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Realizability algebras II : new models of ZF + DC

Jean-Louis Krivine

July 1, 2010

Introduction

The technology of *classical realizability* was developed in [13, 16] in order to extend the proof-program correspondence (*Curry-Howard correspondence*) to the whole of mathematical proofs, with excluded middle, axioms of ZF, dependent choice, existence of a well ordering on $\mathcal{P}(\mathbb{N})$, ...

We show here that this technology is also a new method in order to build models of ZF and to obtain relative consistency results.

The main tools are :

- The structure of *standard realizability algebra* [16], which plays a role similar to a set of forcing conditions.
- The theory ZF_ε [11] which is a conservative extension of ZF, with a notion of *strong membership*, denoted as ε .

The theory ZF_ε is essentially ZF without the extensionality axiom. We note an analogy with the Fraenkel-Mostowski models with “urelements” : we obtain a non well orderable set, which is a Boolean algebra denoted $\mathfrak{J}2$, all elements of which except the unity are empty. But we also notice two important differences :

- The final model of $\text{ZF} + \neg \text{AC}$ is obtained directly, without taking a suitable submodel.
- There exists an injection from the “pathological set” $\mathfrak{J}2$ into \mathbb{R} , and therefore \mathbb{R} is *also not well orderable*.

We show the consistency, relatively to the consistency of ZF, of the theory $\text{ZF} + \text{DC}$ (dependent choice) with the following properties :

- there exists a sequence \mathcal{X}_n of infinite subsets of \mathbb{R} , the “cardinals” of which are strictly decreasing (this means that there is an injection but no surjection from \mathcal{X}_{n+1} to \mathcal{X}_n) ;
- there exists a sequence \mathcal{X}_n of infinite subsets of \mathbb{R} , the “cardinals” of which are strictly increasing, and such that $\mathcal{X}_m \times \mathcal{X}_n$ is equipotent with \mathcal{X}_{mn} .

More detailed properties of \mathbb{R} in this model are given in theorems 35 and 39.

As far as I know, these consistency results are new, and cannot be obtained by forcing. But, in any case, the fact that the simplest non trivial realizability model has a real line with so unusual properties, is of interest in itself. Another aspect of these results, which

is interesting from the point of view of computer science, is the following : in [16], we introduce *read* and *write* instructions in a global memory, in order to realize a form of the axiom of choice (well ordering of \mathbb{R}). Therefore, what we show here, is that these instructions are *indispensable* : without them, we can build a realizability model in which \mathbb{R} is not well ordered.

Standard realizability algebras

The notion of *realizability algebra*, and the particular case of *standard realizability algebra* are defined in [16]. They are variants of the usual notion of *combinatory algebra*. Here, we only need the *standard* realizability algebras, the definition of which we recall below :

We have a countable set Π_0 which is the set of *stack constants*.

We define recursively two sets : Λ (the set of *terms*) and Π (the set of *stacks*). Terms and stacks are finite sequences of elements of the set :

$$\Pi_0 \cup \{B, C, E, I, K, W, \mathbf{cc}, \zeta, \mathbf{k}, (,), [,], \bullet\}$$

which are obtained by the following rules :

- $B, C, E, I, K, W, \mathbf{cc}, \zeta$ are terms ;
- each element of Π_0 is a stack (*empty stacks*) ;
- if ξ, η are terms, then $(\xi)\eta$ is a term (this operation is called *application*) ;
- if ξ is a term and π a stack, then $\xi \bullet \pi$ is a stack (this operation is called *push*) ;
- if π is a stack, then $\mathbf{k}[\pi]$ is a term.

A term of the form $\mathbf{k}[\pi]$ is called a *continuation*. It will also be denoted as \mathbf{k}_π .

A term which does not contain any continuation (i.e. in which the symbol \mathbf{k} does not appear) is called *proof-like*.

Every stack has the form $\pi = \xi_1 \bullet \dots \bullet \xi_n \bullet \pi_0$, where $\xi_1, \dots, \xi_n \in \Lambda$ and $\pi_0 \in \Pi_0$, i.e. π_0 is a stack constant.

If $\xi \in \Lambda$ and $\pi \in \Pi$, the ordered pair (ξ, π) is called a *process* and denoted as $\xi \star \pi$; ξ and π are called respectively the *head* and the *stack* of the process $\xi \star \pi$.

The set of processes $\Lambda \times \Pi$ will also be written $\Lambda \star \Pi$.

Notation. The term $(\dots(((\xi)\eta_1)\eta_2)\dots)\eta_n$ will be also denoted by $(\xi)\eta_1\eta_2\dots\eta_n$ or $\xi\eta_1\eta_2\dots\eta_n$.

For example : $\xi\eta\zeta = (\xi)\eta\zeta = (\xi\eta)\zeta = ((\xi)\eta)\zeta$.

We now choose a recursive bijection from Λ onto \mathbb{N} , which is written $\xi \longmapsto \mathbf{n}_\xi$.

We put $0 = \lambda f \lambda x x$, $\sigma = \lambda n \lambda f \lambda x (f)(n)fx$ (the successor in λ -calculus). For each $n \in \mathbb{N}$, we define $\underline{n} \in \Lambda$ recursively, by putting : $\underline{0} = 0$; $\underline{n+1} = (\sigma)\underline{n}$.

We define a preorder relation \succ , on $\Lambda \star \Pi$. It is the least reflexive and transitive relation such that, for all $\xi, \eta, \zeta \in \Lambda$ and $\pi, \varpi \in \Pi$, we have :

- $(\xi)\eta \star \pi \succ \xi \star \eta \bullet \pi$.
- $I \star \xi \bullet \pi \succ \xi \star \pi$.
- $K \star \xi \bullet \eta \bullet \pi \succ \xi \star \pi$.
- $E \star \xi \bullet \eta \bullet \pi \succ (\xi)\eta \star \pi$.
- $W \star \xi \bullet \eta \bullet \pi \succ \xi \star \eta \bullet \eta \bullet \pi$.
- $C \star \xi \bullet \eta \bullet \zeta \bullet \pi \succ \xi \star \zeta \bullet \eta \bullet \pi$.

$B \star \xi \cdot \eta \cdot \zeta \cdot \pi \succ (\xi)(\eta)\zeta \star \pi.$
 $cc \star \xi \cdot \pi \succ \xi \star k_\pi \cdot \pi.$
 $k_\pi \star \xi \cdot \varpi \succ \xi \star \pi.$
 $\varsigma \star \xi \cdot \eta \cdot \pi \succ \xi \star \underline{n}_\eta \cdot \pi.$

Finally, we have a subset \perp of $\Lambda \star \Pi$ which is a final segment for this preorder, which means that : $\mathfrak{p} \in \perp, \mathfrak{p}' \succ \mathfrak{p} \Rightarrow \mathfrak{p}' \in \perp.$

In other words, we ask that \perp has the following properties :

$(\xi)\eta \star \pi \notin \perp \Rightarrow \xi \star \eta \cdot \pi \notin \perp.$
 $I \star \xi \cdot \pi \notin \perp \Rightarrow \xi \star \pi \notin \perp.$
 $K \star \xi \cdot \eta \cdot \pi \notin \perp \Rightarrow \xi \star \pi \notin \perp.$
 $E \star \xi \cdot \eta \cdot \pi \notin \perp \Rightarrow (\xi)\eta \star \pi \notin \perp.$
 $W \star \xi \cdot \eta \cdot \pi \notin \perp \Rightarrow \xi \star \eta \cdot \eta \cdot \pi \notin \perp.$
 $C \star \xi \cdot \eta \cdot \zeta \cdot \pi \notin \perp \Rightarrow \xi \star \zeta \cdot \eta \cdot \pi \notin \perp.$
 $B \star \xi \cdot \eta \cdot \zeta \cdot \pi \notin \perp \Rightarrow (\xi)(\eta)\zeta \star \pi \notin \perp.$
 $cc \star \xi \cdot \pi \notin \perp \Rightarrow \xi \star k_\pi \cdot \pi \notin \perp.$
 $k_\pi \star \xi \cdot \varpi \notin \perp \Rightarrow \xi \star \pi \notin \perp.$
 $\varsigma \star \xi \cdot \eta \cdot \pi \notin \perp \Rightarrow \xi \star \underline{n}_\eta \cdot \pi \notin \perp.$

Remark. Thus, the only arbitrary part in a standard realizability algebra is the set \perp of processes.

c-terms and λ -terms

We call *c-term* a term which is built with variables, the elementary combinators $B, C, E, I, K, W, cc, \varsigma$ and the application (binary function). A closed c-term is exactly what we have called a proof-like term.

Given a c-term t and a variable x , we define inductively on t , a new c-term denoted by $\lambda x t$. To this aim, we apply the first possible case in the following list :

1. $\lambda x t = (K)t$ if t does not contain x .
2. $\lambda x x = I$.
3. $\lambda x tu = (C\lambda x(E)t)u$ if u does not contain x .
4. $\lambda x tx = (E)t$ if t does not contain x .
5. $\lambda x tx = (W)\lambda x(E)t$ (if t contains x).
6. $\lambda x(t)(u)v = \lambda x(B)tuv$ (if uv contain x).

In [16], it is shown that this definition is correct. This allows us to translate every λ -term into a c-term. In the following, almost every c-term will be written as a λ -term. The fundamental property of this translation is given by theorem 1, which is proved in [16] :

Theorem 1. *Let t be a c-term with the only variables x_1, \dots, x_n ; let $\xi_1, \dots, \xi_n \in \Lambda$ and $\pi \in \Pi$. Then $\lambda x_1 \dots \lambda x_n t \star \xi_1 \cdot \dots \cdot \xi_n \cdot \pi \succ t[\xi_1/x_1, \dots, \xi_n/x_n] \star \pi.$*

The formal system

We write formulas and proofs in the language of first order logic. This formal language consists of :

- *individual variables* x, y, \dots ;
- *function symbols* f, g, \dots ; each one has an arity, which is an integer ; function symbols of arity 0 are called *constant symbols*.
- *relation symbols* ; each one has an arity ; relation symbols of arity 0 are called *propositional constants*. We have two particular propositional constants \top, \perp and three particular binary relation symbols \neq, \notin, \subseteq .

The *terms* are built in the usual way with individual variables and function symbols.

Remark. We use the word “term” with two different meanings : here as a term in a first order language, and also as an element of the set Λ of a realizability algebra. I think that, with the help of the context, no confusion is possible.

The *atomic formulas* are the expressions $R(t_1, \dots, t_n)$, where R is a n -ary relation symbol, and t_1, \dots, t_n are terms.

Formulas are built as usual, from atomic formulas, *with the only logical symbols* \rightarrow, \forall :

- each atomic formula is a formula ;
- if A, B are formulas, then $A \rightarrow B$ is a formula ;
- if A is a formula and x an individual variable, then $\forall x A$ is a formula.

Notations.

The formula $A_1 \rightarrow (A_2 \rightarrow (\dots (A_n \rightarrow B) \dots))$ will be written $A_1, A_2, \dots, A_n \rightarrow B$.

The usual logical symbols are defined as follows :

$\neg A \equiv A \rightarrow \perp$; $A \vee B \equiv (A \rightarrow \perp), (B \rightarrow \perp) \rightarrow \perp$; $A \wedge B \equiv (A, B \rightarrow \perp) \rightarrow \perp$; $\exists x F \equiv \forall x (F \rightarrow \perp) \rightarrow \perp$.

More generally, we shall write $\exists x\{F_1, \dots, F_k\}$ for $\forall x(F_1, \dots, F_k \rightarrow \perp) \rightarrow \perp$.

We shall sometimes write \vec{F} for a finite sequence of formulas F_1, \dots, F_k ;

Then, we shall also write $\vec{F} \rightarrow G$ for $F_1, \dots, F_k \rightarrow G$ and $\exists x\{\vec{F}\}$ for $\forall x(\vec{F} \rightarrow \perp) \rightarrow \perp$.

$A \leftrightarrow B$ is the pair of formulas $\{A \rightarrow B, B \rightarrow A\}$.

The rules of natural deduction are the following (the A_i 's are formulas, the x_i 's are variables of \mathbf{c} -term, t, u are \mathbf{c} -terms, written as λ -terms) :

1. $x_1 : A_1, \dots, x_n : A_n \vdash x_i : A_i$.
2. $x_1 : A_1, \dots, x_n : A_n \vdash t : A \rightarrow B, \quad x_1 : A_1, \dots, x_n : A_n \vdash u : A$
 $\Rightarrow x_1 : A_1, \dots, x_n : A_n \vdash tu : B$.
3. $x_1 : A_1, \dots, x_n : A_n, x : A \vdash t : B \Rightarrow x_1 : A_1, \dots, x_n : A_n \vdash \lambda x t : A \rightarrow B$.
4. $x_1 : A_1, \dots, x_n : A_n \vdash t : A \Rightarrow x_1 : A_1, \dots, x_n : A_n \vdash t : \forall x A$ where x is an individual variable which does not appear in A_1, \dots, A_n .
5. $x_1 : A_1, \dots, x_n : A_n \vdash t : \forall x A \Rightarrow x_1 : A_1, \dots, x_n : A_n \vdash t : A[\tau/x]$ where x is an individual variable and τ is a term.
6. $x_1 : A_1, \dots, x_n : A_n \vdash \mathbf{cc} : ((A \rightarrow B) \rightarrow A) \rightarrow A$ (law of Peirce).

The theory \mathbf{ZF}_ε

We write below a set of axioms for a theory called \mathbf{ZF}_ε . Then :

- We show that \mathbf{ZF}_ε is a conservative extension of \mathbf{ZF} .
- We define the realizability models and we show that each axiom of \mathbf{ZF}_ε is realized by a proof-like \mathbf{c} -term, in every realizability model.

It follows that the axioms of ZF are also realized by proof-like c-terms in every realizability model.

We write the axioms of ZF_ε with the predicate constants $\notin, \notin, \subseteq$. Of course, $x \varepsilon y$ and $x \in y$ are the formulas $x \notin y \rightarrow \perp$ and $x \notin y \rightarrow \perp$.

The notation $x \simeq y \rightarrow F$ means $x \subseteq y, y \subseteq x \rightarrow F$. Thus $x \simeq y$, which represents the usual (extensional) equality of sets, is the pair of formulas $\{x \subseteq y, y \subseteq x\}$.

We use the notations $(\forall x \varepsilon a)F(x)$ for $\forall x(\neg F(x) \rightarrow x \notin a)$ and $(\exists x \varepsilon a)\vec{F}(x)$ for $\forall x(\vec{F}(x) \rightarrow x \notin a) \rightarrow \perp$.

For instance, $(\exists x \varepsilon y)t \simeq u$ is the formula $\forall x(t \subseteq u, u \subseteq t \rightarrow x \notin y) \rightarrow \perp$.

The axioms of ZF_ε are the following :

0. Extensionality axioms.

$$\forall x \forall y [x \in y \leftrightarrow (\exists z \varepsilon y)x \simeq z] ; \forall x \forall y [x \subseteq y \leftrightarrow (\forall z \varepsilon x)z \in y].$$

1. Foundation scheme.

$$\forall x((\forall y \varepsilon x)F[y, x_1, \dots, x_n] \rightarrow F[x, x_1, \dots, x_n]) \rightarrow \forall x F[x, x_1, \dots, x_n]$$

for every formula $F[x, x_1, \dots, x_n]$.

The intuitive meaning of axioms 0 and 1 is that ε is a well founded relation, and that the relation \in is obtained by “collapsing” ε into an extensional binary relation.

The following axioms essentially express that the relation ε satisfies the axioms of Zermelo-Fraenkel *except extensionality*.

2. Comprehension scheme.

$$\forall x_1 \dots \forall x_n \forall a \exists b \forall x (x \varepsilon b \leftrightarrow (x \varepsilon a \wedge F[x, x_1, \dots, x_n]))$$

for every formula $F[x, x_1, \dots, x_n]$.

3. Pairing axiom.

$$\forall a \forall b \exists x \{a \varepsilon x, b \varepsilon x\}.$$

4. Union axiom.

$$\forall a \exists b (\forall x \varepsilon a)(\forall y \varepsilon x) y \varepsilon b.$$

5. Power set axiom.

$$\forall a \exists b \forall x (\exists y \varepsilon b) \forall z (z \varepsilon y \leftrightarrow (z \varepsilon a \wedge z \varepsilon x)).$$

6. Collection scheme.

$$\forall x_1 \dots \forall x_n \forall a \exists b (\forall x \varepsilon a) (\exists y F[x, y, x_1, \dots, x_n] \rightarrow (\exists y \varepsilon b) F[x, y, x_1, \dots, x_n])$$

for every formula $F[x, y, x_1, \dots, x_n]$.

7. Infinity scheme.

$$\forall x_1 \dots \forall x_n \forall a \exists b \{a \varepsilon b, (\forall x \varepsilon b) (\exists y F[x, y, x_1, \dots, x_n] \rightarrow (\exists y \varepsilon b) F[x, y, x_1, \dots, x_n])\}$$

for every formula $F[x, y, x_1, \dots, x_n]$.

The usual Zermelo-Fraenkel set theory is obtained from ZF_ε by identifying the predicate symbols \notin and \notin . Thus, the axioms of ZF are written as follows, with the predicate symbols \notin, \subseteq (recall that $x \simeq y$ is the conjunction of $x \subseteq y$ and $y \subseteq x$) :

0. Equality and extensionality axioms.

$$\forall x \forall y [x \in y \leftrightarrow (\exists z \in y)x \simeq z] ; \forall x \forall y [x \subseteq y \leftrightarrow (\forall z \in x)z \in y].$$

1. Foundation scheme.

$$\forall x((\forall y \in x)F[y, x_1, \dots, x_n] \rightarrow F[x, x_1, \dots, x_n]) \rightarrow \forall x F[x, x_1, \dots, x_n]$$

for every formula $F[x, x_1, \dots, x_n]$ written with the only relation symbols \notin, \subseteq .

2. Comprehension scheme.

$$\forall a \exists b \forall x (x \in b \leftrightarrow (x \in a \wedge F[x, x_1, \dots, x_n]))$$

for every formula $F[x, x_1, \dots, x_n]$ written with the only relation symbols \notin, \subseteq .

3. Pairing axiom.

$$\forall a \forall b \exists x \{a \in x, b \in x\}.$$

4. Union axiom.

$$\forall a \exists b (\forall x \in a) (\forall y \in x) y \in b.$$

5. Power set axiom.

$$\forall a \exists b \forall x (\exists y \in b) \forall z (z \in y \leftrightarrow (z \in a \wedge z \in x)).$$

6. Collection scheme.

$$\forall a \exists b (\forall x \in a) (\exists y F[x, y, x_1, \dots, x_n] \rightarrow (\exists y \in b) F[x, y, x_1, \dots, x_n])$$

for every formula $F[x, y, x_1, \dots, x_n]$ written with the only relation symbols \notin, \subseteq .

7. Infinity scheme.

$$\forall a \exists b \{a \in b, (\forall x \in b) (\exists y F[x, y, x_1, \dots, x_n] \rightarrow (\exists y \in b) F[x, y, x_1, \dots, x_n])\}$$

for every formula $F[x, y, x_1, \dots, x_n]$ written with the only relation symbols \notin, \subseteq .

Remark. The usual statement of the axiom of infinity is the particular case of this scheme, where $a = \emptyset$, and $F(x, y)$ is the formula $y = x \cup \{x\}$.

Let us show that ZF_ε is a conservative extension of ZF. First, it is clear that, if $ZF_\varepsilon \vdash F$, where F is a formula of ZF (i.e. written only with \notin and \subseteq), then $ZF \vdash F$; indeed, it is sufficient to replace \notin with \notin in any proof of $ZF_\varepsilon \vdash F$.

Conversely, we must show that each axiom of ZF is a consequence of ZF_ε .

Theorem 2.

i) $ZF_\varepsilon \vdash \forall a (a \subseteq a)$ (and thus $a \simeq a$).

ii) $ZF_\varepsilon \vdash \forall a \forall x (x \varepsilon a \rightarrow x \in a)$.

i) Using the foundation axiom, we assume $\forall x (x \varepsilon a \rightarrow x \subseteq x)$, and we must show $a \subseteq a$; therefore, we add the hypothesis $x \varepsilon a$. It follows that $x \subseteq x$, then $x \simeq x$, and therefore: $\exists y \{x \simeq y, y \varepsilon a\}$, that is to say $x \in a$. Thus, we have $\forall x (x \varepsilon a \rightarrow x \in a)$, and therefore $a \subseteq a$.

ii) Just shown.

Q.E.D.

Corollary 3. $ZF_\varepsilon \vdash \forall x (x \in a \rightarrow x \in b) \rightarrow a \subseteq b$.

We must show $x \varepsilon a \rightarrow x \in b$, which follows from $x \in a \rightarrow x \in b$ and $x \varepsilon a \rightarrow x \in a$ (theorem 2(ii)).

Q.E.D.

Lemma 4. $ZF_\varepsilon \vdash a \subseteq b, \forall x (x \in b \rightarrow x \in c) \rightarrow a \subseteq c$.

We must show $x \varepsilon a \rightarrow x \in c$, which follows from $x \varepsilon a \rightarrow x \in b$ and $x \in b \rightarrow x \in c$.

Q.E.D.

Theorem 5. $ZF_\varepsilon \vdash \forall y \forall z (y \simeq a, a \varepsilon z \rightarrow y \varepsilon z)$; $ZF_\varepsilon \vdash \forall y \forall z (a \subseteq y, z \varepsilon a \rightarrow z \in y)$.

Call $F(a)$, $F'(a)$ these two formulas. We show $F(a)$ by foundation :

thus, we suppose $(\forall x \varepsilon a)F(x)$ and we first show $F'(a)$: by hypothesis, we have $a \subseteq y$, $z \in a$; thus, there exists a' such that $z \simeq a'$ and $a' \varepsilon a$, and thus $F(a')$. From $a' \varepsilon a$ and $a \subseteq y$, we deduce $a' \in y$. From $z \simeq a'$ and $a' \in y$, we deduce $z \in y$ by $F(a')$.

Then, we show $F(a)$: by hypothesis, we have $y \simeq a$, $a \in z$, thus $a \simeq y'$ and $y' \varepsilon z$ for some y' . In order to show $y \in z$, it is sufficient to show $y \simeq y'$.

Now, we have $y \simeq a$, $a \simeq y'$, and thus $y' \subseteq a$, $a \subseteq y$. From $F'(a)$, we get $\forall z(z \in a \rightarrow z \in y)$; from $y' \subseteq a$, we deduce $y' \subseteq y$ by lemma 4.

We have also $y \subseteq a$, $a \subseteq y'$. From $F'(a)$, we get $\forall z(z \in a \rightarrow z \in y')$; from $y \subseteq a$, we deduce $y \subseteq y'$ by lemma 4.

Q.E.D.

With corollary 3, we obtain :

Corollary 6. $ZF_\varepsilon \vdash b \subseteq c \leftrightarrow \forall x(x \in b \rightarrow x \in c)$.

It is now easy to deduce the equality and extensionality axioms of ZF :

$\forall x(x \simeq x)$; $\forall x \forall y(x \simeq y \rightarrow y \simeq x)$; $\forall x \forall y \forall z(x \simeq y, y \simeq z \rightarrow x \simeq z)$;
 $\forall x \forall x' \forall y \forall y'(x \simeq x', y \simeq y', x \notin y \rightarrow x' \notin y')$; $\forall x \forall y(\forall z(z \notin x \leftrightarrow z \notin y) \rightarrow x \simeq y)$;
 $\forall x \forall y(x \subseteq y \leftrightarrow \forall z(z \notin y \rightarrow z \notin x))$.

Remark. This shows that \simeq is an equivalence relation which is compatible with the relations \in and \subseteq ; but, in general, it is *not compatible with ε* . It is the equality relation for ZF ; it will be called *extensional equivalence*.

Notation. The formula $\forall z(z \notin y \rightarrow z \notin x)$ will be written $x \subset y$. The ordered pair of formulas $x \subset y, y \subset x$ will be written $x \sim y$.

By theorem 2, we get $ZF_\varepsilon \vdash \forall x \forall y(x \subset y \rightarrow x \subseteq y)$. Thus \subset will be called *strong inclusion*, and \sim will be called *strong extensional equivalence*.

- Foundation scheme.

Let $F[x]$ be written with only \notin, \subseteq and let $G[x]$ be the formula $\forall y(y \simeq x \rightarrow F[y])$. Clearly, $\forall x G[x]$ is equivalent to $\forall x F[x]$. Therefore, from axiom scheme 1 of ZF_ε , it is sufficient to show : $\forall b(\forall x(x \in b \rightarrow F[x]) \rightarrow F[b]) \rightarrow (\forall x(x \varepsilon a \rightarrow G[x]) \rightarrow G[a])$, i.e. :

$\forall b(\forall x(x \in b \rightarrow F[x]) \rightarrow F[b]), \forall x \forall y(x \varepsilon a, y \simeq x \rightarrow F[y]), a \simeq b \rightarrow F[b]$.

Therefore, it is sufficient to prove : $\forall x \forall y(x \varepsilon a, y \simeq x \rightarrow F[y]), a \simeq b \rightarrow \forall x(x \in b \rightarrow F[x])$.

From $x \in b, a \simeq b$, we deduce $x \in a$ and therefore (by axiom 0), $x' \varepsilon a$ for some $x' \simeq x$. Finally, we get $F[x]$ from $\forall x \forall y(x \varepsilon a, y \simeq x \rightarrow F[y])$.

- Comprehension scheme : $\forall a \exists b \forall x(x \in b \leftrightarrow (x \in a \wedge F[x]))$

for every formula $F[x, x_1, \dots, x_n]$ written with \notin, \subseteq .

From the axiom scheme 2 of ZF_ε , we get $\forall x(x \varepsilon b \leftrightarrow (x \varepsilon a \wedge F[x]))$. If $x \in b$, then $x \simeq x'$, $x' \varepsilon b$ for some x' . Thus $x' \varepsilon a$ and $F[x']$. From $x \simeq x'$ and $x' \varepsilon a$, we deduce $x \in a$. Since \subseteq and \in are compatible with \simeq , it is the same for F ; thus, we obtain $F[x]$.

Conversely, if we have $F[x]$ and $x \in a$, we have $x \simeq x'$ and $x' \varepsilon a$ for some x' . Since F is compatible with \simeq , we get $F[x']$, thus $x' \varepsilon b$ and $x \in b$.

- Pairing axiom : $\forall x \forall y \exists z \{x \in z, y \in z\}$.

Trivial consequence of axiom 3 of ZF_ε , and theorem 2(ii).

- Union axiom : $\forall a \exists b \forall x \forall y(x \in a, y \in x \rightarrow y \in b)$.

From $x \in a$ we have $x \simeq x'$ and $x' \varepsilon a$ for some x' ; we have $y \in x$, therefore $y \in x'$, thus $y \simeq y'$ and $y' \varepsilon x'$. From axiom 4 of ZF_ε , $x' \varepsilon a$ and $y' \varepsilon x'$, we get $y' \varepsilon b$; therefore $y \in b$, by $y \simeq y'$.

- Power set axiom : $\forall a \exists b \forall x \exists y \{y \in b, \forall z (z \in y \leftrightarrow (z \in a \wedge z \in x))\}$

Given a , we obtain b by axiom 5 of ZF_ε ; given x , we define x' by the condition : $\forall z (z \varepsilon x' \leftrightarrow (z \varepsilon a \wedge z \in x))$ (comprehension scheme of ZF_ε). By definition of b , there exists $y \varepsilon b$ such that $\forall z (z \varepsilon y \leftrightarrow z \varepsilon a \wedge z \varepsilon x')$, and therefore $\forall z (z \varepsilon y \leftrightarrow z \varepsilon a \wedge z \in x)$. It follows easily that $\forall z (z \in y \leftrightarrow z \in a \wedge z \in x)$.

- Collection scheme : $\forall a \exists b (\forall x \in a) (\exists y F[x, y] \rightarrow (\exists y \in b) F[x, y])$

for every formula $F[x, y, x_1, \dots, x_n]$ written with the only relation symbols \notin, \subseteq .

From $x \in a$ and $\exists y F[x, y]$, we get $x \simeq x'$, $x' \varepsilon a$ for some x' , and thus $\exists y F[x', y]$ since F is compatible with \simeq . From axiom scheme 6 of ZF_ε , we get $\exists y (y \varepsilon b \wedge F[x', y])$, and therefore :

$\exists y (y \in b \wedge F[x, y])$, because $y \varepsilon b \rightarrow y \in b$ and F is compatible with \simeq .

- Infinity scheme : $\forall a \exists b \{a \varepsilon b, (\forall x \in b) (\exists y F[x, y] \rightarrow (\exists y \in b) F[x, y])\}$

for every formula $F[x, y, x_1, \dots, x_n]$ written with the only relation symbols \notin, \subseteq .

Same proof.

Q.E.D.

Realizability models of ZF_ε

We start with an (ordinary) model \mathcal{M} of ZFC, called *the ground model* or *the standard model*. In particular, the integers of \mathcal{M} are called *the standard integers*.

The elements of \mathcal{M} will be called *individuals*.

We define a *realizability model* \mathcal{N} , with the same set of individuals. But \mathcal{N} is not a model in the usual sense, because its truth values are subsets of Π instead of $\{0, 1\}$. Therefore, although \mathcal{M} and \mathcal{N} have the same domain (the quantifier $\forall x$ describes the same domain for both), the model \mathcal{N} may (and will, in all non trivial cases) have much more individuals than \mathcal{M} , because it has individuals which are *not named*. In particular, it will have *non standard integers*.

Remark. This is a great difference between *realizability* and *forcing* models of ZF. In a forcing model, each individual is named in the ground model; it follows that integers, and even ordinals, are not changed.

For each closed formula F with parameters in \mathcal{M} , we define two truth values :

$\|F\| \subseteq \Pi$ and $|F| \subseteq \Lambda$.

$|F|$ is defined immediately from $\|F\|$ as follows :

$$\xi \in |F| \Leftrightarrow (\forall \pi \in \|F\|) \xi \star \pi \in \perp.$$

Notation. We shall write $\xi \Vdash F$ for $\xi \in |F|$.

$\|F\|$ is now defined by recurrence on the length of F :

- F is atomic ;

then F has one of the forms $\top, \perp, a \notin b, a \subseteq b, a \notin b$ where a, b are parameters in \mathcal{M} .

We set :

$\|\top\| = \emptyset$; $\|\perp\| = \Pi$; $\|a \not\subseteq b\| = \{\pi \in \Pi; (a, \pi) \in b\}$.

$\|a \subseteq b\|$, $\|a \not\subseteq b\|$ are defined simultaneously by induction on $(\text{rk}(a) \cup \text{rk}(b), \text{rk}(a) \cap \text{rk}(b))$ ($\text{rk}(a)$ being the rank of a).

$\|a \subseteq b\| = \bigcup_c \{\xi \cdot \pi; \xi \in \Lambda, \pi \in \Pi, (c, \pi) \in a, \xi \Vdash c \not\subseteq b\}$;

$\|a \not\subseteq b\| = \bigcup_c \{\xi \cdot \xi' \cdot \pi; \xi, \xi' \in \Lambda, \pi \in \Pi, (c, \pi) \in b, \xi \Vdash a \subseteq c, \xi' \Vdash c \subseteq a\}$.

- $F \equiv A \rightarrow B$; then $\|F\| = \{\xi \cdot \pi; \xi \Vdash A, \pi \in \|B\|\}$.
- $F \equiv \forall x A$: then $\|F\| = \bigcup_a \|A[a/x]\|$.

The following theorem is an essential tool :

Theorem 7 (Adequacy lemma).

Let A_1, \dots, A_n, A be closed formulas of ZF_ε , and suppose that $x_1 : A_1, \dots, x_n : A_n \vdash t : A$.

If $\xi_1 \Vdash A_1, \dots, \xi_n \Vdash A_n$ then $t[\xi_1/x_1, \dots, \xi_n/x_n] \Vdash A$.

In particular, if $\vdash t : A$, then $t \Vdash A$.

We need to prove a (seemingly) more general result, that we state as a lemma :

Lemma 8. Let $A_1[\vec{z}], \dots, A_n[\vec{z}], A[\vec{z}]$ be formulas of ZF_ε , with $\vec{z} = (z_1, \dots, z_k)$ as free variables, and suppose that $x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}] \vdash t : A[\vec{z}]$.

If $\xi_1 \Vdash A_1[\vec{a}], \dots, \xi_n \Vdash A_n[\vec{a}]$ for some parameters (i.e. individuals in \mathcal{M})

$\vec{a} = (a_1, \dots, a_k)$, then $t[\xi_1/x_1, \dots, \xi_n/x_n] \Vdash A[\vec{a}]$.

Proof by recurrence on the length of the derivation of $x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}] \vdash t : A[\vec{z}]$.

We consider the last used rule.

1. $x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}] \vdash x_i : A_i[\vec{z}]$. This case is trivial.

2. We have the hypotheses :

$x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}] \vdash u : B[\vec{z}] \rightarrow A[\vec{z}]$; $x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}] \vdash v : B[\vec{z}]$;
 $t = uv$.

By the induction hypothesis, we have $u[\vec{\xi}/\vec{x}] \Vdash B[\vec{a}/\vec{z}] \rightarrow A[\vec{a}/\vec{z}]$ and $v[\vec{\xi}/\vec{x}] \Vdash B[\vec{a}/\vec{z}]$.

Therefore $(uv)[\vec{\xi}/\vec{x}] \Vdash A[\vec{a}/\vec{z}]$ which is the desired result.

3. We have the hypotheses :

$x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}], y : B[\vec{z}] \vdash u : C[\vec{z}]$; $A[\vec{z}] \equiv B[\vec{z}] \rightarrow C[\vec{z}]$; $t = \lambda y u$.

We want to show that $(\lambda y u)[\vec{\xi}/\vec{x}] \Vdash B[\vec{a}/\vec{z}] \rightarrow C[\vec{a}/\vec{z}]$. Thus, let :

$\eta \Vdash B[\vec{a}/\vec{z}]$ and $\pi \in \|C[\vec{a}/\vec{z}]\|$. We must show :

$(\lambda y u)[\vec{\xi}/\vec{x}] \star \eta \cdot \pi \in \perp$ or else $u[\vec{\xi}/\vec{x}, \eta/y] \star \pi \in \perp$.

Now, by the induction hypothesis, we have $u[\vec{\xi}/\vec{x}, \eta/y] \Vdash C[\vec{a}/\vec{z}]$,

which gives the result.

4. We have the hypotheses :

$x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}] \vdash t : B[\vec{z}]$; $A[\vec{z}] \equiv \forall z_1 B[\vec{z}]$; $\xi_i \Vdash A_i[a_1/z_1, a_2/z_2, \dots, a_k/z_k]$;
the variable z_1 is not free in $A_1[\vec{z}], \dots, A_n[\vec{z}]$.

We have to show that $t[\vec{\xi}/\vec{x}] \Vdash \forall z_1 B[\vec{a}/\vec{z}]$ i.e. $t[\vec{\xi}/\vec{x}] \Vdash \forall z_1 B[a_2/z_2, \dots, a_k/z_k]$. Thus,

we take an arbitrary set b in \mathcal{M} and we show $t[\vec{\xi}/\vec{x}] \Vdash B[b/z_1, a_2/z_2, \dots, a_k/z_k]$.

By the induction hypothesis, it is sufficient to show that $\xi_i \Vdash A_i[b/z_1, a_2/z_2, \dots, a_k/z_k]$. But this follows from the hypothesis on ξ , because z_1 is not free in the formulas A_i .

5. We have the hypotheses :

$x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}] \vdash t : \forall y B[y, \vec{z}]$; $A[\vec{z}] \equiv B[\tau[\vec{z}]/y, \vec{z}]$.

By the induction hypothesis, we have $t[\vec{\xi}/\vec{x}] \Vdash \forall y B[y, \vec{a}/\vec{z}]$; therefore $t[\vec{\xi}/\vec{x}] \Vdash B[b, \vec{a}/\vec{z}]$ for every parameter b . We get the desired result by taking $b = \tau[\vec{a}]$.

6. The result follows from the following :

Theorem 9. *For every formulas A, B , we have $\text{cc} \Vdash ((A \rightarrow B) \rightarrow A) \rightarrow A$.*

Let $\xi \Vdash (A \rightarrow B) \rightarrow A$ and $\pi \in \|A\|$. Then $\text{cc} \star \xi \cdot \pi \succ \xi \star k_\pi \cdot \pi$ which is in \perp , because $k_\pi \Vdash A \rightarrow B$ by lemma 10.

Q.E.D.

Lemma 10. *If $\pi \in \|A\|$, then $k_\pi \Vdash A \rightarrow B$.*

Indeed, let $\xi \Vdash A$; then $k_\pi \star \xi \cdot \pi' \succ \xi \star \pi \in \perp$ for every stack $\pi' \in \|B\|$.

Q.E.D.

This completes the proof of lemma 8 and theorem 7.

Q.E.D.

Realized formulas and coherent models

In the ground model \mathcal{M} , we interpret the formulas of the *language of ZF* : this language consists of $\in, =$; we add some function symbols, but these functions are always defined, in \mathcal{M} , by some formulas written with $\in, =$. We suppose that this ground model satisfies ZFC. The value, in \mathcal{M} , of a closed formula F of the language of ZF, with parameters in \mathcal{M} , is of course 1 or 0. In the first case, we say that \mathcal{M} *satisfies* F , and we write $\mathcal{M} \models F$.

In the realizability model \mathcal{N} , we interpret the formulas of the *language of ZF_ε* , which consists of $\notin, \subseteq, \notin$ and the same function symbols as in the language of ZF. The domain of \mathcal{N} and the interpretation of the function symbols are the same as for the model \mathcal{M} .

The value, in \mathcal{N} , of a closed formula F of ZF_ε with parameters (in \mathcal{M} or in \mathcal{N} , which is the same thing) is an element of $\mathcal{P}(\Pi)$ which is denoted as $\|F\|$, the definition of which has been given above.

Thus, we can no longer say that \mathcal{N} satisfies (or not) a given closed formula F . But we shall say that \mathcal{N} *realizes* F (and we shall write $\mathcal{N} \Vdash F$), if there exists a proof-like term θ such that $\theta \Vdash F$. We say that two closed formulas F, G are *interchangeable* if $\mathcal{N} \Vdash F \leftrightarrow G$.

Notice that, if $\|F\| = \|G\|$, then F, G are interchangeable (indeed $I \Vdash F \rightarrow G$), but the converse is far from being true.

The model \mathcal{N} allows us to make relative consistency proofs, since it is clear, from the adequacy lemma (theorem 7), that the class of formulas which are realized in \mathcal{N} is closed by deduction in classical logic. Nevertheless, we must check that the realizability model \mathcal{N} is *coherent*, i.e. that it does not realize the formula \perp . We can express this condition in the following form :

For every proof-like term θ , there exists a stack $\pi \in \Pi$ such that $\theta \star \pi \notin \perp$.

When the model \mathcal{N} is coherent, it is not *complete*, except in trivial cases. This means that there exist closed formulas F of ZF_ε such that $\mathcal{N} \not\models F$ and $\mathcal{N} \not\models \neg F$.

The axioms of ZF_ε are realized in \mathcal{N}

- Extensionality axioms.

We have $\|\forall z(z \notin b \rightarrow z \notin a)\| = \bigcup \{\xi \cdot \pi; \xi \Vdash c \notin b, \pi \in \|c \notin a\|\}$

by definition of the value of $\|\forall z(z \notin b \rightarrow z \notin a)\|$;

and $\|a \subseteq b\| = \bigcup \{\xi \cdot \pi; (c, \pi) \in a, \xi \Vdash c \notin b\}$ by definition of $\|a \subseteq b\|$.

Therefore, we have $\|a \subseteq b\| = \|\forall z(z \notin b \rightarrow z \notin a)\|$, so that :

$I \Vdash \forall x \forall y (x \subseteq y \rightarrow \forall z (z \notin y \rightarrow z \notin x))$ and $I \Vdash \forall x \forall y (\forall z (z \notin y \rightarrow z \notin x) \rightarrow x \subseteq y)$.

In the same way, we have :

$\|\forall z(a \subseteq z, z \subseteq a \rightarrow z \notin b)\| = \bigcup \{\xi \cdot \xi' \cdot \pi; \xi \Vdash a \subseteq c, \xi' \Vdash c \subseteq a; \pi \in \|c \notin b\|\}$

by definition of the value of $\|\forall z(a \subseteq z, z \subseteq a \rightarrow z \notin b)\|$;

and $\|a \notin b\| = \bigcup \{\xi \cdot \xi' \cdot \pi; (c, \pi) \in b, \xi \Vdash a \subseteq c, \xi' \Vdash c \subseteq a\}$ by definition of $\|a \notin b\|$.

Therefore, we have $\|a \notin b\| = \|\forall z(a \subseteq z, z \subseteq a \rightarrow z \notin b)\|$, so that :

$I \Vdash \forall x \forall y (x \notin y \rightarrow \forall z (x \subseteq z, z \subseteq x \rightarrow z \notin y))$;

$I \Vdash \forall x \forall y (\forall z (x \subseteq z, z \subseteq x \rightarrow z \notin y) \rightarrow x \notin y)$.

Notation. We shall write $\vec{\xi}$ for a finite sequence (ξ_1, \dots, ξ_n) of terms. Therefore, we shall write $\vec{\xi} \Vdash \vec{A}$ for $\xi_i \Vdash A_i$ ($i = 1, \dots, n$).

In particular, the notation $\vec{\xi} \Vdash a \simeq b$ means $\xi_1 \Vdash a \subseteq b, \xi_2 \Vdash b \subseteq a$;

the notation $\vec{\xi} \Vdash A \leftrightarrow B$ means $\xi_1 \Vdash A \rightarrow B, \xi_2 \Vdash B \rightarrow A$.

- Foundation scheme.

Theorem 11. $\mathbf{Y} \Vdash \forall x (\forall y (\vec{F}[y] \rightarrow y \notin x), \vec{F}[x] \rightarrow \perp) \rightarrow \forall x (\vec{F}[x] \rightarrow \perp)$

for every finite sequence $\vec{F}[x, x_1, \dots, x_n]$ of formulas.

Let $\xi \Vdash \forall x (\forall y (\vec{F}[y] \rightarrow y \notin x), \vec{F}[x] \rightarrow \perp)$ and $\vec{\eta} \Vdash \vec{F}[a]$. We show that $\mathbf{Y} \star \xi \cdot \vec{\eta} \cdot \pi \in \perp$, for every $\pi \in \Pi$, by induction on the rank of a . It suffices to show $\xi \star \mathbf{Y} \xi \cdot \vec{\eta} \cdot \pi \in \perp$.

Now, $\xi \Vdash \forall y (\vec{F}[y] \rightarrow y \notin a), \vec{F}[a] \rightarrow \perp$, so that it suffices to show $\mathbf{Y} \xi \Vdash \forall y (\vec{F}[y] \rightarrow y \notin a)$, in other words $\mathbf{Y} \xi \Vdash \vec{F}[b] \rightarrow b \notin a$ for every b . Let $\vec{\zeta} \Vdash \vec{F}[b]$ and $\varpi \in \|b \notin a\|$. Thus, we have $(b, \varpi) \in a$, therefore $\text{rk}(b) < \text{rk}(a)$ so that $\mathbf{Y} \star \xi \cdot \vec{\zeta} \cdot \varpi \in \perp$ by induction hypothesis. It follows that $\mathbf{Y} \xi \star \vec{\zeta} \cdot \varpi \in \perp$, which is the desired result.

Q.E.D.

It follows from theorem 11 that the axiom scheme 1 of ZF_ε (foundation) is realized.

- Comprehension scheme.

Let a be a set, and $F[x]$ a formula with parameters. We put :

$b = \{(x, \xi \cdot \pi); (x, \pi) \in a, \xi \Vdash F[x]\}$; then, we have trivially $\|x \notin b\| = \|F(x) \rightarrow x \notin a\|$.

Therefore $I \Vdash \forall x [x \notin b \rightarrow (F(x) \rightarrow x \notin a)]$ and $I \Vdash \forall x [(F(x) \rightarrow x \notin a) \rightarrow x \notin b]$.

- Pairing axiom.

We consider two sets a and b , and we put $c = \{a, b\} \times \Pi$. We have $\|a \notin c\| = \|b \notin c\| = \|\perp\|$, thus $I \Vdash a \varepsilon c$ and $I \Vdash b \varepsilon c$.

- Union axiom.

Given a set a , let $b = \text{Cl}(a)$ (the transitive closure of a , i.e. the least transitive set which contains a). We show $\|y \notin x \rightarrow x \notin a\| \supset \|y \notin b \rightarrow x \notin a\|$: indeed, let $\xi \cdot \pi \in \|y \notin b \rightarrow x \notin a\|$, i.e. $\xi \Vdash y \notin b$ and $(x, \pi) \in a$. Therefore, $\text{Cl}(a) \supset x$, i.e. $b \supset x$ and thus $\|y \notin b\| \supset \|y \notin x\|$.

Thus, we have $\xi \Vdash y \notin x$, which gives the result.

It follows that $I \Vdash \forall x \forall y [(y \notin x \rightarrow x \notin a) \rightarrow (y \notin b \rightarrow x \notin a)]$.

- Power set axiom.

Given a set a , let $b = \mathcal{P}(\text{Cl}(a) \times \Pi) \times \Pi$. For every set x , we put :

$y = \{(z, \xi \cdot \pi); (z, \pi) \in a, \xi \Vdash F[z, x]\}$, where $F[z, x]$ is a formula with parameters. Then, as we have seen above (comprehension scheme), we have $\|z \notin y\| = \|F[z, x] \rightarrow z \notin a\|$ and therefore $(I, I) \Vdash \forall z (z \notin y \leftrightarrow (F[z, x] \rightarrow z \notin a))$.

Now, it is obvious that $y \in \mathcal{P}(\text{Cl}(a) \times \Pi)$, and therefore $(y, \pi) \in b$ for every $\pi \in \Pi$.

Thus, we have $\|y \notin b\| = \Pi = \|\perp\|$. Therefore $I \Vdash y \varepsilon b$, and finally :

$(I, (I, I)) \Vdash y \varepsilon b \wedge \forall z (z \notin y \leftrightarrow (F[z, x] \rightarrow z \notin a))$.

The power set axiom is the particular case when the formula $F[z, x]$ is $z \varepsilon x$.

- Collection scheme.

Given a set a , and a formula $F[x, y]$ with parameters, let :

$b = \bigcup \{\Phi(x, \xi) \times \text{Cl}(a); x \in \text{Cl}(a), \xi \in \Lambda\}$ with

$\Phi(x, \xi) = \{y \text{ of minimum rank}; \xi \Vdash F[x, y]\}$ or $\Phi(x, \xi) = \emptyset$ if there is no such y .

We show that $\|\forall y (F[x, y] \rightarrow y \notin b)\| \supset \|\forall y (F[x, y] \rightarrow x \notin a)\|$:

Suppose indeed that $\xi \cdot \pi \in \|\forall y (F[x, y] \rightarrow x \notin a)\|$, i.e. $(x, \pi) \in a$ and $\xi \Vdash F[x, y]$ for some y . By definition of $\Phi(x, \xi)$, there exists $y' \in \Phi(x, \xi)$. Moreover, we have :

$x \in \text{Cl}(a)$, $\pi \in \text{Cl}(a)$, and therefore $(y', \pi) \in b$; it follows that $\|y' \notin b\| \supset \|x \notin a\|$. But $y' \in \Phi(x, \xi)$, and therefore $\xi \Vdash F[x, y']$; thus, we have $\xi \cdot \pi \in \|F[x, y'] \rightarrow y' \notin b\|$, which gives the result.

We have proved that $I \Vdash \forall y (F[x, y] \rightarrow y \notin b) \rightarrow \forall y (F[x, y] \rightarrow x \notin a)$.

- Infinity scheme.

Given a set a , we define b as the least set such that :

$\{a\} \times \Pi \subseteq b$ and $\forall x (\forall \pi \in \Pi) (\forall \xi \in \Lambda) ((x, \pi) \in b \Rightarrow \Phi(x, \xi) \times \{\pi\} \subseteq b)$

where $\Phi(x, \xi)$ is defined as above.

We have $\{a\} \times \Pi \subseteq b$, thus $\|a \notin b\| = \|\perp\|$, and therefore $I \Vdash a \varepsilon b$.

We now show that $\|\forall y (F[x, y] \rightarrow y \notin b)\| \supset \|\forall y (F[x, y] \rightarrow x \notin b)\|$:

Suppose indeed that $\xi \cdot \pi \in \|\forall y (F[x, y] \rightarrow x \notin b)\|$, i.e. $(x, \pi) \in b$ and $\xi \Vdash F[x, y]$ for some y . By definition of $\Phi(x, \xi)$, there exists $y' \in \Phi(x, \xi)$. By definition of b , we have $(y', \pi) \in b$, i.e. $\pi \in \|y' \notin b\|$. Now, $y' \in \Phi(x, \xi)$, and therefore $\xi \Vdash F[x, y']$; thus, we have :

$\xi \cdot \pi \in \|F[x, y'] \rightarrow y' \notin b\|$, which gives the result.

It follows that $I \Vdash \forall y (F[x, y] \rightarrow y \notin b) \rightarrow \forall y (F[x, y] \rightarrow x \notin b)$ and therefore :

$(I, I) \Vdash \{a \varepsilon b, \forall x (\forall y (F[x, y] \rightarrow y \notin b) \rightarrow \forall y (F[x, y] \rightarrow x \notin b))\}$.

Function symbols and equality

Following our needs, we shall add to the language of ZF_ε , some *function symbols* f, g, \dots of any arity. A k -ary function symbol f will be interpreted, in the realizability model \mathcal{N} , by a *functional relation*, which is defined in the ground model \mathcal{M} by a formula $F[x_1, \dots, x_k, y]$ of ZF. Thus, we assume that $\mathcal{M} \models \forall x_1 \dots \forall x_k \exists! y F[x_1, \dots, x_k, y]$ ($\exists! y F[y]$ is the conjunction of $\forall y \forall y' (F[y], F[y'] \rightarrow y = y')$ and $\exists y F[y]$).

The axiom schemes of ZF_ε , written in the extended language, are still realized in the model \mathcal{N} , because the above proofs remain valid.

On the other hand, in order to make sure that the axiom schemes of ZF, which use a k -ary function symbol f , are still realized, one must check that this symbol is *compatible with* \simeq , i.e. that the following formula is realized in \mathcal{N} :

$$\forall x_1 \dots \forall x_k (x_1 \simeq y_1, \dots, x_k \simeq y_k \rightarrow f x_1 \dots x_k \simeq f y_1 \dots y_k).$$

We now add a new rule to build formulas of ZF_ε :

If t, u are two terms and F is a formula of ZF_ε , then $t = u \mapsto F$ is a formula of ZF_ε .

The formula $t = u \mapsto \perp$ is denoted $t \neq u$.

The formula $t \neq u \rightarrow \perp$, i.e. $(t = u \mapsto \perp) \rightarrow \perp$ is denoted $t = u$.

The truth value of these new formulas is defined as follows, assuming that t, u, F are closed, with parameters in \mathcal{N} :

$$\|t = u \mapsto F\| = \emptyset \text{ if } t \neq u ; \|t = u \mapsto F\| = \|F\| \text{ if } t = u.$$

It follows that :

$$\begin{aligned} \|t \neq u\| &= \emptyset = \|\top\| \text{ if } t \neq u ; \|t \neq u\| = \Pi = \|\perp\| \text{ if } t = u ; \\ \|t = u\| &= \|\top \rightarrow \perp\| \text{ if } t \neq u ; \|t = u\| = \|\perp \rightarrow \perp\| \text{ if } t = u. \end{aligned}$$

Proposition 12 shows that $t = u \mapsto F$ and $t = u \rightarrow F$ are interchangeable.

Proposition 12.

- i) $\lambda x(x)I \Vdash (t = u \rightarrow F) \rightarrow (t = u \mapsto F)$;
- ii) $\lambda x \lambda y (\text{cc}) \lambda k (y)(k)x \Vdash (t = u \mapsto F), t = u \rightarrow F$.

i) Let $\xi \Vdash t = u \rightarrow F$ and $\pi \in \|t = u \mapsto F\|$. Thus, we have $t = u$ and $\pi \in \|F\|$.

We must show $\lambda x(x)I \star \xi \cdot \pi \in \perp$, that is $\xi \star I \cdot \pi \in \perp$. This is immediate, by hypothesis on ξ , since $I \Vdash t = u$.

ii) Let $\xi \Vdash t = u \mapsto F$, $\eta \Vdash t = u$ and $\pi \in \|F\|$. We must show that :

$$\lambda x \lambda y (\text{cc}) \lambda k (y)(k)x \star \xi \cdot \eta \cdot \pi \in \perp, \text{ soit } \eta \star \kappa_\pi \xi \cdot \pi \in \perp.$$

If $t \neq u$, then $\eta \Vdash \top \rightarrow \perp$, hence the result.

If $t = u$, then $\xi \Vdash F$, thus $\xi \star \pi \in \perp$, therefore $\kappa_\pi \xi \Vdash \perp$.

But we have $\eta \Vdash \perp \rightarrow \perp$, and therefore $\eta \star \xi \cdot \pi \in \perp$.

Q.E.D.

Proposition 13 shows that the formulas $t = u$ and $\forall x (u \notin x \rightarrow t \notin x)$ (*Leibniz equality*) are interchangeable.

Proposition 13.

- i) $I \Vdash t = u \mapsto \forall x (u \notin x \rightarrow t \notin x)$;
- ii) $I \Vdash \forall x (x \notin u \rightarrow x \notin t) \rightarrow t = u$.

i) It suffices to check that $I \Vdash \forall x(u \notin x \rightarrow t \notin x)$ when $t = u$, which is obvious.
ii) We must show that $I \Vdash \forall x(u \notin x \rightarrow t \notin x), t \neq u \rightarrow \perp$. Thus let $\xi \Vdash \forall x(u \notin x \rightarrow t \notin x)$, $\eta \Vdash t \neq u$ and $\pi \in \Pi$; we must show that $\xi \star \eta \bullet \pi \in \perp$.
We have $\xi \Vdash u \notin a \rightarrow t \notin a$ for every a ; we take $a = \{t\} \times \Pi$, thus $\|t \notin a\| = \Pi$, hence $\pi \in \|t \notin a\|$.
If $t = u$, we have $\eta \Vdash \perp$, thus $\eta \Vdash u \notin a$, hence the result.
If $t \neq u$, we have $\|u \notin a\| = \emptyset = \|\top\|$, thus $\eta \Vdash u \notin a$, hence the result.
Q.E.D.

We now show that the axioms of equality are realized.

Proposition 14. $I \Vdash \forall x(x = x)$; $I \Vdash \forall x \forall y(x = y \mapsto y = x)$;

$I \Vdash \forall x \forall y \forall z(x = y \mapsto (y = z \mapsto x = z))$;

$I \Vdash \forall x \forall y(x = y \mapsto (F[x] \rightarrow F[y]))$ for every formula F with one free variable, with parameters.

Trivial, by definition of \mapsto .

Q.E.D.

Conservation of well-foundedness

Theorem 15 says that every well founded relation in the ground model \mathcal{M} , gives a well founded relation in the realizability model \mathcal{N} .

Theorem 15. Let f be a binary function such that $f(x, y) = 1$ is a well founded relation in the ground model \mathcal{M} . Then, for every formula $F[x]$ of ZF_ε with parameters in \mathcal{M} :
 $\mathbf{Y} \Vdash \forall x(\forall y(f(y, x) = 1 \mapsto F[y]) \rightarrow F[x]) \rightarrow \forall x F[x]$.

Let us fix a and let $\xi \Vdash \forall x(\forall y(f(y, x) = 1 \mapsto F[y]) \rightarrow F[x])$. We show, by induction on a , following the well founded relation $f(x, y) = 1$, that $\mathbf{Y} \star \xi \bullet \pi \in \perp$ for every $\pi \in \|F[a]\|$. Thus, suppose that $\pi \in \|F[a]\|$; we need to show that $\xi \star \mathbf{Y} \xi \bullet \pi \in \perp$. By hypothesis, we have :

$\xi \Vdash \forall y(f(y, a) = 1 \mapsto y \notin z) \rightarrow F[a]$; thus, it suffices to show that :

$\mathbf{Y} \xi \Vdash f(y, a) = 1 \mapsto F[y]$ for every y . This is clear if $f(y, a) \neq 1$, by definition of \mapsto .

If $f(y, a) = 1$, we must show $\mathbf{Y} \xi \Vdash F[y]$, i.e. $\mathbf{Y} \star \xi \bullet \rho \in \perp$ for every $\rho \in \|F[y]\|$. But this follows from the induction hypothesis.

Q.E.D.

Sets in \mathcal{M} give type-like sets in \mathcal{N}

We define a unary function symbol \mathbf{J} by putting $\mathbf{J}(a) = a \times \Pi$ for every individual a (element of the ground model \mathcal{M}).

For each set E of the ground model \mathcal{M} , we also introduce the unary function 1_E with values in $\{0, 1\}$, defined as follows :

$1_E(a) = 1$ if $a \in E$; $1_E(a) = 0$ if $a \notin E$.

The formula $1_E(x) = 1 \mapsto A$ will also be denoted as $x \varepsilon \mathbf{J}E \mapsto A$.

In particular, $a \notin \mathbf{J}E$ is identical with $a \varepsilon \mathbf{J}E \mapsto \perp$ that is $1_E(a) \neq 1$.

We shall write $(\forall x \varepsilon \mathbf{J}E) A[x]$ for $\forall x(x \varepsilon \mathbf{J}E \mapsto A[x])$.

Proposition 12 shows that $x \varepsilon \mathfrak{J}E \mapsto A$ and $x \varepsilon \mathfrak{J}E \rightarrow A$ are interchangeable. Therefore $(\forall x \varepsilon \mathfrak{J}E) A[x]$ and $\forall x(x \varepsilon \mathfrak{J}E \rightarrow A[x])$ are also interchangeable. We have :

$$\|(\forall x \varepsilon \mathfrak{J}E) A[x]\| = \bigcup_{a \in E} \|A[a/x]\| \quad \text{and} \quad |(\forall x \varepsilon \mathfrak{J}E) A[x]| = \bigcap_{a \in E} |A[a/x]|.$$

As already said, we shall add to the language of ZF_ε , some function symbols of any arity, which will be interpreted in the ground model \mathcal{M} by some functional relations. Then every formula of the form $\forall \vec{x}(t_1[\vec{x}] = u_1[\vec{x}], \dots, t_k[\vec{x}] = u_k[\vec{x}] \rightarrow t[\vec{x}] = u[\vec{x}])$ which is satisfied in the model \mathcal{M} , is *realized* in the model \mathcal{N} ($t_1, u_1, \dots, t_k, u_k, t, u$ are terms of the language).

Indeed, we verify immediately that :

$$I \Vdash \forall \vec{x}(t_1[\vec{x}] = u_1[\vec{x}] \mapsto (\dots \mapsto (t_k[\vec{x}] = u_k[\vec{x}] \mapsto t[\vec{x}] = u[\vec{x}])) \dots).$$

It follows that if, for instance, $t[x, x']$ sends $E \times E'$ into D in the model \mathcal{M} , then it sends $\mathfrak{J}E \times \mathfrak{J}E'$ into $\mathfrak{J}D$ in the model \mathcal{N} . Indeed, we have then :

$\mathcal{M} \models \forall x \forall x'(1_E(x) = 1, 1_{E'}(x') = 1 \rightarrow 1_D(t[x, x']) = 1)$ and therefore, we have :

$$I \Vdash \forall x \forall x'(1_E(x) = 1 \mapsto (1_{E'}(x') = 1 \mapsto 1_D(t[x, x']) = 1)), \text{ in other words :}$$

$$I \Vdash (\forall x \varepsilon \mathfrak{J}E)(\forall x' \varepsilon \mathfrak{J}E')(t[x, x'] \varepsilon \mathfrak{J}D).$$

Notice, in particular, that the characteristic function 1_E , which takes its values in the set $2 = \{0, 1\}$ in the model \mathcal{M} , takes its values in $\mathfrak{J}2$ in the realizability model \mathcal{N} .

We shall denote \wedge, \vee, \neg the (trivial) Boolean algebra operations in $\{0, 1\}$ (they should not be confused with the logical connectives \wedge, \vee, \neg). In this way, we have defined three function symbols of the language of ZF_ε ; thus, in the realizability model \mathcal{N} , they define a *Boolean algebra structure* on the set $\mathfrak{J}2$.

The set $\tilde{\mathfrak{N}}$ of integers in \mathcal{N}

We add to the language of ZF_ε a constant symbol 0 and a unary function symbol s . Their interpretation in the model \mathcal{M} is as follows :

0 is \emptyset ; $s(a)$ is $\{a\} \times \Pi$ for every set a , in other words $s(a) = \mathfrak{J}(\{a\})$.

Remark. In the definition of the set of integers in the realizability model \mathcal{N} , we are using the singleton as the successor function s , instead of the usual one $x \mapsto x \cup \{x\}$, which is more complicated to define in the realizability model. It would give :

$$s(a) = \{(a, \underline{1} \cdot \pi); \pi \in \Pi\} \cup \{(x, \underline{0} \cdot \pi); x \in a, \pi \in \Pi\}.$$

Theorem 16. *The following formulas are realized in \mathcal{N} :*

- i) $\forall x \forall y (sx = sy \mapsto x = y)$;
- ii) $\forall x (sx \neq 0)$;
- iii) $\forall x \forall y (x \simeq y \rightarrow sx \simeq sy)$;
- iv) $\forall x \forall y (sx \simeq sy \rightarrow x \simeq y)$.

This shows, in particular, that the function s is *compatible with the extensional equivalence* \simeq .

i) We check that $I \Vdash sa = sb \mapsto a = b$. We may suppose $sa = sb$, because $\|sa = sb \mapsto a = b\| = \emptyset$ if $sa \neq sb$. But, in this case, we have $a = b$, by definition of sa, sb .

ii) We have $\|a \notin sa\| = \|\perp\|$, thus $I \Vdash a \varepsilon sa$. Since $\mathcal{N} \Vdash \forall x \forall y (y \notin x \rightarrow y \notin x)$, we have : $\mathcal{N} \Vdash a \in sa$. But $I \Vdash a \notin 0$, and therefore $\mathcal{N} \Vdash (a \notin 0 \rightarrow a \notin sa) \rightarrow \perp$; thus $\mathcal{N} \Vdash \forall x (x \notin 0 \rightarrow x \notin sa) \rightarrow \perp$, i.e. $\mathcal{N} \Vdash sa \not\subseteq 0$. Therefore, $\mathcal{N} \Vdash sa \neq 0$.

iii) We show that the formula $a \simeq b \rightarrow sa \simeq sb$ is realized ; it suffices to realize the formula $a \simeq b \rightarrow sa \subseteq sb$. We prove it by means of already realized sentences.

We need to prove $a \simeq b, x \notin sb \rightarrow x \notin sa$. But $x \notin sa$ has the same truth value as $x \neq a$. Thus, we simply have to prove $a \simeq b \rightarrow a \in sb$. But $a \in sb$ follows from $b \varepsilon sb$ and $a \simeq b$.

iv) In the same way, we prove the formula $sa \simeq sb \rightarrow a \simeq b$ and, in fact $sa \subseteq sb \rightarrow a \simeq b$. The formula $sa \subseteq sb$ is $\forall x (x \notin sb \rightarrow x \notin sa)$; but $x \notin sa$ is the same as $x \neq a$. Thus, from $sa \subseteq sb$ we obtain $a \in sb$, i.e. $(\exists x \varepsilon sb) x \simeq a$. But $x \varepsilon sb$ is the same as $x = b$, so that we obtain $a \simeq b$.

Q.E.D.

The individuals $s^n 0$ are obviously distinct, for $n \in \mathbb{N}$. Therefore, we can define :

$$\tilde{\mathbb{N}} = \{(s^n 0, \underline{n} \cdot \pi); n \in \mathbb{N}, \pi \in \Pi\}$$

and we have :

$\|a \notin \tilde{\mathbb{N}}\| = \emptyset$ if a is not of the form $s^n 0$, with $n \in \mathbb{N}$;

$\|s^n 0 \notin \tilde{\mathbb{N}}\| = \{\underline{n} \cdot \pi; \pi \in \Pi\}$.

The formula $x \varepsilon \tilde{\mathbb{N}}$ will also be written $\text{ent}(x)$.

In the sequel, we shall use the restricted quantifier $\forall x \varepsilon \tilde{\mathbb{N}}$, which we also write $\forall x^{\text{ent}}$, with the following meaning :

$\|\forall x^{\text{ent}} F[x]\| = \|(\forall x \varepsilon \tilde{\mathbb{N}}) F[x]\| = \{\underline{n} \cdot \pi; n \in \mathbb{N}, \pi \in \|F[s^n 0]\|\}$.

The restricted existential quantifier $\exists x \varepsilon \tilde{\mathbb{N}}$ or $\exists x^{\text{ent}}$ is defined as :

$\exists x^{\text{ent}} F[x] \equiv (\exists x \varepsilon \tilde{\mathbb{N}}) F[x] \equiv \neg \forall x^{\text{ent}} \neg F[x]$.

Proposition 17 shows that these quantifiers have indeed the intended meaning : the formulas $\forall x^{\text{ent}} F[x]$ and $\forall x (x \varepsilon \tilde{\mathbb{N}} \rightarrow F[x])$ are interchangeable.

Proposition 17.

i) $\lambda x \lambda y \lambda z (y)(x)z \Vdash \forall x^{\text{ent}} F[x] \rightarrow \forall x (\neg F[x] \rightarrow x \notin \tilde{\mathbb{N}})$;

ii) $\lambda x \lambda y (\text{cc}) \lambda k(x) k y \Vdash \forall x (\neg F[x] \rightarrow x \notin \tilde{\mathbb{N}}) \rightarrow \forall x^{\text{ent}} F[x]$.

i) Let $\xi \Vdash \forall x^{\text{ent}} F[x]$, $\eta \Vdash \neg F[a]$ and $\varpi \in \|a \notin \tilde{\mathbb{N}}\|$. Thus, we have $a = s^n 0$ for some $n \in \mathbb{N}$ (else $\|a \notin \tilde{\mathbb{N}}\| = \emptyset$) and $\varpi = \underline{n} \cdot \pi$. We must show that $\eta \star \xi \underline{n} \cdot \pi \in \perp$.

Now, by hypothesis on ξ , we have $\xi \star \underline{n} \cdot \rho \in \perp$ for any $\rho \in \|F[s^n 0]\|$; thus $\xi \underline{n} \Vdash F[s^n 0]$. Since $\eta \Vdash \neg F[s^n 0]$, we have $\eta \star \xi \underline{n} \cdot \pi \in \perp$, which is the desired result.

ii) Let $\xi \Vdash \forall x (\neg F[x] \rightarrow x \notin \tilde{\mathbb{N}})$ and $\underline{n} \cdot \pi \in \|\forall x^{\text{ent}} F[x]\|$, with $n \in \mathbb{N}$ and $\pi \in \|F[s^n 0]\|$.

We have : $\lambda x \lambda y (\text{cc}) \lambda k(x) k y \star \xi \cdot \underline{n} \cdot \pi \succ \xi \star k_\pi \cdot \underline{n} \cdot \pi$.

Now, we have $k_\pi \Vdash \neg F[s^n 0]$ and $\underline{n} \cdot \pi \in \|s^n 0 \notin \tilde{\mathbb{N}}\|$. Therefore $\xi \star k_\pi \cdot \underline{n} \cdot \pi \in \perp$.

Q.E.D.

Theorem 18 (Recurrence scheme). *For every formula $F[\vec{x}, y]$:*

i) $I \Vdash \forall \vec{x} (\forall y (F[\vec{x}, y] \rightarrow F[\vec{x}, sy]), F[\vec{x}, 0] \rightarrow (\forall n \varepsilon \tilde{\mathbb{N}}) F[\vec{x}, n])$.

ii) $I \Vdash \forall \vec{x} (\forall n \varepsilon \tilde{\mathbb{N}}) (\forall y (F[\vec{x}, sy] \rightarrow F[\vec{x}, y]), F[\vec{x}, n] \rightarrow F[\vec{x}, 0])$.

i) This can be written $I \Vdash (\forall n \varepsilon \tilde{\mathbb{N}}) \forall \vec{x} (\forall y (F[\vec{x}, y] \rightarrow F[\vec{x}, sy]), F[\vec{x}, 0] \rightarrow F[\vec{x}, n])$.

Thus, let $n \in \mathbb{N}$, \vec{a} a sequence of d'individuals, $\xi \Vdash \forall y (F[\vec{a}, y] \rightarrow F[\vec{a}, sy])$, $\alpha \Vdash F[\vec{a}, 0]$.

We must show that, for every $\pi \in \|F[\vec{a}, n]\|$, we have $I \star \underline{n} \cdot \xi \cdot \alpha \cdot \pi \in \perp$.

In fact, we show, by recurrence on n , that $\underline{n} \star \xi \cdot \alpha \cdot \pi \in \perp$.

This is immediate if $n = 0$. In order to go from n to $n+1$, we suppose now $\pi \in \|F[\vec{a}, sn]\|$; we have $\underline{n+1} \star \xi \cdot \alpha \cdot \pi \succ \underline{n+1} \star \xi \cdot \alpha \cdot \pi \succ \sigma \star \underline{n} \cdot \xi \cdot \alpha \cdot \pi \succ \xi \star \underline{n} \xi \alpha \cdot \pi$.

But, by the recurrence hypothesis, we have $\underline{n} \xi \alpha \Vdash F[\vec{a}, n]$; by hypothesis on ξ , we have : $\xi \Vdash F[\vec{a}, n] \rightarrow F[\vec{a}, sn]$. Hence the result, since $\pi \in \|F[\vec{a}, sn]\|$.

ii) Almost the same proof : take now $\xi \Vdash \forall y (F[\vec{a}, sy] \rightarrow F[\vec{a}, y])$, $\pi \in \|F[\vec{a}, 0]\|$ and show, by recurrence on n , that $\underline{n} \star \xi \cdot \alpha \cdot \pi \in \perp$, for every $\alpha \Vdash F[\vec{a}, n]$.

Q.E.D.

Definition. We denote by $\text{int}(n)$ the formula $\forall x (\forall y (sy \not\leq x \rightarrow y \not\leq x), n \not\leq x \rightarrow 0 \not\leq x)$.

Theorem 20 shows that the formulas $\text{int}(n)$ and $n \in \tilde{\mathbb{N}}$ are interchangeable, i.e. the formula $\forall n (\text{int}(n) \leftrightarrow n \in \tilde{\mathbb{N}})$ is realized by a proof-like term : this is the *storage theorem for integers*.

Lemma 19. $\lambda g \lambda x (g)(\sigma)x \Vdash \forall y (sy \not\leq \tilde{\mathbb{N}} \rightarrow y \not\leq \tilde{\mathbb{N}})$.

We show that $\lambda g \lambda x (g)(\sigma)x \Vdash sb \not\leq \tilde{\mathbb{N}} \rightarrow b \not\leq \tilde{\mathbb{N}}$ for every individual b .

This is obvious if b is not of the form $s^n 0$, since then $\|b \not\leq \tilde{\mathbb{N}}\| = \emptyset$. Thus, it remains to show :

$\lambda g \lambda x (g)(\sigma)x \Vdash s^{n+1} 0 \not\leq \tilde{\mathbb{N}} \rightarrow s^n 0 \not\leq \tilde{\mathbb{N}}$. Thus, let $\xi \Vdash s^{n+1} 0 \not\leq \tilde{\mathbb{N}}$; we must show :

$\lambda g \lambda x (g)(\sigma)x \star \xi \cdot \underline{n} \cdot \pi \in \perp$, i.e. $\xi \star \sigma \underline{n} \cdot \pi \in \perp$, which is clear, since $\sigma \underline{n} = \underline{n+1}$.

Q.E.D.

Theorem 20 (Storage theorem).

i) $I \Vdash (\forall x \in \tilde{\mathbb{N}}) \text{int}(x)$.

ii) $T \Vdash \forall x (\text{int}(x), x \not\leq \tilde{\mathbb{N}} \rightarrow \perp)$ with $T = \lambda n \lambda f ((n) \lambda g \lambda x (g)(\sigma)x) f 0$.

i) It is theorem 18(ii), if we take for $F[x, y]$ the formula $y \not\leq x$.

ii) Let $\nu \Vdash \text{int}(a)$, $\phi \Vdash a \not\leq \tilde{\mathbb{N}}$ and $\pi \in \Pi$. We must show $T \star \nu \cdot \phi \cdot \pi \in \perp$, that is : $\nu \star \lambda g \lambda x (g)(\sigma)x \cdot \phi \cdot \underline{0} \cdot \pi \in \perp$.

By hypothesis, we have $\nu \Vdash \forall y (sy \not\leq \tilde{\mathbb{N}} \rightarrow y \not\leq \tilde{\mathbb{N}})$, $a \not\leq \tilde{\mathbb{N}} \rightarrow 0 \not\leq \tilde{\mathbb{N}}$.

But we have $\underline{0} \cdot \pi \in \|0 \not\leq \tilde{\mathbb{N}}\|$ by definition of $\tilde{\mathbb{N}}$ and, by lemma 19 :

$\lambda g \lambda x (g)(\sigma)x \Vdash \forall y (sy \not\leq \tilde{\mathbb{N}} \rightarrow y \not\leq \tilde{\mathbb{N}})$. Hence the result.

Q.E.D.

From theorem 18(i), it follows immediately that the *recurrence scheme of ZF* is realized in \mathcal{N} ; it is the scheme :

$\forall \vec{x} (\forall y (F[\vec{x}, y] \rightarrow F[\vec{x}, sy]), F[\vec{x}, 0] \rightarrow (\forall n \in \tilde{\mathbb{N}}) F[\vec{x}, n])$ for every formula $F[\vec{x}, y]$ of ZF (i.e. written with $\not\leq, \subseteq, 0, s$).

Then, indeed, the formula F is compatible with the extensional equivalence \simeq .

Since the function s is compatible with \simeq , we deduce from lemma 19 that the formula : $\forall y (y \in \tilde{\mathbb{N}} \rightarrow sy \in \tilde{\mathbb{N}})$ is realized in \mathcal{N} ; the formula $0 \in \tilde{\mathbb{N}}$ is also obviously realized.

From the recurrence scheme just proved, we deduce that :

$\tilde{\mathbb{N}}$ is the set of integers of the model \mathcal{N} , considered as a model of ZF.

Theorem 21.

i) Let $f : \mathbb{N}^k \rightarrow \mathbb{N}$ be a recursive function. Then, the formula :

$(\forall x_1 \in \tilde{\mathbb{N}}) \dots (\forall x_k \in \tilde{\mathbb{N}}) f(x_1, \dots, x_k) \in \tilde{\mathbb{N}}$ is realized in \mathcal{N} .

ii) Let $g : \mathbb{N}^k \rightarrow 2$ be a recursive function. Then, the formula :

$(\forall x_1 \in \tilde{\mathbb{N}}) \dots (\forall x_k \in \tilde{\mathbb{N}}) (g(x_1, \dots, x_k) = 1 \vee g(x_1, \dots, x_k) = 0)$ is realized in \mathcal{N} .

i) This can be written $\forall x_1^{\text{ent}} \dots \forall x_k^{\text{ent}} \text{ent}(f(x_1, \dots, x_k))$. The proof is done in [16, 13].

ii) We have $\mathcal{N} \Vdash (\forall x_1 \in \mathbb{J}\mathbb{N}) \dots (\forall x_k \in \mathbb{J}\mathbb{N}) g(x_1, \dots, x_k) \in \mathbb{J}2$.

Now, since g is recursive, we have, by (i) :

$\mathcal{N} \Vdash (\forall x_1 \in \tilde{\mathbb{N}}) \dots (\forall x_k \in \tilde{\mathbb{N}}) g(x_1, \dots, x_k) \in \tilde{\mathbb{N}}$.

Hence the result, by lemma 22.

Q.E.D.

Lemma 22. $\lambda x \lambda y \lambda f(f)xy \Vdash (\forall x \in \mathbb{J}2)(x \neq 1, x \neq 0 \rightarrow x \notin \tilde{\mathbb{N}})$.

We have to show :

$\lambda x \lambda y \lambda f(f)xy \Vdash \top, \perp \rightarrow 0 \notin \tilde{\mathbb{N}}$ and $\lambda x \lambda y \lambda f(f)xy \Vdash \perp, \top \rightarrow 1 \notin \tilde{\mathbb{N}}$.

Thus let $\xi \Vdash \top$ (i.e. $\xi \in \Lambda$ arbitrary) and $\eta \Vdash \perp$. We have to show :

$\lambda x \lambda y \lambda f(f)xy \star \xi \cdot \eta \cdot \underline{0} \cdot \pi \in \underline{\perp}$ and $\lambda x \lambda y \lambda f(f)xy \star \eta \cdot \xi \cdot \underline{1} \cdot \pi \in \underline{\perp}$

which is trivial.

Q.E.D.

Remarks. i) In the present paper, theorem 21 is used only in trivial particular cases.

ii) Let us recall the difference between $\mathbb{J}\mathbb{N}$ and $\tilde{\mathbb{N}}$ (the set of integers in the model \mathcal{N}) ; we have :

$\xi \Vdash (\forall x \in \mathbb{J}\mathbb{N}) F[x]$ iff $(\forall n \in \mathbb{N})(\forall \pi \in \|F[s^n 0]\|) \xi \star \pi \in \underline{\perp}$.

$\xi \Vdash (\forall x \in \tilde{\mathbb{N}}) F[x]$ iff $(\forall n \in \mathbb{N})(\forall \pi \in \|F[s^n 0]\|) \xi \star \underline{n} \star \pi \in \underline{\perp}$.

Notice that we have $K \Vdash \forall x(x \notin \mathbb{J}\mathbb{N} \rightarrow x \notin \tilde{\mathbb{N}})$, in other words $K \Vdash \tilde{\mathbb{N}} \subset \mathbb{J}\mathbb{N}$. This means that, in \mathcal{N} , the set $\tilde{\mathbb{N}}$ of integers is strongly included in $\mathbb{J}\mathbb{N}$. In the particular realizability model considered below (and, in fact, in every non trivial realizability model) the formula $\mathbb{J}\mathbb{N} \not\subset \tilde{\mathbb{N}}$ is realized.

Non extensional and dependent choice

For each formula $F(x, y_1, \dots, y_m)$ of ZF_ε , we add a function symbol f_F of arity $m + 1$,

with the axiom : $\forall \vec{y}((\forall k \in \tilde{\mathbb{N}}) F[f_F(k, \vec{y}), \vec{y}] \rightarrow \forall x F[x, \vec{y}])$

or else : $\forall \vec{y}(\forall k^{\text{ent}} F[f_F(k, \vec{y}), \vec{y}] \rightarrow \forall x F[x, \vec{y}])$.

It is the *axiom scheme of non extensional choice*, in abbreviated form NEAC.

Remarks. i) The axiom scheme NEAC does not imply the axiom of choice in ZF, because we do not suppose that the symbol f_F is compatible with the extensional equivalence \simeq . It is the reason why we speak about *non extensional* axiom of choice. On the other hand, as we show below, it implies DC (the axiom of dependent choice).

ii) It seems that we could take for f_F a m -ary function symbol and use the following simpler (and logically equivalent) axiom scheme NEAC' : $\forall \vec{y}(F[f_F(\vec{y}), \vec{y}] \rightarrow \forall x F[x, \vec{y}])$.

But this axiom scheme cannot be realized, even though the axiom scheme NEAC is realized by a very simple proof-like term (theorem 23), *provided the instruction ς is present*.

More precisely, we can define a function f_F in \mathcal{M} , such that NEAC is realized in \mathcal{N} , but this is impossible for NEAC'.

Theorem 23 (NEAC).

For each closed formula $\forall x \forall \vec{y} F$, we can define a $(m+1)$ -ary function symbol f_F such that :

$$\lambda x(\zeta)xx \Vdash \forall \vec{y} (\forall k^{\text{ent}} F[f_F(k, \vec{y})/x, \vec{y}] \rightarrow \forall x F[x, \vec{y}]).$$

For each $k \in \mathbb{N}$ we put $P_k = \{\pi \in \Pi; \xi \star \underline{k} \cdot \pi \notin \perp, k = n_\xi\}$.

For each individual x , we have : $\|\forall x F[x, \vec{y}]\| = \bigcup^a \|F[a, \vec{y}]\|$.

Therefore, there exists a function f_F such that, given $k \in \mathbb{N}$ and \vec{y} such that $P_k \cap \|\forall x F[x, \vec{y}]\| \neq \emptyset$, we have $P_k \cap \|F[f_F(k, \vec{y}), \vec{y}]\| \neq \emptyset$.

Now, we want to show $\lambda x(\zeta)xx \Vdash \forall k^{\text{ent}} F[f_F(k, \vec{y}), \vec{y}] \rightarrow F[x, \vec{y}]$, for every individuals x, \vec{y} .

Thus, let $\xi \Vdash \forall k^{\text{ent}} F[f_F(k, \vec{y}), \vec{y}]$ and $\pi \in \|F[a, \vec{y}]\|$; we must show $\lambda x(\zeta)xx \star \xi \cdot \pi \in \perp$.

If this is false, we have $\zeta \star \xi \cdot \xi \cdot \pi \notin \perp$ and therefore $\xi \star \underline{j} \cdot \pi \notin \perp$ with $j = n_\xi$.

It follows that $\pi \in P_j \cap \|F[a, \vec{y}]\|$; thus, there exists $\pi' \in P_j \cap \|F[f_F(j, \vec{y}), \vec{y}]\|$.

Now, we have $\underline{j} \cdot \pi' \in \|\forall k^{\text{ent}} F[f_F(k, \vec{y}), \vec{y}]\|$, and therefore, by hypothesis on ξ , we have : $\xi \star \underline{j} \cdot \pi' \in \perp$. This is a contradiction.

Q.E.D.

NEAC implies DC

Let us call DCS (dependent choice scheme) the following axiom scheme :

$$\forall \vec{z} (\forall x \exists y F[x, y, \vec{z}] \rightarrow \forall n^{\text{ent}} \exists ! y S_F[n, y, \vec{z}] \wedge \forall n^{\text{ent}} \exists y \exists y' \{S_F[n, y, \vec{z}], S_F[sn, y', \vec{z}], F[y, y', \vec{z}]\}).$$

where F is a formula of ZF_ε with free variables x, y, \vec{z} ; the formula S_F is written below.

In the following, we omit the variables \vec{z} (the parameters), for sake of simplicity.

The usual axiom of dependent choice DC is obtained by taking for $F[x, y, z_0, z_1]$ the formula $y \varepsilon z_0 \wedge (x \varepsilon z_0 \rightarrow \langle x, y \rangle \varepsilon z_1)$.

We now show how to define the formula S_F , so that $\text{ZF}_\varepsilon, \text{NEAC} \vdash \text{DCS}$; we shall conclude that DC is realized.

So, let us assume $\forall x \exists y F[x, y]$. By NEAC, there is a function symbol f such that :

$\forall x \exists k^{\text{ent}} F[x, f(k, x)]$. We define the formula $R_F[x, y]$ as follows :

$$R_F[x, y] \equiv \exists k^{\text{ent}} \{F[x, f(k, x)], \forall i^{\text{ent}} (i < k \rightarrow \neg F[x, f(i, x)]), y = f(k, x)\}.$$

This means : “ $y = f(k, x)$ for the first integer k such that $F[x, f(k, x)]$ ”.

Therefore, R_F is functional, i.e. we have $\forall x \exists ! y R_F(x, y)$.

S_F is defined so as to represent a sequence obtained by iteration of the function given by R_F , beginning (arbitrarily) at 0 :

$$S_F(n, x) \equiv \forall z [\forall m \forall y \forall y' (\langle m, y \rangle \varepsilon z, R_F(y, y') \rightarrow \langle sm, y' \rangle \varepsilon z), \langle 0, 0 \rangle \varepsilon z \rightarrow \langle n, x \rangle \varepsilon z].$$

It should be clear that, with this definition of S_F , we obtain :

$$\forall n^{\text{ent}} \exists ! y S_F[n, y] \text{ and } \forall n^{\text{ent}} \exists y \exists y' \{S_F[n, y], S_F[sn, y'], F[y, y']\}.$$

Thus, DCS is provable from ZF_ε and NEAC.

Remark. We have used the binary function symbol $\langle x, y \rangle$ which is defined, in the ground model \mathcal{M} , in the usual way : $\langle a, b \rangle = \{\{a\}, \{a, b\}\}$. Then, the formulas :

$$\forall x \forall x' \forall y \forall y' (\langle x, y \rangle = \langle x', y' \rangle \rightarrow x = x'), \quad \forall x \forall x' \forall y \forall y' (\langle x, y \rangle = \langle x', y' \rangle \rightarrow y = y'),$$

are trivially realized by I .

Properties of the Boolean algebra $\mathfrak{J}2$

Let $(x < y)$ be the binary recursive function defined as follows in \mathcal{M} :
 $(m < n) = 1$ if $m, n \in \mathbb{N}$, $m < n$; else $(m < n) = 0$.

Theorem 24. *For every choice of \perp , the relation $(x < y) = 1$ is, in \mathcal{N} , a strict well founded partial order, which is the usual order on integers (i.e. on $\tilde{\mathbb{N}}$).*

Indeed, the formulas $\forall x((x < x) \neq 1)$; $\forall x \forall y((x < y) = 1 \leftrightarrow (y < x) \neq 1)$ and $\forall x \forall y \forall z((x < y) = 1 \leftrightarrow ((y < z) = 1 \leftrightarrow (x < z) = 1))$ are trivially realized.

Moreover, since the relation $(x < y) = 1$ is well founded, we have (theorem 15) :

$\mathbf{Y} \Vdash \forall x(\forall y((y < x) = 1 \leftrightarrow F[y]) \rightarrow F[x]) \rightarrow \forall x F[x]$

for every formula $F[x]$ with parameters and one free variable.

By theorem 21(ii), the binary recursive function $(x < y)$ sends $\tilde{\mathbb{N}}^2$ into $\{0, 1\}$, in the model \mathcal{N} .

Therefore, it suffices to check that the following formulas are realized in \mathcal{N} :

$(\forall x, y \in \tilde{\mathbb{N}})(y \leq x \rightarrow (x < y) \neq 1)$; $(\forall x, y \in \tilde{\mathbb{N}})(x < y \rightarrow (x < y) = 1)$.

Now the following formulas are trivially realized :

$(\forall x, y, z \in \mathfrak{J}\mathbb{N})(x = y + z \rightarrow (x < y) \neq 1)$; $(\forall x, y, z \in \mathfrak{J}\mathbb{N})(y = x + z + 1 \rightarrow (x < y) = 1)$.

Q.E.D.

Theorem 25.

The following formulas are realized in \mathcal{N} :

i) $(\forall x \in \mathfrak{J}\mathbb{N})(\forall m \in \mathfrak{J}\mathbb{N})((x < m) = 1 \leftrightarrow x \in \mathfrak{J}m)$;

ii) $(\forall m \in \mathfrak{J}\mathbb{N})(\forall n \in \mathfrak{J}\mathbb{N})((m < n) = 1 \rightarrow \mathfrak{J}m \subset \mathfrak{J}n)$.

iii) $(\forall x \in \mathfrak{J}\mathbb{N})(\forall m \in \mathfrak{J}\mathbb{N})((x < m) = 1 \leftrightarrow (\exists y \in \mathfrak{J}\mathbb{N})(m = x + y + 1))$.

Recall that $x \subset y$ is the formula $\forall z(z \notin y \rightarrow z \notin x)$.

i) We have trivially $\|(a < m) \neq 1\| = \|a \notin \mathfrak{J}m\|$ for every $a, m \in \mathbb{N}$.

ii) By transitivity of the relation $(m < n) = 1$ (theorem 24).

iii) We observe that $\|(a < m) \neq 1\| = \|(\forall y \in \mathfrak{J}\mathbb{N})(m \neq a + y + 1)\|$ for every $a, m \in \mathbb{N}$.

Q.E.D.

For each $n \in \mathfrak{J}\mathbb{N}$ (and, in particular, for each $n \in \tilde{\mathbb{N}}$, i.e. for each integer of \mathcal{N}), the set defined, in \mathcal{N} , by $(x < n) = 1$ (the strict initial segment defined by n) is therefore extensionally equivalent to $\mathfrak{J}n$.

Theorem 26. *In \mathcal{N} , the application $(x, y) \mapsto my + x$ is a bijection from $\mathfrak{J}m \times \mathfrak{J}n$ onto $\mathfrak{J}(mn)$. Indeed, the following formulas are realized in \mathcal{N} by I :*

i) $(\forall m, n \in \mathfrak{J}\mathbb{N})(\forall x \in \mathfrak{J}m)(\forall y \in \mathfrak{J}n)(my + x \in \mathfrak{J}(mn))$;

ii) $(\forall m, n \in \mathfrak{J}\mathbb{N})(\forall x, x' \in \mathfrak{J}m)(\forall y, y' \in \mathfrak{J}n)(my + x = my' + x' \leftrightarrow x = x')$;

$(\forall m, n \in \mathfrak{J}\mathbb{N})(\forall x, x' \in \mathfrak{J}m)(\forall y, y' \in \mathfrak{J}n)(my + x = my' + x' \leftrightarrow y = y')$;

iii) $(\forall m, n \in \mathfrak{J}\mathbb{N})(\forall z \in \mathfrak{J}(mn))(\exists x \in \mathfrak{J}m)(\exists y \in \mathfrak{J}n) z = my + x$.

i) and ii) We simply have to replace $(\forall m \in \mathfrak{J}\mathbb{N})$ and $(\forall x \in \mathfrak{J}m)$ with their definitions, which are : $(\forall m \in \mathfrak{J}\mathbb{N}) F \equiv \forall m(1_{\mathbb{N}}(m) = 1 \leftrightarrow F)$; $(\forall x \in \mathfrak{J}m) F \equiv \forall x((x < m) = 1 \leftrightarrow F)$.

We see immediately that these two formulas are realized by I .

iii) We show that :

$$I \Vdash (\forall m, n, z \in \mathbb{N})((\forall x, y \in \mathbb{N})((x < m) = 1 \mapsto ((y < n) = 1 \mapsto z \neq my + x)) \rightarrow (z < mn) \neq 1).$$

Thus, we consider :

$$m, n, z_0 \in \mathbb{N} ; \xi \in \Lambda, \xi \Vdash \forall x \forall y ((x < m) = 1 \mapsto ((y < n) = 1 \mapsto z \neq my + x))$$

and $\pi \in \|(z_0 < mn) \neq 1\|$. We must show $I \star \xi \cdot \pi \in \perp$, that is $\xi \star \pi \in \perp$.

We have $\|(z_0 < mn) \neq 1\| \neq \emptyset$, therefore $z_0 < mn$.

Thus, there exists $x_0, y_0 \in \mathbb{N}, x_0 < m, y_0 < n$ such that $z_0 = mx_0 + y_0$. Now, by hypothesis on ξ , we have :

$$\xi \Vdash (x_0 < m) = 1 \mapsto ((y_0 < n) = 1 \mapsto z_0 \neq my_0 + x_0), \text{ in other words } \xi \Vdash \perp.$$

Q.E.D.

Injection of \mathbb{N} into $\mathcal{P}(\tilde{\mathbb{N}})$

Recall that we have fixed a recursive bijection : $\xi \mapsto n_\xi$ from Λ onto \mathbb{N} . The inverse bijection will be denoted $n \mapsto \xi_n$.

This bijection is used in the execution rule of the instruction ς , which is as follows :

$$\varsigma \star \xi \cdot \eta \cdot \pi \succ \xi \star \eta \cdot \pi.$$

We define, in \mathcal{M} , a function $\Delta : \mathbb{N} \rightarrow 2$ by putting $\Delta(n) = 0 \Leftrightarrow \xi_n \Vdash \perp$.

Thus, we have defined a function symbol Δ , in the language of ZF_ε . In the realizability model \mathcal{N} , the symbol Δ represents a function from \mathbb{N} into $\mathbb{2}$. In particular, the function Δ sends the set $\tilde{\mathbb{N}}$ of integers of the model \mathcal{N} into the Boolean algebra $\mathbb{2}$.

Lemma 27. *For every $n \in \mathbb{N}$, we have $\xi_n \Vdash \Delta(n) \neq 0$.*

We write this as $\Delta(n) = 0 \Rightarrow \xi_n \Vdash \perp$, which follows from the definition of Δ .

Q.E.D.

Theorem 28. *Let us put $\theta = \lambda x \lambda y (\varsigma) y x x$; then, we have :*

$$\theta \Vdash (\forall x \in \mathbb{2})(x \neq 0 \rightarrow \exists n^{\text{ent}} \{\Delta(n) \neq 0, \Delta(n) \leq x\})$$

where \leq is the order relation of the Boolean algebra $\mathbb{2}$: $y \leq x$ is the formula $x = (y \vee x)$.

We must show $\theta \Vdash (\forall x \in \mathbb{2})(x \neq 0, \forall n^{\text{ent}} (\Delta(n) \neq 0 \rightarrow x \neq \Delta(n) \vee x) \rightarrow \perp)$.

Thus, let $a \in \{0, 1\}$, $\xi \Vdash a \neq 0$, $\eta \Vdash \forall n^{\text{ent}} (\Delta(n) \neq 0 \rightarrow a \neq \Delta(n) \vee a)$ and $\pi \in \Pi$.

We must show $\theta \star \xi \cdot \eta \cdot \pi \in \perp$ that is $\varsigma \star \eta \cdot \xi \cdot \xi \cdot \pi \in \perp$, or else $\eta \star \xi \cdot \xi \cdot \pi \in \perp$.

By hypothesis on η , it suffices to show $n_\xi \cdot \xi \cdot \pi \in \|\forall n^{\text{ent}} (\Delta(n) \neq 0 \rightarrow a \neq \Delta(n) \vee a)\|$.

By lemma 27, we have $\xi \Vdash \Delta(n_\xi) \neq 0$. By definition of the quantifier $\forall n^{\text{ent}}$, it remains to show $\pi \in \|a \neq \Delta(n_\xi) \vee a\|$ or else $a = \Delta(n_\xi) \vee a$.

This is obvious if $a = 1$; if $a = 0$, then $\xi \Vdash \perp$, by hypothesis on ξ .

Therefore $\Delta(n_\xi) = 0$ by definition of Δ , hence the result.

Q.E.D.

By theorem 28, the set $\{\Delta(n); n \in \tilde{\mathbb{N}}, \Delta(n) \neq 0\}$ is, in the realizability model \mathcal{N} , a countable dense subset of the Boolean algebra $\mathbb{2}$: this means that each element $\neq 0$ of this Boolean algebra has a lower bound of the form $\Delta(n) \neq 0$ with $n \in \tilde{\mathbb{N}}$.

It follows that the application of $\mathbb{2}$ into $\mathcal{P}(\tilde{\mathbb{N}})$ given by :

$$x \mapsto \{n \in \tilde{\mathbb{N}}; \Delta(n) \leq x, \Delta(n) \neq 0\}$$

is one to one : indeed, if $a, b \in \mathbb{J}2$ with $a \neq b$, then $a + b \neq 0$; thus, there exists an integer $n \in \tilde{\mathbb{N}}$ such that $\Delta(n) \neq 0$ and $\Delta(n) \leq a + b$. Therefore, we have $\Delta(n) \leq a$ iff $(b \wedge \Delta(n)) = 0$.

But, since $\Delta(n) \neq 0$, we get : $\Delta(n) \leq a$ iff $\Delta(n) \not\leq b$.

We have shown :

Theorem 29.

The formula : “ there exists an injection of $\mathbb{J}2$ into $\mathcal{P}(\tilde{\mathbb{N}})$ ” is realized in the model \mathcal{N} .

Corollary 30. *The formula : “ for every integer n there exists an injection of $\mathbb{J}n$ into $\mathcal{P}(\tilde{\mathbb{N}})$ ” is realized in the model \mathcal{N} .*

Using theorem 26 we see, by recurrence on m , that the model \mathcal{N} realizes the formula :

“ $(\forall m \in \tilde{\mathbb{N}}) (\mathbb{J}2)^m$ is equipotent to $\mathbb{J}(2^m)$ ” ; and therefore also the formula :

“ $(\forall m \in \tilde{\mathbb{N}})$ there exists an injection of $\mathbb{J}(2^m)$ into $\mathcal{P}(\tilde{\mathbb{N}})$ ”.

Finally, by theorem 25(ii), we see that the following formula is realized :

“ $(\forall n \in \tilde{\mathbb{N}})$ there exists an injection of $\mathbb{J}n$ into $\mathcal{P}(\tilde{\mathbb{N}})$ ”.

Q.E.D.

Realizability models in which \mathbb{R} is not well ordered

$\mathbb{J}2$ atomless

Theorem 31. *We suppose there exist two proof-like terms ω_0, ω_1 such that, for every $\pi \in \Pi$, we have $\omega_0 k_\pi \Vdash \perp$ or $\omega_1 k_\pi \Vdash \perp$. Then, the Boolean algebra $\mathbb{J}2$ is non trivial. Indeed :*

$\theta \Vdash \forall x (x \neq 1, x \neq 0 \rightarrow x \notin \mathbb{J}2) \rightarrow \perp$ with $\theta = \lambda f (cc) \lambda k ((f)(\omega_1)k)(\omega_0)k$.

Let $\xi \Vdash \forall x (x \neq 1, x \neq 0 \rightarrow x \notin \mathbb{J}2)$ and $\pi \in \Pi$. We must show :

$\theta \star \xi \cdot \pi \in \perp$, that is $\xi \star \omega_1 k_\pi \cdot \omega_0 k_\pi \cdot \pi \in \perp$.

But, by hypothesis on ξ , we have $\xi \Vdash \top, \perp \rightarrow \perp$ and $\xi \Vdash \perp, \top \rightarrow \perp$. Hence the result, by hypothesis on ω_1, ω_0 .

Q.E.D.

Theorem 32. *We suppose that there exists three proof-like terms $\alpha_0, \alpha_1, \alpha_2$ such that, for every $\xi \in \Lambda$ and $\pi \in \Pi$, we have $k_\pi \xi \alpha_0 \Vdash \perp$ or $k_\pi \xi \alpha_1 \Vdash \perp$ or $k_\pi \xi \alpha_2 \Vdash \perp$.*

Then, the Boolean algebra $\mathbb{J}2$ is atomless. Indeed :

$\theta \Vdash \forall x [\forall y (xy \neq 0, xy \neq x \rightarrow y \notin \mathbb{J}2), x \neq 0 \rightarrow x \notin \mathbb{J}2]$

with $\theta = \lambda x \lambda y (cc) \lambda k ((x)(k)y\alpha_0)((x)(k)y\alpha_1)(k)y\alpha_2$.

By a simple computation, we see that we must show :

i) $\theta \Vdash (\perp, \perp \rightarrow \perp), \perp \rightarrow \perp$.

ii) $\theta \Vdash |\top, \perp \rightarrow \perp| \cap |\perp, \top \rightarrow \perp|, \top \rightarrow \perp$.

Proof of (i) : let $\eta \in |\perp, \perp \rightarrow \perp|$ and $\xi \in |\perp|$. We must show $\theta \star \eta \cdot \xi \cdot \pi \in \perp$, that is : $\eta \star k_\pi \xi \alpha_0 \cdot ((\eta)(k_\pi)\xi \alpha_1)(k_\pi)\xi \alpha_2 \cdot \pi \in \perp$.

But, from $\xi \Vdash \perp$, we deduce $k_\pi \xi \zeta \Vdash \perp$ for every $\zeta \in \Lambda_c$.

Since $\eta \Vdash \perp, \perp \rightarrow \perp$, we have $((\eta)(\mathbf{k}_\pi)\xi\alpha_1)(\mathbf{k}_\pi)\xi\alpha_2 \Vdash \perp$ and therefore :
 $\eta \star \mathbf{k}_\pi\xi\alpha_0 \cdot ((\eta)(\mathbf{k}_\pi)\xi\alpha_1)(\mathbf{k}_\pi)\xi\alpha_2 \cdot \pi \in \perp$.

Proof of (ii) : let $\eta \in |\top, \perp \rightarrow \perp| \cap |\perp, \top \rightarrow \perp|$ and $\xi \in \Lambda_c$. Again, we must show that :
 $\eta \star \mathbf{k}_\pi\xi\alpha_0 \cdot ((\eta)(\mathbf{k}_\pi)\xi\alpha_1)(\mathbf{k}_\pi)\xi\alpha_2 \cdot \pi \in \perp$. If this is false, then :

$\mathbf{k}_\pi\xi\alpha_0 \not\Vdash \perp$ (because $\eta \Vdash \perp, \top \rightarrow \perp$) and

$((\eta)(\mathbf{k}_\pi)\xi\alpha_1)(\mathbf{k}_\pi)\xi\alpha_2 \not\Vdash \perp$ (because $\eta \Vdash \top, \perp \rightarrow \perp$).

But, since $\eta \Vdash \perp, \top \rightarrow \perp$ (resp. $\top, \perp \rightarrow \perp$), we have $\mathbf{k}_\pi\xi\alpha_1 \not\Vdash \perp$ (resp. $\mathbf{k}_\pi\xi\alpha_2 \not\Vdash \perp$).

This contradicts the hypothesis of the theorem.

Q.E.D.

\mathbb{R} not well orderable

Theorem 33.

We suppose that there exists a proof-like term ω such that, for every $\xi, \xi' \in \Lambda$, $\xi \neq \xi'$ and $\pi \in \Pi$, we have $\omega\mathbf{k}_\pi\xi \Vdash \perp$ or $\omega\mathbf{k}_\pi\xi' \Vdash \perp$.

Then we have, for every formula F with three free variables :

$\theta \Vdash (\forall m, n \in \mathbb{N}) \forall z [(m < n) = 1 \mapsto$

$(\forall x \forall y \forall y' (F(x, y, z), F(x, y', z), y \neq y' \rightarrow \perp), (\forall y \varepsilon \mathbb{J}n) \neg (\forall x \varepsilon \mathbb{J}m) \neg F(x, y, z) \rightarrow \perp)]$

with $\theta = \lambda x \lambda x' (\mathbf{cc}) \lambda k (x') \lambda z (x z z) (\omega) k z$.

Remark. This shows that, if $(m < n) = 1$, then $(\mathbb{J}m \subset \mathbb{J}n)$ and there is no surjection of $\mathbb{J}m$ onto $\mathbb{J}n$: indeed, it suffices to take, as $F(x, y, z)$, the formula $\langle x, y \rangle \varepsilon z$.

Assume this is false ; then, there exist $m, n \in \mathbb{N}$ with $m < n$, an individual c , two terms $\xi, \xi' \in \Lambda$ and a stack $\pi \in \Pi$ such that :

$\theta \star \xi \cdot \xi' \cdot \pi \notin \perp$;

$\xi \Vdash \forall x \forall y \forall y' [F(x, y, c), F(x, y', c), y \neq y' \rightarrow \perp]$;

$\xi' \Vdash (\forall y \varepsilon \mathbb{J}n) \neg (\forall x \varepsilon \mathbb{J}m) \neg F(x, y, z)$.

Therefore, we have $\xi' \star \eta \cdot \pi \notin \perp$ with $\eta = \lambda z (\xi z z) (\omega) \mathbf{k}_\pi z$. By hypothesis on ξ' we have, for every integer $i < n$: $\eta \not\Vdash (\forall x \varepsilon \mathbb{J}m) \neg F(x, i, c)$. Thus, there exists an integer $m_i < m$ such that $\eta \not\Vdash \neg F(m_i, i, c)$. It follows that there exist $\xi_i \in \Lambda$ and $\pi_i \in \Pi$ such that $\xi_i \Vdash F(m_i, i, c)$ and $\eta \star \xi_i \cdot \pi_i \notin \perp$. By definition of η , we get $\xi \star \xi_i \cdot \xi_i \cdot \omega \mathbf{k}_\pi \xi_i \cdot \pi_i \notin \perp$. By hypothesis on ξ , it follows that $\omega \mathbf{k}_\pi \xi_i \not\Vdash i \neq i$; in other words, we have $\omega \mathbf{k}_\pi \xi_i \not\Vdash \perp$ for every integer $i < n$.

By the hypothesis of the theorem, it follows that we have $\xi_i = \xi_j$ for every $i, j < n$.

But, since $m_i < m < n$ and $i < n$, there exist $i, j < n$, $i \neq j$ such that $m_i = m_j = k$. Then, $\xi_i = \xi_j \Vdash F(k, i, c), F(k, j, c)$ and $\omega \mathbf{k}_\pi \xi_i \Vdash i \neq j$ since $\|i \neq j\| = \emptyset$.

Therefore, by hypothesis on ξ , we have $\xi \star \xi_i \cdot \xi_i \cdot \omega \mathbf{k}_\pi \xi_i \cdot \pi_i \in \perp$, which is a contradiction.

Q.E.D.

Now, we see that, with the hypothesis of theorem 33, there is no surjection from $\mathbb{J}2$ onto $\mathbb{J}2 \times \mathbb{J}2$. Indeed, by theorem 26, there exists a bijection from $\mathbb{J}2 \times \mathbb{J}2$ onto $\mathbb{J}4$ and, by theorem 33, there is no surjection from $\mathbb{J}2$ onto $\mathbb{J}4$. It follows that $\mathbb{J}2$ cannot be well ordered.

Now, by theorem 28, $\mathbb{J}2$ is equipotent with a subset of $\mathcal{P}(\tilde{\mathbb{N}})$, and this subset is infinite, by theorem 32. Therefore, the hypothesis of theorems 32 and 33 are sufficient in order that the following formula be realized in the model \mathcal{N} :

There is no well ordering on the set of reals.

In fact, the hypothesis of theorem 33 is sufficient : this follows from theorem 34.

Theorem 34.

Same hypothesis as theorem 33 : there exists a proof-like term ω such that, for every $\pi \in \Pi$ and $\xi, \xi' \in \Lambda$, $\xi \neq \xi'$, we have $\omega k_\pi \xi \Vdash \perp$ or $\omega k_\pi \xi' \Vdash \perp$.

Then we have, for every formula F with three free variables :

$\theta \Vdash \forall z \{ \forall x [\forall n^{ent} F(n, x, z) \rightarrow x \notin \mathbb{N}], \forall n \forall x \forall y [\neg F(n, x, z) \neg F(n, y, z), x \neq y \rightarrow \perp] \rightarrow \perp \}$
with $\theta = \lambda x \lambda x' (cc) \lambda k(x) \lambda n (cc) \lambda h(x' h h) (\omega k) \lambda f(f) h n$.

Remark. This formula means that, in the realizability model \mathcal{N} , there is no surjection from the set of integers $\tilde{\mathbb{N}}$ onto \mathbb{N} : it suffices to take for $F(x, y, z)$ the formula $\langle x, y \rangle \notin z$ (the graph of an hypothetical surjection being $\langle x, y \rangle \varepsilon z$).

Reasoning by contradiction, we suppose that there is an individual c , a stack $\pi \in \Pi$, and two terms ξ, ξ' such that :

$\xi \Vdash \forall x [\forall n^{ent} F(n, x, c) \rightarrow x \notin \mathbb{N}]$; $\xi' \Vdash \forall n \forall x \forall y [\neg F(n, x, z) \neg F(n, y, z), x \neq y \rightarrow \perp]$ and $\theta \star \xi \cdot \xi' \cdot \pi \notin \perp$.

Therefore, we have $\xi \star \eta \cdot \pi \notin \perp$, with $\eta = \lambda n (cc) \lambda h(\xi' h h) (\omega k_\pi) \lambda f(f) h n$.

By hypothesis on ξ , we have $\eta \not\Vdash \forall n^{ent} F(n, 0, c)$ and $\eta \not\Vdash \forall n^{ent} F(n, 1, c)$. Thus, we see that there exist $n_0, n_1 \in \mathbb{N}$, $\pi_0 \in \|F(n_0, 0, c)\|$ and $\pi_1 \in \|F(n_1, 1, c)\|$ such that $\eta \star n_0 \cdot \pi_0 \notin \perp$ and $\eta \star n_1 \cdot \pi_1 \notin \perp$. By performing these two processes, we obtain :

$\xi' \star k_{\pi_0} \cdot k_{\pi_0} \cdot \zeta_0 \cdot \pi_0 \notin \perp$ et $\xi' \star k_{\pi_1} \cdot k_{\pi_1} \cdot \zeta_1 \cdot \pi_1 \notin \perp$,

with $\zeta_0 = (\omega k_\pi) \lambda f(f) k_{\pi_0} n_0$ and $\zeta_1 = (\omega k_\pi) \lambda f(f) k_{\pi_1} n_1$.

By hypothesis on ξ' , we have $\xi' \Vdash \neg F(n_0, 0, c), \neg F(n_0, 1, c), 0 \neq 0 \rightarrow \perp$.

Since $k_{\pi_0} \Vdash \neg F(n_0, 0, c)$, we see that $\zeta_0 \not\Vdash \perp$ and, in the same way, $\zeta_1 \not\Vdash \perp$.

Thus, by the hypothesis of the theorem, we have :

$\lambda f(f) k_{\pi_0} n_0 = \lambda f(f) k_{\pi_1} n_1$, and therefore $n_0 = n_1$ and $\pi_0 = \pi_1$.

But, we have $\xi' \Vdash \neg F(n_0, 0, c), \neg F(n_0, 1, c), 0 \neq 1 \rightarrow \perp$. Moreover, we have :

$\pi_0 \in \|F(n_0, 0, c)\|$ and $\pi_1 \in \|F(n_1, 1, c)\|$, thus $\pi_0 \in \|F(n_0, 1, c)\|$ since $n_0 = n_1$, $\pi_0 = \pi_1$.

Therefore $k_{\pi_0} \Vdash \neg F(n_0, 0, c)$ and $\neg F(n_0, 1, c)$. Moreover, we have obviously $\zeta_0 \Vdash 0 \neq 1$, since $\|0 \neq 1\| = \emptyset$. Therefore, we have $\xi' \star k_{\pi_0} \cdot k_{\pi_0} \cdot \zeta_0 \cdot \pi_0 \in \perp$, which is a contradiction.

Q.E.D.

Theorems 33 and 34 show that \mathbb{N} is infinite and not equipotent with $\mathbb{N} \times \mathbb{N}$, thus not well orderable. Since \mathbb{N} is equipotent with a subset of $\mathcal{P}(\tilde{\mathbb{N}})$ (theorem 29), we have shown that $\mathcal{P}(\tilde{\mathbb{N}})$ is not well orderable, with the hypothesis of theorem 33.

More precisely, by corollary 30, we know that \mathbb{N} is equipotent with a subset of $\mathcal{P}(\tilde{\mathbb{N}})$ for each integer n . Therefore, we have :

Theorem 35. With the hypothesis of theorem 33, the following formula is realized :

“ There exists a sequence \mathcal{X}_n of infinite subsets of $\mathcal{P}(\tilde{\mathbb{N}})$ such that, for every integers $m, n \geq 2$:

- there is an injection from \mathcal{X}_n into \mathcal{X}_{n+1} ;
- there is no surjection from \mathcal{X}_n onto \mathcal{X}_{n+1} ;
- $\mathcal{X}_m \times \mathcal{X}_n$ and \mathcal{X}_{mn} are equipotent ”.

For each integer n , the set $\{0, 1, \dots, n-1\}$ is a ring : the ring of integers modulo n ; the Boolean algebra $\{0, 1\}$ is a set of idempotents in this ring. These ring operations extend to the realizability model, giving a ring structure on $\mathbb{J}n$, and $\mathbb{J}2$ is a set of idempotents in $\mathbb{J}n$.

For each $a \in \mathbb{J}2$, the equation $ax = x$ defines an ideal in $\mathbb{J}n$, which we denote as $a\mathbb{J}n$. The application $x \mapsto ax$ is a retraction from $\mathbb{J}n$ onto $a\mathbb{J}n$.

Proposition 36. *The following formulas are realized in \mathcal{N} :*

- i) $(\forall n \in \mathbb{J}\mathbb{N})(\forall a \in \mathbb{J}2)$ (the application $x \mapsto (ax, (1-a)x)$ is a bijection
from $\mathbb{J}n$ onto $a\mathbb{J}n \times (1-a)\mathbb{J}n$).
- ii) $(\forall m, n \in \mathbb{J}\mathbb{N})(\forall a \in \mathbb{J}2)$ (the application $(x, y) \mapsto my + x$ is a bijection
from $a\mathbb{J}m \times a\mathbb{J}n$ onto $a\mathbb{J}(mn)$).

i) Trivial : the inverse is $(y, y') \mapsto y + y'$.

ii) By theorem 26, this application is injective ; clearly, it sends $a\mathbb{J}m \times a\mathbb{J}n$ into $a\mathbb{J}(mn)$. Conversely, if $z \in a\mathbb{J}(mn)$, then there exists $x \in \mathbb{J}m$ and $y \in \mathbb{J}n$ such that $z = my + x$; thus, we have $z = az = may + ax$ with $ax \in a\mathbb{J}m$ and $ay \in a\mathbb{J}n$.

Q.E.D.

Theorem 37. *We suppose that, for each $\alpha \in \Lambda$, $\pi \in \Pi$, and every distinct $\zeta_0, \zeta_1, \zeta_2 \in \Lambda$, we have $k_\pi \alpha \zeta_0 \Vdash \perp$ or $k_\pi \alpha \zeta_1 \Vdash \perp$ or $k_\pi \alpha \zeta_2 \Vdash \perp$.*

Then, for each formula $F(x, y, z)$ with three free variables, we have :

$\theta \Vdash \forall z (\forall m, n \in \mathbb{J}\mathbb{N})(\forall a \in \mathbb{J}2)[(2m < n) = 1 \mapsto$
 $(a \neq 0, \forall x \forall y \forall y' (F(x, y, z), F(x, y', z), y \neq y' \rightarrow \perp), (\forall y \in \mathbb{J}n)(\exists x \in \mathbb{J}m) F(x, ay, z) \rightarrow \perp)]$
with $\theta = \lambda a \lambda x \lambda y (\text{cc}) \lambda k (y) \lambda z (x z z) (k) a z$.

Remark. This formula means that, if $n > 2m$, $a \in \mathbb{J}2$, $a \neq 0$, then there is no surjection from $\mathbb{J}m$ onto $a\mathbb{J}n$: it suffices to take $F(x, y, z) \equiv \langle x, y \rangle \varepsilon z$.

Reasoning by contradiction, let us consider $m, n \in \mathbb{N}$ with $n > 2m$, $a \in \{0, 1\}$, an individual c , three terms $\alpha, \xi, \eta \in \Lambda$ and $\pi \in \Pi$ such that :

$\theta \star \alpha \cdot \xi \cdot \eta \cdot \pi \notin \perp$, $\alpha \Vdash a \neq 0$, $\xi \Vdash \forall x \forall y \forall y' (F(x, y, c), F(x, y', c), y \neq y' \rightarrow \perp)$,
 $\eta \Vdash (\forall y \in \mathbb{J}n) \neg (\forall x \in \mathbb{J}m) \neg F(x, ay, c)$.

We have $\theta \star \alpha \cdot \xi \cdot \eta \cdot \pi \succ \eta \star \theta' \cdot \pi$ and therefore $\eta \star \theta' \cdot \pi \notin \perp$ with $\theta' = \lambda z (\xi z z) (k_\pi) \alpha z$. It follows that, for every $y \in \{0, \dots, n-1\}$, we have $\theta' \not\Vdash (\forall x \in \mathbb{J}m) \neg F(x, ay, c)$.

Thus, there exist two functions $y \mapsto x_y$ (resp. $y \mapsto \zeta_y$) from $\{0, \dots, n-1\}$ into $\{0, \dots, m-1\}$ (resp. into Λ), such that $\zeta_y \Vdash F(x_y, ay, c)$ and $\theta' \star \zeta_y \cdot \varpi_y \notin \perp$ (for some suitable stacks ϖ_y).

Now, we have $\theta' \star \zeta_y \cdot \varpi_y \succ \xi \star \zeta_y \cdot \zeta_y \cdot \kappa_y \cdot \varpi_y$ with $\kappa_y = k_\pi \alpha \zeta_y$; therefore, we have : $\xi \star \zeta_y \cdot \zeta_y \cdot \kappa_y \cdot \varpi_y \notin \perp$ for each $y \in \{0, \dots, n-1\}$.

By hypothesis on ξ (with $y = y'$), it follows that $\kappa_y \not\Vdash \perp$ for every $y < n$.

It follows first that $\alpha \not\Vdash \perp$ and therefore, we have $a = 1$; thus $\zeta_y \Vdash F(x_y, y, c)$.

Moreover, since $n > 2m$, there exist $y_0, y_1, y_2 < n$ distinct, such that $x_{y_0} = x_{y_1} = x_{y_2}$.

But, following the hypothesis of the theorem, the terms $\zeta_{y_0}, \zeta_{y_1}, \zeta_{y_2}$ cannot be distinct, because $\kappa_{y_0}, \kappa_{y_1}, \kappa_{y_2} \not\Vdash \perp$. Therefore we have, for instance, $\zeta_{y_0} = \zeta_{y_1}$; then, we apply the hypothesis on ξ with $y = y_0, y' = y_1$, which gives $\xi \star \zeta_{y_0} \cdot \zeta_{y_1} \cdot \kappa \cdot \varpi \in \perp$ for every $\kappa \in \Lambda$ and $\varpi \in \Pi$. But it follows that $\xi \star \zeta_{y_0} \cdot \zeta_{y_0} \cdot \kappa_{y_0} \cdot \varpi_{y_0} \in \perp$ which is a contradiction.

Q.E.D.

Corollary 38. *With the hypothesis of theorem 37, the following formulas are realized :*

- i) $(\forall n \in \tilde{\mathbb{N}})(\forall a \in \mathbb{J}2)(a \neq 0 \rightarrow \text{there is no surjection from } \mathbb{J}n \text{ onto } a\mathbb{J}(n+1)).$
- ii) $(\forall n \in \tilde{\mathbb{N}})(\forall a, b \in \mathbb{J}2)(ab = 0, b \neq 0 \rightarrow \text{there is no surjection from } a\mathbb{J}n \text{ onto } b\mathbb{J}2).$
- iii) $(\forall n \in \tilde{\mathbb{N}})(\forall a, b \in \mathbb{J}2)(ab = a, a \neq b \rightarrow \text{there is no surjection from } a\mathbb{J}n \text{ onto } b\mathbb{J}2).$

i) We prove this formula by contradiction : if there is a surjection from $a\mathbb{J}n$ onto $a\mathbb{J}(n+1)$ then, for each integer $k \in \tilde{\mathbb{N}}$, there exists a surjection from $(\mathbb{J}n)^k$ onto $(a\mathbb{J}(n+1))^k$; and thus also a surjection from $\mathbb{J}(n^k)$ onto $a\mathbb{J}((n+1)^k)$ (theorem 36). But, for $k > n$, we have $(n+1)^k > 2n^k$ and this contradicts theorem 37.

ii) Indeed, we have $(a+b)\mathbb{J}n = a\mathbb{J}n \times b\mathbb{J}n$. Reasoning by contradiction, there would exist a surjection from $(a+b)\mathbb{J}n$ onto $b\mathbb{J}2 \times b\mathbb{J}n$, thus also onto $b\mathbb{J}(2n)$ (theorem 36), thus a surjection from $\mathbb{J}n$ onto $b\mathbb{J}(2n)$, which contradicts (i).

iii) Otherwise, it would exist a surjection from $a\mathbb{J}n$ onto $(b-a)\mathbb{J}2$, which contradicts (ii).

Q.E.D.

Application. By DC, since $\mathbb{J}2$ is atomless, there exists in $\mathbb{J}2$ a strictly decreasing sequence. By corollary 38(iii) and theorem 29, there exists a sequence of infinite subsets of $\mathcal{P}(\tilde{\mathbb{N}})$, the “cardinals” of which are strictly decreasing. More precisely, let \mathcal{B} be the image of $\mathbb{J}2$ by the injection in $\mathcal{P}(\tilde{\mathbb{N}})$ given by theorem 29 ; then we have :

Theorem 39. *With the hypothesis of theorem 37, the following formula is realized in \mathcal{N} :*

“There exists a subset \mathcal{B} of $\mathcal{P}(\tilde{\mathbb{N}})$ (the real line of the model \mathcal{N}), such that \mathcal{B} is an atomless Boolean algebra for the usual order \subseteq on $\mathcal{P}(\tilde{\mathbb{N}})$, with $\emptyset, \mathbb{N} \in \mathcal{B}$; $a, b \in \mathcal{B} \Rightarrow a \cap b \in \mathcal{B}$.

If $a \in \mathcal{B}, a \neq \emptyset$ then $a\mathcal{B}$ is infinite and there is no surjection from \mathcal{B} onto $a\mathcal{B} \times a\mathcal{B}$ (where $a\mathcal{B}$ means $\{x \in \mathcal{B}; x \subseteq a\}$).

If $a, b \in \mathcal{B}, a, b \neq \emptyset$ and $a \cap b = \emptyset$, then there is no surjection from $a\mathcal{B}$ onto $b\mathcal{B}$ (the “cardinals” of $a\mathcal{B}, b\mathcal{B}$ are incomparable).

If $a, b \in \mathcal{B}, a \subseteq b$ and $a \neq b$, then there is no surjection from $a\mathcal{B}$ onto $b\mathcal{B}$ (the “cardinal” of $a\mathcal{B}$ is strictly less than the “cardinal” of $b\mathcal{B}$ ”).

In other words, for $a, b \in \mathcal{B}$, we have : $a \subseteq b \Leftrightarrow$ there exists a surjection from $b\mathcal{B}$ onto $a\mathcal{B}$. The order, in the atomless Boolean algebra \mathcal{B} , is the order on the “cardinals” of its initial segments, like in a finite Boolean algebra.

The model of threads

This model is the canonical instance of a non trivial coherent realizability model. It is defined as follows :

Let $n \mapsto \pi_n$ be an enumeration of the *stack constants* and let $n \mapsto \theta_n$ be a recursive enumeration of the *proof-like terms*. For each $n \in \mathbb{N}$, the *thread of number n* is the set of processes which appear during the execution of the process $\theta_n \star \pi_n$.

Note that every term which appears in the n -th thread contains the only stack constant π_n .

We define \perp^c (the complement of \perp) as the union of the threads.

Therefore, we have $\xi \star \pi \in \perp$ iff the process $\xi \star \pi$ never appears in any thread.

For every term ξ , we have $\xi \Vdash \perp$ iff ξ never appears in head position in any thread.

If ξ is a proof-like term, we have $\xi = \theta_n$ for some integer n , and therefore $\xi \star \pi_n \notin \perp$, by definition of \perp . It follows that *the model of threads is coherent*.

If $\xi \in \Lambda$, $\xi \not\vdash \perp$ then ξ appears in head position in at least one thread. This thread is unique, unless ξ is a proof-like term, because it is determined by the number of any stack constant which appears in ξ .

Theorem 40. *The hypothesis of theorems 31, 32, 33 and 37 are satisfied in the model of threads.*

The hypothesis of theorems 33 and 31 are trivially satisfied if we take :

$$\omega = (\lambda x x x)\lambda x x x, \quad \omega_0 = \omega 0, \quad \text{and} \quad \omega_1 = \omega 1.$$

Moreover, the hypothesis of theorem 37 is obviously stronger than the hypothesis of theorem 32.

We check the hypothesis of theorem 37 by contradiction : we suppose $k_\pi \alpha \zeta_0 \not\vdash \perp$, $k_\pi \alpha \zeta_1 \not\vdash \perp$ and $k_\pi \alpha \zeta_2 \not\vdash \perp$. Therefore, these three terms appear in head position, and moreover in the same thread : indeed, since they contain the stack π , this thread has the same number as the stack constant of π .

Let us consider their first appearance in head position, for instance with the order 0, 1, 2. Therefore we have, in this thread : $k_\pi \alpha \zeta_0 \star \rho_0 \succ \alpha \star \pi \succ \dots \succ k_\pi \alpha \zeta_1 \star \rho_1 \succ \alpha \star \pi \succ \dots$. But, at the second appearance of $\alpha \star \pi$, the thread enters into a loop, and the term $k_\pi \alpha \zeta_2$ can never arrive in head position, since $\zeta_1 \neq \zeta_2$.

Q.E.D.

References

- [1] S. Berardi, M. Bezem, T. Coquand. *On the computational content of the axiom of choice*. J. Symb. Log. 63 (1998), p. 600-622.
- [2] H.B. Curry, R. Feys. *Combinatory Logic*. North-Holland (1958).
- [3] W. Easton. *Powers of regular cardinals*. Ann. Math. Logic 1 (1970), p. 139-178.
- [4] J.Y. Girard. *Une extension de l'interprétation fonctionnelle de Gödel à l'analyse*. Proc. 2nd Scand. Log. Symp. (North-Holland) (1971) p. 63-92.
- [5] T. Griffin. *A formulæ-as-type notion of control*. Conf. record 17th A.C.M. Symp. on Principles of Progr. Languages (1990).
- [6] S. Grigorieff. *Combinatorics on ideals and forcing*. Ann. Math. Logic 3(4) (1971), p. 363-394.
- [7] W. Howard. *The formulas-as-types notion of construction*. Essays on combinatory logic, λ -calculus, and formalism, J.P. Seldin and J.R. Hindley ed., Acad. Press (1980) p. 479-490.
- [8] J. M. E. Hyland. *The effective topos*. The L.E.J. Brouwer Centenary Symposium (Noordwijkerhout, 1981), 165-216, Stud. Logic Foundations Math., 110, North-Holland, Amsterdam-New York, 1982.

- [9] G. Kreisel. *On the interpretation of non-finitist proofs I.*
J. Symb. Log. 16 (1951) p. 248-26.
- [10] G. Kreisel. *On the interpretation of non-finitist proofs II.*
J. Symb. Log. 17 (1952), p. 43-58.
- [11] J.-L. Krivine. *Typed lambda-calculus in classical Zermelo-Fraenkel set theory.*
Arch. Math. Log., 40, 3, p. 189-205 (2001).
http://www.pps.jussieu.fr/~krivine/articles/zf_epsilon.pdf
- [12] J.-L. Krivine. *Dependent choice, 'quote' and the clock.*
Th. Comp. Sc., 308, p. 259-276 (2003).
<http://hal.archives-ouvertes.fr/hal-00154478>
<http://www.pps.jussieu.fr/~krivine/articles/quote.pdf>
- [13] J.-L. Krivine. *Realizability in classical logic.*
In *Interactive models of computation and program behaviour.*
Panoramas et synthèses, Société Mathématique de France, 27 (2009).
<http://hal.archives-ouvertes.fr/hal-00154500>
Version mise à jour à :
<http://www.pps.jussieu.fr/~krivine/articles/Luminy04.pdf>
- [14] J.-L. Krivine. *Realizability : a machine for Analysis and set theory.*
Geocal'06 (fevrier 2006 - Marseille); Mathlogaps'07 (juin 2007 - Aussois).
<http://cel.archives-ouvertes.fr/cel-00154509>
Version mise à jour à :
<http://www.pps.jussieu.fr/~krivine/articles/Mathlog07.pdf>
- [15] J.-L. Krivine. *Structures de réalisabilité, RAM et ultrafiltre sur \mathbb{N} .* (2008)
<http://hal.archives-ouvertes.fr/hal-00321410>
<http://www.pps.jussieu.fr/~krivine/articles/Ultrafiltre.pdf>
- [16] J.-L. Krivine. *Realizability algebras : a program to well order \mathbb{R} .* (2010)
<http://hal.archives-ouvertes.fr/hal-00483232>
http://www.pps.jussieu.fr/~krivine/articles/Well_order.pdf