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Essential self-adjointness for combinatorial Schrödinger operators II- Metrically non complete graphs

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Abstract

We consider weighted graphs, we equip them with a metric structure given by a weighted distance, and we discuss essential self-adjointness for weighted graph Laplacians and Schrödinger operators in the metrically non complete case.

1 Introduction

This paper is a continuation of [To] which contains some statements about essential self-adjointness of Schrödinger operators on graphs. In [To], it was proved that on any metrically complete weighted graph with bounded degree, the Laplacian is essentially self-adjoint and the same holds for the Schrödinger operator provided the associated quadratic form is bounded from below.
Here, we will investigate essential self-adjointness mainly on metrically non complete graphs.

[0] **Keywords:** metrically non complete graph, weighted graph Laplacian, Schrödinger operator, essential selfadjointness.

[0] **Math Subject Classification (2000):** 05C63, 05C50, 05C12, 35J10, 47B25.

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Let us recall that a weighted graph G is a generalization of an electrical network where the set of vertices and the set of edges are respectively weighted with positive functions ω and c. For any given positive function p, a weighted distance d_p can be defined on G. Thus we have the usual notion of completeness for G as a metric space. The main result of Section 3 states that the weighted graph Laplacian $\Delta_{\omega,c}$ (see the definition below) is not essentially self-adjoint if the graph is of finite volume and metrically non complete. The proof is derived from the existence of the solution for a Dirichlet problem at infinity, see Section 2.
In Section 4 we establish some conditions implying essential self-adjointness. More precisely, in the case of metrically non complete graphs, we get the essential self-adjointness of $\Delta_{1,a} + W$ provided the potential W is bounded from below by $N/2D^2$, where N is the maximal degree and D the distance to the boundary. We extend this result to $\Delta_{\omega,c} + W$ for another distance which is defined in terms of the edge weights as well as of the vertex weights. We use for both results a technical tool deduced from Agmon-type estimates and inspired by the nice paper [Nen], see also [Col-Tr].
We discuss in Section 5 the case of tree-like graphs. Under some assumptions on a, we prove that for any potential W, $\Delta_{1,a} + W$ is essentially self-adjoint using an extension of Weyl's theory in the discrete case. In the particular case of the graph $\mathbb{N}$, the same result had been proved in [Ber] (p.504) in the context of Jacobi matrices. We give some examples in Subsection 5.3 to illustrate the links between the previous results. Moreover we establish the sharpness of the conditions of Theorem 4.2.
In the context of essential self-adjointness problems for infinite graphs, we refer to [Go-Sch] and [Ma] for related results.
The last section is an Appendix dealing with Weyl's limit point-limit circle criteria [RS], in the discrete case as well as in the continuous case.

Let us start with the definitions.
$G = (V, E)$ will denote an *infinite graph*, with $V = V(G)$ the set of *vertices* and $E = E(G)$ the set of *edges*. We write $x \sim y$ for $\{x, y\} \in E$.
The graph G is always assumed to be *locally finite*, that is any $x \in V$ has a finite number of neighbors, which we call the *degree* (or valency) of x. If the degree is bounded independently of x in V, we say that the graph G is of *bounded degree*.
The space of real functions on the graph G is denoted

$$C(V) = \{f : V \longrightarrow \mathbb{R}\}$$

and $C_0(V)$ is the subspace of functions with finite support.
We consider, for any weight $\omega : V \longrightarrow \mathbb{R}_+^\star$, the space

$$l^2_\omega(V) = \{f \in C(V);\ \sum_{x\in V} \omega_x^2 f^2(x) < \infty\}.$$

It is a Hilbert space when equipped with the inner product:

$$\langle f, g\rangle_{l^2_\omega} = \sum_{x\in V} \omega_x^2 f(x).g(x) \ .$$

For any $\omega : V \longrightarrow \mathbb{R}_+^\star$, and $c : E \longrightarrow \mathbb{R}_+^\star$, the *weighted graph Laplacian* $\Delta_{\omega,c}$ on the graph G weighted by the *conductance* c on the edges and by the *weigth* ω on the vertices, is defined by:

$$(\Delta_{\omega,c} f)(x) = \frac{1}{\omega_x^2} \sum_{y\sim x} c_{x,y} (f(x) - f(y))$$

for any $f \in C(V)$ and any $x \in V$. If $\omega \equiv 1$, we have

$$(\Delta_{1,a} f)(x) = \sum_{y\sim x} a_{x,y} (f(x) - f(y)) .$$

Definition 1.1 *Let $p : E \longrightarrow \mathbb{R}_+^\star$ be given, the* weighted distance d_p *on the weighted graph G is defined by*

$$d_p(x, y) = \inf_{\gamma\in\Gamma_{x,y}} L(\gamma)$$

where $\Gamma_{x,y}$ is the set of the paths $\gamma : x_1 = x, x_2, \cdots, x_N = y$ from x to y. The length $L(\gamma)$ is computed as the sum of the p-weights for the edges of the path γ:

$$L(\gamma) = \sum_{1\le i\le n} p_{x_i,x_{i+1}}$$

We say that the metric space (G, d_p) is complete when every Cauchy sequence of vertices has its limit in V.

Definition 1.2 *We denote by $\hat{V}$ the metric completion of (G, d_p) and by $V_\infty = \widehat{V} \setminus V$ the metric boundary of V.*

Definition 1.3 *If G is a non finite graph and G_0 a finite sub-graph of G, the* **ends of G relatively to G_0** *are the non finite connected components of $G \setminus G_0$.*

2 The Dirichlet problem at infinity

We will use in this section the distance d_p defined using the weights $p_{x,y} = c_{x,y}^{-\frac{1}{2}}$. Let us consider the quadratic form

$$Q(f) = \langle (\Delta_{\omega,c} + 1)f \mid f\rangle_{l^2_\omega} = \sum_{\{x,y\}\in E} c_{x,y}(f(x) - f(y))^2 + \sum_{x\in V} \omega_x^2 f(x)^2 \ .$$

We have

Lemma 2.1 *For any $f : V \to \mathbb{R}$ so that $Q(f) < \infty$ and for any $a, b \in V$, we have*

$$|f(a) - f(b)| \leq \sqrt{Q(f)} d_p(a, b).$$

Proof.–

For any $\{x, y\} \in E$, $|f(x) - f(y)| \leq \sqrt{Q(f)}/\sqrt{c_{x,y}}$. For any path γ from a to b, defined by the vertices $x_1 = a, x_2, \cdots, x_N = b$, we have $|f(a) - f(b)| \leq \sqrt{Q(f)} L(\gamma)$. Taking the infimum of the righthandside with respect to γ we get the result.

□

Remark 2.1 *Lemma 2.1 implies that any function f with $Q(f) < \infty$ extends to $\hat{V}$ as a Lipschitz function $\hat{f}$. We will denote by f_∞ the restriction of $\hat{f}$ to V_∞.*

Theorem 2.1 *Let $f : V \to \mathbb{R}$ with $Q(f) < \infty$, then there exists a unique continuous function $F : \hat{V} \to \mathbb{R}$ which satisfies both conditions:*

(i) $(F - f)_\infty = 0$

(ii) $(\Delta_{\omega,c} + 1)(F_{|V}) = 0$

Proof.–

We will denote by A_f the affine space of continuous functions $G : \hat{V} \to \mathbb{R}$ which satisfy $Q(G) < \infty$ and $(G - f)_\infty = 0$.

Q is lower semi-continuous for the pointwise convergence on V as defined by $Q = \sup Q_\alpha$ with $Q_\alpha(f) =$ sum of a finite number of terms in Q.

Let $Q_0 := \inf_{G \in A_f} Q(G)$ and G_n be a corresponding minimizing sequence. The G_n's are equicontinuous and pointwise bounded. From Ascoli's Theorem, this implies the existence of a locally uniformly convergent subsequence $G_{n_k} \to F$. Using semi-continuity, we have $Q(F) = Q_0$.

If $x \in V$ and δ_x is the Dirac function at the vertex x, we have

$$\frac{d}{dt}_{|t=0} Q(F + t\delta_x) = 2\omega_x^2[(\Delta_{\omega,c} + 1)F(x)]$$

and this is equal to 0, because F is the minimum of Q restricted to A_f.

Uniqueness is proved using a maximum principle.

□

3 Non essentially self-adjoint Laplacians

Theorem 3.1 *Let $\Delta_{\omega,c}$ be a weighted graph Laplacian and assume the following conditions:*

(i) (G, d_p) with $p_{x,y} = c_{x,y}^{-\frac{1}{2}}$ is NON complete,

(ii) there exists a function $f : V \to \mathbb{R}$ with $Q(f) < \infty$ and $f_\infty \neq 0$.

Then $\Delta_{\omega,c}$ is non essentially self-adjoint.

Proof.–

Because $\Delta_{\omega,c}$ is ≥ 0 on $C_0(V)$, it is enough (see Theorem X.26 [RS]) to build a non zero function $F : V \to \mathbb{R}$ which is in $l^2_\omega(V)$ and satisfies

$$(\Delta_{\omega,c} + 1)F = 0 \ . \tag{1}$$

The function F given by Theorem 2.1 will be the solution of equation (1) the limit of which at infinity is f_∞.

□

Remark 3.1 *The assumptions of Theorem 3.1 are satisfied if (G, d_p) is non complete and $\sum \omega_y^2 < \infty$: it is enough to take $f \equiv 1$.*

They are already satisfied if G has a non complete "end" of finite volume.

Remark 3.2 *Theorem 3.1 is not valid for the Riemannian Laplacian: if X is a closed Riemannian manifold of dimension ≥ 4, $x_0 \in X$ and $Y = X \setminus x_0$, the Laplace operator on Y is essentially self-adjoint (see [Col1]) and Y has finite volume.*

Question 3.1 *In Theorem 3.1, what is the deficiency index of $\Delta_{\omega,c}$ in terms of the geometry of the weighted graph?*

4 Schrödinger operators for metrically non complete graphs

We consider a Schrödinger operator $H = \Delta_{1,a} + W$ such that (G, d_p), with $p_{xy} = a_{xy}^{-\frac{1}{2}}$, is non complete as a metric space; it means that there exists Cauchy sequences of vertices without limit in the set V. We will assume that G is of bounded degree, and we denote the upper bound by N.

Definition 4.1 *For a vertex $x \in V$, we denote by $D(x)$ the distance to the boundary V_∞ defined by*

$$D(x) = \inf_{z \in V_\infty} d_p(x, z).$$

Lemma 4.1 *We have, for any edge $\{x, y\}$,*

$$|D(x) - D(y)| \leq d_p(x, y) \leq a_{x,y}^{-\frac{1}{2}} \ . \tag{2}$$

4.1 Agmon-type estimates

Lemma 4.2 *Let v be a solution of $Hv = 0$ and let $f \in C_0(V)$. Then*

$$\langle fv\ , H(fv)\rangle \ = \sum_{\{x,y\}\in E} a_{x,y} v(x)v(y)(f(x)-f(y))^2\ . \tag{3}$$

Proof.–

For the reader's convenience we recall here the computation made in [To]:

$$\langle fv\ , H(fv)\rangle = \sum_{x\in V} f(x)v(x)\left(\sum_{y\sim x} a_{x,y}(f(x)-f(y))v(y)\right)$$

where we used the fact that $Hv(x) = 0$. An edge $\{x, y\}$ contributes to the sum twice. The total contribution is $f(x)v(x)\ a_{x,y}(f(x) - f(y))v(y) + f(y)v(y)a_{y,x}(f(y) - f(x))v(x)$ so

$$\langle fv\ , H(fv)\rangle = \sum_{\{x,y\}\in E} a_{x,y}(f(x)-f(y))\,(f(x)v(x)v(y) - f(y)v(y)v(x))$$

□

Theorem 4.1 *Let λ a real number and v a solution of $(H - \lambda)v = 0$. Assume that v belongs to $l^2(V)$ and that there exists a constant $c > 0$ such that, for all $u \in C_0(V)$,*

$$\langle u|(H-\lambda)u\rangle \geq \frac{N}{2}\sum_{x\in V}\max\left(\frac{1}{D(x)^2}, 1\right)|u(x)|^2 + c\|u\|^2, \tag{4}$$

then $v \equiv 0$.

Proof.–

This theorem is based on Lemma 4.2 applied to $H - \lambda$. Let us consider ρ and R satisfying $0 < \rho < \frac{1}{2}$ and $1 < R < +\infty$. For any $\varepsilon > 0$, we define the function $f_\varepsilon : V \to \mathbb{R}$ by $f_\varepsilon = F_\varepsilon(D)$ with $F_\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ the continuous piecewise affine function defined by

$$F_\varepsilon(u) = \begin{cases} 0 \text{ for } u \leq \varepsilon \\ \rho(u-\varepsilon)/(\rho-\varepsilon) \text{ for } \varepsilon \leq u \leq \rho \\ u \text{ for } \rho \leq u \leq 1 \\ 1 \text{ for } 1 \leq u \leq R \\ R+1-u \text{ for } R \leq u \leq R+1 \\ 0 \text{ for } u \geq R+1 \end{cases}$$

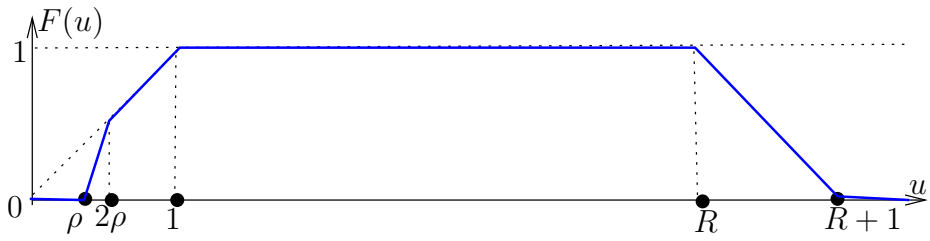

Figure 1: The function F

Using Lemma 4.2, Remark 4.1 and the inequalities

$$v(x)v(y) \leq \frac{1}{2}(v(x)^2 + v(y)^2) \ ,$$

the lefthandside of Equation (3) is bounded as follows

$$\langle f_\varepsilon v | (H-\lambda)(f_\varepsilon v)\rangle \leq \frac{1}{2}\sum_{x\in V} v(x)^2 \Phi_\varepsilon(x) \ ,$$

with

$$\Phi_\varepsilon(x) = \sum_{y\sim x} \frac{(f_\varepsilon(x) - f_\varepsilon(y))^2}{d(x,y)^2} \leq \frac{N\rho^2}{(\rho-\varepsilon)^2}$$

where the second inequality uses the fact that f_ε is $\dfrac{\rho}{\rho-\varepsilon}$−Lipshitz. This implies

$$\langle f_\varepsilon v | (H-\lambda)(f_\varepsilon v)\rangle \leq \frac{N\rho^2}{2(\rho-\varepsilon)^2}\|v\|_{l^2}^2 \ . \tag{5}$$

On the other hand, due to assumption (4) the lefthandside of Equation (3) is bounded from below as follows:

$$\langle f_\varepsilon v | (H-\lambda)(f_\varepsilon v)\rangle \geq \frac{N}{2}\sum_{\rho\leq D(x)\leq R} v(x)^2 + c\|f_\varepsilon v\|_{l^2}^2 \ . \tag{6}$$

We put together Equation (5) and (6). Then we do $\varepsilon \to 0$. After that, we do $\rho \to 0$ and $R \to \infty$. We get $v \equiv 0$.

□

4.2 Essential self-adjointness

Theorem 4.2 *Consider the Schrödinger operator $H = \Delta_{1,a} + W$ on a graph $G = (V, E)$ such that (G, d_p), with $p_{x,y} = a_{x,y}^{-\frac{1}{2}}$, is* non complete *as a metric space. For a vertex $x \in V$, we denote by $D(x)$ the distance from x to the boundary V_∞. We assume the two following conditions:*

(i) G is of bounded degree *and we denote the upper bound by N,*

(ii) there exists $M < \infty$ so that

$$\forall x \in V,\ W(x) \geq \frac{N}{2D(x)^2} - M\ . \tag{7}$$

Then the Schrödinger operator H is essentially self-adjoint.

Remark 4.1 *In the case where (V, d_p) is complete, we have $D(x) = \infty$ and so $W \geq -M$, this result is already proved in [To].*

Proof.–

We have, for any $u \in C_0(V)$

$$\langle u|Hu\rangle \geq \sum_{x\in V} W(x)|u(x)|^2,$$

so using assumption (7) we get:

$$\langle u|(H-\lambda)u\rangle - \frac{N}{2}\sum_{x\in V}\frac{1}{D(x)^2}|u(x)|^2 \geq \sum_{x\in V} -(M+\lambda)\|u\|^2\ .$$

Then choosing for example

$$\lambda = -M - 1$$

we get the inequality (4) with $c = 1$, and the proof follows from Theorem 4.1.

□

Theorem 4.3 *Consider the Schrödinger operator $H = \Delta_{\omega,c} + W$ on a graph G, define $\alpha_{x,y} = \min\{\omega_x, \omega_y\}$ and assume that (G, d_p), with $p_{xy} = \dfrac{\alpha_{x,y}}{\sqrt{c_{x,y}}}$, is* non complete *as a metric space.*

We assume the following conditions:

(i) G is of bounded degree and we denote the upper bound by N,

(ii) there exists $M < \infty$ so that

$$\forall x \in V,\ W(x) \geq \frac{N}{2D(x)^2} - M\ . \tag{8}$$

Then the Schrödinger operator H is essentially self-adjoint.

Remark 4.2 *In the particular case when $\sum \omega_x^2 < \infty$, the Laplacian $H = \Delta_{\omega,c}$ does not satisfy the assumption (8) so this result is coherent with Theorem 3.1.*

Remark 4.3 *The exponent of $D(x)$ in (8) is sharp. In fact, one can find a potential W such that $W(x) \geq \dfrac{k}{D(x)^2}$ where $k < \dfrac{N}{2}$ and weights ω and c such that $H = \Delta_{\omega,c} + W$ is non essentially self-adjoint. See Example 5.3.2 .*

Proof.–

The proof follows the previous scheme. Lemma 4.2 still holds provided we change $a_{x,y}$ into $c_{x,y}$ and $\langle , | , \rangle$ into $\langle , | , \rangle_{l^2_\omega}$. Theorem 4.1 is replaced by

Theorem 4.4 *Let v be a solution of $(H - \lambda)v = 0$. Assume that v belongs to $l^2_\omega(V)$ and that there exists a constant $c > 0$ such that, for all $u \in C_0(V)$,*

$$\langle u|(H-\lambda)u\rangle_{l^2_\omega} \geq \frac{N}{2} \sum_{x \in V} \max\left(\frac{1}{D(x)^2}, 1\right) \omega_x^2 |u(x)|^2 + c\|u\|^2_{l^2_\omega} \ , \quad (9)$$

then $v \equiv 0$.

Proof.–

We construct the function $f_\varepsilon = F_\varepsilon(D)$ as previously, the distance D is now associated to the new metric d_p.
Using Lemma 4.2, Remark 4.1, the inequalities $v(x)v(y) \leq \frac{1}{2}(v(x)^2 + v(y)^2)$ and the definition of $\alpha_{x,y}$ we get

$$\langle f_\varepsilon v|(H-\lambda)(f_\varepsilon v)\rangle_{l^2_\omega} \leq \frac{1}{2} \sum_{x \in V} v(x)^2 \omega_x^2 \Phi_\varepsilon(x) \ ,$$

with

$$\Phi_\varepsilon(x) = \sum_{y \sim x} \frac{(f_\varepsilon(x) - f_\varepsilon(y))^2}{d_p(x,y)^2} \leq \frac{N\rho^2}{(\rho - \varepsilon)^2}$$

where the second inequality uses the fact that f_ε is $\dfrac{\rho}{\rho - \varepsilon}$–Lipshitz.
This implies

$$\langle f_\varepsilon v|(H-\lambda)(f_\varepsilon v)\rangle_{l^2_\omega} \leq \frac{N\rho^2}{2(\rho-\varepsilon)^2} \|v\|^2_{l^2_\omega} \ . \quad (10)$$

On the other hand, due to assumption 4 the lefthandside of Equation 3 is bounded from below as follows:

$$\langle f_\varepsilon v|(H-\lambda)(f_\varepsilon v)\rangle_{l^2_\omega} \geq \frac{N}{2}\sum_{\rho\leq D(x)\leq R}\omega_x^2 v(x)^2 + c\|f_\varepsilon v\|^2_{l^2_\omega} \ . \tag{11}$$

We put together Equation (10) and (11). Then we do $\varepsilon \to 0$. After that, we do $\rho \to 0$ and $R \to \infty$. We get $v \equiv 0$.

□

Then Theorem 4.4 follows from the following inequality, for any $u \in C_0(V)$:

$$\langle u|Hu\rangle_{l^2_\omega} \geq \sum_{x\in V} W(x)\omega_x^2|u(x)|^2,$$

so

$$\langle u|(H-\lambda)u\rangle_{l^2_\omega} - \frac{N}{2}\sum_{x\in V}\frac{1}{D(x)^2}\omega_x^2|u(x)|^2 \geq \sum_{x\in V} -(M+\lambda)\|u\|^2_{l^2_\omega} \ .$$

□

5 Schrödinger operators on "tree-like" graphs

5.1 Introduction

Definition 5.1 *The graph $\mathbb{N}$ is the graph defined by $V = \{0, 1, 2, \cdots\}$ and $E = \{\{n, n+1\} \mid n = 0, 1, \cdots\}$.*

Definition 5.2 *We will call an infinite graph $G = (V, E)$* tree-like *if there exists a finite sub-graph G_0 of G so that $G\backslash G_0$ is the union of a finite number of disjoint copies G_α of the graph $\mathbb{N}$ (the* ends *of G relatively to G_0 according to Definition 1.3).*

For example, the graph $\mathbb{Z}$, defined similarly to $\mathbb{N}$, is tree-like.

A Laplace operator $L = \Delta_{1,a}$ on G is defined on each end G_α of G by

$$L^\alpha f_n \ = \ -a^\alpha_{n,n+1}f_{n+1} + (a^\alpha_{n-1,n} + a^\alpha_{n,n+1})f_n - a^\alpha_{n-1,n}f_{n-1} \ ,$$

where the $a^\alpha_{n-1,n}$'s are > 0. If $W : V \to \mathbb{R}$, we will consider Schrödinger operators H on $C_0(G)$ defined by $H = \Delta_{1,a} + W$.

Lemma 5.1 *Let G_0 be a finite sub-graph of G. The operator $H = \Delta_{1,a} + W$ on G is essentially self-adjoint if and only if it is essentially self-adjoint on each end of G relatively to G_0. More precisely, the deficiency indices $n_\pm$ are the sum of the corresponding deficiency indices of the ends.*

We will need the following Lemma which is a consequence of Kato-Rellich Theorem, see [Go-Sch], Proposition 2.1 :

Lemma 5.2 *If A and B are 2 unbounded symmetric operators with the same domains and $A - B$ is bounded, then the deficiency indices of A and B are the same.*

Proof.–

We give here an alternative proof to this result.

The domains of both closures coincide: the "graph-norms" $\|Au\|_{l^2} + \|u\|_{l^2}$ and $\|Bu\|_{l^2} + \|u\|_{l^2}$ are equivalent on $\mathcal{H}$. The domains of the adjoints coincide too. Let $K = D(A^\star)/D(\bar{A})$ and $Q_A(u,v) = i\left(\langle A^\star u|v\rangle - \langle u|(A^\star v\rangle\right)$ which is well defined on K. We know that this Hermitian form is non degenerate and hence the dimension of the positive eigenspace is constant along the curve $A + t(B - A)$.

□

Using Lemma 5.2, we can prove Lemma 5.1:

Proof.–

We will consider the operator H_{red} where we replace the entries $a_{x,y}$ of H with $\{x, y\} \in E(G_0)$ by 0. The claim of the Lemma is clear for H_{red} because it is the direct orthogonal sum of the Schrödinger operators of the ends and a finite rank l^2-bounded matrix. We can then use Lemma 5.2 because $H - H_{\text{red}}$ is bounded.

□

Remark 5.1 *It follows from Lemma 5.1 that, concerning essential self-adjointness questions for tree-like graphs, it is enough to work on the graph $\mathbb{N}$. We have*

$$(Hf)_0 = -a_{0,1}f_1 + a_{0,1}f_0 + W_0 f_0.$$

This implies that the space of solutions of $(H - \lambda)u = 0$ on $\mathbb{N}$ is of dimension 1 and any solution so that f_0 vanishes is $\equiv 0$. We will consider also solutions "near infinity", i.e. $(f_n)_{n\geq 0}$ satisfies $((H - \lambda)f)_n = 0$ for $n \geq 1$; this space is of dimension 2.

5.2 Main result

It is known ([Dod]) that $H = \Delta_{1,a} + W$ is essentially self-adjoint provided $\Delta_{1,a}$ is bounded as an operator on $l^2(G)$ and W bounded from below. For tree-like graphs, we have the following result, which holds for *any* potential W :

Theorem 5.1 *1. If*

$$\frac{1}{a_{n-1,n}} \notin l^1(\mathbb{N}) \ , \tag{12}$$

the Schrödinger operator $H = \Delta_{1,a} + W$ with domain $C_0(\mathbb{N})$ is essentially self-adjoint for any potential W .

2. If G is tree-like and if for each end G_α, $1/a^\alpha_{n-1,n} \notin l^1(\mathbb{N})$, then $H = \Delta_{1,a}+W$ with domain $C_0(V)$ is essentially self-adjoint for any potential W.

Remark 5.2 *The condition (12) is sufficient but not necessary. See Example 5.3.2 .*

Proof.–

The proof is by contradiction using Corollary 6.1 which is an analog of Weyl's limit point-limit circle criteria in the discrete case.

Let us consider an operator $\Delta_{1,a}$ such that (12) is fulfilled. We assume that any sequence u, such that $(H - i)u = 0$ near infinity, is in $l^2(\mathbb{N})$. In particular, there exists a basis f, g of solutions of $(H - i)f = 0$ with $f \in l^2(\mathbb{N})$ and $g \in l^2(\mathbb{N})$.

We have

$$-a_{n,n+1}\ f_{n+1} + (a_{n-1,n} + a_{n,n+1} + (W_n - i))\ f_n - a_{n-1,n}\ f_{n-1} \ = \ 0 \ ,$$

and the same holds for g. The Wronskian of f and g is the sequence $\mathcal{W}_n = f_n g_{n-1} - f_{n-1} g_n$. We have:

$$\mathcal{W}_{n+1} = \frac{a_{n-1,n}}{a_{n,n+1}}\ \mathcal{W}_n \ ,$$

which implies

$$\mathcal{W}_n \ = \ \frac{a_{0,1}}{a_{n-1,n}} \mathcal{W}_1 \quad \forall n \in \mathbb{N} \ .$$

But since the Wronskian is in $l^1(\mathbb{N})$ according to the assumption that f and g are in $l^2(\mathbb{N})$, we get a contradiction with the hypothesis (12).

□

5.3 Examples of Schrödinger operators

5.3.1 Example 1

Let us consider the Laplacian $\Delta_{\omega,c}$ on $\mathbb{N}$, with, $\forall n > 0$, $c_{n-1,n} = n^3$ and, $\forall n \geq 0$, $\omega_n = \dfrac{1}{n+1}$. We deduce from Theorem 3.1 that $\Delta_{\omega,c}$ is not essentially self-adjoint. This operator (see [To] Lemma 2.1) is unitarily equivalent

to the Schrödinger operator $H = \Delta_{1,a} + W$ with $a_{n-1,n} = \dfrac{c_{n-1,n}}{\omega_{n-1}\omega_n} \sim n^5$ and $W_n = \dfrac{1}{\omega_n}\left[c_{n,n+1}\left(\dfrac{1}{\omega_n} - \dfrac{1}{\omega_{n+1}}\right) + c_{n-1,n}\left(\dfrac{1}{\omega_n} - \dfrac{1}{\omega_{n-1}}\right)\right] \sim -3n^3$, which is therefore not essentially self-adjoint.

According to Theorem 5.1, such an operator must verify $\dfrac{1}{a_{n-1,n}} \in l^1(\mathbb{N})$, which is indeed the case. We notice that this operator gives also an example for the consistency of Theorem 4.2, since the potential W does not satisfy the inequality (7).

5.3.2 Example 2: discretization of a Schrödinger operator on $\mathbb{R}^+$

Let us consider the Schrödinger operator on $]0, +\infty[$ defined on smooth compactly supported functions by $Lf := -f" + \dfrac{A}{x^2}f$. This operator is essentially self-adjoint if and only if $A > 3/4$ (see [RS] theorem X 10).

We discretize this operator in the following way: let us consider the graph $\Gamma = (V, E)$ resulting of the following dyadic subdivision of the interval $(0, 1)$: the vertices are the $x_n = 2^{-n}$ and the edges are the pairs $\{2^{-n}, 2^{-n+1}\}$ which correspond to the intervals $[2^{-n}, 2^{-n+1}]$ of length $\omega_n^2 = 2^{-n}$.

Then we define, for any

$$f \in l^2_\omega(V) = \{f \in C(V) \mid \sum_{n\in\mathbb{N}} 2^{-n} f_n^2 < +\infty\}$$

where we set $f = (f_n)$, the quadratic form

$$Q(f) = \sum_{n\in\mathbb{N}} 2^{-n}\left[\left(\frac{f_{n+1} - f_n}{2^{-n}}\right)^2 + A2^{2n} f_n^2\right] .$$

According to the previous definitions and if we set $c_{n,n+1} = 2^n$, this quadratic form is associated to the Schrödinger operator $H = \Delta_{\omega,c} + W$ on $\mathbb{N}$ with the potential $W_n := A2^{2n}$.

Let us set $a_{n,n+1} = \dfrac{c_{n,n+1}}{\omega_n\omega_{n+1}} = 2^{2n+\frac{1}{2}}$; then ([To]) H is unitarily equivalent to

$$\widehat{H} = \Delta_{1,a} + \widehat{W} + W$$

with

$$\widehat{W_n} = \frac{1}{\omega_n}\left[c_{n,n+1}\left(\frac{1}{\omega_n} - \frac{1}{\omega_{n+1}}\right) + c_{n-1,n}\left(\frac{1}{\omega_n} - \frac{1}{\omega_{n-1}}\right)\right] = 2^{2n}\left(\frac{3}{2} - \frac{5\sqrt{2}}{4}\right) .$$

We have $\widehat{H} = \Delta_{1,a} + (A - A_0)4^n$ with $A_0 = \dfrac{5\sqrt{2}}{4} - \dfrac{3}{2}$ (> 0). The metric graph $(\mathbb{N}, d_p)$ with $p_{n,n+1} = a_{n,n+1}^{-1/2}$ is non complete. The solutions u of $Hu = 0$ verify

$$4u_{n+1} - \left(5 + 2\sqrt{2}(A - A_0)\right) u_n + u_{n-1} = 0 .$$

The solutions are generated by α_1^n and α_2^n where α_1 and α_2 are the roots of

$$4\alpha^2 - \left(5 + 2\sqrt{2}(A - A_0)\right)\alpha + 1 = 0 \ .$$

We have $|\alpha_1| < 1$ and $|\alpha_2| < 1$ if and only if $A_0 - \dfrac{5}{\sqrt{2}} < A < A_0$.

Hence H is essentially self-adjoint (by Corollary 6.1) if and only if $A \geq A_0$ or $A \leq A_0 - \dfrac{5}{\sqrt{2}}$.

Proposition 5.1 *The discretized operator H is essentially self-adjoint if and only if*

$$(\star)\ A \geq \frac{5\sqrt{2}}{4} - \frac{3}{2}$$

or $A \leq -\dfrac{5\sqrt{2}}{4} - \dfrac{3}{2}$.

From this result we can deduce several informations:

1. The condition $(\star)$ is analogous to the condition $A > 3/4$ in the continuous case.

2. Proposition 5.1 implies that for $A = 0$ the operator $H = \Delta_{\omega,c}$ is non essentially self-adjoint, which is a result predicted by Theorem 3.1.

3. This gives examples of essentially self-adjoint operators with $1/a_n \in l_1$.

4. Sharpness of the assumption (8) in Theorem 4.3

   In this context, the distance d_p is associated to

   $$p_{x,y} = \frac{\alpha_{x,y}}{\sqrt{c_{x,y}}}$$

   with $\alpha_{x,y} = \min\{\omega_x, \omega_y\}$ so we get

   $$D(n) = \sum_{p \geq n} \frac{\alpha_{p,p+1}}{\sqrt{c_{p,p+1}}} = \sum_{p \geq n} \left(\frac{2^{-p-1}}{2^p}\right)^{1/2} \ = \ 2^{-\frac{1}{2}} 2^{-n} 2$$

   so

   $$\frac{1}{D(n)^2} = 2^{2n-1} \ .$$

   If the operator $H = \Delta_{\omega,c} + A4^n$ satisfies the assumption (8), then $A > \dfrac{1}{2}$ which implies condition $(\star)$, since $\dfrac{1}{2} > \dfrac{5\sqrt{2}}{4} - \dfrac{3}{2}$, so Theorem 4.3 is coherent with proposition 5.1. Moreover the operator $H = \Delta_{\omega,c} + A4^n$ with $A = \dfrac{5\sqrt{2}}{4} - \dfrac{3}{2}$ is non essentially self-adjoint, which implies that the estimate (8) on the growth of the potential in Theorem 4.3 is sharp.

5.3.3 Example 3

Let us consider the Laplacian $\Delta_{\omega,c}$ on $\mathbb{N}$, where the coefficients verify $c_{n-1,n} = n^\gamma$ with $\gamma > 2$ and $\omega_n = (n+1)^{-\beta}$ with $\beta > \frac{1}{2}$. We deduce from Theorem 3.1 that $\Delta_{\omega,c}$ is non essentially self-adjoint.

From [To] Proposition 2.1, this operator is unitarily equivalent to the Schrödinger operator $H = \Delta_{1,a} + W$, with $a_{n-1,n} \sim n^{\gamma+2\beta}$ and $W_n \sim -\beta(\beta+\gamma-1)n^{2\beta+\gamma-2}$, which is therefore also not essentially self-adjoint. We emphasize that W is not bounded from below, which is predicted in [To], Theorem 3.2.

Furthermore, according to Theorem 5.1, such an operator must verify $\frac{1}{a_{n-1,n}} \in l_1(\mathbb{N})$, which is indeed the case. Following the terminology of the previous sections, it means the non completeness of $(\mathbb{N}, d_p)$ with the weights $p_{n-1,n} = a_{n-1,n}^{-1/2}$.

5.3.4 Example 4

Let us consider the Laplacian $H = \Delta_{\omega,c}$ on a radial tree $G = (V, E)$. For any vertex x, we denote by $\mathrm{dist}(x)$ the distance from x to the center 0 and define $\omega_x = 2^{-n}$, $c_{x,y} = 2^n$, with $y \sim x$, $\mathrm{dist}(x) = n, \mathrm{dist}(y) = n+1$. We assume that the graph G has a uniform degree $N+1$.

Let us set $a_{x,y} = \frac{c_{x,y}}{\omega_x \omega_y} = 2^{3n+1}$ for any edge so that $\mathrm{dist}(x) = n, \mathrm{dist}(y) = n+1$; then H is unitarily equivalent to

$$\widehat{H} = \Delta_{1,a} + W$$

with

$$W(x) = 2^{3n}\left(-N + \frac{1}{4}\right)$$

for any x such that $\mathrm{dist}(x) = n$.

The radial solutions u of $Hu = 0$ can be seen as sequences (u_n) and satisfy the equation:

$$u_{n+1} - \left(\frac{5}{4} - \frac{N}{2}\right) u_n + \frac{1}{8} u_{n-1} = 0 \ .$$

The solutions are generated by α_1^n and α_2^n where α_1 and α_2 are the roots of

$$\alpha^2 - \left(\frac{5}{4} - \frac{N}{2}\right)\alpha + \frac{1}{8} = 0 \ .$$

We have $|\alpha_1| < 1$ and $|\alpha_2| < 1$ if and only if $\frac{1}{4} < N < \frac{19}{4}$.

Hence (by Corollary 6.1) we get the following result:

Proposition 5.2 *If $N \leq 4$, then H is non essentially self-adjoint .*

Remark 5.3 *We have*

$$\sum_x \omega_x^2 = \sum_n \omega_n^2 N^n = \sum_n (\frac{N}{4})^n \ .$$

If $N < 4$, then $\sum_x \omega_x^2 < \infty$ so Theorem 3.1 can also be applied to get the result since the graph is non complete with respect to the metric d_p , with $p_{x,y} = c_{x,y}^{-\frac{1}{2}}$.

6 Appendix: Weyl's "limit point-limit circle" criteria

6.1 The discrete case

The goal of this section is to prove the discrete version of the Weyl's "limit point-limit circle" criterium. Our presentation is simpler than the classical presentation for the continuous case (see [RS], Appendix to section X.1).

Let us consider the Hilbert space $\mathcal{H} := l^2(\mathbb{N}, \mathbb{C}^N)$ and the formally symmetric differential operator P defined by

$$Pf(0) = P_{0,0}f(0) + P_{0,1}f(1),\ \forall l \geq 1, Pf(l) = P_{l,l-1}f(l-1) + P_{l,l}f(l) + P_{l,l+1}f(l+1)$$

where

1. $\forall l \geq 1,\ P^\star_{l-1,l} = P_{l,l-1}$
2. $\forall l \geq 0,\ P^\star_{l,l} = P_{l,l}$
3. $\forall l \geq 0,\ P_{l,l+1}$ is invertible
4. $\exists M \in \mathbb{R}$ so that for any $f \in C_0(\mathbb{N}, \mathbb{C}^N)$, $Q_P(f) = \langle Pf \mid f \rangle \geq -M\|f\|^2$.

Let us define the subspace $\mathcal{E}$ of $\mathcal{H}$ as the set of l^2 sequences f so that, for all $l \geq 1$, $(P - i)f(l) = 0$; the space $\mathcal{E}$ is isomorphic to the space of germs at infinity of l^2 solutions of $(P - i)f = 0$. Assumption 3. implies that $\dim \mathcal{E} \leq 2N$. Let us denote by $\mathcal{K} = \ker(P - i) \cap l^2$ and consider the following sequence

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathbb{C}^N \to 0 \ , \tag{13}$$

where the non trivial arrow is given by $f \to (P - i)f(0)$. We have the

Theorem 6.1 *The sequence (13) is exact and the deficiency indices $n_\pm = \dim \mathcal{K}$ of P are given by $n_\pm = \dim \mathcal{E} - N$.*

Proof.–

Assumption 4. implies (using Corollary of Theorem X.1 in [RS]) that the deficiency indices are equal. The only non trivial point is to prove that the arrow $p : \mathcal{E} \to \mathbb{C}^N$ is surjective. Let us consider $\tilde{P}$ a self-adjoint extension of P which exists because $n_+ = n_-$. Let us consider the map $\rho : \mathbb{C}^N \to \mathcal{E}$ defined by $\rho(x) = (\tilde{P} - i)^{-1}(x, 0, 0, \cdots)$. Then $p \circ \rho = \mathrm{Id}_{\mathbb{C}^N}$.

□

Corollary 6.1 *The Schrödinger operator $H = \Delta_{1,a} + W$ defined on $C_0(\mathbb{N})$ is essentially self-adjoint if and only if there exists a sequence u such that $(H-i)u = 0$ near infinity (i.e. $((H-i)u)_n = 0$ for n large enough) which is not in $l^2(\mathbb{N})$.*

6.2 The continuous case

A similar method works for the continuous case. Let $H = -\dfrac{d^2}{dx^2} + A(x)$ be a system of differential operators where $A(x)$ is Hermitian for every x and is continuous on $[0, a[$ as a function of x. The differential operator H is L^2-symmetric on the Dirichlet domain

$$D = C_0^\infty([0, a[, \mathbb{C}^N) \cap \{u \mid u(0) = 0\} \ .$$

We denote H_D the closure of (H, D). Let us assume that $n_+(H_D) = n_-(H_D)$ which is true for example if A is bounded from below or if A is real-valued. Then

Theorem 6.2 *If $\mathcal{E}$ is the space of solutions u of the differential equation $(H - i)u = 0$ which are L^2 near a, then $n_\pm(H_D) = \dim \mathcal{E} - N$.*

Proof.–

Let us consider the sequence

$$0 \to \ker(H_D - i) \to \mathcal{E} \to \mathbb{C}^N \to 0 \ , \tag{14}$$

where the only non trivial arrow is given by $u \to u(0)$. This sequence is exact: we have only to prove the surjectivity of the non trivial arrow. Let $\tilde{H}$ be a self-adjoint extension of H_D and $\chi \in C_0^\infty([0, a[, \mathbb{R})$ with $\chi(0) = 1$. For any $X \in \mathbb{C}^N$, let us consider

$$u = \chi X - (\tilde{H} - i)^{-1} \left((H - i)(\chi X)\right) \ .$$

Then $(H - i)u = 0$, $u(0) = V$ and u is L^2 near a.

□

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