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A RESOLVENT ESTIMATE FOR OPERATORS WITH FINITE SPECTRUM

RACHID ZAROUF

Abstract

Let T be a *power bounded* operator on a Banach space E : $P(T) = \sup_{k \geq 0} \|T^k\|_{E \rightarrow E} < \infty$. The condition “ $P(T) < \infty$ ” is called *power boundedness condition*. Then, setting $\rho(T) = \sup_{|z| > 1} (|z| - 1) \|R(z, T)\|_{E \rightarrow E}$, where $R(z, T)$ stands for the resolvent of T at point z , the inequality $\rho(T) \leq P(T)$ is obvious and the condition “ $\rho(T) < \infty$ ” is known as the classical *resolvent condition*.

Here, we suppose that $n \geq 1$ is an integer and T is an algebraic operator acting on a Banach space E , which minimal annihilating polynomial is of degree less or equal than n . We also suppose that the spectrum $\sigma(T)$ of T is included in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Then, T is of course *power bounded*. But what happens to the inequality $\rho(T) \leq P(T)$? We set $\rho_{\frac{1}{2}}(T) = \sup_{|z| > 1} \sqrt{|z|^2 - 1} \|R(z, T)\|_{E \rightarrow E}$, and show that under our hypothesis on T :

$$\rho_{\frac{1}{2}}(T) \leq \delta \frac{n}{\sqrt{1-r}} P(T) \text{ and } \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \frac{\rho_{\frac{1}{2}}(T)}{P(T)} \leq \pi \sqrt{1+r},$$

where $\delta = \pi(1 + \sqrt{2}) + 1$, and $r = \max_{\lambda \in \sigma(T)} |\lambda|$. This result is a consequence of a Bernstein-type inequality for rational functions in \mathbb{D} having at most n poles all outside of $\frac{1}{r}\mathbb{D}$, involving the Hardy norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H^2}$. We show this Bernstein-type inequality and its asymptotic sharpness as $n \rightarrow \infty$ and $r \rightarrow 1^-$.

We also show the asymptotic sharpness of our upper bound $\delta \frac{n}{\sqrt{1-r}} P(T)$ as $n \rightarrow \infty$ and $r \rightarrow 1^-$ but this time in the following sense : there exists a contraction A_r on the Hilbert space $(\mathbb{C}^n, |\cdot|_2)$ of spectrum $\{r\}$ such that

$$\underline{\lim}_{r \rightarrow 1} (1-r)^{\frac{1}{2}-\beta} \rho_{\frac{1}{2}}(A_r) \geq \cot\left(\frac{\pi}{4n}\right) = P(A_r) \cdot \cot\left(\frac{\pi}{4n}\right),$$

for all $\beta \in (0, \frac{1}{2})$. We finally link our result to the well-known Kreiss Matrix Theorem which has been also proved using a Bernstein-type inequality for the same class of rational functions but involving this time the Hardy norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H^\infty}$.

Introduction

Let $(E, \|\cdot\|)$ be a Banach space and T be an operator on E . Recall that T is algebraic if there exists a polynomial $p \neq 0$ such that $p(T) = 0$. We write the minimal annihilating polynomial in a monic form,

$$m_T(z) = m_\sigma(z) = \prod_{1 \leq k \leq n} (\lambda_k - z),$$

where $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is the spectrum $\sigma(T)$ of T ($\sigma(T)$ consists of the eigenvalues of T , with possible multiplicities as they occur in the minimal polynomial).

Recall also that T is *power bounded* if the following *power boundedness condition* is satisfied :

$$(PBC) \quad P(T) = \sup_{k \geq 0} \|T^k\|_{E \rightarrow E} < \infty.$$

In this case, by a power series expansion it is readily verified that T satisfies the well-known inequality

$$(1) \quad \rho(T) \leq P(T),$$

where $\rho(T) = \sup_{|z| > 1} (|z| - 1) \|R(z, T)\|_{E \rightarrow E}$, and $R(z, T) = (zI - T)^{-1}$ (I being the identity operator on E). In particular the *resolvent condition* :

$$(RC) \quad \rho(T) < \infty,$$

implies the *power boundedness* one :

$$(PBC) \implies (RC).$$

Now let us remark that for z in the corona $\{1 < |z| < 2\}$, we have $\sqrt{|z|^2 - 1} \gg |z| - 1$ and then

$$\sqrt{|z|^2 - 1} \|R(z, T)\|_{E \rightarrow E} \gg (|z| - 1) \|R(z, T)\|_{E \rightarrow E}.$$

So how tall is the quantity $\sqrt{|z|^2 - 1} \|R(z, T)\|_{E \rightarrow E}$? Here, we define

$$\rho_{\frac{1}{2}}(T) = \sup_{|z| > 1} \sqrt{|z|^2 - 1} \|R(z, T)\|_{E \rightarrow E},$$

and the condition :

$$(RC_{\frac{1}{2}}) \quad \rho_{\frac{1}{2}}(T) < \infty.$$

Now, what happens if we replace $\rho(T)$ by $\rho_{\frac{1}{2}}(T)$ in inequality (1)? Recalling the following well-known inequality :

$$(\star) \quad \|R(z, T)\|_{E \rightarrow E} \geq \frac{1}{\text{dist}(z, \sigma(T))}, \quad \forall z \notin \sigma(T),$$

we get that if $P(T) < \infty$ and $\sigma(T) \cap \mathbb{T} \neq \emptyset$, then $\rho_{\frac{1}{2}}(T) = \infty$. A natural question arises : under which hypothesis on T could we hope the condition $(RC_{\frac{1}{2}})$ to be realised? It is necessary to suppose that $\sigma(T) \cap \mathbb{T} = \emptyset$. This leads to define the following set \mathcal{F}_n .

Definition. Let $n \geq 1$ be an integer. We denote by \mathcal{F}_n the set of all algebraic operators T acting on a Banach space E such that $\deg(m_T) \leq n$, and the spectrum $\sigma(T)$ of T is included in the unit disc \mathbb{D} .

We prove that if $T \in \mathcal{F}_n$ (that is to say if we add the condition that the spectrum $\sigma(T)$ of T is included in the open unit disc \mathbb{D}), then conditions

$$(RC_{\frac{1}{2}}) \text{ and } (PBC),$$

are both satisfied (the fact that (PBC) is satisfied is obvious). (Conversely using (\star) , if T is algebraic and satisfies both $(RC_{\frac{1}{2}})$ and (PBC) , then $\sigma(T) \subset \mathbb{D}$). More precisely, we show in Section 2 that if $T \in \mathcal{F}_n$, then

$$(2) \quad \rho_{\frac{1}{2}}(T) \leq \delta \frac{n}{\sqrt{1-r}} P(T) \text{ and } \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \frac{\rho_{\frac{1}{2}}(T)}{P(T)} \leq \pi \sqrt{1+r},$$

where $\delta = \pi(1 + \sqrt{2}) + 1$, and $r = \max_{\lambda \in \sigma(T)} |\lambda|$. This result is a consequence of a Bernstein-type inequality for rational functions in \mathbb{D} having at most n poles all outside of $\frac{1}{r}\mathbb{D}$, involving the Hardy norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H^2}$. In Section 1, we prove this Bernstein-type inequality and its asymptotic sharpness as $n \rightarrow \infty$ and $r \rightarrow 1^-$. Then in Section 2, we apply the inequality proved in Section 1 in order to get the statement (2) above.

Finally in Section 3, we link our result to Kreiss Matrix Theorem which has been recently proved by Leveque and Trefethen [LeTr], and improved by Spijker [Sp], also as a consequence of a Bernstein-type inequality for the same class of rational functions, but involving this time the Hardy norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H^\infty}$.

1. Bernstein-type inequalities for rational functions

Let $\mathcal{P}_n = \{p = \sum_{k=0}^n a_k z^k : a_k \in \mathbb{C}\}$, be the complex space of polynomials of degree less or equal than $n \geq 1$. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\mathbb{D}}$ its closure. Given $r \in (0, 1)$, we define

$$\mathcal{R}_{n,r} = \left\{ \frac{p}{q}, : p \in \mathcal{P}_{n-1}, q \in \mathcal{P}_n, q(\zeta) = 0 \implies \zeta \notin \frac{1}{r}\mathbb{D} \right\},$$

the set of all rational functions in \mathbb{D} of degree less or equal than $n \geq 1$, having at most n poles all outside of $\frac{1}{r}\mathbb{D}$.

1.1 Statement of the problem. Given two Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ of holomorphic functions in \mathbb{D} , we are searching for the “best possible” constant $C_{n,r}(X, Y)$ such that

$$\|f'\|_X \leq C_{n,r}(X, Y) \|f\|_Y,$$

$\forall f \in \mathcal{R}_{n,r}$.

The proper definition of $C_{n,r}(X, Y)$ is the following one. Let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} , and $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$ be the corresponding finite Blaschke product, where $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$ is an elementary Blaschke factor for $\lambda \in \mathbb{D}$. Let also K_{B_σ} be the n -dimensional space defined by

$$K_{B_\sigma} = \mathcal{L}in(k_{\lambda_i} : i = 1..n),$$

where σ is a family of distincts elements of \mathbb{D} , and where $k_\lambda = \frac{1}{1-\bar{\lambda}z}$ is the Szegő kernel associated to λ . An obvious modification allows to generalize the definition of K_{B_σ} in the case where the sequence σ admits multiplicities.

Notice that using the scalar product $(\cdot, \cdot)_{H^2}$ on H^2 , an equivalent description of this space is :

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2,$$

where H^2 stands for the standard Hardy space of the unit disc \mathbb{D} ,

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(rz)|^2 dm(z) < \infty \right\},$$

m being the Lebesgue normalized measure on \mathbb{T} . Let D be the operator of differentiation on $(K_{B_\sigma}, \|\cdot\|_Y)$:

$$\begin{aligned} D : (K_{B_\sigma}, \|\cdot\|_Y) &\rightarrow (X, \|\cdot\|_X) \\ f &\mapsto f'. \end{aligned}$$

For $r \in [0, 1)$ and $n \geq 1$, we have

$$C_{n,r}(X, Y) = \sup \left\{ \|D\|_{K_{B_\sigma} \rightarrow X} : 1 \leq \#\sigma \leq n, |\lambda| \leq r \forall \lambda \in \sigma \right\}.$$

1.2 The case $X = H^1$, $Y = H^\infty$, known results.

1.2.1. Dolzhenko proved in [Dol] (see also [Pek], p.560 - inequality (11)) that

$$(1.2.1) \quad C_{n,r}(H^1, H^\infty) \leq c_1 n,$$

where c_1 is a numerical constant (c_1 is not explicitly given in [Dol]).

1.2.2. Leveque and Trefethen improved Dolzhenko's result in [LeTr] :

$$(1.2.2) \quad C_{n,r}(H^1, H^\infty) \leq 2en,$$

1.2.3. Spijker in its turn improved Leveque and Trefethen's result in [Sp] :

$$(1.2.3) \quad C_{n,r}(H^1, H^\infty) \leq en,$$

In fact Leveque, Trefethen [LeTr] and Spijker [Sp] proved and used the fact that

$$(1.2.4) \quad P(T) \leq C_{n,r}(H^1, H^\infty) \rho(T),$$

in order to apply their Bernstein-type estimates to Kreiss Matrix Theorem, see Section 3 below.

In paragraph 1.3 below, we replace the algebra H^∞ by the Hardy space H^2 and we give an estimate for the Bernstein-type constant $C_{n,r}(H^1, H^2)$ in order to apply it (in Section 2) to our matricial problem.

1.3 The case $X = H^1$, $Y = H^2$.

Theorem 1. Let $r \in [0, 1)$ and $n \geq 1$. Then,

$$(1.3.1) \quad \theta_{n,r} \frac{n}{\sqrt{1-r}} \leq C_{n,r}(H^1, H^2) \leq \gamma_{n,r} \frac{n}{\sqrt{1-r}},$$

where $\gamma_{n,r} = \left(\frac{r}{\sqrt{n(1+r)}} + \sqrt{1+r} \right)$ and $\theta_{n,r} = \frac{1}{\pi\sqrt{1+r}} \left(\left(1 - \frac{1}{n}\right) + r \left(1 - \frac{1}{2n}\right) + \frac{r^2}{3} \right)$. In particular,

$$(1.3.2) \quad \frac{1+r+\frac{r^2}{3}}{\pi\sqrt{1+r}} \frac{1}{\sqrt{1-r}} \leq \underline{\lim}_{n \rightarrow \infty} \frac{C_{n,r}(H^1, H^2)}{n} \leq \overline{\lim}_{n \rightarrow \infty} \frac{C_{n,r}(H^1, H^2)}{n} \leq \sqrt{1+r} \frac{1}{\sqrt{1-r}},$$

and

$$(1.3.3) \quad \frac{7}{3\pi\sqrt{2}} \leq \underline{\lim}_{r \rightarrow 1^-} \underline{\lim}_{n \rightarrow \infty} \frac{\sqrt{1-r}}{n} C_{n,r}(H^1, H^2) \leq \overline{\lim}_{r \rightarrow 1^-} \overline{\lim}_{n \rightarrow \infty} \frac{\sqrt{1-r}}{n} C_{n,r}(H^1, H^2) \leq \sqrt{2}.$$

Proof. We first prove the right-hand side inequality from (1.3.1). Let P_B be the orthogonal projection of the Hardy space H^2 onto its subspace $K_B = H^2 \ominus BH^2$. Since $f \in K_B$,

$$f = P_B f = \sum_{k=1}^n (f, e_k)_{H^2} e_k,$$

where the family $(e_k)_{k=1}^n$ (known as Malmquist basis associated to σ , see [N3] p.117) defined by,

$$e_1 = (1 - |\lambda_1|^2)^{\frac{1}{2}} f_1, \quad e_k = \left(1 - |\lambda_k|^2\right)^{\frac{1}{2}} \left(\prod_{j=1}^{k-1} b_{\lambda_j}\right) f_k = (f_k / \|f_k\|_2) \prod_{j=1}^{k-1} b_{\lambda_j}, \quad k \geq 2,$$

where $f_k(z) = \frac{1}{1 - \lambda_k z}$. Noticing that

$$e'_k = \sum_{i=1}^{k-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} e_k + \overline{\lambda_k} \frac{1}{(1 - \overline{\lambda_k} z)} e_k,$$

for all $k = 2..n$, we get

$$(P_B f)' = \sum_{i=1}^n \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^{n-1} (f, e_k)_{H^2} e_k + \sum_{k=1}^n (f, e_k)_{H^2} \overline{\lambda_k} \frac{1}{(1 - \overline{\lambda_k} z)} e_k,$$

for all $f \in K_B$. Now using both Cauchy-Schwarz inequality and the fact that family $(e_k)_{k=1}^n$ is orthonormal in H^2 , we get

$$\begin{aligned} & \left\| \sum_{k=1}^n \overline{\lambda_k} (f, e_k)_{H^2} \frac{1}{(1 - \overline{\lambda_k} z)} e_k \right\|_{H^1} \leq \sum_{k=1}^n |(f, e_k)_{H^2}| \left\| \lambda_k \frac{1}{(1 - \overline{\lambda_k} z)} \right\|_{H^2} \|e_k\|_{H^2} \leq \\ & \leq \sum_{k=1}^n |(f, e_k)_{H^2}| \frac{|\lambda_k|}{\sqrt{1 - |\lambda_k|^2}} \leq \left(\sum_{k=1}^n |(f, e_k)_{H^2}|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n \frac{|\lambda_k|^2}{1 - |\lambda_k|^2} \right)^{\frac{1}{2}} \leq \frac{r}{\sqrt{1+r}} \sqrt{\frac{n}{1-r}} \|f\|_{H^2}. \end{aligned}$$

Finally,

$$\left\| \sum_{i=1}^{n-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^n (f, e_k)_{H^2} e_k \right\|_{H^1} \leq \sum_{i=1}^{n-1} \left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{L^2} \left(\sum_{k=i+1}^n |(f, e_k)_{H^2}|^2 \right)^{\frac{1}{2}}.$$

Moreover, since $b'_{\lambda_i}/b_{\lambda_i} = 1/(\lambda_i - z) + \overline{\lambda_i}/(1 - \overline{\lambda_i} z)$, we get that $\left\| b'_{\lambda_i}/b_{\lambda_i} \right\|_{L^2} \leq \frac{1+|\lambda_i|}{\sqrt{1-|\lambda_i|^2}}$, and

$$\left\| \sum_{i=1}^{n-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^n (f, e_k)_{H^2} e_k \right\|_{H^1} \leq (1+r) \sum_{i=1}^{n-1} \frac{1}{(1 - |\lambda_i|^2)^{\frac{1}{2}}} \|f\|_{H^2} \leq$$

$$\leq \frac{1+r}{\sqrt{1+r}} \frac{n-1}{\sqrt{1-r}} \|f\|_{H^2}.$$

As a consequence,

$$\begin{aligned} \|(P_B f)'\|_{H^1} &\leq \frac{r}{\sqrt{1+r}} \sqrt{\frac{n}{1-r}} \|f\|_{H^2} + \sqrt{1+r} \frac{n-1}{\sqrt{1-r}} \|f\|_{H^2} \leq \\ &\leq \left(\frac{r}{\sqrt{n(1+r)}} + \sqrt{1+r} \right) \frac{n}{\sqrt{1-r}} \|f\|_{H^2} = \gamma_{n,r} \frac{n}{\sqrt{1-r}} \|f\|_{H^2}, \end{aligned}$$

which completes the proof of the right-hand side inequality.

We now prove the left-hand side inequality from (1.3.1). Let $e_n = \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r^{n-1}$. Then $e_n \in K_{b_r^n}$ and $\|e_n\|_{H^2} = 1$, (see [N1], Malmquist-Walsh Lemma, p.116). Moreover,

$$\begin{aligned} e_n' &= \frac{r(1-r^2)^{\frac{1}{2}}}{(1-rz)^2} b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r' b_r^{n-2} = \\ &= -\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r' b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r' b_r^{n-2}, \end{aligned}$$

since $b_r' = \frac{r^2-1}{(1-rz)^2}$. Then,

$$e_n' = b_r' \left[-\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r^{n-2} \right],$$

and

$$\begin{aligned} \|e_n'\|_{H^1} &= \frac{1}{2\pi} \int_{\mathbb{T}} |b_r'(w)| \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} (b_r(w))^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rw} (b_r(w))^{n-2} \right| dw = \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} |b_r'(w)| \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r(w) + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rw} \right| dw, \end{aligned}$$

which gives, using the variables $u = b_r(w)$,

$$\|e_n'\|_{H^1} = \frac{1}{2\pi} \int_{\mathbb{T}} \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rb_r(u)} \right| du.$$

But $1-rb_r = \frac{1-rz-r(r-z)}{1-rz} = \frac{1-r^2}{1-rz}$ and $b_r' \circ b_r = \frac{r^2-1}{(1-rb_r)^2} = -\frac{(1-rz)^2}{1-r^2}$. This implies

$$\begin{aligned} \|e_n'\|_{H^1} &= \frac{1}{2\pi} \int_{\mathbb{T}} \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-r^2} (1-ru) \right| du = \\ &= \frac{1}{(1-r^2)^{\frac{1}{2}}} \frac{1}{2\pi} \int_{\mathbb{T}} |(1-ru)(-ru + (n-1)(1-ru))| du. \end{aligned}$$

Without loss of generality we can replace r by $-r$, which gives

$$\|e_n'\|_{H^1} = \frac{1}{(1-r^2)^{\frac{1}{2}}} \|\varphi_n\|_{H^1},$$

where $\varphi_n = (1+rz)(rz + (n-1)(1+rz))$. Expanding, we get

$$\varphi_n = (1+rz)(nrz + (n-1)) = (n-1) + (2n-1)rz + nr^2z^2.$$

Taking $\Phi_n = (n-1)z + (2n-1)r\frac{z^2}{2} + nr^2\frac{z^3}{3}$, we have $\Phi_n' = \varphi_n$ and $\Phi_n(0) = 0$. Now using Hardy's inequality

$$(1.3.4) \quad \|f\|_W \leq \pi \|f'\|_{H^1} + |f(0)|,$$

(see N. Nikolski, [N2] p. 370 8.7.4 -(c)), W being the Wiener algebra of absolutely convergent Fourier series

$$(1.3.5) \quad W = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_W = \sum_{k \geq 0} |\hat{f}(k)| < \infty \right\},$$

we get

$$\|\Phi_n\|_W \leq \pi \|\Phi_n'\|_{H^1} + |\Phi_n(0)| = \pi \|\varphi_n\|_{H^1}.$$

As a result,

$$\|\varphi_n\|_{H^1} \geq \frac{1}{\pi} \left[(n-1) + \frac{(2n-1)r}{2} + \frac{nr^2}{3} \right],$$

and

$$\begin{aligned} \|\varphi_n'\|_{H^1} &\geq \frac{1}{(1-r^2)^{\frac{1}{2}}} \frac{1}{\pi} \left[(n-1) + \frac{(2n-1)r}{2} + \frac{nr^2}{3} \right] = \frac{1}{\pi} \frac{1}{(1-r^2)^{\frac{1}{2}}} \left[n \left(1 + r + \frac{r^2}{3} \right) - 1 - \frac{r}{2} \right] = \\ &= \frac{1}{\pi} \frac{n}{(1-r^2)^{\frac{1}{2}}} \left[\left(1 - \frac{1}{n} \right) + r \left(1 - \frac{1}{2n} \right) + \frac{r^2}{3} \right] = \theta_{n,r} \frac{n}{\sqrt{1-r}}, \end{aligned}$$

which completes the proof. (1.3.2) and (1.3.3) follow. \square

2. An upper bound for $\rho_{\frac{1}{2}}(T)$

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \frac{\rho_{\frac{1}{2}}(T)}{P(T)} \leq \pi \sqrt{1+r}$$

The aim of this section is to prove the Theorem 2 below. In point (i) of Theorem 2, we give an upper bound for $\rho_{\frac{1}{2}}(T)$ if $T \in \mathcal{F}_n$. In point (ii) of the same theorem, we prove the asymptotic sharpness of this upper estimate as n tends to infinity and the spectral radius $r(T)$ of T tends to 1^- .

Theorem 2. (i) Let $T \in \mathcal{F}_n$ and $\sigma(T)$ be the spectrum of T . Then,

$$\rho_{\frac{1}{2}}(T) \leq (\pi C_{n,r}(H^1, H^2) + 1 - \Pi) P(T),$$

where $\Pi = \Pi_{\lambda \in \sigma(T)} |\lambda|^2$. In particular,

$$\rho_{\frac{1}{2}}(T) \leq \delta \frac{n}{\sqrt{1-r}} P(T),$$

and

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \frac{\rho_{\frac{1}{2}}(T)}{P(T)} \leq \pi \sqrt{1+r},$$

where $\delta = \pi(1 + \sqrt{2}) + 1$, and $r = \max_{\lambda \in \sigma(T)} |\lambda|$ stands for the spectral radius of T .

(ii) Let $|\cdot| = |\cdot|_2$ be the Hilbert norm of \mathbb{C}^n and $r \in (0, 1)$. There exists a contraction A_r on the Hilbert space $(\mathbb{C}^n, |\cdot|_2)$ of spectrum $\{r\}$ such that

$$\underline{\lim}_{r \rightarrow 1^-} (1-r)^{\frac{1}{2}-\beta} \rho_{\frac{1}{2}}(A_r) \geq \cot\left(\frac{\pi}{4n}\right) \geq P(A_r) \cot\left(\frac{\pi}{4n}\right),$$

for all $\beta \in (0, \frac{1}{2})$.

Remark. It may also be of interest to consider the quantity $\rho_{\frac{1}{2}}(T) = \sup_{|z| > 1} (|z|^p - 1)^{\frac{1}{p}} \|R(z, T)\|_{E \rightarrow E}$ for $p \geq 1$. As for $p = 2$, if $P(T) < \infty$ and $\sigma(T) \cap \mathbb{T} \neq \emptyset$, using (\star) we get $\rho_{\frac{1}{2}}(T) = \infty$ for all $p \geq 1$.

Proof of Theorem 2. *Proof of (i).* We set $\sigma = \sigma(T) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ its spectrum (λ_j being taken with their multiplicities). We define the Blaschke product $B = \prod_{k=1}^n b_{\lambda_k}$, where, $b_{\lambda_i} = \frac{\lambda_i - z}{1 - \bar{\lambda}_i z}$, for all $i = 1..n$. First,

$$\|R(\lambda, T)\| \leq P(T) \left\| \frac{1}{\lambda - z} \right\|_{W/BW},$$

(see [N1] Theorem 3.24, p.31), where the Wiener algebra W of absolutely convergent Fourier series is defined in (1.3.5) above, and

$$\left\| \frac{1}{\lambda - z} \right\|_{W/BW} = \inf \left\{ \|f\|_W : f(\lambda_j) = \frac{1}{\lambda - \lambda_j}, j = 1..n \right\}.$$

Let $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$. As before, if P_B is the orthogonal projection of the Hardy space H^2 onto its subspace $K_B = H^2 \ominus BH^2$, then the function $f = P_B \left(\frac{1}{\lambda} k_{1/\bar{\lambda}} \right)$ satisfies $f - \frac{1}{\lambda - z} \in BW, \forall j = 1..n$. In particular,

$$\left\| \frac{1}{\lambda - z} \right\|_{W/BW} \leq \left\| \frac{1}{\lambda} P_B k_{1/\bar{\lambda}} \right\|_W.$$

We are now going to apply Hardy's inequality (recalled in (1.3.4) above) to $P_B k_{1/\bar{\lambda}}$. Using both this inequality and Theorem 1, we get

$$\begin{aligned} \left\| P_B k_{1/\bar{\lambda}} \right\|_W &\leq \pi \left\| \left(P_B k_{1/\bar{\lambda}} \right)' \right\|_{H^1} + \left| \left(P_B k_{1/\bar{\lambda}} \right) (0) \right| \leq \\ &\leq \pi C_{n,r} (H^1, H^2) \left\| P_B k_{1/\bar{\lambda}} \right\|_{H^2} + \sum_{k=1}^n \left| \left(k_{1/\bar{\lambda}}, e_k \right)_{H^2} \right| |e_k(0)| \leq \\ &\leq \pi C_{n,r} (H^1, H^2) \left\| P_B k_{1/\bar{\lambda}} \right\|_{H^2} + \left(\sum_{k=1}^n \left| \left(k_{1/\bar{\lambda}}, e_k \right)_{H^2} \right|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n |e_k(0)|^2 \right)^{\frac{1}{2}} = \\ &= \pi C_{n,r} (H^1, H^2) \left\| P_B k_{1/\bar{\lambda}} \right\|_{H^2} + \left\| P_B k_{1/\bar{\lambda}} \right\|_{H^2} \left(\sum_{k=1}^n |e_k(0)|^2 \right)^{\frac{1}{2}} \leq \\ &\leq \left[\pi C_{n,r} (H^1, H^2) + \left(\sum_{k=1}^n |e_k(0)|^2 \right)^{\frac{1}{2}} \right] \left\| k_{1/\bar{\lambda}} \right\|_{H^2} = \\ &= \left[\pi C_{n,r} (H^1, H^2) + \left(\sum_{k=1}^n |e_k(0)|^2 \right)^{\frac{1}{2}} \right] \frac{|\lambda|}{\sqrt{|\lambda|^2 - 1}} \leq \\ &\leq \left[\pi C_{n,r} (H^1, H^2) + 1 - |\lambda_1|^2 + \sum_{k=2}^n \left(1 - |\lambda_k|^2 \right) \prod_{i=1}^{k-1} |\lambda_i|^2 \right] \frac{|\lambda|}{\sqrt{|\lambda|^2 - 1}} = \\ &= \left[\pi C_{n,r} (H^1, H^2) + 1 - |\lambda_1|^2 + \sum_{k=2}^n \left(\prod_{i=1}^{k-1} |\lambda_i|^2 - \prod_{i=1}^k |\lambda_i|^2 \right) \right] \frac{|\lambda|}{\sqrt{|\lambda|^2 - 1}} = \\ &= \left(\pi C_{n,r} (H^1, H^2) + 1 - |\lambda_1|^2 + |\lambda_1|^2 - \prod_{i=1}^n |\lambda_i|^2 \right) \frac{|\lambda|}{\sqrt{|\lambda|^2 - 1}} = \\ &= \left(\pi C_{n,r} (H^1, H^2) + 1 - \prod_{i=1}^n |\lambda_i|^2 \right) \frac{|\lambda|}{\sqrt{|\lambda|^2 - 1}} = \end{aligned}$$

$$= (\pi C_{n,r}(H^1, H^2) + 1 - \Pi) \frac{|\lambda|}{\sqrt{|\lambda|^2 - 1}}.$$

Finally,

$$\left\| \frac{1}{\lambda} P_B k_{1/\bar{\lambda}} \right\|_W \leq (\pi C_{n,r}(H^1, H^2) + 1 - \Pi) \frac{1}{\sqrt{|\lambda|^2 - 1}},$$

and in particular,

$$\sqrt{|\lambda|^2 - 1} \|R(\lambda, T)\| \leq (\pi C_{n,r}(H^1, H^2) + 1 - \Pi),$$

for all $\lambda \notin \mathbb{D}$, and it remains to use Theorem 1 and the right-hand side inequality from (1.3.1) to complete the proof of (i).

Proof of (ii). Let M_n be the $n \times n$ nilpotent Toeplitz matrix defined by

$$M_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \cdot & 0 & 1 & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & 0 \\ 0 & \cdot & \cdot & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}.$$

Let also $f_r = \frac{z+r}{1+rz}$ with $r \in (0, 1)$. For every $\lambda \in \mathbb{C}$,

$$\lambda - f_r = \lambda - \frac{z+r}{1+rz} = \frac{\lambda - r - z(1 - \lambda r)}{1 + rz}.$$

So if we set $A_r = f_r(M_n)$, then

$$\begin{aligned} (\lambda I_n - A_r)^{-1} &= (\lambda I_n - f_r(M_n))^{-1} = ((\lambda - f_r)(M_n))^{-1} \\ &= (\lambda - f_r)^{-1}(M_n) = \left(\frac{1 + rz}{\lambda - r - z(1 - \lambda r)} \right) (M_n) = \frac{1}{\lambda - r} \left(\frac{1 + rz}{1 + z \frac{\lambda r - 1}{\lambda - r}} \right) (M_n). \end{aligned}$$

We suppose $\lambda \in \mathbb{R}$, $\lambda > 1$ and set $\nu = \frac{\lambda r - 1}{\lambda - r}$, which means $\lambda = \frac{1 - r\nu}{r - \nu}$. Then

$$\lambda - 1 = \frac{(1 - r)(1 + \nu)}{r - \nu} \text{ and } \lambda - r = \frac{1 - r^2}{r - \nu}.$$

In particular,

$$\begin{aligned} \frac{\sqrt{|\lambda|^2 - 1}}{|\lambda - r|} &= \sqrt{\lambda + 1} \frac{\sqrt{\lambda - 1}}{\lambda - r} = \frac{\sqrt{\lambda + 1}}{1 + r} \frac{\sqrt{(1 - r)(1 + \nu)} r - \nu}{\sqrt{r - \nu}} \frac{r - \nu}{1 - r} = \\ &= \frac{\sqrt{\lambda + 1}}{1 + r} \sqrt{\frac{1 + \nu}{1 - r}} \sqrt{r - \nu}. \end{aligned}$$

Now let $\alpha \in (0, 1)$ and

$$\lambda = \lambda(r) = \frac{1 + r - r(1 - r)^\alpha}{1 + r - (1 - r)^\alpha},$$

then $\lambda > 1$ and the corresponding $\nu = \nu(r)$ is given (after calculation) by

$$\nu(r) = (1 - r)^\alpha - 1.$$

As a consequence taking $\lambda = \lambda(r)$ we get,

$$\begin{aligned} \sup_{|\lambda| > 1} \sqrt{|\lambda|^2 - 1} \left\| (\lambda I_n - f_r(M_n))^{-1} \right\| &\geq \\ &\geq \frac{\sqrt{\lambda + 1}}{1 + r} \sqrt{\frac{1 + \nu}{1 - r}} \sqrt{r - \nu} \left\| \left(\frac{1 + rz}{1 + \nu z} \right) (M_n) \right\| = \\ &= \frac{\sqrt{(\lambda + 1)(r - \nu)}}{1 + r} (1 - r)^{\frac{\alpha}{2} - \frac{1}{2}} \left\| \left(\frac{1 + rz}{1 + \nu z} \right) (M_n) \right\|, \end{aligned}$$

and

$$\begin{aligned} & (1-r)^{\frac{1}{2}-\frac{\alpha}{2}} \sup_{|\lambda|>1} \sqrt{|\lambda|^2-1} \left\| (\lambda I_n - f_r(M_n))^{-1} \right\| \geq \\ & \geq \frac{\sqrt{(\lambda+1)(r-\nu)}}{1+r} \left\| \left(\frac{1+rz}{1+\nu z} \right) (M_n) \right\|, \end{aligned}$$

and taking finally the limit as r tends to 1^- , we get

$$\begin{aligned} & \underline{\lim}_{r \rightarrow 1} (1-r)^{\frac{1}{2}-\frac{\alpha}{2}} \sup_{|\lambda|>1} \sqrt{|\lambda|^2-1} \left\| (\lambda I_n - f_r(M_n))^{-1} \right\| \geq \\ & \geq \frac{\sqrt{(1+1)(1+1)}}{1+1} \left\| \left(\frac{1+z}{1-z} \right) (M_n) \right\| = \\ & = \left\| \left(\frac{1+z}{1-z} \right) (M_n) \right\| = \left\| \begin{pmatrix} 1 & 2 & \cdot & \cdot & 2 \\ & 1 & 2 & \cdot & 2 \\ & & \cdot & \cdot & \cdot \\ & & & 1 & 2 \\ & & & & 1 \end{pmatrix} \right\| = \cot \left(\frac{\pi}{4n} \right), \end{aligned}$$

see [DS] , Theorem 2 - p.4 for the last equality. \square

3. The link with Kreiss Matrix Theorem

Recalling that the set of $n \times n$ matrices T which spectrum $\sigma(T)$ are included in \mathbb{D} , is a subset of \mathcal{F}_n , we link our result to Kreiss Matrix Theorem. Indeed in this theorem, E is finite dimensional : it stands for \mathbb{C}^n endowed with the Hilbert norm $|\cdot|_2$.

A result of Leveque and Trefethen [LeTr] improved by Spijker [Sp], shows that we have

$$\rho(T) \leq P(T) \leq en \cdot \rho(T).$$

As it is recalled in the Introduction, the fact that $\rho(T) \leq P(T)$ (inequality (1)) is obvious whereas inequality $P(T) \leq en \cdot \rho(T)$ is called Kreiss Matrix Theorem. The two conditions “ $P(T) < \infty$ ” and “ $\rho(T) < \infty$ ” are equivalent (but this is already not the case for infinite matrices).

The main tool used by Leveque, Trefethen [LeTr] and Spijker [Sp] is the same : a Bernstein-type inequality for rational functions, (see Section 1). They show it and then apply it to their problem using (1.2.4). More precisely, Leveque and Trefethen proved in [LeTr] that $P(T) \leq 2en \cdot \rho(T)$ (using the fact that $C_{n,r}(H^1, H^\infty) \leq 2en$, see Section 1 - 1.2.2), and later Spijker in [Sp] improved their result avoiding the extra-factor 2 : $P(T) \leq en \cdot \rho(T)$ (using and proving the fact that $C_{n,r}(H^1, H^\infty) \leq en$, see Section 1 - 1.2.3) .

An open question. Since we know that $P(T) \leq en \cdot \rho(T)$, and that the constant en is sharp in this inequality, a natural question arises looking at our Theorem 2 : conversely, is it possible to find the “best possible” constant K such that

$$P(T) \leq K \rho_{\frac{1}{2}}(T),$$

for all $T \in \mathcal{F}_n$?

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