $n \geq 1$, we have*

$$\mathrm{Var}\left(I^{(l)}_{\phi,n}\right) \leq C_\alpha \begin{cases} \max(n^{-1}, n^{-2\alpha}) & \text{if } \alpha \neq \frac{1}{2} \\ n^{-1}\log(n) & \text{if } \alpha = \frac{1}{2} \end{cases}$$

Proof. Going back to the last proof and its notations, it is sufficient to prove that

$$\sum_{|j|\leq n-1} (\rho_{1,n} * \rho_{2,n}(j))^2 \leq C \max(1, n^{1-2\alpha}).$$

We will only consider the case $p = 0$ and leave the reader see that the proof is the same for the other terms. In this case, $\rho_{2,n}$ is uniformly in $\ell^{\frac{l+1}{l}}(\mathbb{Z})$. By Hausdorff-Young Inequality, one has the

inclusion $\ell^{\frac{l+1}{l}}(\mathbb{Z}) * \ell^q(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$ when $\frac{1}{q} = \frac{1}{2} + \frac{1}{l+1}$, with the corresponding norm inequality. So it is sufficient to compute the norm of $\rho_{1,n}$ in $\ell^q(\mathbb{Z})$, which is elementary by Hölder Inequality. For this, we use the fact that

$$\sum_{|k|\leq n} (1+|k|)^{-2\alpha} \leq C_\alpha \max(n^{1-2\alpha}, 1) \text{ when } \alpha \neq \frac{1}{2} \text{ and } \sum_{|k|\leq n} (1+|k|)^{-1} \leq C_{\frac{1}{2}} \log(n).$$

□

This kind of proof can be generalized to other assumptions on data. We give now one computation that leads to a bound for the speed of convergence in Breuer major Theorem.

Proposition 2.8. *Assume that* $\sum_{k\in\mathbb{Z}} |r(k)|^{l+1}(1+|k|)^{\alpha(l+1)} < \infty$ *for some* $\alpha > 0$. *Then, when* $l = 1$, *for some* $C_\alpha > 0$ *and for all* $n \geq 1$ *we have the uniform estimate*

$$\mathrm{Var}\left(I^{(1)}_{f_X,n}\right) \leq C_\alpha \max(n^{-1}, n^{-4\alpha}).$$

For $l \geq 2$, *for some* $C_\alpha > 0$ *and for all* $n \geq 1$ *we have*

$$\mathrm{Var}\left(I^{(l)}_{f_X,n}\right) \leq C_\alpha \begin{cases} n^{-2\alpha(l+1)} & : \quad \alpha < \frac{1}{l(l+1)} \\ n^{-2\alpha-\frac{2}{l+1}} & : \quad \frac{1}{l(l+1)} < \alpha < \frac{1}{2} - \frac{1}{l+1} \\ n^{-1} & : \quad \alpha > \frac{1}{2} - \frac{1}{l+1} \end{cases}.$$

Proof. Again, we go back to the previous notations and estimate $\sum_{|j|\leq n-1} (\rho_{1,n} * \rho_{2,n}(j))^2$. Let us first consider $l = 1$. The only case to consider is $p = 0$, and we want to prove the estimate

$$\sum_{|j|\leq n-1} (\rho_{1,n} * \rho_{2,n}(j))^2 \leq C \max(n^{1/2-2\alpha}, n^{1/2}).$$

Here $\rho_{1,n} = \rho_{2,n}$ coincides with $|r|$ for $|k| \leq n-1$. Assume first that $\alpha < 1/4$. It follows from Hölder inequality that $\|\rho_{1,n}\|_{\ell^{4/3}(\mathbb{Z})} \leq C_\alpha n^{1/4-\alpha}$. Now the convolution of two sequences in $\ell^{4/3}(\mathbb{Z})$ is in $\ell^2(\mathbb{Z})$, which allows to conclude in this case. For $\alpha > 1/4$, the sequence ρ_1 is in $\ell^{4/3}(\mathbb{Z})$ and we conclude in the same way.
It remains to conclude for $\alpha = 1/4$. We want to prove that $\sum_{|j|\leq n-1} (\rho_{1,n} * \rho_{2,n}(j))^2$ is uniformly bounded under the assumption that $\sum_{k\in\mathbb{Z}} |r(k)|^2(1+|k|)^{1/2} < \infty$. Let h_n be the trigonometric polynomial with $\rho_{1,n}$ as Fourier coefficients. Then $\rho_{1,n} * \rho_{2,n} = \rho_{1,n} * \rho_{1,n}$ are the Fourier coefficients of the function h_n^2. The function h_n is uniformly in the Sobolev space $\mathcal{H}^{1/4}$. Now it follows from Sobolev Theorem (see [20] for instance) that such functions are uniformly in $L^4(\mathbb{T})$. By Plancherel Identity,

$$\sum_{j\in\mathbb{Z}} (\rho_{1,n} * \rho_{1,n}(j))^2 = \frac{1}{2\pi} \int_{\mathbb{T}} |h_n(x)|^4 d\mu(x) \leq C.$$

Let us now consider $l \geq 2$ and estimate again the norm of $\rho_{1,n} * \rho_{2,n}$ in $\ell^2(\mathbb{Z})$. The worst case is obtained for $p = 0$. Then $\rho_{1,n}$ coincides with $|r|$ while ρ_2 is equal to $|r|^l$. Then

$$\|\rho_{1,n} * \rho_{2,n}\|_{\ell^2(\mathbb{Z})} \leq \|\rho_{1,n}\|_{\ell^2(\mathbb{Z})} \|\rho_{2,n}\|_{\ell^1(\mathbb{Z})}.$$

The first estimate is obtained by taking the norm of $\rho_{1,n}$ in $\ell^2(\mathbb{Z})$ and the norm of $\rho_{2,n}$ in $\ell^1(\mathbb{Z})$, as long as this last one is not uniformly bounded. For larger values of α, ρ_2 in $\ell^1(\mathbb{Z})$ and the bound is given by the the norm of $\rho_{1,n}$ in $\ell^2(\mathbb{Z})$, as long as this last one is not uniformly bounded. □

2.4. **Variable spectral densities.** In practice, the spectral density may change at each step of computation of the mean. This is what happens for instance when we look at increments of a Gaussian process at different scales. It is important to have still Central Limit Theorems in this context, as well as methods to compute the speed of convergence. Let us first state a CLT in this context.

Theorem 2.9. *Let $X_n = (X_n(k))_{k\in\mathbb{Z}}$ be centered stationary Gaussian time series. Let $l \geq 2$. We assume that there exists a sequence (g_n) of complex functions in $L^{\frac{2l}{l-1}}(\mathbb{T}) \cap L^2_e(\mathbb{T})$ that converges in this space to a function g and such that $f_{X_n} = |g_n|^2$. We call $r(k) := \frac{1}{2\pi}\int_{\mathbb{T}} e^{-ikx}|g(x)|^2 d\mu(x)$ and assume that $r(0) = 1$. Then we have the following asymptotic properties for n tending to infinity:*

(i)
$$\mathrm{Var}\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_l(X_n)(k)\right) \longrightarrow \sigma_l^2,$$

(ii)
$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_l(X_n)(k) \xrightarrow{d} \mathcal{N}(0,\sigma_l^2),$$

with

$$\sigma_l^2 = l!\sum_{k\in\mathbb{Z}} r(k)^l. \tag{14}$$

Proof. Remark that we can always assume that

$$X_n(k) = \int_{-\pi}^{+\pi} e^{-ikx} g_n(x) dW(x).$$

This is a particular case of the following situation. We consider triangular arrays that are obtained in the following way. We assume that we are given random centered Gaussian variables $X_n(k)$ for $k = 0, \cdots, n-1$. We assume that $\mathrm{Cov}(X_n(j), X_n(j+k)) = r_n(k)$.

Proposition 2.10. *Let $l \geq 1$. Assume that we are given such a triangular array and that, moreover, there exists a stationary centered Gaussian sequence X with given covariance $r \in \ell^l(\mathbb{Z})$ such that $r(0) = 1$, defined on the same probability space than X_n with $\mathrm{Cov}(X_n(j), X(j')) = h_n(j-j')$ and the two following assumptions are satisfied:*

(a) $\sum_{|k|<n-1} |r_n(k) - r(k)|^l$ *tends to 0 for n tending to infinity.*

(b) $\sum_{|k|<n-1} |h_n(k) - r(k)|^l$ *tends to 0 for n tending to infinity.*

Then, the conclusion of Theorem 2.9 holds true.

Proof. If we define Y_n as before, we can apply Breuer Major Theorem to Y_n. Let us define

$$Z_n := \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_l(X_n)(k).$$

We will prove that $\|Y_n - Z_n\|_{L^2(\Omega),\mathbb{P}} \to 0$, as $n \to +\infty$, from which we can conclude directly according to Slutsky's Theorem. Since Y_n and Z_n are centered variables, $\|Y_n - Z_n\|^2_{L^2(\Omega,\mathbb{P})} = \mathrm{Var}(Y_n - Z_n)$, with

$$\mathrm{Var}(Y_n - Z_n) = \mathrm{Var}(Y_n) + \mathrm{Var}(Z_n) - 2\mathrm{Cov}(Y_n, Z_n).$$

Similar computations as for Y_n imply that

$$\mathrm{Var}(Z_n) = l!\sum_{k=-(n-1)}^{n-1}\left(1 - \frac{|k|}{n}\right) r_n(k)^l.$$

It follows from Assumption (a) that $\mathrm{Var}(Y_n)$ and $\mathrm{Var}(Z_n)$ have the same limit.
Let us now consider the covariance,

$$\begin{aligned}\mathrm{Cov}(Y_n,Z_n) &= \frac{1}{n}\sum_{k=0}^{n-1}\sum_{k'=0}^{n-1}\mathrm{Cov}(H_l(X_n)(k),H_l(X)(k')\\ &= \frac{l!}{n}\sum_{k=0}^{n-1}\sum_{k'=0}^{n-1}h_n(k-k')^l\\ &= l!\sum_{k=-(n-1)}^{n-1}\left(1-\frac{|k|}{n}\right)h_n(k)^l.\end{aligned}$$

Again $\mathrm{Cov}(Y_n,Z_n)$ and $\mathrm{Var}(Y_n)$ have the same limit from Assumption (b). □

Let us finish the proof of Theorem 2.9 for $l\geq 2$. It is sufficient to remark that convergence of g_n to g in $L^{\frac{2l}{l-1}}(\mathbb{T})$ implies convergence of $|g_n|^2$ to $|g|^2$ in $L^{\frac{l}{l-1}}(\mathbb{T})$, which, in turn, by the theorem of Hausdorff-Young, implies the convergence of the sequence of Fourier coefficients in $\ell^l(\mathbb{Z})$. We conclude for the convergence of r_n to r. We have also the convergence of $g_n\overline{g}$ to $|g|^2$ in $L^{\frac{l}{l-1}}(\mathbb{T})$. Since the Fourier coefficients of $g_n\overline{g}$ give the sequence h_n, this implies its convergence to r. □

2.5. **Speed of convergence in Breuer Major Central Limit Theorem.** We are now able to bound the speed of convergence in Theorem 2.1 under the assumption that $\sum|r(k)|^l(1+|k|)^{l\alpha}$ is finite for some $\alpha>0$, as well as in Theorem 2.9. We recall that the distance of Kolmogorov between the random variables Y and Z is defined as

$$d_{Kol}(Y,Z)=\sup_{z\in\mathbb{R}}|P(Y<z)-P(Z<z)|. \tag{15}$$

We will be particularly interested by the distance of Kolomogorov to some normal random variable σN, where $N\sim\mathcal{N}(0,1)$. We recall that (see [22] for instance), for Z a centered random variable with variance 1 in the l-th Wiener chaos,

$$d_{Kol}(Z,N)\leq\sqrt{\left(\mathrm{Var}\left(\frac{1}{l}\|DZ\|_2^2\right)\right)}. \tag{16}$$

The following lemma will allow us to compute the required Kolmogorov distances.

Lemma 2.11. *For Y and Z two centered random variables in the l-th Wiener chaos, for all $\epsilon>0$ we have the inequality*

$$d_{Kol}(Y,Z)\leq C_\epsilon(\mathrm{Var}(Y-Z))^{\frac{1}{2}-\epsilon}+2d_{Kol}(Z,N).$$

Proof. We have

$$|P(Y<z)-P(Z<z)|\leq\max\{P(Y<z,Z\geq z),P(Y\geq z,Z<z)\}.$$

So let us give a bound for $P(Y<z,Z\geq z)$, the other term being treated by the same method. We have, for all $m\geq 1$, and for $\eta>0$ to choose later on,

$$\begin{aligned}P(Y<z,Z\geq z) &\leq P(z\leq Z<z+\eta)+\frac{\mathbb{E}(|Y-Z|^{2m})}{\eta^{2m}}\\ &\leq 2d_{Kol}(Z,N)+P(z\leq N<z+\eta)+C_m\frac{\mathrm{Var}(Y-Z)^m}{\eta^{2m}}\\ &\leq 2d_{Kol}(Z,N)+\eta+C_m\frac{\mathrm{Var}(Y-Z)^m}{\eta^{2m}}.\end{aligned}$$

We have used the fact that all $L^p(\Omega,\mathbb{P})$ norms are equivalent in a Wiener chaos. We choose $m=\epsilon^{-1}-\frac{1}{2}$. We conclude by minimizing the right hand side as a function of η. □

We do not know whether this inequality is far from sharp or not.

In the particular case when Z is itself a Gaussian random variable, we have a better estimate. This leads to the following lemma.

Lemma 2.12. *For Y a centered random variable in the l-th Wiener chaos we have the inequality*

$$d_{Kol}(Y,\sigma N) \leq \frac{2}{\sigma^2}|\mathrm{Var}(Y)) - \sigma^2| + \sqrt{\left(\mathrm{Var}\left(\frac{1}{l\sigma^2}\|DY\|_2^2\right)\right)}.$$

Proof. When $\frac{|\mathrm{Var}(Y)-\sigma^2|}{\sigma^2} > \frac{1}{2}$ there is nothing to prove. Otherwise we write

$$d_{Kol}(Y,\sigma N) \leq d_{Kol}(Y,\sqrt{\mathrm{Var}(Y)}N) + d_{Kol}(\sqrt{\mathrm{Var}(Y)}N,\sigma N).$$

We use the Malliavin derivative for the first distance, then a direct computation of the distance between N and a multiple of N. More precisely, for example for $z > 0$ and $\sigma > 1$, one has the inequality $P(z \leq N \leq \sigma z) \leq (\sigma-1)ze^{-z^2/2} \leq \sigma - 1$. □

We can now state the first theorem of this subsection, which gives the speed of convergence in Breuer Major Theorem.

Theorem 2.13. *Let $(X(k))_{k\in\mathbb{Z}}$ be a centered stationary Gaussian time series with an absolutely continuous spectral measure. Assume that r satisfies $r(0) = 1$ and the assumption*

$$\sum_{k\in\mathbb{Z}} |r(k)|^l (1+|k|)^{l\alpha} < \infty \tag{17}$$

for some $\alpha > 0$. Then, for $l = 2$, for some constant $C_\alpha > 0$ and for all $n \geq 1$,

$$d_{Kol}\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_2(X)(k), \sigma_2 N\right) \leq C_\alpha \max(n^{-2\alpha}, n^{-1/2}), \tag{18}$$

while, for $l \geq 3$, for some constant $C_\alpha > 0$ and for all $n \geq 1$,

$$d_{Kol}\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_l(X)(k), \sigma_l N\right) \leq C_\alpha \begin{cases} n^{-\alpha l} & : \quad \alpha < \frac{1}{l(l-1)} \\ n^{-\alpha-\frac{1}{l}} & : \quad \frac{1}{l(l-1)} < \alpha < \frac{1}{2} - \frac{1}{l} \\ n^{-\frac{1}{2}} & : \quad \alpha > \frac{1}{2} - \frac{1}{l} \end{cases}. \tag{19}$$

with $\sigma_l^2 = l! \sum_{k\in\mathbb{Z}} r(k)^l$.

Proof. Let us first prove that

$$|\sigma_l^2 - \mathrm{Var}(Y_n)| \leq C \max(n^{-\alpha l}, n^{-1}). \tag{20}$$

From the expression of $\mathrm{Var}(Y_n)$ given above we deduce that

$$l!^{-1}\left|\sigma_l^2 - \mathrm{Var}(Y_n)\right| \leq \frac{1}{n}\sum_{k=-(n-1)}^{n-1} |k||r(k)|^l + \sum_{|k|\geq n} |r(k)|^l.$$

We conclude directly, by Hölder Inequality, using the fact that $\sum_{|k|\geq n} |r(k)|^l (1+|k|)^{l\alpha} < \infty$. Now the required estimate for the Malliavin derivative is given by Proposition 2.8. □

This bound for the speed of convergence can be compared to the ones given in [23], which it improves. In particular, the speed of convergence for the fractional Brownian Noise given in Example 2.7 in [23], which one can obtain through the same method as above, can be improved to the following bounds.

Remark 2.14. *Assume that $r(k) = O(|k|^{-a})$. Then, for $a > \frac{1}{2}$, $l = 2$ and $a \neq \frac{3}{4}$,*

$$d_{Kol}\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_2(X)(k), \sigma_2 N\right) \leq C \max(n^{1-2a}, n^{-1/2}), \tag{21}$$

while, for $l \geq 2$ and $a > \frac{1}{l}$

$$d_{Kol}\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_l(X)(k), \sigma_l N\right) \leq C \begin{cases} n^{-la+1} & : \quad a < \frac{1}{l-1} \\ n^{-a} & : \quad \frac{1}{l-1} < a < \frac{1}{2} \\ n^{-\frac{1}{2}} & : \quad a > \frac{1}{2} \end{cases} . \tag{22}$$

It may be seen that these estimates are the best possible that one can obtain by the Malliavin derivative. Remark that for $l = 2$ and $a = 3/4$, explicit computations can be done in terms of the Riesz potential, which show that one cannot have the rate $n^{-1/2}$. For $a = 1/2$, one can also improve the logarithmic rate given in [23].

Next we give the speed of convergence in Theorem 2.9, which depends on the speed of convergence of g_n to g.

Theorem 2.15. *Let $X_n = (X_n(k))_{k\in\mathbb{Z}}$ be centered stationary Gaussian time series, which can be written as*

$$X_n(k) = \int_{-\pi}^{+\pi} e^{-ikx} g_n(x) dW(x),$$

and satisfy the assumptions of Theorem 2.9 with $r(k) := \frac{1}{2\pi}\int_{\mathbb{T}} e^{-ikx}|g(x)|^2 d\mu(x)$. We assume moreover that the two following properties are satisfied.

$$\sum_{k\in\mathbb{Z}} |r(k)|^l (1+|k|)^{\alpha l} < \infty. \tag{23}$$

$$\|g_n - g\|_{\frac{2l}{l-1}} \leq C n^{-\beta}. \tag{24}$$

Then, for $\epsilon > 0$, for $l = 2$, for some constant C_α and for all $n \geq 1$, we have

$$d_{Kol}\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_2(X_n)(k), \sigma_2 N\right) \leq C_\alpha \max(n^{-\beta/2+\epsilon}, n^{-2\alpha}, n^{-1/2}), \tag{25}$$

while, for $l \geq 3$, for some constant C_α and for all $n \geq 1$, we have

$$d_{Kol}\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_l(X_n)(k), \sigma_l N\right) \leq C_\alpha \begin{cases} \max(n^{-\beta/2+\epsilon}, n^{-\alpha l}) & : \quad \alpha < \frac{1}{l(l-1)} \\ \max(n^{-\beta/2+\epsilon}, n^{-\alpha-\frac{1}{l}}) & : \quad \frac{1}{l(l-1)} < \alpha < \frac{1}{2} - \frac{1}{l} \\ \max(n^{-\beta/2+\epsilon}, n^{-\frac{1}{2}}) & : \quad \alpha > \frac{1}{2} - \frac{1}{l} \end{cases} . \tag{26}$$

Proof. We go back to the notations used in the proof of Proposition 2.10. From the last theorem we already have a bound for the speed of convergence of Y_n. We will use Lemma 2.11 to obtain a bound for the speed of convergence of Z_n. We want to have the speed of convergence of $\mathrm{Var}(Y_n - Z_n)$ to 0, or, which is equivalent, the speed of convergence of $\mathrm{Var}(Z_n) - \mathrm{Var}(Y_n)$ and $\mathrm{Cov}(Y_n, Z_n) - \mathrm{Var}(Y_n)$ to 0, that is, the speed of convergence of $|\sum |r_n(k)|^l - \sum |r(k)|^l|$ on one side, $|\sum |h_n(k)|^l - \sum |r(k)|^l|$ on the other one, with $h_n(k) = \mathrm{Cov}(X_n(k), X(0)) = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-ikx} g_n(x)\overline{g(x)} d\mu(x)$. The first one is bounded (up to some constant) by $\|r_n - r\|_{\ell^l(\mathbb{Z})} \leq \||g_n|^2 - |g|^2\|_{\frac{l}{l-1}}$, by Hausdorff-Young Inequality. Now this last quantity is bounded (up to some constant) by $\|g_n - g\|_{\frac{2l}{l-1}}$. We do the same with h_n. □

We do not know whether one can get rid of the ϵ, or improve this bound of the speed of convergence.

3. Vector-valued central limit theorem and generalizations

3.1. **Vector-valued central limit theorem.** We now describe a very useful expansion of Theorem 2.9 to the vectorial case. In our setting of multiple Itô-Wiener integrals, our main tool is [27] who proves that vectorial Central Limit Theorems follow from Central Limit Theorems for marginals and convergence of covariance matrix. For this purpose let us introduce several notations. Let $d \geq 2$. Let us consider $\overrightarrow{g_n} = (g_{1,n}, \ldots, g_{d,n})$ for $n \in \mathbb{N}$ and $\overrightarrow{g} = (g_1, \ldots, g_d)$ such that all marginals are complex

valued functions in $L^2_e(\mathbb{T})$ and normalized so that $\frac{1}{2\pi}\int_{\mathbb{R}}|g_j(x)|^2 d\mu(x) = 1$, for all $1 \leq j \leq d$. Then one can consider the vector-valued centered stationary Gaussian time series defined by

$$\overrightarrow{X}(k) = \int_{-\pi}^{+\pi} e^{-ikx}\overrightarrow{g}(x)dW(x) \text{ and } \overrightarrow{X_n}(k) = \int_{-\pi}^{+\pi} e^{-ikx}\overrightarrow{g_n}(x)dW(x),\ k \in \mathbb{Z}.$$

Let us recall that $\overrightarrow{X}(k) = (X_1(k), \ldots, X_d(k))$ with $X_j(k) = I_1(g_{j,k})$, writing $g_{j,k}(x) = e^{-ikx}g_j(x)$, and consider the stationary vector-valued processes

$$\overrightarrow{H_l(X)}(k) = (H_l(X_1)(k), \ldots, H_l(X_d)(k)) = \left(I_l\left(g_{1,k}^{\odot_l}\right), \ldots, I_l\left(g_{d,k}^{\odot_l}\right)\right),\ k \in \mathbb{Z},\ l \geq 1.$$

Similarly we define $\overrightarrow{H_l(X_n)}$, replacing $g_{j,k}$ by $g_{j,n,k}(x) = e^{-ikx}g_{j,n}(x)$. Then we write

$$r_{i,j}(k) = \mathrm{Cov}(X_i(k), X_j(0)) = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-ikx} g_i(x)\overline{g_j(x)}d\mu(x).$$

Theorem 3.1. *Let $l \geq 2$. We assume that $\overrightarrow{g_n}$ is a sequence of vectorial complex functions in $L^{\frac{2l}{l-1}}(\mathbb{T}) \cap L^2_e(\mathbb{T})$ that converges in this space to the function $\overrightarrow{g}$. Then, for n tending to infinity,*

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\overrightarrow{H_l(X_n)(k)} \xrightarrow{d} \mathcal{N}(0, \Sigma_l),$$

with $(\Sigma_l)_{i,j} = l!\sum_{k\in\mathbb{Z}} r_{i,j}(k)^l$.

Proof. Following the proof of Theorem 2.9 we introduce

$$\overrightarrow{Z_n} = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\overrightarrow{H_l(X_n)}(k) \text{ and } \overrightarrow{Y_n} = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\overrightarrow{H_l(X)}(k).$$

Assumptions ensure that $\overrightarrow{Z_n} - \overrightarrow{Y_n} \to 0$, as $n \to +\infty$, in $L^2(\Omega, \mathbb{P})$ so that, using Slutsky's Theorem as in the proof of Proposition 2.10, it is sufficient to prove that $\overrightarrow{Y_n} \xrightarrow{d} \mathcal{N}(0, \Sigma_l)$. First, let us denote by $\Sigma_{n,l}$ the covariance matrix of this vector and remark that, for $1 \leq i, j \leq d$, $(\Sigma_{n,l})_{i,j}$ is given by

$$\mathrm{Cov}(Y_{n,i}, Y_{n,j}) = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{k'=0}^{n-1}\mathbb{E}(I_l(g_{n,i,k}^{\odot_l})I_l(g_{n,j,k'}^{\odot_l})) = \frac{l!}{n}\sum_{k=0}^{n-1}\sum_{k'=0}^{n-1}\langle g_{n,i,k}, g_{n,j,k'}\rangle_\mu^l,$$

with

$$\langle g_{n,i,k}, g_{n,j,k'}\rangle_\mu = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-i(k-k')x} g_{n,i}(x)\overline{g_{n,j}(x)}\mu(dx) := r_{n,i,j}(k-k').$$

Assumption on $\overline{g_n}$ implies that $g_{n,i}\overline{g_{n,j}} \in L^{\frac{l}{l-1}}(\mathbb{T})$ such that $r_{n,i,j} \in \ell^l(\mathbb{Z})$, according to Hausdorff-Young Inequality. Similarly $r_{i,j} \in \ell^l(\mathbb{Z})$. Moreover, up to a constant C that may change from a line to another one

$$\begin{aligned}\left|\sum_{|k|\leq n-1}\left(1-\frac{|k|}{n}\right)\left(r_{n,i,j}(k)^l - r_{i,j}(k)^l\right)\right| &\leq C\|r_{n,i,j} - r_{i,j}\|_{\ell^l(\mathbb{Z})}\\ &\leq C\|g_{n,i}\overline{g_{n,j}} - g_i\overline{g_j}\|_{\frac{l}{l-1}}\\ &\leq C\left(\|g_{n,i} - g_i\|_{\frac{2l}{l-1}} + \|g_{n,j} - g_j\|_{\frac{2l}{l-1}}\right).\end{aligned}$$

This proves that

$$\Sigma_{n,l} \underset{n\to+\infty}{\longrightarrow} \Sigma_l, \tag{27}$$

and therefore Σ_l is a covariance matrix such that Assumption 6 of Proposition 2 of [27] is satisfied. Moreover, we already know by Theorem 2.9 that the real random variables $Y_{n,j}$ converge in distribution to $\mathcal{N}(0, (\Sigma_l)_{j,j})$. According to Proposition 2 of [27], this is sufficient to conclude for the proof. □

Remark 3.1. *One can also have a bound for the speed of convergence as in Section 2, based on results of* [25, 24]*.One considers now the distance of Wasserstein*

$$d_W(\overrightarrow{Y},\overrightarrow{Z}) = \sup|\mathbb{E}(\Phi(\overrightarrow{Y}) - \mathbb{E}(\Phi(\overrightarrow{Z}))|,$$

where the supremum is taken on all Lipschitz functions with Lipschitz constant bounded by 1*, under the assumption that the matrix* Σ_l *is positive definite. When it is not the case, the function* Φ *is taken of class* $\mathcal{C}^2$*, with bounded second derivatives. Mutatis mutandis, the bounds obtained for the speed of convergence are the same as in the last section for* $\overrightarrow{Y_n}$*. For* $\overrightarrow{Z_n}$*, they rely on the speed of convergence of the functions* $\overrightarrow{g_n}$ *towards* $\overrightarrow{g}$*. One has to adapt Lemma 2.11 and 2.12 in this context. Lemma 2.11 can be replaced by the fact that, for* Φ *with Lipschitz constant bounded by* 1*, the distance of Wasserstein between* $\overrightarrow{Y}$ *and* $\overrightarrow{Z}$ *is bounded by* $\mathbb{E}(|\overrightarrow{Y} - \overrightarrow{Z}|) \le \mathbb{E}(|\overrightarrow{Y} - \overrightarrow{Z}|^2)^{\frac{1}{2}}$*. So Theorem 2.15 holds with obvious changes in the assumptions, with* $\beta/2 - \epsilon$ *replaced by* $\beta/2$ *in the conclusion.*

3.2. **Extension to stationary centered Gaussian fields.** Until now, we have chosen to restrict our study to stationary centered Gaussian processes, essentially for notational sake of simplicity. However, all previous results have their counterpart in the framework of Gaussian random fields that are indexed by $\mathbb{Z}^\nu$ for some integer $\nu \ge 2$ instead of $\mathbb{Z}$. The Hilbert space $L^2_e(\mathbb{T})$ is replaced by $L^2_e(\mathbb{T}^\nu)$ such that when $g \in L^2_e(\mathbb{T}^\nu)$, one can define a stationary random field by the stochastic integral

$$X(k) = \int_{[-\pi,\pi)^\nu} e^{-ik\cdot x} g(x) dW^\nu(x),\ k \in \mathbb{Z}^\nu,$$

where $k \cdot x$ is the Euclidean scalar product of k and x in $\mathbb{R}^\nu$ and W^ν is a complex centered Gaussian measure on $[-\pi,\pi)^\nu$ with Lebesgue control measure $\frac{1}{(2\pi)^\nu}\mu^\nu$, satisfying $W^\nu(-A) = \overline{W^\nu(A)}$ a.s. for any Borel set A of $[-\pi,\pi)^\nu$. By isometry the covariance function of such a field is still given by the Fourier coefficients of the function $|g|^2 \in L^1(\mathbb{T}^\nu)$ defined by

$$r(k) = \frac{1}{(2\pi)^\nu}\int_{\mathbb{T}^\nu} e^{-ik\cdot x}|g(x)|^2 d\mu^\nu(x),\ k \in \mathbb{Z}^\nu.$$

Then Theorems 2.9 and 3.1 are generalized in the following setting.

Theorem 3.2. *Let* $\nu, d \ge 1$ *integers. Let* $\overrightarrow{X_n} = \left(\overrightarrow{X_n}(k)\right)_{k\in\mathbb{Z}^\nu}$ *be centered stationary Gaussian fields with values in* $\mathbb{R}^d$*, which can be written as*

$$\overrightarrow{X_n}(k) = \int_{[-\pi,\pi)^\nu} e^{-ik\cdot x}\overrightarrow{g_n}(x) dW^\nu(x).$$

We assume that $\overrightarrow{g_n}$ *is a sequence of vectorial complex functions in* $L^{\frac{2l}{l-1}}(\mathbb{T}^\nu) \cap L^2_e(\mathbb{T}^\nu)$ *that converges in this space to a function* $\overrightarrow{g} = (g_1, \dots, g_d)$*. We call* $r_j(k) := \frac{1}{(2\pi)^\nu}\int_{\mathbb{T}^\nu} e^{-ik\cdot x}|g_j(x)|^2 d\mu^\nu(x)$ *and assume that* $r_j(0) = 1$ *for* $1 \le j \le d$*. Then, for* n *tending to infinity,*

(i)

$$\mathrm{Var}\left(\frac{1}{n^{\nu/2}}\sum_{k_1,\dots,k_d=0}^{n-1} H_l(X_{n,j})(k)\right) \longrightarrow (\Sigma_l)_{jj},$$

(ii)

$$\frac{1}{n^{\nu/2}}\sum_{k_1,\dots,k_d=0}^{n-1} H_l(\overrightarrow{X_n})(k) \xrightarrow{d} \mathcal{N}(0,\Sigma_l),$$

with

$$(\Sigma_l)_{i,j} = l!\sum_{k\in\mathbb{Z}^\nu} r_{i,j}(k)^l \text{ where } r_{i,j}(k) = \frac{1}{(2\pi)^\nu}\int_{\mathbb{T}^\nu} e^{-ik\cdot x} g_i(x)\overline{g_j(x)} d\mu^\nu(x). \tag{28}$$

Remark 3.3. *One can also have a bound for the speed of convergence (written in terms of the distance of Wasserstein if $d > 1$) as in the last subsection, but with different exponents coming from the generalization of Proposition 2.8. The new bound is* $\max(n^{-\nu}, n^{-4\alpha})$ *when $l = 1$, due to the fact that the Sobolev space $\mathcal{H}^\alpha$ is contained in $L^4(\mathbb{T}^\nu)$ for $\alpha = \nu/4$. When $l \geq 2$, the new bound is*

$$\max(n^{-2(l+1)\alpha}, n^{-2\alpha-\frac{2\nu}{l+1}}, n^{-\nu}).$$

4. Application to generalized quadratic variations

In this section we consider a continuous time real-valued centered Gaussian field with stationary increments, defined through a spectral representation

$$Y(t) = \int_{\mathbb{R}^\nu} \left(e^{-it\cdot x} - 1\right) F(x)^{1/2} d\widetilde{W}^\nu(x), t \in \mathbb{R}^\nu, \tag{29}$$

where $\widetilde{W}^\nu$ is a complex centered Gaussian measure on $\mathbb{R}^\nu$ with Lebesgue control measure μ^ν, such that $\widetilde{W}^\nu(-A) = \overline{\widetilde{W}^\nu(A)}$ a.s. for any Borel set A of $\mathbb{R}^\nu$. We refer to the introduction for more notations and comments.

We assume that only $\{Y(k/n); k = (k_1, \ldots, k_\nu) \in \mathbb{Z}^\nu$ with $0 \leq k_1, \ldots, k_\nu \leq n-1\}$ are known for some large n. We now define generalized quadratic variations. We first define generalized increments. More precisely, our first step is to consider a stationary field induced by these observations and this is obtained through a filtering of this sequence. In particular we consider the discrete time stationary field

$$Z_{n,\mathbf{a}}(k) = \sum_{m_1,\ldots,m_\nu=0}^{p} a_1(m_1)\ldots a_\nu(m_\nu) Y\left(\frac{k_1+m_1}{n}, \ldots, \frac{k_\nu+m_\nu}{n}\right), \quad \text{for } k = (k_1, \ldots, k_\nu) \in \mathbb{Z}^\nu$$

and $\mathbf{a} = (\mathbf{a_1}, \ldots, \mathbf{a}_\nu)$ with $\mathbf{a_j} = (a_j(0), \ldots, a_j(p)) \in \mathbb{R}^{p+1}$ a discrete filter of length $p+1$ and of order K_j ($p, K_j \in \mathbb{N}$ with $p \geq K_j$), which means that $\sum_{m_j=0}^{p} a_j(m_j) \neq 0$ when $K_j = 0$ and otherwise

$$\sum_{m_j=0}^{p} a_j(m_j) m_j^r = 0 \quad \text{for } 0 \leq r \leq K_j - 1 \quad \text{and} \quad \sum_{m_j=0}^{p} a_j(m_j) m_j^{K_j} \neq 0.$$

For $\nu = 1$, examples are given

- the increments of Y: $Z_{n,\mathbf{a}}(k) = Y\left(\frac{k+1}{n}\right) - Y\left(\frac{k}{n}\right)$ for $\mathbf{a} = (-1, 1)$, which is a filter of order 1.
- the second order increments of Y: $Z_{n,\mathbf{a}}(k) = Y\left(\frac{k+2}{n}\right) - 2Y\left(\frac{k+1}{n}\right) + Y\left(\frac{k}{n}\right)$ for $\mathbf{a} = (1, -2, 1)$, which is a filter of order 2.

In dimension $\nu = 2$, following the works of Chan & Wood [9] and Zu & Stein [30] we can also consider the following types of increments:

- Vertical $Z_{n,\mathbf{a}}(k) = Y\left(\frac{k_1}{n}, \frac{k_2+2}{n}\right) - 2Y\left(\frac{k_1}{n}, \frac{k_2+1}{n}\right) + Y\left(\frac{k_1}{n}, \frac{k_2}{n}\right)$ for $\mathbf{a_1} = (1)$ filter of order 0 and $\mathbf{a_2} = (1, -2, 1)$ filter of order 2.
- Horizontal $Z_{n,\mathbf{a}}(k) = Y\left(\frac{k_1+2}{n}, \frac{k_2}{n}\right) - 2Y\left(\frac{k_1+1}{n}, \frac{k_2}{n}\right) + Y\left(\frac{k_1}{n}, \frac{k_2}{n}\right)$ for $\mathbf{a_1} = (1, -2, 1)$ filter of order 2 and $\mathbf{a_2} = (1)$ filter of order 0.
- Superficial $Z_{n,\mathbf{a}}(k) := \square^n_{k_1,k_2}(Y) = Y\left(\frac{k_1+1}{n}, \frac{k_2+1}{n}\right) - Y\left(\frac{k_1+1}{n}, \frac{k_2}{n}\right) - Y\left(\frac{k_1}{n}, \frac{k_2+1}{n}\right) + Y\left(\frac{k_1}{n}, \frac{k_2}{n}\right)$ for $\mathbf{a_1} = \mathbf{a_2} = (-1, 1)$ filter of order 1.

Coming back to the general case, let us associate to the filter $\mathbf{a_j}$ the real polynomial

$$P_{\mathbf{a_j}}(x_j) = \sum_{m_j=0}^{p} a_j(m_j) x_j^{m_j}, \text{ for } x_j \in \mathbb{R},$$

then $\mathbf{a_j}$ is a filter of order $K_j \geq 1$ if and only if $P^{(r)}_{\mathbf{a_j}}(1) = 0$, for $0 \leq r \leq K_j - 1$ and $P^{(K_j)}_{\mathbf{a_j}}(1) \neq 0$. By Taylor formula, this implies that there exists $c_j > 0$ such that

$$\left|P_{\mathbf{a_j}}\left(e^{-ix_j}\right)\right| \leq c_j \min\left(|x_j|^{K_j}, 1\right), x_j \in \mathbb{R}. \tag{30}$$

Moreover using the spectral representation of Y, one has

$$Z_{n,\mathbf{a}}(k) = \int_{\mathbb{R}^\nu} e^{-i\frac{k\cdot x}{n}} \prod_{j=1}^{\nu} P_{\mathbf{a_j}}\left(e^{-i\frac{x_j}{n}}\right) F(x)^{1/2} d\widetilde{W^\nu}(x).$$

We will note $P_{\mathbf{a}}(x) := \prod_{j=1}^{\nu} P_{\mathbf{a_j}}(x_j)$. The only assumption that we will use is the fact that $P_{\mathbf{a}}$ has a zero of order $K := K_1 + K_2 + \cdots K_\nu$ at $(1, \cdots, 1)$. We say that the filter $\mathbf{a}$ has order K. Then we have

$$\begin{aligned}
\mathrm{Cov}(Z_{n,\mathbf{a}}(k), Z_{n,\mathbf{a}}(k')) &= \int_{\mathbb{R}^\nu} e^{-i\frac{(k-k')\cdot x}{n}} \left|P_{\mathbf{a}}\left(e^{-i\frac{x_1}{n}}, \cdots, e^{-i\frac{x_\nu}{n}}\right)\right|^2 F(x) d\mu^\nu(x) \\
&= \frac{1}{(2\pi)^\nu} \int_{[-\pi,\pi)^\nu} e^{-i(k-k')\cdot x} \left|P_{\mathbf{a}}\left(e^{-ix_1}, \cdots, e^{-ix_\nu}\right)\right|^2 \sum_{k\in\mathbb{Z}^\nu} (2\pi n)^\nu F(nx + 2n\pi k) d\mu^\nu(x).
\end{aligned}$$

So the spectral density of the $Z_{n,\mathbf{a}}$ is given by

$$f_{n,\mathbf{a}}(x) = \left|P_{\mathbf{a}}\left(e^{-ix_1}, \cdots, e^{-ix_\nu}\right)\right|^2 \sum_{k\in\mathbb{Z}^\nu} (2\pi n)^\nu F(nx + 2n\pi k),\ x \in [-\pi,\pi)^\nu. \tag{31}$$

Because of the assumption on $\mathbf{a}$, one can find a positive constant $c > 0$ such that

$$\left|P_{\mathbf{a}}\left(e^{-ix_1}, \cdots, e^{-ix_\nu}\right)\right| \leq c|x|^K, x = (x_1, \ldots, x_\nu) \in [-\pi,\pi)^\nu, \tag{32}$$

Then, the generalized quadratic variations of Y are defined as

$$V_{n,\mathbf{a}} = \frac{1}{(n-p+1)^\nu} \sum_{k_1,\ldots,k_\nu=0}^{n-p} \left(Z_{n,\mathbf{a}}(k)\right)^2. \tag{33}$$

Such quantities are very helpful to estimate the H parameter as explained below.

4.1. **Central Limit Theorem for quadratic variations.** We now consider random fields Y for which Assumption 4 is valid. More precisely, let Ω be a strictly positive homogeneous function of degree 0 that is continuous on the sphere $S^{\nu-1}$. We assume that

$$F(x) = \frac{\Omega(x)}{|x|^{2H+\nu}} + R(x), \tag{34}$$

where the rest R satisfies the estimate

$$|R(x)| \leq \frac{\kappa}{|x|^{2H+\nu}(1+|x|^\gamma)}. \tag{35}$$

We will prove a Central Limit Theorem for the generalized quadratic variations related to H. We will see that the limit does not depend on the rest. We use the notations given above.

Theorem 4.1. *Let us assume that F, the spectral density of Y satisfies* (34) *and* (35) *for some $H > 0$ and $\gamma > 0$. Let $\mathbf{a}$ be a filter of order K. If $K > H + \frac{\nu}{4}$, then for n tending to infinity,*

(i) $(n-p+1)^\nu \mathrm{Var}\left(\frac{V_{n,\mathbf{a}}}{\mathbb{E}(V_{n,\mathbf{a}})} - 1\right) \longrightarrow \sigma^2_{\mathbf{a}}(H)$

(ii) $(n-p+1)^{\nu/2}\left(\frac{V_{n,\mathbf{a}}}{\mathbb{E}(V_{n,\mathbf{a}})} - 1\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2_{\mathbf{a}}(H))$,

with

$$\sigma^2_{\mathbf{a}}(H) = \frac{2(2\pi)^\nu}{C_{\mathbf{a}}(H)^2} \int_{[-\pi,\pi)^\nu} \left|P_{\mathbf{a}}\left(e^{-ix_1}, \cdots, e^{-ix_\nu}\right)\right|^4 \left|\sum_{k\in\mathbb{Z}^\nu} \frac{\Omega(x+2\pi k)}{|x+2\pi k|^{2H+\nu}}\right|^2 d\mu^\nu(x), \tag{36}$$

where

$$C_{\mathbf{a}}(H) = \int_{\mathbb{R}^\nu} \left|P_{\mathbf{a}}\left(e^{-ix_1}, \cdots, e^{-ix_\nu}\right)\right|^2 \frac{\Omega(x)}{|x|^{2H+\nu}} d\mu^\nu(x). \tag{37}$$

Proof. This will be a direct consequence of the previous sections. We can write

$$\frac{V_{n,\mathbf{a}}}{\mathbb{E}\left(V_{n,\mathbf{a}}\right)} - 1 = H_2(X_{n,\mathbf{a}}(k)),$$

with $\{X_{n,\mathbf{a}}(k),\, k \in \mathbb{Z}^\nu\}$ a stationary Gaussian time series, given by

$$X_{n,\mathbf{a}}(k) := \frac{Z_{n,\mathbf{a}}(k)}{\sqrt{\mathrm{Var}\left(Z_{n,\mathbf{a}}(k)\right)}}.$$

The spectral density of $X_{n,\mathbf{a}}$ is easily deduced from the one of $Z_{n,\mathbf{a}}$ given in (31), using the fact that

$$C_{n,\mathbf{a}} := \mathrm{Var}\left(Z_{n,\mathbf{a}}(k)\right) = \int_{\mathbb{R}^\nu} \left|P_{\mathbf{a}}\left(e^{-ix_1}, \cdots, e^{-ix_\nu}\right)\right|^2 n^\nu F(nx) d\mu^\nu(x).$$

This means that we can write $f_{X_{n,\mathbf{a}}}$ as $|g_{n,\mathbf{a}}|^2$, with

$$g_{n,\mathbf{a}}(x) := \frac{1}{\sqrt{C_{n,\mathbf{a}}}} P_{\mathbf{a}}\left(e^{-ix_1}, \cdots, e^{-ix_\nu}\right) \sqrt{\sum_{k\in\mathbb{Z}^\nu} (2\pi n)^\nu F(nx + 2n\pi k)}. \tag{38}$$

We are in position to apply Theorem 2.9 or Theorem 3.2 depending on the dimension. It is sufficient to prove that $g_{n,\mathbf{a}}$ is uniformly in $L^4(\mathbb{T})$ and converges to $g_{\mathbf{a}}$, with

$$g_{\mathbf{a}}(x) = \frac{1}{\sqrt{C_{\mathbf{a}}(H)}} P_{\mathbf{a}}\left(e^{-ix_1}, \cdots, e^{-ix_\nu}\right) \sqrt{\sum_{k\in\mathbb{Z}^\nu} (2\pi)^\nu \frac{\Omega(x+2\pi k)}{|x+2\pi k|^{2H+\nu}}} \tag{39}$$

and $C_{\mathbf{a}}(H)$ given by (37). The required convergence properties are contained in the following lemma.

Lemma 4.2. *We have the following.*

$$n^{2H}\mathbb{E}\left(V_{n,\mathbf{a}}\right) - C_{\mathbf{a}}(H) = \begin{cases} \underset{n\to+\infty}{O}\left(n^{-2(K-H)}\right) & \text{if } K-H < \gamma/2 \\ \underset{n\to+\infty}{O}\left(n^{-\gamma}\log n\right) & \text{if } K-H = \gamma/2 \\ \underset{n\to+\infty}{O}\left(n^{-\gamma}\right) & \text{if } K-H > \gamma/2 \end{cases}.$$

Moreover $g_{n,\mathbf{a}}$ and $g_{\mathbf{a}}$ are in $L^4(\mathbb{T})$ and

$$\|g_{n,\mathbf{a}} - g_{\mathbf{a}}\|_4 = \begin{cases} \underset{n\to+\infty}{O}\left(n^{-(K-H-\nu/4)}\right) & \text{if } K-H < \gamma+\nu/4 \\ \underset{n\to+\infty}{O}\left(n^{-\gamma}\log n\right) & \text{if } K-H = \gamma+\nu/4 \\ \underset{n\to+\infty}{O}\left(n^{-\gamma}\right) & \text{if } K-H > \gamma+\nu/4 \end{cases}.$$

Proof. For the first estimates, we have to bound

$$\int_{\mathbb{R}^\nu} \frac{\min(|x|^{2H}, 1)}{|x|^{2H+\nu}(1+|nx|^\gamma)} dx = \int_{|x|<1/n} + \int_{1/n<|x|<1} + \int_{|x|>1}.$$

Next, let us prove that $g_{\mathbf{a}}$ is in $L^4(\mathbb{T})$. We write that

$$\begin{aligned} \int_{\mathbb{T}^\nu} |g_{\mathbf{a}}(x)|^4 \, d\mu^\nu(x) &= c\int_{[-\pi,\pi)^\nu} \left|P_{\mathbf{a}}\left(e^{-ix_1}, \cdots, e^{-ix_\nu}\right)\right|^4 \left(\sum_{k\in\mathbb{Z}^\nu} \frac{\Omega(x+2\pi k)}{|x+2\pi k|^{2H+\nu}}\right)^2 d\mu^\nu(x) \\ &\le c\int_{[-\pi,\pi)^\nu} |x|^{4H} \frac{\Omega(x)^2}{|x|^{4H+2\nu}} d\mu^\nu(x) + C \\ &< +\infty. \end{aligned}$$

We have used the fact that, for $|x| \le \pi$, the sum $\displaystyle\sum_{k\neq 0} \frac{1}{|x+2\pi k|^{2H+\nu}}$ is uniformly bounded since $K > H + \frac{\nu}{4}$. Next, in order to bound the norm of $g_{n,\mathbf{a}} - g_{\mathbf{a}}$, we have to consider the quantity

$$\Delta_n(x) := \left| n^{\frac{\nu}{2}+H} \left(\sum_{k\in\mathbb{Z}^\nu} F(n(x+2k\pi))\right)^{\frac{1}{2}} - \left(\sum_{k\in\mathbb{Z}^\nu} \frac{\Omega(x+2\pi k)}{|x+2\pi k|^{2H+\nu}}\right)^{\frac{1}{2}} \right|.$$

By Minkowsky Inequality,

$$\Delta_n(x) \leq \left(\sum_{k\in\mathbb{Z}^\nu} \left| n^{\frac{\nu}{2}+H} F(n(x+2k\pi))^{\frac{1}{2}} - \frac{\Omega(x+2\pi k)^{\frac{1}{2}}}{|x+2\pi k|^{H+\frac{\nu}{2}}} \right|^2 \right)^{\frac{1}{2}}.$$

From Assumptions (34) and (35) and from the fact that Ω is bounded below on the unit sphere, we deduce that (with some constant C that varies from line to line)

$$|R(x)| \leq \frac{C}{|x|^\gamma} \frac{\Omega(x)}{|x|^{2H+\nu}}.$$

It follows that, for $|x| > \frac{2C}{n}$, we have the inequality

$$\Delta_n(x) \leq \frac{C}{n^\gamma |x|^\gamma} \left(\sum_{k\in\mathbb{Z}^\nu} \frac{\Omega(x+2\pi k)}{|x+2\pi k|^{2H+\nu}} \right)^{\frac{1}{2}}.$$

Using this inequality, as well as the fact that for all x we have

$$\Delta_n(x) \leq C \left(\sum_{k\in\mathbb{Z}^\nu} \frac{\Omega(x+2\pi k)}{|x+2\pi k|^{2H+\nu}} \right)^{\frac{1}{2}},$$

we estimate easily the integral of $|x|^{4K}|\Delta_n(x)|^4$ on $(-\pi,+\pi)$. The inequalities for the norm of $g_{n,\mathbf{a}} - g_{\mathbf{a}}$ follow at once. □

This finishes the proof of the theorem. □

4.2. **Remark on the speed of convergence.** We keep the notations of the last subsection and interest ourselves to the speed of convergence towards a Gaussian law. We want to give a bound for the Kolmogorov distance in one variable, or the distance of Wasserstein in general, between $\frac{V_{n,\mathbf{a}}}{\mathbb{E}(V_{n,\mathbf{a}})}$ and a Gaussian random variable of law $\mathcal{N}(0,\sigma^2_{\mathbf{a}}(H))$. By Lemma 4.2, we have a bound for the speed of convergence of $g_{n,\mathbf{a}}$ towards $g_{\mathbf{a}}$. So we can use Theorem 2.15 or Remark 3.3 as soon as $|g_{\mathbf{a}}|^2$ belongs to some Sobolev space $\mathcal{H}^\alpha$. Or, if the Fourier coefficients of $|g_{\mathbf{a}}|^2$ behave like a power $|k|^{-a}$, then we can use Remark 2.14 to have sharp bound for the speed of convergence. This is what we discuss now.

Let us start with the dimension one, where Ω is a constant. Then the Fourier coefficients of $|g_{\mathbf{a}}|^2$ are also, up to a constant, the values of the Fourier transform of the function

$$\frac{|P_{\mathbf{a}}(e^{-ix})|^2}{|x|^{2H+\nu}}.$$

It is classical that these last ones may be written as

$$r(k) := \sum_{m=-p}^{p} b_m |k+m|^{2H}$$

where b_m are coefficients of the polynomial $Q := |P|^2$. Using the fact that Q vanishes at order $2K$ and Taylor's Formula, one sees that $|r(k)| = O(|k|^{2H-2K})$, which is the kind of estimate that we wanted.

In higher dimension, we will show that we can conclude with some regularity assumption on Ω. Specifically, if we assume that Ω is of class $\mathcal{C}^1$ on the unit sphere, then the first partial derivatives of

$$h_{\mathbf{a}}(x) := \frac{|P_{\mathbf{a}}(e^{-ix})|^2 \Omega(x)}{|x|^{2H+\nu}}$$

satisfy the same kind of estimates as the function itself, apart from the loss of 1 in the power of $|x|$ in one term, and the fact that P or $\overline{P}$ has been replaced by its derivative in another one. If again $r(k) := \int_{\mathbb{R}^\nu} e^{-ik\cdot x} h_{\mathbf{a}}(x) d\mu^\nu(x)$, then $k_j r(k)$ appears as the Fourier coefficients of the jth partial derivative of $h_{\mathbf{a}}$. Remark first that we have proved in Lemma 4.2 that the sequence $r(k)$ is in $\ell^2(\mathbb{Z}^\nu)$ by proving that the periodization of $h_{\mathbf{a}}$ is in $L^2(\mathbb{T}^\nu)$, which is equivalent by Plancherel Theorem. For the same reason, to prove that $k_j r(k)$ is a sequence in $\ell^2(\mathbb{Z}^\nu)$, it is equivalent to prove that the

periodization of the jth derivative of $h_{\mathbf{a}}$ is in $L^2(\mathbb{T}^\nu)$. Under the assumption that Ω is of class $\mathcal{C}^1$ on the unit sphere, we do this by the same method, but on the stronger assumption that $K-1-H>\nu/4$. The stronger assumption is linked to the loss of 1 in the power of $|x|$. Finally, under these two assumptions, we conclude that $\sum_{k\in\mathbb{Z}^\nu}|k|^2|r(k)|^2<\infty$.

One can weaken or strengthen these assumptions on Ω to obtain the full range of Sobolev spaces. In all cases we have a speed of convergence towards a Gaussian law in $|n|^{-\epsilon}$ with ϵ depending on γ, K, H, and the regularity of Ω.

4.3. **Application to the identification of H.** As we said before, a central question is the estimation of H from real data. As an application of generalized quadratic variations, we obtain Proposition 1.3 of [8] without additional assumption of regularity on the spectral density. Actually, let us fix the dimension $\nu=1$ and, following [18], let us consider the filtered process of Y with a dilated filter. More precisely, let $U\geq 2$ an integer. For an integer $u\in\{1,\dots,U\}$, the dilation $\mathbf{a}^u$ of $\mathbf{a}$ is defined by, for $0\leq m\leq pu$,

$$a^u_m=\begin{cases} a_{m'} & \text{if } m'=mu\\ 0 & \text{otherwise.}\end{cases}$$

Since $\sum_{m=0}^{pu} m^r a^u_m = u^r\sum_{m=0}^{p} m^r a_m$, the filter $\mathbf{a}^u$ has the same order than $\mathbf{a}$ but length pu. Then,

$$\{X_{n,\mathbf{a}^u}(k)\,;\,k\in\mathbb{Z}^\nu,u\in\{1,\dots,U\}\}\overset{fdd}{=}\left\{\frac{Z_{n,\mathbf{a}^u}(k)}{\sqrt{\mathrm{Var}\left(Z_{n,\mathbf{a}^u}(k)\right)}}\,;\,k\in\mathbb{Z}^\nu,u\in\{1,\dots,U\}\right\},$$

so that $\left(\frac{V_{n,\mathbf{a}^u}}{\mathbb{E}\left(V_{n,\mathbf{a}^v}\right)},\frac{V_{n,\mathbf{a}^2}}{\mathbb{E}\left(V_{n,\mathbf{a}^2}\right)}\right)\underset{n\to+\infty}{\longrightarrow}(1,1)$ almost surely with asymptotic normality for $K>H+\frac{1}{4}$, according to Theorems 3.2 and 4.1, for any $u,v\in\{1,\dots,U\}$. According to Proposition 1.1 of [8] (see also Lemma 4.2 above), Assumption (4) implies that

$$n^{2H}\mathbb{E}\left(V_{n,\mathbf{a}^u}\right)=u^{2H}C_{\mathbf{a}}(H)+\begin{cases}\underset{n\to+\infty}{O}\left(n^{-2(K-H)}\right) & \text{if } K-H<\gamma/2\\ \underset{n\to+\infty}{O}\left(n^{-\gamma}\log n\right) & \text{if } K-H=\gamma/2\\ \underset{n\to+\infty}{O}\left(n^{-\gamma}\right) & \text{if } K-H>\gamma/2\end{cases},$$

so that for $u,v\in\{1,\dots,U\}$,

$$\widehat{H_{n,\mathbf{a}}}(u,v):=\frac{1}{2\log(u/v)}\log\left(\frac{V_{n,\mathbf{a}^u}}{V_{n,\mathbf{a}^v}}\right)\underset{n\to+\infty}{\longrightarrow}H\text{ a.s.}$$

with asymptotic normality when $K>H+1/4$ and $\gamma>1/2$.

4.4. **Functional Central Limit Theorem for Quadratic variations of a stationary Gaussian random process.** Up to now, we have only considered finite distributions. In this last subsection we want to prove that one can have convergence for continuous time processes as well. For sake of simplicity we restrict our study to the case $\nu=1$. Let us consider the random process Y given by (3) in dimension $\nu=1$. Then Assumption (4) on the spectral density F of Y can be written as

$$F(x)=\frac{c}{|x|^{2H+1}}+\underset{|x|\to+\infty}{O}\left(\frac{1}{|x|^{2H+1+\gamma}}\right). \tag{40}$$

We keep the notations of the previous section and consider for a discrete filter $\mathbf{a}$ of length $p+1$ and order $K\geq 1$ the filtered process of discrete observations of Y defined for $n\geq p$ by

$$Z_{n,\mathbf{a}}(k)=\sum_{m=0}^{p}a_m Y\left(\frac{k+m}{n}\right),\quad\text{for } k\in\mathbb{Z}.$$

Following Donsker's Theorem we consider a functional version of the Central Limit Theorem obtained in Theorem 4.1. For this purpose let us introduce the continuous time random process defined for

$t \in [\frac{p}{n}, 1]$ by

$$S_{n,\mathbf{a}}(t) = \frac{1}{\sqrt{n-p+1}} \sum_{k=0}^{[nt]-p} \left(\frac{Z_{n,\mathbf{a}}(k)^2}{C_{n,\mathbf{a}}} - 1 \right),$$

and $S_{n,\mathbf{a}}(t) = 0$ for $0 \leq t < \frac{p}{n}$, where $C_{n,\mathbf{a}} = \mathbb{E}\left(Z_{n,\mathbf{a}}(k)^2\right)$. Then $S_{n,\mathbf{a}}(1) = \sqrt{n-p+1}\left(\frac{V_{n,\mathbf{a}}}{\mathbb{E}(V_{n,\mathbf{a}})} - 1\right)$. Moreover, $S_{n,\mathbf{a}}$ is a.s. a càdlàg process on $[0,1]$ and we denote as usual $\mathcal{D}([0,1])$ the set of such processes. We also introduce the càdlàg random process defined on $[0,1]$ by $Y_n(t) = Y\left(\frac{[nt]}{n}\right)$. Let us recall that according to (40) and Proposition 1 of [7] we can assume that Y is a.s. continuous on $[0,1]$. It follows that Y_n converges in law to Y in the space $\mathcal{D}([0,1])$ equipped with the Skorohod topology. Then, Theorem 6 of [4] has the following counterpart in our setting.

Theorem 4.3. *We keep notations introduced in the previous section. Let assume that F, the spectral density of Y satisfies (40) for some $H > 0$ and $\gamma > 0$. Let $\mathbf{a}$ be a filter of order K. If $K > H + \frac{1}{4}$ and $\int_{\mathbb{R}} \min\left(1, |x|^{4K}\right) F(x)^2 d\mu(x) < +\infty$, then for n tending to infinity, we obtain the weak convergence (in the space $\mathcal{D}([0,1])^2$ equipped with the Skorohod topology)*

$$(Y_n(t), S_{n,\mathbf{a}}(t)) \longrightarrow \left(Y(t), \sigma_{\mathbf{a}}^2(H) B(t)\right),$$

where B is a standard Brownian motion on $[0,1]$ that is defined on the same probability space than Y, independent of Y and

$$\sigma_{\mathbf{a}}^2(H) = \frac{4\pi}{C_{\mathbf{a}}(H)^2} \int_{[-\pi,\pi)} \left|P_{\mathbf{a}}\left(e^{-ix}\right)\right|^4 \left| \sum_{k\in\mathbb{Z}} \frac{c}{|x+2\pi k|^{2H+1}} \right|^2 d\mu(x), \tag{41}$$

where

$$C_{\mathbf{a}}(H) = \int_{\mathbb{R}} \left|P_{\mathbf{a}}\left(e^{-ix}\right)\right|^2 \frac{c}{|x|^{2H+1}} d\mu(x). \tag{42}$$

Proof. Let us first consider the convergence of $S_{n,\mathbf{a}}(t)$. As previously we introduce the function

$$g_{\mathbf{a}}(x) = \frac{1}{C_{\mathbf{a}}(H)} P_{\mathbf{a}}\left(e^{-ix}\right) \sqrt{\sum_{k\in\mathbb{Z}} \frac{2\pi c}{|x+2\pi k|^{2H+1}}},$$

and denote by $r_{\mathbf{a}}$ the Fourier coefficients of $|g_{\mathbf{a}}|^2$. Let us consider the centered stationary discrete Gaussian time series $X_{\mathbf{a}}$ which admits $|g_{\mathbf{a}}|^2$ for spectral density, and therefore $r_{\mathbf{a}}$ as covariance function. Then, let us define the random process

$$\widetilde{S}_{n,\mathbf{a}}(t) = \frac{1}{\sqrt{n-p+1}} \sum_{k=0}^{[nt]-p} H_2(X_{\mathbf{a}})(k),$$

for $t \in [\frac{p}{n}, 1]$ and $\widetilde{S}_{n,\mathbf{a}}(t) = 0$ otherwise. It follows from the proof of Theorem 2.9 that, for all $n \geq p$,

$$S_{n,\mathbf{a}}(t) \overset{fdd}{=} \widetilde{S}_{n,\mathbf{a}}(t) + o_{L^2(\Omega,\mathbb{P})}(1).$$

Then according to Slutsky's Theorem the convergence in finite dimensional distributions of $S_{n,\mathbf{a}}$ toward $\sigma_{\mathbf{a}}^2(H)B$ will follow from the one of $\widetilde{S}_{n,\mathbf{a}}$.
Note that $g_{\mathbf{a}} \in L^4(\mathbb{T})$ when $K > H + \frac{1}{4}$, such that $r_{\mathbf{a}} \in \ell^2(\mathbb{Z})$. Then, one has for all $t > 0$,

$$\widetilde{S}_{n,\mathbf{a}}(t) = \sqrt{\frac{[nt]-p+1}{n-p+1}} \frac{1}{\sqrt{[nt]-p+1}} \sum_{k=0}^{[nt]-p} H_2(X_{\mathbf{a}})(k) \xrightarrow[n\to+\infty]{d} \mathcal{N}\left(0, t\sigma_{\mathbf{a}}^2(H)\right) = \sigma_{\mathbf{a}}^2(H) B(t),$$

according to Theorem 2.1.
Therefore, according to Proposition 2 of [27], it is sufficient to prove that for any $0 < t < s$,

$$\mathrm{Cov}\left(\widetilde{S}_{n,\mathbf{a}}(t), \widetilde{S}_{n,\mathbf{a}}(s)\right) \underset{n\to+\infty}{\longrightarrow} \sigma_{\mathbf{a}}^2(H) \mathrm{Cov}\left(B(t), B(s)\right).$$

So let us write

$$\mathrm{Cov}\left(\widetilde{S}_{n,\mathbf{a}}(t),\widetilde{S}_{n,\mathbf{a}}(s)\right)=\mathrm{Var}\left(\widetilde{S}_{n,\mathbf{a}}(t)\right)+\mathrm{Cov}\left(\widetilde{S}_{n,\mathbf{a}}(t),\widetilde{S}_{n,\mathbf{a}}(s)-\widetilde{S}_{n,\mathbf{a}}(t)\right).$$

The first term converges to $\sigma^2_{\mathbf{a}}(H)t=\sigma^2_{\mathbf{a}}(H)\mathrm{Var}\left(B(t)\right)$, according to Theorem 2.1. Then

$$\begin{aligned}\mathrm{Cov}\left(\widetilde{S}_{n,\mathbf{a}}(t),\widetilde{S}_{n,\mathbf{a}}(s)-\widetilde{S}_{n,\mathbf{a}}(t)\right) &= \frac{2}{n-p+1}\sum_{k=0}^{[nt]-p}\sum_{l=[nt]-p+1}^{[ns]-p} r^2_{\mathbf{a}}(l-k)\\ &\leq \frac{2}{n-p+1}\sum_{j=1}^{[ns]} j r^2_{\mathbf{a}}(j)+\sum_{j=\min([nt]-p,[ns]-[nt]-1)}^{[ns]} r^2_{\mathbf{a}}(j).\end{aligned}$$

The second term tends to zero as a rest of a convergent series, since $t<s$. For the first term, recall that

$$|r_{\mathbf{a}}(k)|\leq C(1+|k|)^{-2(K-H)},$$

so that for $\alpha\in(0,\min(2\left(K-H-1/4\right),1/2))$

$$\frac{2}{n-p+1}\sum_{j=1}^{[ns]} j r^2_{\mathbf{a}}(j)\leq Cn^{-2\alpha}s^{1-2\alpha},$$

which tends to zero as n tends to infinity. This ends the proof of the convergence in finite dimensional distributions of $\widetilde{S}_{n,\mathbf{a}}$ and thus $S_{n,\mathbf{a}}$ to $\sigma^2_{\mathbf{a}}(H)B$.
Let us prove the tightness. We clearly have for $0<t\leq s$ and n sufficiently large

$$\begin{aligned}\mathbb{E}\left((S_{n,\mathbf{a}}(t)-S_{n,\mathbf{a}}(s))^2\right) &\leq C\|g_{n,\mathbf{a}}\|_4^4\left(\frac{[ns]-[nt]}{n}\right)\\ &\leq C\left(\|g_{\mathbf{a}}\|_4^4+1\right)(s-t).\end{aligned}$$

Finally for $t\leq s\leq r$, by Hölder inequality and using the equivalence of $L^p(\Omega,\mathbb{P})$ norms in the second chaos,

$$\begin{aligned}\mathbb{E}\left((S_{n,\mathbf{a}}(t)-S_{n,\mathbf{a}}(s))^2(S_{n,\mathbf{a}}(r)-S_{n,\mathbf{a}}(s))^2\right) &\leq \mathbb{E}\left((S_{n,\mathbf{a}}(t)-S_{n,\mathbf{a}}(s))^4\right)^{1/2}\mathbb{E}\left((S_{n,\mathbf{a}}(r)-S_{n,\mathbf{a}}(s))^4\right)^{1/2}\\ &\leq C\mathbb{E}\left((S_{n,\mathbf{a}}(t)-S_{n,\mathbf{a}}(s))^2\right)\mathbb{E}\left((S_{n,\mathbf{a}}(r)-S_{n,\mathbf{a}}(s))^2\right)\\ &\leq C(s-t)(r-s)\leq C(r-t)^2.\end{aligned}$$

The tightness of $S_{n,\mathbf{a}}$ follows from Theorem 13.5 of [6].
Now, let us consider the sequence of vectorial processes $(Y_n,S_{n,\mathbf{a}})$ which belong to $\mathcal{D}([0,1])^2$. Each coordinate is tight, thus $(Y_n,S_{n,\mathbf{a}})$ is also tight. It remains to study the finite dimensional convergence. Any linear combination of the coordinates of the above vector belongs to the order one and order two Chaos respectively. Moreover they have both a Gaussian limit. The Theorem 1 (item (iv)) of [27] allows to conclude of the vector itself and that the two Gaussian limit are independent. Summarizing we have

$$(Y_n,S_{n,\mathbf{a}})\xrightarrow{d}(Y,\sigma^2_{\mathbf{a}}B),$$

where the convergence is in distribution in the space $\mathcal{D}^2([0,1])$ and the two processes coordinates are Gaussian and independent. This also implies that the convergence is stable in law. □

Such a result is a fundamental tool when one deals with non linear function of a Gaussian field, see [10] for instance. The observed field is now $U(t)=g(Y(t))$, $t\in\mathbb{R}$, for a non-linear function g with extra assumptions of smoothness and integrability. A Central Limit Theorem can be obtained for the suitably normalized quadratic variations of U instead of Y, using Taylor expansion and similar computations than in Theorem 7 of [4]. In this case the limit variable is no more Gaussian and is given by the stochastic integral $\sigma^2_{\mathbf{a}}(H)\int_0^1\left(g'(Y(t))\right)^2dB(t)$.

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