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ENHANCED GAUSSIAN PROCESSES AND APPLICATIONS

L. COUTIN AND N. VICTOIR

ABSTRACT. We propose some construction of enhanced Gaussian processes using Karhunen-Loeve expansion. We obtain a characterization and some criterion of existence and uniqueness. Using rough-path theory, we derive some Wong-Zakai theorem.

1. GENERALITIES

In [17] Lyons developed a general theory of differential equations of the form

$$(1.1) \quad dy_t = f(y_t)dx_t.$$

Classical integration/ODE theory gives a meaning to such differential equations when x has bounded variation. Lyons extended this notion to the case when x is a path with values in a Banach space B , and of finite p -variation, $p \geq 1$. To do so, one needs first to lift x to a path of finite p -variation in the free nilpotent group of B . In other words, one needs to define and make a choice for the "iterated integrals" of order less than or equal to $[p]$ of x . We refer the reader to, for example, [15, 17, 18].

In this paper, our aim is to work towards the study of a "natural" p -rough path process lying above an arbitrary Gaussian process. We simplify the problem by only looking at lift in the free nilpotent group of step 2, i.e. we are just looking at the Lévy area of Gaussian processes. This was already done by Lévy in 1950 for Brownian motion, see [16] or more recently [14] and [6], and for fractional Brownian motion, see [7] or [23]. Moreover, Biane and Yor, in [1] have constructed the Lévy area using the expansion of Brownian motion in the basis of Legendre polynomials.

Karhunen- Loeve expansion Theorem provide a natural way to approximate paths of a Gaussian process by a smooth process. This paper is devoted to study how its expansion allow to lift \mathbb{R}^d -valued Gaussian process x to a path \mathbf{x} with values in some free nilpotent of step 2 group over \mathbb{R}^d (or in other words, how to construct the Lévy area of x , i.e. the second iterated integral of x). We also show that if the process x with some area process satisfies some quite natural conditions, then \mathbf{x} will be the limit of the lift of the Karhunen-Loeve approximations of x .

The proof of the convergence of Karhunen- Loeve expansion Theorem or of some properties on Gaussian processes relies on the convex property of the vector spaces. The free nilpotent group of step 2 do not share this property. In the first part of this paper, we give a proof of a weak version the Karhunen- Loeve expansion Theorem using a discrete martingale. Some basic results on the free nilpotent group of step 2 are given. Then, in the second part, using again some martingales, we lift the process x to a path \mathbf{x} with values in some free nilpotent of step 2 group over \mathbb{R}^d . A

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characterisation and a result of uniqueness is also given. For the Brownian motion and the fractional Brownian motion, this definition coincides with the one obtained by dyadic linear approximation as in [7]. In the third part the case of Volterra Gaussian processes is studied. We conclude with a Wong-Zakai Theorem.

1.1. Gaussian Processes. We define on the measure space $\Omega = C_0([0, 1], \mathbb{R}^d)$ and its Borel σ -algebra denoted by \mathbb{F} , the probability measure \mathbb{P} corresponding to the law of a d -dimensional centered Gaussian process with covariance function C . We let (H, \langle, \rangle) the associated Cameron-Martin space associated to \mathbb{P} . We assume that the process has continuous sample paths, then it is continuous in $L^2(\Omega, \mathbb{F}, \mathbb{P})$ and the covariance function is continuous. Following [10] Theorem 2.8.2, the space (H, \langle, \rangle) is separable. Let $e = (e_i)_{i \in \mathbb{N}}$ be an orthonormal basis on (H, \langle, \rangle) . One can always represents X under \mathbb{P} with the formula

$$(1.2) \quad X^k = \sum_{i=0}^{\infty} (N_i^e)^k e_i^k$$

where $N_i^e = \langle X, e_i \rangle$ are independent standard d dimensional normal random variables. Here \langle, \rangle is the duality bracket. We let $\mathcal{F}_n^e = \sigma(N_i^e, 0 \leq i \leq n)$.

We warm up with the following two propositions. Their results (and stronger results) are well known, see Theorem 2.4.2 of [10], but the proof given here allow us to generalize in the next section to the "natural lift" of X to a process with values in some free nilpotent group.

Proposition 1. *For all $t \in [0, 1]$,*

$$X_n^e(t) := E(X(t) | \mathcal{F}_n^e) = \sum_{i=0}^n N_i^e e_i(t).$$

Proof. It is just the observation that $\sum_{i=n+1}^{\infty} N_i^e e_i$ is mean 0 and independent of \mathcal{F}_n^e . \square

Proposition 2. *For all $q \geq 1$, and for all $t \in [0, 1]$, $X_n^e(t)$ converges to $X(t)$ almost surely and in L^q .*

Proof. Since X has continuous sample paths, then almost surely $\|X\|_{\infty} := \sup_{t \in [0, 1]} \|X(t)\| < \infty$. Note that $\|X\|_{\infty} < \infty$ a.s. implies that the r.v. $\|X\|_{\infty}$ has a Gaussian tail (from Borell's inequality), and therefore is in L^q for all $1 \leq q < \infty$. For all $t \in [0, 1]$

$$\begin{aligned} |X_n^e(t)| &= |E(X(t) | \mathcal{F}_n^e)| \\ &\leq E(\|X\|_{\infty} | \mathcal{F}_n^e). \end{aligned}$$

Taking the suppreum over all t , we obtain that $\|X_n^e\|_{\infty} \leq E(\|X\|_{\infty} | \mathcal{F}_n^e)$. Therefore, by Doob's inequality, $\sup_n \|X_n^e\|_{\infty}$ is in L^q for all $1 \leq q < \infty$. By the martingale convergence theorem, $X_n^e(t) \rightarrow X(t)$ for all t , where the convergence is in L^q and a.s. \square

1.2. Free Nilpotent Group of Step 2.

1.2.1. *Definitions.* We define $G^2(\mathbb{R}^d)$ to be the space $\{(x, y) \in \mathbb{R}^d \oplus M_d(\mathbb{R}), y^{i,j} + y^{j,i} = x^i x^j\}$ together with the product

$$(x_1, y_1) \otimes (x_2, y_2) = \left(x_1 + x_2, y_1 + y_2 + \left(x_1^i x_2^j \right)_{i,j} \right).$$

Indeed, $(G^2(\mathbb{R}^d), \otimes)$ is the free nilpotent group of step 2 over \mathbb{R}^d .

We define for a \mathbb{R}^d -valued path x of finite q -variation for $q < 2$, the canonical lift of x to a $G^2(\mathbb{R}^d)$ -valued path:

$$S(x)_t = \left(x_t, \int_0^t x_u^i dx_u^j \right), \quad t \in [0, 1].$$

Observe that $G^2(\mathbb{R}^d) = \{S(x)_1, x \text{ smooth } \mathbb{R}^d\text{-valued path}\}$. That allows us to define a homogeneous norm on $G^2(\mathbb{R}^d)$:

$$\|g\| = \sup_{\substack{x \text{ smooth} \\ S(x)_1 = g}} \int_0^1 |\dot{x}_u| du,$$

and from this homogeneous norm, a left invariant distance on $G^2(\mathbb{R}^d)$:

$$d(g, h) = \|g^{-1} \otimes h\|.$$

If $g = (x, y) \in G^2(\mathbb{R}^d)$, we define $\pi_i(g)$ to be the projection of x on the i^{th} component of \mathbb{R}^d , and $\pi_{j,k}(g)$ the $(j, k)^{\text{th}}$ component of y on $\mathbb{R}^d \times \mathbb{R}^d$. In particular, if x is a smooth \mathbb{R}^d -valued path then $\pi_i(S(x)_1) = x_1^i$ and $\pi_{j,k}(S(x)_1) = \int_0^1 x_u^j dx_u^k$.

We have an equivalence of homogeneous norm result: there exists some constant $c, C > 0$ such that for all $g \in G^2(\mathbb{R}^d)$,

$$(1.3) \quad c \|g\| \leq \max_{i,j,k} \left\{ |\pi_i(g)|, \sqrt{|\pi_{j,k}(g)|} \right\} \leq C \|g\|.$$

1.2.2. *Paths with Values in $G^2(\mathbb{R}^d)$.* When x is a path in $C_0([0, 1], G^2(\mathbb{R}^d))$, the space of continuous functions from $[0, 1]$ into $G^2(\mathbb{R}^d)$ starting at 0, we let as a notation

$$x_{s,t} = x_s^{-1} \otimes x_t, \quad (s, t) \in [0, 1]^2.$$

On $C_0([0, 1], G^2(\mathbb{R}^d))$, we define the following distances:

$$\begin{aligned} d_\infty(x, y) &= \sup_{0 \leq t \leq 1} d(x_t, y_t), \\ \|x\|_\infty &= d_\infty(0, x). \end{aligned}$$

For a given control¹ ω and $p \geq 1$, we define

$$\begin{aligned} d_{p,\omega}(x, y) &= \sup_{0 \leq s < t \leq 1} \frac{d(x_{s,t}, y_{s,t})}{\omega(s, t)^{1/p}}, \\ \|x\|_{p,\omega} &= d_{p,\omega}(0, x). \end{aligned}$$

The applications $\|\cdot\|_\infty$ and $\|\cdot\|_{p,\omega}$ are not some pseudo norms since $x, y \in C_0([0, 1], G^2(\mathbb{R}^d))$ does not imply $x \otimes y \in C_0([0, 1], G^2(\mathbb{R}^d))$.

¹i.e. a continuous map from $\{s \leq t, s, t \in [0, 1]\}$ such that $\omega(t, s) + \omega(s, u) \leq \omega(t, u)$, $\forall t \leq s \leq u$, null on the diagonal

1.2.3. *Translation operator on Path Space.* We will want to "add" two paths with values in $G^2(\mathbb{R}^d)$. This can be done when between these two paths make sense. This addition or translation operation will be denoted T .

The map T can be understood in the following way: for some smooth paths x and h , we define

$$T_h(S(x)) := S(x + h).$$

Then if $(S(x_n))$ converges to X in the uniform topology associated to d_∞ and h_n converges to h in bounded variation, then $(T_{h_n}(S(x_n)))$ converges in the uniform topology to a continuous $G^2(\mathbb{R}^d)$ -valued path denoted $T_h(X)$. One can check that $T_h(X)$ satisfies

$$(1.4) \quad \begin{aligned} \pi_i(T_h(X)) &= h^i + x^i, \\ \pi_{i,j}(T_h(X)) &= \pi_{i,j}(X) + \int_0^\cdot h_u^i dh_u^j + \int_0^\cdot x_u^i dh_u^j \\ &\quad + h^i x^j - h_0^i x_0^j - \int_0^\cdot x_u^j dh_u^i. \end{aligned}$$

2. NATURAL LIFT OF A GAUSSIAN PROCESS TO A $G^2(\mathbb{R}^d)$ -VALUED PROCESS

2.1. **Definition and First Property.** We denote by BV the set of continuous paths of bounded variation.

Assumption 1. (1) *There exists an orthonormal basis $e = (e_i)_{i \geq 0}$ of (H, \langle, \rangle) which is in $H \cap BV$;*
 (2) *the components of X are independent.*

Example 1. *Point (1) of Assumption 1 is fulfilled if C is continuous for the usual distance on $[0, 1]^2$, and for all $t \in [0, 1]$, $C(t, \cdot) \in BV$. Indeed, the vector space generated by $\{C(t, \cdot), t \in [0, 1]\}$ is dense in (H, \langle, \rangle) and Assumption 1 follows from an orthonormalisation procedure.*

Example 2. *In particular, Assumption 1 is satisfied for fractional Brownian motion, for any Hurst parameter $h > 0$.*

All the orthonormal basis $(e_i)_{i \in \mathbb{N}}$ of H that we will consider will be implicitly assumed to be in BV .

Definition 1. *We say that $\mathbf{X} : [0, 1] \rightarrow G^2(\mathbb{R}^d)$ defines a natural lift of the Gaussian process X , associated to the orthonormal basis e (to a $G^2(\mathbb{R}^d)$ -valued process) if*

- (1) *$S(X_{0,n}^e)_t$ converges in probability to \mathbf{X}_t for all $t \in [0, 1]$,*
- (2) *\mathbf{X} has a continuous sample paths.*

Note from the definition of the canonical lift on smooth \mathbb{R}^d valued path and Proposition 2, that

$$(2.1) \quad \begin{aligned} \pi_i(\mathbf{X}) &= X^i, \quad i \in \{1, \dots, d\} \\ \pi_{i,i}(\mathbf{X}) &= \frac{(X^i)^2}{2}. \end{aligned}$$

Lemma 1. *Let e be an orthonormal basis on H , such that $e_n \in BV$ for $n \in \mathbb{N}$. Let $t \in [0, 1]$, The random variable $(S(X_n^e)_t)$ converges almost surely if and only if*

$$\max_{i,j} \sum_{0 \leq l < k < \infty} \left[\int_0^t \left(e_l^i(s) \dot{e}_k^j(s) - e_k^j(s) \dot{e}_l^i(s) \right) ds \right]^2 < +\infty.$$

Proof. From equality (2.1) and Proposition 2, we only have to study the convergence of $((\pi_{i,j} - \pi_{j,i})(S(X_n^e)_t))$. First observe that $((\pi_{i,j} - \pi_{j,i})(S(X_n^e)_t))$ is \mathcal{F}_n^e -martingale. Moreover, because it belongs to the second Wiener chaos, the convergence of $((\pi_{i,j} - \pi_{j,i})(S(X_n^e)_t))$ in probability is equivalent to the convergence in L^2 . By martingale convergence theorem, $((\pi_{i,j} - \pi_{j,i})(S(X_n^e)_t))$ converges in L^2 and almost surely if and only if $\lim_{n \rightarrow \infty} E \left(|(\pi_{i,j} - \pi_{j,i})(S(X_n^e)_t)|^2 \right) < \infty$. But

$$\begin{aligned} E \left(|(\pi_{i,j} - \pi_{j,i})(S(X_n^e)_t)|^2 \right) &= E \left(\left| \sum_{0 \leq l, k \leq n} (N_l^i N_k^j) \int_0^t \left(e_l^i(s) \dot{e}_k^j(s) - e_k^j(s) \dot{e}_l^i(s) \right) ds \right|^2 \right) \\ &= \sum_{0 \leq l, k \leq n} \left[\int_0^t \left(e_l^i(s) \dot{e}_k^j(s) - e_k^j(s) \dot{e}_l^i(s) \right) ds \right]^2. \end{aligned}$$

Observe that we have used the independence of the coordinates of the Gaussian process X . \square

A kind of 0 – 1 law is also available.

Lemma 2. *Let e be an orthonormal basis on H , such that $e_n \in BV$ for $n \in \mathbb{N}$. Let $t \in [0, 1]$. If $\mathbb{P}(\{\omega, (S(X_n^e)_t(\omega))_{n \in \mathbb{N}} \text{ converges}\}) > 0$, then $\mathbb{P}(\{\omega, (S(X_n^e)_t(\omega))_{n \in \mathbb{N}} \text{ converges}\}) = 1$.*

Proof. Assume that $\mathbb{P}(\{\omega, (S(X_n^e)_t(\omega))_{n \in \mathbb{N}} \text{ converges}\}) > 0$, and denote for $i, j \in \{1, \dots, d\}$

$$\Gamma^{i,j} = \{\omega, (S(X_n^e)_t^{i,j}(\omega) - S(X_n^e)_t(\omega)^{j,i})_{n \in \mathbb{N}} \text{ converges}\}.$$

For $i \in \{1, \dots, d\}$, observe that $S(X_n^e)_t^i = \frac{(X_n^e(t)^i)^2}{2}$. Theorem 1.1.1 of [10] applied to the Gaussian vector $(X_n^e(t)^i)_{n \in \mathbb{N}}$ yields

$$\mathbb{P}(\{(X_n^e(t)^i)_{n \in \mathbb{N}} \text{ and } (X_n^e(t)^{i,i})_{n \in \mathbb{N}} \text{ converge}\}) = 1.$$

For $i \neq j$, conditionally to $\sigma(N_l^i, l \in \mathbb{N})$, $(S(X_n^e)_t^{i,j} - S(X_n^e)_t^{j,i})_{n \in \mathbb{N}}$ is a Gaussian vector, and using the same arguments, almost surely

$$\mathbb{E}(\mathbf{1}_{\Gamma^{i,j}} / \sigma(N_l^i, l \in \mathbb{N})) = \mathbf{1}_{\Gamma^{i,j}}.$$

But the role of i and j in the conditioning are symmetric and the following equality holds

$$(2.2) \quad \mathbb{E}(\mathbf{1}_{\Gamma^{i,j}} / \sigma(N_l^i, l \in \mathbb{N})) = \mathbf{1}_{\Gamma^{i,j}} = \mathbb{E}(\mathbf{1}_{\Gamma^{i,j}} / \sigma(N_l^j, l \in \mathbb{N})).$$

Since the σ fields $\sigma(N_l^j, l \in \mathbb{N})$ and $\sigma(N_l^i, l \in \mathbb{N})$ are independant, conditioning all terms of equality (2.2) by $\sigma(N_l^j, l \in \mathbb{N})$ yields

$$\mathbf{1}_{\Gamma^{i,j}} = \mathbb{P}(\Gamma^{i,j}) > 0.$$

Then, $\mathbf{1}_{\Gamma^{i,j}} = 1$ almost surely. This achieves the proof, since

$$S(X_n^e)^{i,j} = \frac{1}{2} [(X_n^e)^i (X_n^e)^j + S(X_n^e)^{i,j} - S(X_n^e)^{j,i}].$$

□

2.2. A Characterization of a Natural Lift, and a Uniqueness Result. We will use the maps

$$\begin{aligned} \phi_i & : C_0([0, 1], \mathbb{R}^d) \rightarrow C_0([0, 1], \mathbb{R}^d) \\ (x_1, \dots, x_d) & \rightarrow (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_d). \end{aligned}$$

Observe that $\mathbb{P} \circ \phi_i = \mathbb{P}$ for all i . (\mathbb{P} is the probability measure introduced in the previous section).

Theorem 1. *Assume that Assumption 1 is fulfilled.*

The path $\mathbf{X} : [0, 1] \rightarrow G^2(\mathbb{R}^d)$ is a natural lift of X for some orthonormal basis e , if and only if there exists a measurable map

$$\Psi : C_0([0, 1], \mathbb{R}^d) \rightarrow C_0([0, 1], G^2(\mathbb{R}^d)) \cup \{\delta\}$$

(where δ is a cimentary point) such that

- Definition 2.**
- (1) $\mathbf{X} = \Psi(X)$ a.s. and $\Psi(X) \neq \delta$ a.s.
 - (2) The projection of $\Psi(X)$ onto \mathbb{R}^d is equal to X a.s.. (Lifting property)
 - (3)
$$\begin{cases} \pi_{j,k} \Psi(\phi_i(X)) = -\pi_{j,k} \Psi(X) & \text{if } i \in \{j, k\}, \\ \pi_{j,k} \Psi(\phi_i(X)) = \pi_{j,k} \Psi(X) & \text{if } i \notin \{j, k\}. \end{cases}$$
 (Symmetry property)
 - (4) For all path $h \in H \cap BV$, $\Psi(h) = S(h)$ (Definition on "smooth" paths).
 - (5) For all path $h \in H \cap BV$, $\Psi(X + h) = T_h \Psi(X)$ almost surely. (Stability of translations property).
 - (6) The r.v. $\|\mathbf{X}_{s,t}\|$ is in $L^2(\Omega, \mathbb{F}, \mathbb{P})$ for $0 \leq s < t \leq 1$. (Integrability property)

Proof. If \mathbf{X} is a natural lift (associated to an orthonormal basis f), it is easy to check that it satisfies properties (1) to (6).

Conversely, we want to check that if Ψ is a measurable map satisfying the above condition, then $\Psi(X)$ is the natural lift associated to e . The proof will be complete once we prove that for all n , for all $0 \leq s \leq t \leq 1$,

$$(2.3) \quad E \left(\log \Psi(X)_{s,t} | \mathcal{F}_n^e \right) = \log S(X_n^e)_{s,t}.$$

Indeed, the martingale $\log S(X_{0,n}^e)_{s,t}$ converges to $\log \mathbf{X}_{s,t}$, and the above equality plus the fact that $\log \Psi(X)_{s,t}$ is $\mathcal{F}_{0,\infty}^e$ -measurable would prove that $\mathbf{X}_{s,t} = \Psi(X)_{s,t}$. Here, \log is defined by its power serie.

The first level of equality (2.3) was proved in Proposition 1. We therefore only need to prove that for $i \neq j$, $0 \leq s \leq t \leq 1$, $n \in \mathbb{N}$ using (1.4),

$$E \left((\pi_{i,j} - \pi_{j,i}) \left(\Psi(X)_{s,t} \right) | \mathcal{F}_n^e \right) = (\pi_{i,j} - \pi_{j,i}) \left(S(X_n^e)_{s,t} \right).$$

From the stability of translations property, $\Psi(X) = T_{X_n^e}(\Psi(X - X_n^e))$. In particular, for all $0 \leq s < t \leq 1$, and $1 \leq i < j \leq d$,

$$\begin{aligned}
 (\pi_{i,j} - \pi_{j,i}) \left(\Psi(X)_{s,t} \right) &= (\pi_{i,j} - \pi_{j,i}) \left(\Psi(X - X_n^e)_{s,t} \right) + (\pi_{i,j} - \pi_{j,i}) \left(S(X_n^e)_{s,t} \right) \\
 (2.4) \quad &+ \int_s^t (X - X_n^e)_{s,u}^i d(X_n^e)_u^j - \int_s^t (X - X_n^e)_{s,u}^j d(X_n^e)_u^i \\
 &- \int_s^t (X - X_n^e)_{s,u}^j d(X_n^e)_u^i + \int_s^t (X - X_n^e)_{s,u}^i d(X_n^e)_u^j \\
 &+ (X_n^e)_{s,t}^i (X - X_n^e)_{s,t}^j - (X_n^e)_{s,t}^j (X - X_n^e)_{s,t}^i.
 \end{aligned}$$

It is easy to check that all the expressions in equality (2.4) are in $L^2(\Omega, \mathbb{F}, \mathbb{P})$. As $(X - X_n^e)$ is independent of \mathcal{F}_n^e while N_k^e is \mathcal{F}_n^e -measurable,

$$\begin{aligned}
 E \left(\int_s^t (X - X_n^e)_{s,u}^i d(X_n^e)_u^j \mid \mathcal{F}_n^e \right) &= \sum_{k=0}^n N_k^j \int_s^t E \left((X - X_n^e)_{s,u}^i \mid \mathcal{F}_n^e \right) de_k^j(u) \\
 &= \sum_{k=0}^n N_k^j \int_s^t E \left((X - X_n^e)_{s,u}^i \right) de_k^j(u) \\
 &= 0.
 \end{aligned}$$

The same equality and argument applies to $\int_s^t (X - X_n^e)_{s,u}^j d(X_n^e)_u^i$ and to $(X_n^e)_{s,t}^i (X - X_n^e)_{s,t}^j$ and the reverse expressions. Therefore,

$$E \left((\pi_{i,j} - \pi_{j,i}) \left(\Psi(X)_{s,t} \right) \mid \mathcal{F}_n^e \right) = (\pi_{i,j} - \pi_{j,i}) \left(S(X_n^e)_{s,t} \right) + E \left((\pi_{i,j} - \pi_{j,i}) \left(\Psi(X - X_n^e)_{s,t} \right) \mid \mathcal{F}_n^e \right).$$

From the symmetry assumption, $(\pi_{i,j} - \pi_{j,i}) \left(\Psi(X - X_n^e)_{s,t} \right) = -(\pi_{i,j} - \pi_{j,i}) \left(\Psi \circ \Phi_i(X - X_n^e)_{s,t} \right)$.

Since X and $\phi_i(X)$ has the same law, $(\pi_{i,j} - \pi_{j,i}) \left(\Psi(X - X_n^e)_{s,t} \right)$ is a centered random variable. Hence, as $X - X_n^e$ is independent of \mathcal{F}_n^e , we obtain,

$$E \left((\pi_{i,j} - \pi_{j,i}) \left(\Psi(X - X_n^e)_{s,t} \right) \mid \mathcal{F}_n^e \right) = 0.$$

Therefore,

$$E \left((\pi_{i,j} - \pi_{j,i}) \left(\Psi(X)_{s,t} \right) \mid \mathcal{F}_n^e \right) = (\pi_{i,j} - \pi_{j,i}) \left(S(X_n^e)_{s,t} \right).$$

□

As a simple corollary of the previous result, we obtain the important result of uniqueness of the natural lift.

Corollary 1. *Let X be a Gaussian process, and assume that there exists a natural lift \mathbf{X} of X associated to some orthonormal basis e of H in BV . Then, for all orthonormal basis f of H in BV , there exists a natural lift \mathbf{X}^f associated to X . Moreover, almost surely, for all such orthonormal basis, $\mathbf{X}^f = \mathbf{X}^e$.*

2.3. Other Constructions.

Theorem 2. *Assume that there exists linear measurable maps $\Delta_n : C_0([0, 1], \mathbb{R}^d) \rightarrow H \cap BV$ such that*

- (1) *Almost surely, $S \circ \Delta_n(X)$ converges in uniform topology.*
- (2) *$\Delta_n(h)$ converges pointwise to h and $\sup_n |\Delta_n(h)|_{BV} < \infty \forall h \in H \cap BV$;*
- (3) *For all $1 \leq i \leq d$, $\Delta_n \circ \phi_i = \phi_i \circ \Delta_n$.*

Then, there exists a (unique up to indistinguishability) natural lift of X , and it is $\mathbf{X} := \lim_{n \rightarrow \infty} S \circ \Delta_n(X)$.

Proof. We define $\Psi(X) = \lim_{n \rightarrow \infty} S \circ \Delta_n(X)$. Condition 1 clearly implies that $\Psi(X)$ has almost surely continuous paths. Hence, $T_h(\Psi(X))$ exists for all $h \in BV$. Moreover,

$$\begin{aligned} S \circ \Delta_n(X + h) &= S(\Delta_n(X) + \Delta_n(h)) \\ &= T_{\Delta_n(h)}(S \circ \Delta_n(X)), \end{aligned}$$

and by property of the translation operator, we see that $T_{\Delta_n(h)}(S \circ \Delta_n(X))$ converges in uniform topology to variation topology to $T_h(\Psi(X))$. Hence, $\Psi(X + h)$ is well defined a.s. and equal a.s. to $T_h(\Psi(X))$. The other conditions of theorem 1 are easily checked to be true. \square

Corollary 2. *The level n dyadic piecewise linear approximation of a continuous path, i.e.*

$$\Delta_n(x)_t = x_{\frac{k}{2^n}} + (2^n t - k) \left(x_{\frac{k+1}{2^n}} - x_{\frac{k}{2^n}} \right) \text{ for } t \in \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right].$$

Assume that $S \circ \Delta_n(X)$ converges almost in uniform topology. Then, $\mathbf{X} := \lim_{n \rightarrow \infty} S \circ \Delta_n(X)$ is the unique natural lift associated to X .

The above corollary is obvious. It proves in particular that the lift of fractional Brownian motion constructed in [7] is a natural one.

2.4. Convergence in $d_{p,\omega}$ topology.

Theorem 3. *Assume that there exists a natural lift associated \mathbf{X} to a Gaussian process X , and assume that for some control ω , such that $\|\mathbf{X}\|_{p,\omega}$ is in $L^q(\Omega, \mathbb{F}, \mathbb{R})$ for $q \geq 2$. Let us fix a orthonormal basis e of H which in BV . Then,*

$$\sup_n \|S(X_{0,n}^e)\|_{p,\omega}$$

is in L^q . In particular, for all $p' > p$, $d_{p',\omega}(S(X_{0,n}^e), \mathbf{X})$ converges to 0 almost surely and in L^q .

Proof. We define $A = ((\pi_{i,j} - \pi_{j,i})(\mathbf{X}))_{(i,j) \in \{1,\dots,d\}}$ to be the area of \mathbf{X} . For all $s < t \in [0, 1]$,

$$\begin{aligned} \|S(X_{0,n}^e)_{s,t}\| &\leq C |E(X_{s,t} | \mathcal{F}_n^e)| + C \sqrt{|E(A_{s,t} | \mathcal{F}_n^e)|} \\ &\leq 2C \sqrt{E(\|\mathbf{X}_{s,t}\|^2 | \mathcal{F}_n^e)} \\ &\leq 2\omega(s,t)^{1/p} \sqrt{E(\|\mathbf{X}\|_{p,\omega}^2 | \mathcal{F}_n^e)}. \end{aligned}$$

Hence, since $q \geq 2$,

$$\sup_n \|S(X_{0,n}^e)\|_{p,\omega} \leq C \sup_n E(\|\mathbf{X}\|_{p,\omega}^q | \mathcal{F}_{0,n}^e)^{1/q},$$

which in L^q by Doob's inequality. By interpolation, we obtain the convergence of $d_{p',\omega}(S(X_{0,n}^e), \mathbf{X})$ to 0 both almost surely and in L^q . \square

3. THE PARTICULAR CASE OF A VOLTERRA GAUSSIAN PROCESS

This section is devoted to apply the previous results to Volterra Gaussian processes. There are a lot of work about integration with respect to these processes see [9], [4] or [5] for more details and references therein. Since we are only interesting in the construction of enhanced Gaussian Volterra processes, we work in a more simpler framework.

Let K be a measurable kernel $K : [0, 1]^2 \rightarrow \mathbb{R}$ such that for all $t \in [0, 1]$, $K(t, \cdot) \in L^2([0, 1], \mathbb{R}, dr)$, and for all $0 \leq t \leq s \leq 1$, $K(t, s) = 0$. Let $B = (B^1, \dots, B^d)$ be a d dimensional Brownian motion, then the Gaussian Volterra process associated to B and K is the process $(X(t), t \in [0, 1])$ defined by :

$$X^i(t) = \int_0^t K(t, s) dB_s^i, \quad t \in [0, 1], \quad i = 1, \dots, d.$$

Its covariance function is

$$C(t, s) = c(t, s)I_{\mathbb{R}}^d, \quad (s, t) \in [0, 1]$$

where $I_{\mathbb{R}}^d$ is the identity matrix and

$$c(t, s) = \int_0^1 K(t, u)K(s, u)du.$$

In order to construct the natural lift we may assume the following

- Assumption 2.** (1) *There exists $\alpha > 0$ such that the map $t \mapsto K(t, \cdot)$ is α Hölder continuous from $[0, 1]$ to $L^2([0, 1], \mathbb{R}, dr)$,*
 (2) *The function $t \mapsto \int_0^t K(t, s)ds$ is of bounded variation.*
 (3) *The function $t \mapsto K(t, s)$ has a differential with respect to t on $]s, 1]$ denoted by $\partial K(t, s)$, $\partial K(t, \cdot)$ belongs to $L_{loc}^1([0, t], \mathbb{R}, du)$ and $\sup_{0 \leq s < t \leq 1} |\partial K(t, s)|(t - s)^{\frac{3}{2}} < +\infty$.*

Under point (1) of Assumption 2, X has a modification with β Hölder continuous sample path for any $\beta < \alpha$. In the sequel, we only consider this modification. Indeed, the variance of the increments is

$$\sum_{i=1}^d \mathbb{E}(|X^i(t) - X^i(s)|^2) = d \int_0^1 [K(t, u) - K(s, u)]^2 du.$$

Therefore using (1) of Assumption 2 there exists a constant C_α such that

$$\sum_{i=1}^d E(|X_t^i - X_s^i|^2) \leq C_\alpha |t - s|^{2\alpha},$$

and the existence of a continuous modification is a consequence of the Kolmogorov Theorem.

The Cameron-Martin space associated to X , is

$$H = \{h, h(t) = \int_0^t K(t, s)\dot{h}(s)ds, t \in [0, 1], \dot{h} \in L^2([0, 1], \mathbb{R}^d, ds)\},$$

endowed with the scalar product $\langle h, g \rangle = \langle \dot{h}, \dot{g} \rangle_{L^2([0, 1], \mathbb{R}^d, ds)}$. Let us recall the proof given in [8]. Indeed, in one hand, if $h(t) = \int_0^t K(t, s)\dot{h}(s)ds$, $\dot{h} \in L^2([0, 1], \mathbb{R}^d, ds)$,

$t \in [0, 1]$, then for any $n \in \mathbb{N}^*$, $\alpha_i \in \mathbb{R}$, $t_i \in [0, 1]$, $i = 1, \dots, n$,

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i h(t_i) \right\|^2 &= \left\| \int_{[0,1]} \sum_{i=1}^n \alpha_i K(t_i, s) \dot{h}(s) ds \right\|^2 \\ &\leq \|\dot{h}\|_{L^2([0,1], \mathbb{R}^d, ds)}^2 \left[\sum_{i,j=1}^n \alpha_i \alpha_j c(t_i, t_j) \right], \end{aligned}$$

that means that h belongs to H and $|h|_H \leq \|\dot{h}\|_{L^2([0,1], \mathbb{R}^d, ds)}$. On the other hand, let $h, g \in H$, there exists two Gaussian random vectors Φ_h, Φ_g , belonging to the Gaussian space associated to X such that for all $t \in [0, 1]$, $j = 1, \dots, d$, $h^j(t) = \mathbb{E}(\Phi_h^j X^j(t))$ and $g^j(t) = \mathbb{E}(\Phi_g^j X^j(t))$. Then, Φ_h and Φ_g belong to the Gaussian space associated to B and there exists \dot{h} and \dot{g} in $L^2([0, 1], \mathbb{R}^d, ds)$ such that $\Phi_h^j = \int_0^1 \dot{h}^j(s) dB_s^j$ and $\Phi_g^j = \int_0^1 \dot{g}^j(s) dB_s^j$ for $j = 1, \dots, d$. We derive that, for $t \in [0, 1]$, $h(t) = \int_0^1 K(t, s) \dot{h}(s) ds$, $g(t) = \int_0^t \dot{g}(s) K(t, s) ds$ and

$$\langle h, g \rangle = E(\langle \Phi_h, \Phi_g \rangle_{\mathbb{R}^d}) = \langle \dot{h}, \dot{g} \rangle_{L^2([0,1], \mathbb{R}^d, ds)}.$$

Let $(h_n)_{n \in \mathbb{N}}$ be an orthonormal basis of $L^2([0, 1], \mathbb{R}, dr)$ belonging to $C^\infty([0, 1], \mathbb{R})$, and set

$$(3.1) \quad e_n(t) = \int_0^t K(t, s) h_n(s) ds, \quad t \in [0, 1].$$

Then under Assumption 2, (2) and (3), (e_n) is an orthonormal basis of $(H, \langle \cdot, \cdot \rangle)$ which is $H \cap BV$. Indeed, for $n \in \mathbb{N}$, $t \in [0, 1]$,

$$e_n(t) = \int_0^1 K(t, s) ds h_n(t) + \int_0^1 K(t, s) (h_n(s) - h_n(t)) ds$$

is the sum of a function in VB and a function absolutely continuous with respect to the Lebesgue measure with derivative given by

$$\int_0^t \partial K(t, s) (h_n(s) - h_n(t)) ds - \int_0^t K(t, s) ds \dot{h}_n(t), \quad t \in [0, 1].$$

Let us introduce some notation: for $\Pi = (t_i)_{i=0}^{|\Pi|}$ a subdivision of $[0, 1]$, and $t \in [0, 1]$,

$$(3.2) \quad K_2^\Pi(t, u, v) = \sum_{t_i \in \Pi, t_i \leq t} (K(t_i, u)[K(t_{i+1}, v) - K(t_i, v)] - K(t_i, v)[K(t_{i+1}, u) - K(t_i, u)]),$$

and

$$\begin{aligned} K_2(t, u, v) &= 2 \int_u^t K(r, u) \partial K(r, v) dr - K(t, u) K(t, v) \quad \text{if } u > v, \\ &= -2 \int_v^t K(r, v) \partial K(r, u) dr + K(t, u) K(t, v) \quad \text{if } v > u. \end{aligned}$$

Lemma 3. *Let K be a measurable kernel fulfilling Assumption 2. The sequel $(S_t(X_n^e))_{n \in \mathbb{N}}$ converges in probability if and only if $K_2(t, \cdot, \cdot)$ belongs to $L^2([0, 1], dudv)$.*

Proof. Let $k, l \in \mathbb{N}$, since e_k and e_l have finite variation, the integral of e_l with respect to e_k is limit of the Riemann sums. Then using the integral representation given in (3.1), Fubini's lemma and the definition of $K_2^{\Pi^n}(t, \cdot, \cdot)$ given in (3.2), we have

$$(3.3) \quad \int_0^t (e_l(s)\dot{e}_k(s) - e_k(s)\dot{e}_l(s)) ds = \lim_{n \rightarrow \infty} \langle h_l \otimes h_k, K_2^{\Pi^n}(t, \cdot, \cdot) \rangle.$$

Note that $(u, v) \mapsto K_2^{\Pi}(t, u, v)$ is antisymmetric, so we deal only with $u > v$. Then using a change of variable, with $t_{i_t} \leq t < t_{i_t+1}$,

$$\begin{aligned} K_2^{\Pi}(t, u, v) &= -K(t_{i_t+1}, u)K(t_{i_t+1}, v) + \sum_{t_i \in \Pi, t_i \leq t} 2K(t_i, u)[K(t_{i+1}, v) - K(t_i, v)] \\ &+ \sum_{t_i \in \Pi, t_i \leq t} [K(t_{i+1}, u) - K(t_i, u)][K(t_{i+1}, v) - K(t_i, v)]. \end{aligned}$$

Since $(h_n)_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2([0, 1], \mathbb{R}, du)$, then $(h_k \otimes h_l)_{(l, k) \in \mathbb{N}^2}$ is an orthonormal basis of $L^2([0, 1]^2, \mathbb{R}, dudv)$. According Lemma 1, the sequence of random variables $(S(X_n^e)_t)_n$ converges almost surely if and only if

$$\sum_{0 \leq l < k < \infty} \left[\int_0^t (e_l(s)\dot{e}_k(s) - e_k(s)\dot{e}_l(s)) ds \right]^2 < +\infty.$$

In other words,

$$\sum_{0 \leq k < l < \infty} \lim_{n \rightarrow \infty} \langle h_k \otimes h_l, K_2^{\Pi^n}(t, \cdot, \cdot) \rangle^2 < \infty.$$

If under Assumption 2, $(K_2^{\Pi^n}(t, \cdot, \cdot))_n$ converges to $K_2(t, \cdot, \cdot)$ in $L^1([0, 1]^2, dudv)$ then using the fact that the function h_k are bounded on $[0, 1]$ for all $k \in \mathbb{N}$, $\lim_{n \rightarrow \infty} \langle h_k \otimes h_l, K_2^{\Pi^n}(t, \cdot, \cdot) \rangle = \langle h_k \otimes h_l, K_2(t, \cdot, \cdot) \rangle$ for all $l, k \in \mathbb{N}$; and $(S(X_n^e)_t)_n$ converges almost surely if and only if $K_2(t, \cdot, \cdot)$ belongs to $L^2([0, 1], dudv)$.

Then, it remains to prove that $(K_2^{\Pi^n}(t, \cdot, \cdot))_n$ converge to $K_2(t, \cdot, \cdot)$ in $L^1([0, 1]^2, dudv)$.

We split $K_2^{\Pi^n}(t, \cdot, \cdot) - K_2(t, \cdot, \cdot) = S_1^{\Pi^n}(t, \cdot, \cdot) + 2S_2^{\Pi^n}(t, \cdot, \cdot) + S_3^{\Pi^n}(t, \cdot, \cdot)$, where for $0 \leq v < u \leq 1$,

$$S_1^{\Pi^n}(t, u, v) = K(t, v)K(t, u) - K(t_{i_t+1}, u)K(t_{i_t}, v),$$

$$S_2^{\Pi^n}(t, u, v) = \sum_{t_i \in \Pi^n, t_i \leq t} K(t_i, u)(K(t_{i+1}, v) - K(t_i, v)) - \int_u^t K(r, u)\partial K(r, v)dr,$$

and

$$S_3^{\Pi^n}(t, \cdot, \cdot) = \sum_{t_i \in \Pi, t_i \leq t} [K(t_{i+1}, u) - K(t_i, u)][K(t_{i+1}, v) - K(t_i, v)].$$

First, observe that for $v < u$

$$\left| S_1^{\Pi^n}(t, u, v) \right| \leq |K(t_{i_t+1}, u)| |K(t_{i_t}, v) - K(t, v)| + |K(t, u) - K(t_{i_t+1}, u)| |K(t, v)|$$

and use Fubini's Theorem and Cauchy Schwartz inequality to derive :

$$\|S_1^{\Pi^n}(t, \cdot, \cdot)\|_{L^1([0, 1]^2, \mathbb{R}, dudv)} \leq 2 \left(\sqrt{\mathbb{E}((X_{t_{i_t+1}}^1)^2)\mathbb{E}((X_{t_{i_t}}^1 - X_t^1)^2)} + \sqrt{\mathbb{E}((X_t^1)^2)\mathbb{E}((X_{t_{i_t+1}}^1 - X_t^1)^2)} \right).$$

Since X is a Gaussian process with continuous sample path $\|S_1^{\Pi^n}(t, \cdot, \cdot)\|_{L^1([0,1]^2, \mathbb{R}, dudv)}$ converge to 0 when n goes to infinity.

Second, we observe that for $0 \leq v < u \leq 1$,

$$(3.4) \quad S_2^{\Pi^n}(t, u, v) = \int_{t_{i_{u+1}}}^t (K(t_{i_r}, u) - K(r, u)) \partial K(r, v) dr \\ - \int_u^{t_{i_{u+1}}} K(r, u) \partial K(r, v) dr + K(t_{i_t}, u) (K(t_{i_{t+1}}, v) - K(t, v)).$$

The convergence of the last term of the right member of (3.4) to 0 in $L^1([0, 1]^2, dudv)$ follows the same lines as the convergence of $\|S_1^{\Pi^n}(t, \cdot, \cdot)\|_{L^1([0,1]^2, \mathbb{R}, dudv)}$ to 0.

For the first term of the right member of (3.4) note that

$$\int_{0 \leq v \leq u \leq 1} dudv \int_{t_{i_{u+1}}}^t |K(t_{i_r}, u) - K(r, u)| |\partial K(r, v)| dr \\ = \int_{t_1}^t dr \int_0^{t_{i_r}} dv |\partial K(r, v)| \int_v^{t_{i_r}} |K(t_{i_r}, u) - K(r, u)| du.$$

Using Cauchy Schwarz inequality in the integral with respect to du and

$$\mathbb{E}((X_z - X_y)^2) = \int_0^1 [K(z, r) - K(y, r)]^2 dr$$

we derive

$$\int_{0 \leq v \leq u \leq 1} dudv \int_{t_{i_{u+1}}}^t |K(t_{i_r}, u) - K(r, u)| |\partial K(r, v)| dr \\ \leq \sqrt{C_\alpha} \int_0^1 dr \int_0^{t_{i_r}} |\partial K(r, v)| \sqrt{t_{i_r} - v} |r - t_{i_r}|^\alpha dv.$$

Then $\int_{0 \leq v \leq u \leq 1} dudv \int_{t_{i_{u+1}}}^t |K(t_{i_r}, u) - K(r, u)| |\partial K(r, v)| dr$ converges to 0 when n goes to ∞ since $t_r \leq r$. For the second term of the right member of (3.4), the same kind of computations yields

$$\int_{0 \leq v \leq u \leq 1} dudv \int_u^{t_{i_{u+1}}} |K(r, u)| |\partial K(r, v)| dr \\ \leq \int_0^1 dr \int_0^r |\partial K(r, v)| dv \int_{\max(v, t_r)}^r |K(r, u)| du \\ \leq \sqrt{C_\alpha} \int_0^1 dr \int_0^r |\partial K(r, v)| |r - \max(v, t_r)|^{1/2} |r - v|^\alpha dv.$$

We conclude that $\|S_2^{\Pi^n}(t, \cdot, \cdot)\|_{L^1([0,1]^2, \mathbb{R}, dudv)}$ converge to 0 when n goes to infinity. Using the same arguments, $\|S_3^{\Pi^n}(t, \cdot, \cdot)\|_{L^1([0,1]^2, \mathbb{R}, dudv)}$ converge to 0 when n goes to infinity. □

Corollary 3. *Let K be a mesurable kernel fulfilling Assumption 2. Assume that*

- $(K_2^{\Pi^n}(t, \cdot, \cdot))_{n \in \mathbb{N}}$ converges in $L^2([0, 1]^2, dudv)$ to $K_2(t, \cdot, \cdot)$,
- $t \mapsto K_2(t, \cdot, \cdot)$ is β Hölder continuous in $L^2([0, 1]^2, \mathbb{R}, dudv)$,

then \mathbf{X} is natural lift of X .

Proof. We define $A = ((\pi_{i,j} - \pi_{j,i})(\mathbf{X}))_{(i,j) \in \{1, \dots, d\}^2}$ to be the area of \mathbf{X} . For all $s < t \in [0, 1]$,

In order to establish the continuity of the paths of \mathbf{X} , according to the expression of $\pi_{i,i}(\mathbf{X})$ given in (2.1) it only remains to prove that A has a continuous version. Just observe that for $t, s \in [0, 1]^2$

$$\begin{aligned} \mathbb{E}((A_t^{i,j} - A_s^{i,j})^2) &= \|K^2(t, \cdot, \cdot) - K^2(s, \cdot, \cdot)\|_{L^2([0,1], \mathbb{R}, dudv)}^2 \\ &\leq |t - s|^{2\beta}. \end{aligned}$$

Then using the fact (see [2]) that there exists a constant C_p such that for all variable Y in the second Wiener chaos of X ,

$$\mathbb{E}(Y^p) \leq C_p \mathbb{E}(Y^2)^{p/2}$$

and the Kolmogorov Lemma we obtain the continuity of $A^{i,j}$ and then of \mathbf{X} in $G^2(\mathbb{R}^d)$. \square

As it is pointed out in the pioneering paper of Decreusefond-Üstünel, [8], a now celebrate example of Volterra process which satisfies the previous assumptions is the fractional Brownian motion with Hurst parameter $h \in (0, 1]$. The associated kernel is ([24]) : for $s < t$,

$$\begin{aligned} K_h(t, s) &= c_h s^{\frac{1}{2}-h} \int_s^t (u-s)^{h-\frac{3}{2}} u^{h-\frac{1}{2}} du, \quad \text{for } h > \frac{1}{2}, \\ &= c_h \left[\frac{(t-s)^{h-\frac{1}{2}} t^{h-\frac{1}{2}}}{h-\frac{1}{2}} - \int_s^t (u-s)^{h-\frac{1}{2}} u^{h-\frac{3}{2}} du \right] s^{\frac{1}{2}-h}, \quad \text{for } h < \frac{1}{2}, \\ &= \mathbf{1}_{[0,t]}(s) \quad \text{for } h = \frac{1}{2} \end{aligned}$$

where c_h is a suitable constant such that the covariance function is

$$c(t, s) = \frac{1}{2} [s^{2h} + t^{2h} - |t - s|^{2h}].$$

Remark 1. *As it pointed out in Proposition 32 of [7], fractional Brownian motion fulfills the existence condition of Corollary 3 if and only if is $h > \frac{1}{4}$.*

4. APPLICATION: A GENERALISED WONG-ZAKAI THEOREM

For simplicity, we will work with the Gaussian process Brownian Motion B , together with its natural lift $\mathbf{B}_t = (B_t, \int_0^t B_u \circ dB_u)$ (it is a natural lift from theorem 2 for example). It is clear that we can extend the following result to more general Gaussian processes. We fix a orthonormal basis e of the Cameron-Martin space of B , i.e. $(\dot{e}_n)_n$ is a orthonormal basis of $L^2([0, 1], du)$. Then, $B_t = \sum_{i=0}^{\infty} N_i^e e_i(t)$, and define $B_{0,n}^e(t) = \sum_{i=0}^n N_i^e e_i(t)$. Then, from the continuity of the Ito map and the results in this paper, we obtain the following theorem:

Theorem 4. *Assume that $p \in [2, 3)$. Let $V = (V_i)_{1 \leq i \leq d}$ be some vector fields on \mathbb{R}^n which are $C^{p+\varepsilon}$, $\varepsilon > 0$. Define $Y_{0,n}$ to be the solution of the ODE*

$$\begin{cases} dY_{0,n}(t) = V(Y_{0,n}(t)) dB_{0,n}(t) \\ Y_0 = y_0. \end{cases}$$

Almost surely, $Y_{0,n}$ converges in p -variation topology to the solution of the Stratonovich SDE

$$\begin{cases} dY(t) = V(Y(t)) \circ dB(t) \\ Y_0 = y_0. \end{cases}$$

Observe that if we take the Haar basis for the orthonormal basis e of $L^2([0, 1], du)$, then we fall back on the classical Wong-Zakai theorem.

5. DIRECTION OF FURTHER RESEARCH

It would be nice to extend our result to lift to the free nilpotent group of step 3, or to a general step n . Things there get harder, as the martingale arguments fails to work for integrals of the type $\int |B_u^1|^2 dB_u^2$. The condition to check whether we can have a lift is quite neat and easy to read on Volterra process.

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