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► **To cite this version:**

René Ndoundam, Maurice Tchunte, Claude Tadonki. Parallel Chip Firing Game associated with n-cube orientations. International Conference on Computational Science 2004, ICCS 2004, Jun 2004, Krakov, Poland. pp.610,611,612,613. hal-00496371

**HAL Id: hal-00496371**

**<https://hal.science/hal-00496371>**

Submitted on 30 Jun 2010

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# Parallel Chip Firing Game associated with *n-cube* orientations

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June 30, 2010

## Abstract

We study the cycles generated by the chip firing game associated with *n-cube* orientations. We show the existence of the cycles generated by *parallel evolutions* of even lengths from 2 to  $2^n$  on  $H_n$  ( $n \geq 1$ ), and of odd lengths different from 3 and ranging from 1 to  $2^{n-1} - 1$  on  $H_n$  ( $n \geq 4$ ).

**Keywords :** Graph, chip firing game, parallel evolution, cycle, transient, *n-cube* orientation.

## 1 Introduction

Consider a digraph  $G = (V, A)$ , where  $V = \{1, \dots, n\}$  is the set of vertices and  $A \subseteq V \times V$  is the set of arcs. The *out-degree* ( resp. *in-degree* ) of a vertex  $i$ ,

hereafter denoted by  $d^+(i)$  (resp.  $d^-(i)$ ), is the number of vertices  $j$  such that  $(i, j) \in A$  (resp.  $(j, i) \in A$ ). A vertex with *out-degree* zero is called a *sink*. All these notions apply to an undirected graph  $G = (V, E)$  by considering an edge  $e = [i, j]$  as two opposite arcs  $(i, j)$  and  $(j, i)$ .

In the *parallel chip firing game* played on  $G$ , a state is a mapping  $x : V \rightarrow N$  which can be viewed as a distribution of chips onto the vertices of  $G$ . A vertex is said to be *active* in a state  $x$  if  $x(i) \geq d^+(i)$ , otherwise it is said to be *passive*. In a move of the game, a state  $x$  is transformed into a new state as follows : every vertex tries to send one chip to every *out-neighbor*.

- If it is not possible, i.e. if  $i$  is *passive*, then it resigns ;
- Otherwise, vertex  $i$  is *active* and sends the chips.

It is easily seen that the number of chips remains constant. Therefore, the evolution is ultimately *periodic*. More precisely, if  $x^t, t \geq 0$ , denotes the state of the system at time  $t$ , then there exists an integer  $q$  called *transient length* and another integer  $p$  called *period* or *cycle length* such that

$$x^{t+p} = x^t \text{ for } t \geq q, \text{ and } x^{t+p'} \neq x^t \text{ for } p' < p. \quad (1)$$

The sequence  $x^0, x^1, \dots, x^{q-1}$  is called the *transient* and every sequence of  $p$  consecutive states  $x^t, x^{t+1}, \dots, x^{t+p-1}$ , such that  $t \geq q$ , is called a *cycle* of the evolution.

Following Spencer's introductory paper [SPE 86] which was devoted to the chip firing game on chains, many authors have been interested in this problem. The most interesting questions concern the relationships between the structure of the graph on one hand, and the transients and periods generated by the chip firing game on the other hand. Concerning the *period*, Bitar and Goles have shown that if  $G$  is a tree, then only periods one and two occur [BIT 92]. Later, Prisner has studied a generalization of the game by considering multigraphs, i.e.

digraphs with multiplicities on the arcs. He has then shown that there is a sharp contrast in the behavior for eulerian digraphs (i.e. digraphs where the *in-degree* of each vertex equals its *out-degree*). More precisely, he has proved that in every strongly connected eulerian multigraph, any divisor of every dicycle length occurs as a period [PRI 94]. He has also shown that there is no polynomial  $h(n)$  such that the periods generated by the chip firing game on digraphs of order  $n$  are bounded by  $h(n)$ . Readers interested by other results on periods and transients of the chip firing game may refer to [TAR 88, AND 89, BIT 89, BJO 91, ERI 91, GOL 93]. Readers interested by combinatorial games may refer to [Gol 02, GM 02, Gol 04, Sjo 05, Fra 09].

There is a particular case where the chip firing game is related to graph orientations. Indeed, let us consider an undirected graph  $G = (V, E)$ , and let us assume that initially, the edges of  $G$  can be oriented in such a way that the number of chips of every vertex equals the *in-degree* of that vertex. If this property is true in the initial configuration, then it remains true throughout the game. One step of the game then consists in reversing the orientations of all edges going into sinks. Goles and Prisner [GOL 00] have studied *gardens of Eden*, i.e. states that can appear only at time  $t = 0$ . They have also studied the relationships between graph orientations and evolutions induced by states with  $|E|$  chips. Moreover, Kiwi, Ndoundam, Tehuente and Goles [KIW 94] have exhibited cycles of exponential length  $e^{\Omega(n \log n)}$  generated by the chip firing game associated with the orientations of cascades of rings. Other results on this particular case may be found in [ERI 94].

In this paper, we study the dynamics generated by the chip firing game associated with *n-cube* orientations. More precisely, using a recurrent approach, we show that for  $n \geq 4$ , there exists cycles of even lengths from 2 to  $2^n$  on  $H_n$  ( $n \geq 1$ ), and of odd lengths different from 3 and ranging from 1 to  $2^{n-1} - 1$  on  $H_n$  ( $n \geq 4$ ).

The remainder of this paper is organized as follows. In the next section,

we present some basic notations and definitions related to  $n$ -cubes. Section 3 is devoted to the recurrent construction of *left cyclic partitions* and possible period lengths whereas section 4 presents some concluding remarks.

## 2 Basic notations and definitions

An  $n$ -dimensional hypercube (or  $n$ -cube) is an undirected graph  $H_n = (V, E)$ , where  $V = \{0, 1\}^n$  is the set of vertices and two nodes  $u = (u_0, u_1, \dots, u_{n-1})$  and  $v = (v_0, v_1, \dots, v_{n-1})$  are neighbors if and only if they differ in only one bit in their binary representations, i.e. there is an integer  $i$  such that  $u_i \neq v_i$  and  $u_j = v_j$  for  $j \neq i$ . One can define recursively the  $n$ -cube as follows :

- The 0-cube is reduced to one vertex ;
- $H_{n+1}$  is obtained by taking two copies of  $H_n$  and connecting all pairs of equivalent vertices.

Fig. 1 illustrates this constructions for  $n = 0, 1, 2, 3$ .

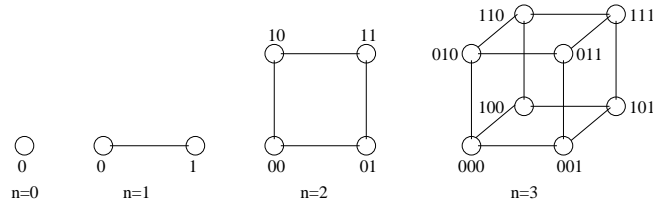


Figure 1:  $n$ -cubes for  $n \leq 3$ .

Hereafter, given a set  $W$  and a boolean value  $x$ , we denote  $xW = \{xu : u \in W\}$ . With this notation, we can write  $H_{n+1} = 0H_n \cup 1H_n$ .

**Definition 1.** A *block-sequential evolution* of the chip firing game associated with graph orientations and played on an  $n$ -cube is obtained as follows. Consider a sequence of non empty subsets  $\{W_i; i \geq 0\}$  of  $\{0, 1\}^n$ . At time  $t$ , every vertex  $u$  of  $W_t$  is considered. If  $u$  is a sink then the orientation of all its *in-going* arcs are reversed, otherwise no action is undertaken. Hereafter, we say

that a vertex *fires* at time  $t$  if it belongs to  $W_t$  and is a *sink* at time  $t$ .

The *parallel evolution* is therefore a particular case of the general scheme described above. Another classical evolution scheme is the *sequential evolution* where  $W_t$  is reduced to one vertex (i.e.  $|W_t| = 1$ ) and there is a permutation  $\sigma$  of  $\{0, \dots, 2^n - 1\}$  such that  $\{W_i; i \geq 0\}$  is periodic of period  $W_{\sigma(0)}, W_{\sigma(1)}, \dots, W_{\sigma(2^n - 1)}$ . Both parallel and sequential evolutions are particular cases of the so-called *serial-parallel* evolutions where the sequence  $\{W_i; i \geq 0\}$  is periodic of period  $W_0, W_1, \dots, W_{k-1}$ , with the constraint that  $W_0 \cup W_1 \cup \dots \cup W_{k-1}$  is a partition of  $\{0, 1, \dots, 2^n - 1\}$ .

**Definition 2.** A partition  $S_0 \cup S_1 \cup \dots \cup S_{k-1}$  of the vertices of an  $n$ -cube is called a *left cyclic* partition if the two following statements hold.

- For all  $i$  from 0 to  $k-1$ , every vertex of  $S_i$  has a neighbor in  $S_{i-1}$ , where index operations are performed modulo  $k$ .
- For all  $i$  from 0 to  $k-1$ , there is no edge between two vertices of  $S_i$ .

**Comment 2.** Canonical decompositions defined in [GOL 00] for acyclic digraphs are obtained from *left cyclic partitions* by orienting the edges such that all arcs from the set  $S_i$  go to sets  $S_j$  such that  $j > i$ . On the other hand, *left cyclic partitions* are more restrictive than the partitions introduced in [PRI 94] since we do not allow edges joining two vertices of the same subset. Indeed, in the chip firing game associated with graph orientations, two neighbors cannot fire simultaneously, whereas this situation is possible for the general chip firing game.

We present an important property of left cyclic partitions on an  $n$ -cube.

**Theorem 1** *If a partition  $S_0 \cup S_1 \cup \dots \cup S_{k-1}$  of the vertices of an  $n$ -cube  $H_n$*

is a left cyclic partition then there is a cyclic evolution of the chip firing game associated with graph orientations and played on  $H_n$ , such that for every  $t \geq 0$ ,  $S_t$  is the set of vertices which are fired at time  $t$ .

**Proof.** Let  $S_0, S_1, \dots, S_{k-1}$  be a left cyclic partition. Consider an orientation where every edge  $e = [u, v]$  such that  $u \in S_i$  and  $v \in S_j, i < j$ , is oriented from  $v$  to  $u$ . It is easily seen that in the parallel chip firing game starting with such a configuration, the subsets of vertices which fire at successive steps correspond to a periodic sequence of period  $S_0, S_1, \dots, S_{k-1}$ .

□

### 3 Recurrent construction of left cyclic partitions

In this section, we first present the construction of *left cyclic partitions* of even lengths.

**Lemma 1** *An  $n$ -cube admits left cyclic partitions of all even lengths from 2 to  $2^n$ .*

**Proof.** Let  $H_n = (V, E)$  be an  $n$ -cube and let  $p$  be an even integer between 2 and  $2^n$ . It is well known that, since  $p$  is even, there is a cycle  $[x_0, x_1, \dots, x_{p-1}, x_0]$  of length  $p$  in  $H_n$ . Now, for every vertex  $u$ , let  $\Gamma(u)$  denote the set of all neighbors of  $u$  in  $H_n$ . This notation is naturally extended to a set of vertices. A *left cyclic partition* of order  $p$  is obtained as follows.

```

For  $i = 0, \dots, p - 1$  do
   $S_i \leftarrow \{x_i\}$ 
endfor
 $S = V - \{x_0, x_1, \dots, x_{p-1}\}$ 
while ( $S \neq \emptyset$ ) do
  For  $i \leftarrow 0$  to  $p - 1$  do
     $S_{i+1} \leftarrow S_{i+1} \cup (\Gamma(S_i) \cap S)$ 
  
```

$$S \leftarrow S - (\Gamma(S_i) \cap S)$$

**endfor**

**endwhile**

It is obvious that  $S_0, \dots, S_{p-1}$  is a partition of  $V$  and that every vertex in  $S_i$  has at least one neighbor in  $S_{i-1}$ . So we just need to show that two vertices of the same subset  $S_i$  cannot be neighbors. Let  $a$  and  $b$  be two vertices of  $S_i$ .

- There is a path from  $a$  to  $x_0$  of length  $\ell_1$  such that  $\ell_1 = i \bmod p$ ,
- There is a path from  $b$  to  $x_0$  of length  $\ell_2$  such that  $\ell_2 = i \bmod p$ ,

Since  $p$  is even, it follows that  $\ell_1 = \ell_2 \bmod 2$ . Hence, if  $a$  and  $b$  were neighbors, there would exist a cyclic path of odd length  $\ell_1 + \ell_2 + 1$  joining  $a$  and  $b$  in  $H_n$ , which is not possible since  $H_n$  is a bipartite graph. This shows that two vertices of the same subset cannot be neighbors.

□.

The following figure displays the partition of order 4 in  $H_3$  obtained by the previous procedure starting with the cycle  $[000, 001, 011, 010]$ .

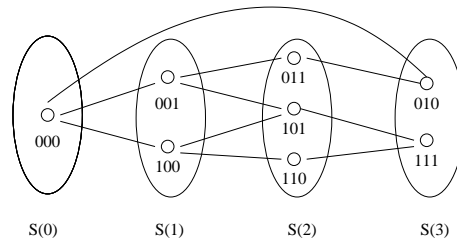


Figure 2: *Left cyclic partition* of orders 4 generated in  $H_3$ .

Let us now turn to the construction of *left cyclic partitions* of odd lengths.

**Lemma 2** *If  $S_0, S_1, S_2$  is a left cyclic partition of  $H_n, n \geq 2$ , then every vertex of  $S_i$  has at least two neighbors in  $S_{i-1}$  for  $i = 0, 1, 2$ .*

**Proof.** Because of symmetry considerations, we can assume that  $i = 2$ . So let  $x$  be a vertex of  $S_2$ . From the definition of *left cyclic partitions*,

- $x$  has a neighbor  $x \oplus e_j$  in  $S_1$ , where  $\oplus$  is the XOR operator and  $e_j$  is a



vector of the canonical basis.

- similarly,  $x \oplus e_j$  has a neighbor  $x \oplus e_j \oplus e_k$  in  $S_0$ .

Now consider the vertex  $x \oplus e_k$ .

- It is a neighbor of  $x$ , hence it does not belong to  $S_2$ .
- It is a neighbor of  $x \oplus e_j \oplus e_k$ , hence it does not belong to  $S_0$ .

It then follows that  $x \oplus e_k$  belongs to  $S_1$ , hence  $x$  admits two neighbors  $x \oplus e_j$  and  $x \oplus e_k$  which are both in  $S_1$ .

□

**Lemma 3** *If  $H_n$ ,  $n \geq 3$  admits a left cyclic partition of order 3, then  $H_{n-1}$  admits a left cyclic partition of order 3.*

**Proof.** Let  $S_0, S_1, S_2$  be a *left cyclic partition* of order 3 of  $H_n$ . Let  $x$  be a vertex of  $S_i$ . We can assume without loss of generality that  $x = 1a$ . Since  $H_n = 0H_{n-1} \cup 1H_{n-1}$ ,  $y = 0a$  is the unique neighbor of  $x$  in  $0H_{n-1}$ . Consequently, from lemma 2,  $x$  admits a neighbor in  $1H_{n-1} \cap S_{i-1}$ . This shows that the subgraph  $1H_{n-1}$  which is isomorphic to  $H_{n-1}$ , contains a *left cyclic partition* of order 3.

□

**Proposition 1**  *$n$ -cubes do not admit left cyclic partitions of order 3.*

**Proof.** An  $n$ -cube with  $n \leq 1$  has less than 3 vertices and cannot admit a *left cyclic partition* of order 3. On the other hand, from lemma 2, if  $S_0, S_1, S_2$  is a *left cyclic partition* of an  $n$ -cube,  $n \geq 2$ , then every  $S_i$  contains at least two elements (i.e.  $|S_i| \geq 2$ ). Consequently, the 2-cube  $H_2$  which is of cardinality 4 cannot admit a *left cyclic partition* of order 3. By application of lemma 3, we deduce that no  $n$ -cube,  $n \geq 3$  admits a *left cyclic partition* of order 3.

□

Proposition 1 gives the lower bound for *left cyclic partitions* of odd lengths. Let now study the upper bound.

**Proposition 2** *If  $S_0, \dots, S_{p-1}$  is a left cyclic partition of odd order  $p$  of  $H_n$ , then  $p \leq 2^{n-1} - 1$ .*

**Proof.** We just have to show that in such a case,  $|S_i| \geq 2$  for  $i = 0, \dots, p-1$ . Indeed, starting from a vertex  $a_{p-1} \in S_{p-1}$ , we construct a chain  $[a_{p-1}, a_{p-2}, \dots, a_0, b_{p-1}, b_{p-2}, \dots, b_0]$  such that  $a_i, b_i \in S_i$  for  $i = 0, \dots, p-1$ . It is clear that  $a_i \neq b_i, i = 0, \dots, p-1$ , otherwise we would have displayed a closed path of odd length in  $H_n$  which is not possible.

□

Now that we have established lower and upper bounds for *left cyclic partitions* of odd lengths, let us show that all intermediate lengths are admissible.

**Lemma 4** *If  $H_n$  admits a left cyclic partition of order  $p$ , then  $H_{n+1}$  admits left cyclic partition of order  $p$ .*

**Proof.** If  $S_0, \dots, S_{p-1}$  is a left cyclic partition of order  $p$  in  $H_n$ , then it is easily checked that  $1S_i \cup 0S_{i-1}, i = 0, \dots, p-1$  is a *left cyclic partition* of order  $p$  in  $H_{n+1}$

□.

**Lemma 5** *If  $H_n$  admits a left cyclic partition of odd order  $p, p \geq 5$  then  $H_{n+1}$  admits a left cyclic partition of order  $2p-1$ . Moreover, if  $p \geq 7$ , then  $H_{n+1}$  admits a left cyclic partition of order  $2p-3$ .*

**Proof.** Let  $S_0, S_1, \dots, S_{p-1}$  be a *left cyclic partition* of odd order  $p$ .

- Case  $p \geq 5$

The following sequence is a *left cyclic partition* of order  $2p-1$  in  $H_{n+1}$ .

$0S_0, 1S_0 \cup 0S_1, 1S_1, 1S_2, 0S_2, 0S_3, 1S_3 \dots, 1S_{2i}, 0S_{2i}, 0S_{2i+1}, 1S_{2i+1}, \dots, 1S_{p-3}, 0S_{p-3}, 0S_{p-2}, 1S_{p-2}, 1S_{p-1}, 0S_{p-1},$

- Case  $p \geq 7$

A *left cyclic partition* of order  $2p-3$  in  $H_{n+1}$  is obtained from the *left cyclic*

partition exhibited in the case  $p \geq 5$  by replacing the subsequence  $1S_2, 0S_2, 0S_3, 1S_3, 1S_4, 0S_4, 0S_5, 1S_5$  by  $1S_2, 0S_2 \cup 1S_3, 0S_3, 0S_4, 1S_4 \cup 0S_5, 1S_5$ .

□

**Lemma 6**  $H_4$  admits left cyclic partitions of orders 5 and 7.

**Proof.**

• A left cyclic partition of order 5 in  $H_4$  is the following :

$\{0000, 1101\}, \{0001, 1100, 0010, 1111\}, \{0110, 1011\}, \{0100, 0111, 1001, 1010\}, \{0011, 0101, 1000, 1110\}$ .

• A left cyclic partition of order 7 in  $H_4$  is the following :

$\{0000, 1101\}, \{0001, 1100\}, \{0011, 1110\}, \{0010, 1111\}, \{0110, 1011\}, \{0100, 0111, 1001, 1010\}, \{0101, 1000\}$ .

Fig. 3 displays the partitions.

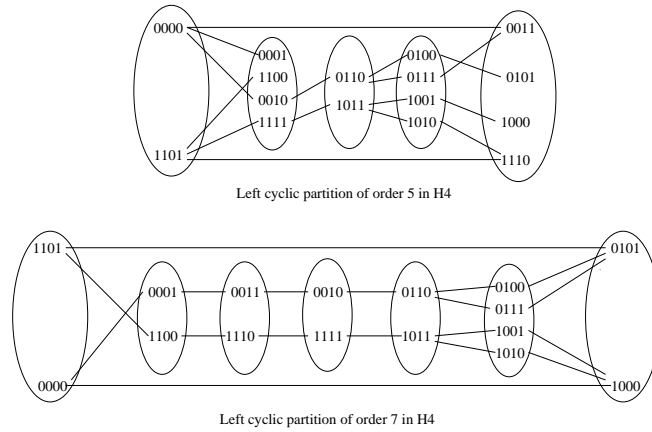


Figure 3: Left cyclic partitions of orders 5 and 7 in  $H_4$ .

□

**Lemma 7**  $H_n, n \geq 4$ , admits a left cyclic partition of order  $2^{n-1} - 1$ .

**Proof.** Consider the sequence  $\{u_i; 0 \leq i \leq 2^{n-1} - 1\}$ , defined by  $u_i = \text{bin}(i) \oplus \text{bin}(i/2)$ , where  $\text{bin}(x)$  is the  $n$ -position binary representation of the integer

$x$ , and symbol  $/$  denotes integer division. It can be easily checked that this sequence corresponds to a *hamiltonian cycle* in  $H_{n-1}$ . Now, let us denote  $v_i = u_i \oplus 1 \oplus 2^{n-2}$  (i.e.  $v_i$  is obtained from  $u_i$  by changing the first and last bits) and  $N = 2^n$ . It is also easy to check that  $\{v_i; 0 \leq i \leq 2^{n-1} - 1\}$  is a *hamiltonian cycle* of  $H_{n-1}$ . Let us now consider the following sets

$$\{0u_0, 1v_0\}, \dots, \{0u_{N-4}, 1v_{N-4}\}, \{0u_{N-3}, 0u_{N-1}, 1v_{N-3}, 1v_{N-1}\}, \{0u_{N-2}, 1v_{N-2}\}. \quad (2)$$

At this step, it is important to recall that two vertices referenced by  $i$  and  $j$  are neighbors in the hypercube if and only if there is an integer  $k$  such that  $i \oplus j = 2^k$ . Observe that  $0u_i \oplus 1v_i = 2^{n-1} \oplus (u_i \oplus v_i) = 2^{n-1} \oplus 1 \oplus 2^{n-2}$ . Hence,  $0u_i$  and  $1v_i$  are not neighbors in the hypercube  $H_n$ . On the other hand,  $u_{N-4} = 100\dots010$ ,  $u_{N-2} = 10\dots01$ ,  $u_{N-1} = 10\dots0$  and  $v_0 = u_0 \oplus 1 \oplus 2^{n-2} = 10\dots01 = u_{N-2}$ . Hence,

$$0u_{N-4}, 0u_{N-1}, 0u_{N-2}, 1v_0 \text{ is a chain of } H_n. \quad (3)$$

Moreover,  $v_{N-4} = 0\dots011$ ,  $v_{N-2} = 0\dots0 = u_0$  and  $v_{N-1} = 0\dots01 = u_1$ . Hence

$$1v_{N-4}, 1v_{N-1}, 1v_{N-2}, 0u_0 \text{ is a chain of } H_n. \quad (4)$$

Properties 3 and 4 together with the fact that  $\{u_i; 0 \leq i \leq 2^{n-1} - 1\}$  and  $\{v_i; 0 \leq i \leq 2^{n-1} - 1\}$  are both *hamiltonian cycles* of  $H_n$ , imply that the partition exhibited in 2 is a *left cyclic partition*.

□

**Proposition 3**  $H_n$ ,  $n \geq 4$ , admits left cyclic partitions of all odd orders from 5 to  $2^{n-1} - 1$ .

**Proof.** We proceed by induction on  $n$ . For  $n = 4$  the result follows from lemma 6.

Assuming that the result holds for  $n \geq 4$ , let us consider an  $(n+1)$ -cube together with an odd integer  $p \in [5, 2^n - 1]$ .

- Case 1 :  $5 \leq p \leq 2^{n-1} - 1$ . The result follows from the induction hypothesis by application of lemma 4.

- Case 2 :  $2^{n-1} - 1 < p < 2^n - 1$ . There is an odd integer  $q$ ,  $7 < q < 2^{n-1} - 1$ , such that  $p = 2q - 1$  or  $p = 2q - 3$ . The result follows from the induction hypothesis by application of lemma 5.

- Case 3 :  $p = 2^n - 1$ . The result follows from lemma 7.

□

We are now ready to state the main theorem.

**Theorem 2** *There exists cycles generated by the parallel chip firing game associated with  $n$ -cube orientations,  $n \geq 4$ , are of even lengths from 2 to  $2^n$ , and of odd lengths different from 3 and ranging from 1 to  $2^{n-1} - 1$ .*

**Proof.** We just need to show that this property holds for *left cyclic partitions* of vertices of  $n$ -cubes. The existence of *left cyclic partitions* of all even lengths from 2 to  $2^n$  follows from lemma 1. Let us turn to odd periods  $p$ .

- Case 1 :  $p = 1$ . Consider an orientation which contains a *hamiltonian cycle*. Clearly, such a configuration is a *fixed point* for the chip firing game associated with graph orientations.

- Case 2 :  $p = 3$ . The non existence of period 3 follows from proposition 1.

- Case 3 :  $5 \leq p \leq 2^{n-1} - 1$ . The existence of this period follows from proposition 3.

□

## 4 Conclusion

We show in the particular case of *parallel evolutions* on *n-cube*, the existence of cycles of even lengths from 2 to  $2^n$ , and of odd lengths different from 3 and ranging from 1 to  $-1 + 2^{n-1}$ . In case of *parallel evolutions* on *n-cube*, the existence of cycles of lengths greater than  $2^n$  remains an open question.

**Acknowledgement.** This work was supported by the French Agency *Aire développement* through the project *Calcul Parallèle* and by the Project NTIC of the University of Ngaoundéré.

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