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FACTORIZATION FOR NON-SYMMETRIC OPERATORS AND EXPONENTIAL H -THEOREM

M.P. GUALDANI, S. MISCHLER, C. MOUHOT

ABSTRACT. We present a factorization method for estimating resolvents of non-symmetric operators in Banach or Hilbert spaces in terms of estimates in another (typically smaller) “reference” space. This applies to a class of operators writing as a “regularizing” part (in a broad sense) plus a dissipative part. Then in the Hilbert case we combine this factorization approach with an abstract Plancherel identity on the resolvent into a method for enlarging the functional space of decay estimates on semigroups. In the Banach case, we prove the same result however with some loss on the norm. We then apply these functional analysis approach to several PDEs: the Fokker-Planck and kinetic Fokker-Planck equations, the linear scattering Boltzmann equation in the torus, and, most importantly the linearized Boltzmann equation in the torus (at the price of extra specific work in the latter case). In addition to the abstract method in itself, the main outcome of the paper is indeed the first proof of exponential decay towards global equilibrium (e.g. in terms of the relative entropy) for the full Boltzmann equation for hard spheres, conditionnally to some smoothness and (polynomial) moment estimates. This improves on the result in [12] where the rate was “almost exponential”, that is polynomial with exponent as high as wanted, and solves a long-standing conjecture about the rate of decay in the H -theorem for the nonlinear Boltzmann equation, see for instance [10, 27].

Mathematics Subject Classification (2000): 35P15 Estimation of eigenvalues, upper and lower bounds, 35Q84 Fokker-Planck equations, 76P05 Rarefied gas flows, Boltzmann equation [See also 82B40, 82C40, 82D05].

Keywords: spectral gap; semigroup; Plancherel theorem; coercivity; hypocoercivity; Fokker-Planck equation; Boltzmann equation; non cutoff; fractional derivative; exponential rate; thermalization; entropy.

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1. INTRODUCTION

This paper deals with the study of decay properties of linear semigroups and their link with spectral properties. Let us give a brief sketch of the question we address. Consider two Banach spaces $E \subset \mathcal{E}$, and two unbounded closed linear operator L and \mathcal{L} resp. on E and \mathcal{E} with spectrum $\Sigma(L), \Sigma(\mathcal{L}) \subsetneq \mathbb{C}$. They generate two C_0 -semigroups $S(t)$ and $\mathcal{S}(t)$ resp. in E and \mathcal{E} . Also assume that $\mathcal{L}|_E = L$, and E is dense in \mathcal{E} .

The theoretical question we address in this work is the following: **Can one deduce (quantitative) informations on $\Sigma(\mathcal{L})$ and $\mathcal{S}(t)$ in terms of informations on $\Sigma(L)$ and $S(t)$?**

More precisely, we provide here an answer where (i) the spectral gap property of L in E can be shown to hold for \mathcal{L} in the space \mathcal{E} and consequently (ii) explicit estimates on the rate of decay of the semigroup $\mathcal{S}(t)$ can be computed from the ones on $S(t)$. Our results hold for a class of operators \mathcal{L} which split as $\mathcal{L} = \mathcal{A} + \mathcal{B}$, where \mathcal{A} maps \mathcal{E} into E and \mathcal{B} 's spectrum is well localized. We then show that linear Boltzmann-like and Fokker-Planck equations fall in that class and therefore their spectral-gap property can be extended from the linearization space (the weighted L^2 space, where the strong decay is prescribed by the equilibrium) to larger Hilbert or Banach spaces (for example with polynomial decay). It is worth mentioning that we provide results in the non spatially homogeneous situation, and that the method is optimal in terms of the rate of decay (no loss on the way from E to \mathcal{E} , as would be the case in, say, some interpolation approach).

Enlarging the functional space for the spectral properties of linear operators is useful in nonlinear PDE analysis when the stability theory for the associated linearized equation is not compatible (that it, is established in a space too small) with the functional space in which the nonlinear PDE is well posed. The applications concern existence and uniqueness of stationary solution (thanks to perturbative or implicit function argument, see for instance [19, 2]) as well as the proof of long time convergence to the equilibrium (see for instance [22, 19]). In this paper we construct a general abstract approach whereas previous papers were dealing with particular cases. Within this general framework we give new results for the space homogeneous Fokker-Planck and linear Boltzmann equations, and more importantly we are able to tackle the corresponding space inhomogeneous (or kinetic) equations as well as the full (we mean again space inhomogeneous) Boltzmann equation. We also refer to the note [23] where some results of this paper were announced.

The main outcome of this paper is thus a proof of the exponential convergence to equilibrium for the full Boltzmann equation in the torus. This question goes back to the famous H -theorem of Boltzmann, and a functional inequality version of it (linear control of the relative entropy in terms of the entropy production functional) was conjectured by Cercignani [10] in the spatially homogeneous case (see also [11]). This conjecture has motivated important works from the early 1990's on: see [9, 8, 29, 6, 30] (in the spatially homogeneous case). While it has been shown to be false in general [6], it has given a formidable impulse to the works on the Boltzmann equation in the last two decades. It has been shown to be almost true in

many cases in the spatially homogeneous cases in [30] (in the sense that polynomial inequalities relating the relative entropy and the entropy production hold for a power as close to 1 as wanted), and was an important inspiration for the work [12] in the inhomogeneous case. However, due to the fact Cercignani's conjecture is false even in the homogeneous case [6], these important progresses in the far from equilibrium regime were unable to prove the natural conjecture about the *exponential decay of the relative entropy* in order to prove H -theorem with the correct time scale. This has motivated the work [22] which answers this question, but only in the spatially homogeneous case.

Here we finally answer this question for the full Boltzmann equation for hard spheres (or cutoff hard potentials), in the same setting as in [12], that is under some *a priori* regularity assumptions (Sobolev norms and polynomial moments bounds). This is based on the idea of connecting the nonlinear theory in [12] with the linearized theory developed in [24] and it makes crucial use of our new method for enlarging the space of decay estimates on semigroups. This hence answers the main question raised by Villani in [27, Subsection 1.8, page 62].

Let us mention that there is a huge gap between the spatially homogeneous case (where the equation is coercive and the linearized semigroup is self-adjoint or sectorial depending on the functional space) and the spatially inhomogeneous case (where the equation is *hypocoercive* and its linearized is neither sectorial, nor even hypoelliptic).

The plan of the paper is the following. In section 2 we state and prove the abstract factorization theorem for unbounded closed operators. Section 3 is devoted to the links between estimates on the resolvent and decay estimates on the semigroup and only in section 4 we can assemble the preceding elements into new estimates showing enlargement of the functional space of the decay of semigroups. In sections 5 and 6 we apply the abstract method to the linear Boltzmann and Fokker-Planck equations (either spatially homogeneous or in the torus). Finally in section 7 we consider the nonlinear Boltzmann equation. Using the abstract factorization approach for the linearized Boltzmann equation in the torus plus some additional specific work, we prove that one can “connect” the nonlinear theory and the linearized theory in order to prove the exponential decay of the relative entropy.

If the reader is searching for a quick overview of our method, a possibility is to go directly to the statement of Corollary 4.2 which is the “concrete” version of the abstract method for most applications, and then to the section 6 for an easy and very short application of the method to the linear Boltzmann equation.

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2. THE FACTORIZATION THEOREM

Let us start with some notations. For some given Banach spaces X and Y we denote by $\mathcal{B}(Y \rightarrow X)$ the space of bounded linear operators from Y to X , we denote by $\mathcal{B}(X) = \mathcal{B}(X \rightarrow X)$ in the case when $X = Y$, and we denote by $\mathcal{C}(X)$ the space of closed unbounded linear operators on X with dense domain.

For $\Lambda \in \mathcal{C}(X)$ we denote by $D(\Lambda)$ its domain, by $N(\Lambda)$ its null space, by $M(\Lambda) = \lim_{k \rightarrow \infty} N(\Lambda^k)$ its algebraic null space, by $R(\Lambda)$ its range and by $\Sigma(\Lambda)$ its spectrum, so that for any $\xi \in \mathbb{C} \setminus \Sigma(\Lambda)$ the operator $\Lambda - \xi$ is invertible and the resolvent operator $(\Lambda - \xi)^{-1}$ is well defined, belongs to $\mathcal{B}(X)$ and has range equal to $D(\Lambda)$.

For a given real number $a \in \mathbb{R}$, we define the half complex plane $\Delta_a := \{z \in \mathbb{C}, \Re z > a\}$. We recall that for any isolated eigenvalue $\eta \in \Sigma(\Lambda)$ in the sense that $\Sigma(\Lambda) \cap \{\zeta \in \mathbb{C}, |\zeta - \eta| \leq r\} = \{\eta\}$ for some $r > 0$, we define $\Pi_{\Lambda, \eta}$ the projection on closure of the algebraic eigenspace associated to η by (see [18, III-(6.19)])

$$(2.1) \quad \Pi_{\Lambda, \eta} := \frac{i}{2\pi} \int_{|\zeta - \eta| = r} (\Lambda - \zeta)^{-1} d\zeta.$$

Remind that $E_\eta := R(\Pi_{\Lambda, \eta})$ is finite dimensional iff there exists $\alpha_0 \in \mathbb{N}^*$ such that $N((\Lambda - \eta)^\alpha) = N((\Lambda - \eta)^{\alpha_0})$ for any $\alpha \geq \alpha_0$, so that $M(\Lambda - \eta) = N((\Lambda - \eta)^{\alpha_0})$. In that case, we say that η is a discrete eigenvalue.

In this section we consider $E \subset \mathcal{E}$ two Banach spaces such that E is dense in \mathcal{E} , and two operators $L \in \mathcal{C}(E)$ and $\mathcal{L} \in \mathcal{C}(\mathcal{E})$ such that $\mathcal{L}|_E = L$. The associated resolvent operators are denoted by $R(\xi) := (L - \xi)^{-1}$ for $\xi \notin \Sigma(L)$ and $\mathcal{R}(\xi) := (\mathcal{L} - \xi)^{-1}$ for $\xi \notin \Sigma(\mathcal{L})$.

Theorem 2.1. *Assume there are some real numbers $a \in \mathbb{R}$, some complex numbers $\xi_1, \dots, \xi_k \in \Delta_a$, $k \in \mathbb{N}$ (with the convention $\{\xi_1, \dots, \xi_k\} = \emptyset$ if $k = 0$) such that one has*

- (H1) Localization of the spectrum of L :** $\Sigma(L) \cap \Delta_a = \{\xi_1, \dots, \xi_k\}$;
- (H2) Decomposition of \mathcal{L} :** *there exist two closed unbounded operators \mathcal{A} and \mathcal{B} on \mathcal{E} (with domains containing $\text{Dom}(\mathcal{L})$) such that $\mathcal{L} = \mathcal{A} + \mathcal{B}$, and*
 - (i) $\Sigma(\mathcal{B}) \cap \Delta_a = \emptyset$;
 - (ii) $\mathcal{A} \in \mathcal{B}(\mathcal{E} \rightarrow E)$;
 - (iii) *there is some $\xi_0 \in \Delta_a$ such that $\mathcal{L} - \xi_0$ is invertible in \mathcal{E} .*

Then \mathcal{L} satisfies

- (i) $\Sigma(\mathcal{L}) \cap \Delta_a = \{\xi_1, \dots, \xi_k\}$;
- (ii) *for any $\xi \in \Delta_a \setminus \{\xi_1, \dots, \xi_k\}$ the following bound for the resolvent operator holds:*

$$(2.2) \quad \|\mathcal{R}(\xi)\|_{\mathcal{B}(\mathcal{E})} \leq \|(\mathcal{B} - \xi)^{-1}\|_{\mathcal{B}(\mathcal{E})} + \|R(\xi)\|_{\mathcal{B}(E)} \|\mathcal{A}(\mathcal{B} - \xi)^{-1}\|_{\mathcal{B}(\mathcal{E} \rightarrow E)}.$$

Remarks 2.2. (1) *In words, assumption **(H2)** means that one may decompose \mathcal{L} as $\mathcal{L} = \mathcal{A} + \mathcal{B}$ where \mathcal{A} has a “good localization/regularization property” (in the generalized sense of the “change of space” **(H2)-(ii)**) and \mathcal{B} has a “good spectral property” (in the sense that **(H2)-(i)** holds).*

- (2) One may relax **(H2)-(i)** into $\Sigma(\mathcal{B}) \cap \Delta_a \subset \{\xi_1, \dots, \xi_k\}$.
- (3) One may relax **(H2)-(ii)** into the fact that $\mathcal{A}(\mathcal{B} - \xi)^{-1}$ is bounded from \mathcal{E} to E for any $\xi \in \Delta_a \setminus \{\xi_1, \dots, \xi_k\}$.
- (4) In the theorem (and the previous remarks) one may replace $\Delta_a \setminus \{\xi_1, \dots, \xi_k\}$ by any nonempty open connected set $\Omega \subset \mathbb{C}$.
- (5) One may replace **(H2)-(iii)** by
- (H2)-(iii')** $\|(\mathcal{B} - \lambda)^{-1}\|_{\mathcal{B}(\mathcal{E})} \leq K/(\lambda - a)$ for some $K, a > 0$ and for any $\lambda \in \mathbb{R}$, $\lambda > a$.

Indeed, assuming **(H2)-(iii')** one may find $\xi_0 \in \mathbb{R}$, $\xi_0 > a$ such that $\|\mathcal{A}(\mathcal{B} - \xi_0)^{-1}\|_{\mathcal{B}(\mathcal{E})} < 1$ and then $\mathcal{L} - \xi_0 = (\text{Id}_{\mathcal{E}} + \mathcal{A}(\mathcal{B} - \xi_0)^{-1})(\mathcal{B} - \xi_0)$ is invertible as the product of two invertible operators, which proves **(H2)-(iii)**.

Proof of Theorem 2.1. Define $\Omega := \Delta_a \setminus \{\xi_1, \dots, \xi_k\}$ and for $\xi \in \Omega$

$$U(\xi) := \mathcal{B}(\xi)^{-1} - R(\xi)\mathcal{A}\mathcal{B}(\xi)^{-1},$$

where $R(\xi) = (L - \xi)^{-1}$ and $\mathcal{B}(\xi) := \mathcal{B} - \xi$.

By assumptions **(H1)**, **(H2)-(i)** and **(H2)-(ii)**, $\mathcal{B}(\xi)^{-1} : \mathcal{E} \rightarrow \mathcal{E}$, $\mathcal{A}\mathcal{B}(\xi)^{-1} : \mathcal{E} \rightarrow E$ and $R(\xi) : E \rightarrow E$ are bounded operators. Introduce now the canonical injection $J : E \rightarrow \mathcal{E}$. Taking into account that $R = JR$, $\mathcal{A} = J\mathcal{A}$, $\mathcal{L}J = JL$, the following identity holds:

$$\begin{aligned} (\mathcal{L} - \xi)R(\xi)\mathcal{A}\mathcal{B}(\xi)^{-1} &= (\mathcal{L} - \xi)JR(\xi)\mathcal{A}\mathcal{B}(\xi)^{-1} = J(L - \xi)R(\xi)\mathcal{A}\mathcal{B}(\xi)^{-1} \\ &= J\text{Id}_E\mathcal{A}\mathcal{B}(\xi)^{-1} = J\mathcal{A}\mathcal{B}(\xi)^{-1} = \mathcal{A}\mathcal{B}(\xi)^{-1}. \end{aligned}$$

The operator U is a right-inverse of $\mathcal{L} - \xi$ since $U(\xi) : \mathcal{E} \rightarrow \mathcal{E}$ is a well-defined bounded operator, and

$$\begin{aligned} (\mathcal{L} - \xi)U(\xi) &= (\mathcal{A} + \mathcal{B}(\xi))\mathcal{B}(\xi)^{-1} - (\mathcal{L} - \xi)R(\xi)\mathcal{A}\mathcal{B}(\xi)^{-1} \\ &= \mathcal{A}\mathcal{B}(\xi)^{-1} + \text{Id}_{\mathcal{E}} - (\mathcal{L} - \xi)R(\xi)\mathcal{A}\mathcal{B}(\xi)^{-1} \\ &= \mathcal{A}\mathcal{B}(\xi)^{-1} + \text{Id}_{\mathcal{E}} - \mathcal{A}\mathcal{B}(\xi)^{-1} = \text{Id}_{\mathcal{E}}. \end{aligned}$$

As a consequence, $\mathcal{L} - \xi$ is onto. Moreover, if $\mathcal{L} - \xi$ is invertible we have $\mathcal{R}(\xi) := (\mathcal{L} - \xi)^{-1} = U(\xi)$ and then $\mathcal{R}(\xi)$ satisfies the announced estimate (2.2). In order to prove that $\mathcal{L} - \xi$ is invertible we may argue as follows using assumption **(H2)-(iii)**. It is well known that $\mathcal{L} - \eta$ invertible implies that $\mathcal{L} - \xi$ is invertible for any $\xi \in \mathbb{C}$ such that

$$|\xi - \eta| < \|\mathcal{R}(\eta)\|_{\mathcal{B}(\mathcal{E})}^{-1}$$

and

$$(2.3) \quad \mathcal{R}(\xi) = \mathcal{R}(\eta) \sum_{n=0}^{\infty} (\eta - \xi)^n \mathcal{R}(\eta)^n.$$

For a given $\xi \in \Omega$ we fix a path γ from ξ_0 to ξ , that is a continuous function $\gamma : [0, 1] \rightarrow \Omega$ such that $\gamma(0) = \xi_0$, $\gamma(1) = \xi$. Thanks to assumption **(H2)-(ii)** and the summation representation (2.3) for the operator $(\mathcal{B} - \xi)^{-1}$ we first deduce that

$$\sup \{ \|\mathcal{A}(\mathcal{B} - \gamma(t))^{-1}\|_{\mathcal{B}(\mathcal{E} \rightarrow E)}; t \in [0, 1] \} < \infty,$$

and then next that

$$\inf \left\{ \|U(\gamma(t))\|_{\mathcal{B}(\mathcal{E})}^{-1}; t \in [0, 1] \right\} > 0.$$

We conclude that $\mathcal{L} - \xi$ is invertible, so that $\Sigma(\mathcal{L}) \cap \Delta_a \subset \{\xi_1, \dots, \xi_k\}$.

Finally, since $N(L - \xi_j) \subset N(\mathcal{L} - \xi_j)$ for any $j = 1, \dots, k$, we also have $\{\xi_1, \dots, \xi_k\} \subset \Sigma(\mathcal{L}) \cap \Delta_a$. \square

Let us give quickly a sufficient condition in the spirit of this abstract section for answering the natural question about whether the geometric eigenspaces of L and \mathcal{L} associated with ξ_1, \dots, ξ_k are the same.

Proposition 2.3. *Assume that L satisfies **(H1)** with $k \geq 1$, **(H2)-(i)** and **(H2)-(ii)**, as well as*

$$(\mathcal{B} - \xi_j)[\text{Dom}(L)] \supset \mathcal{A}[\mathcal{E}_j] \quad \text{for some } j \in \{1, \dots, k\}.$$

Then one has $N(\mathcal{L} - \xi_j) = N(L - \xi_j)$.

Proof of Proposition 2.3. The equation

$$g \in \text{Dom}(\mathcal{L}), \quad (\mathcal{L} - \xi_j)g = 0,$$

implies $(\mathcal{B} - \xi_j)g = -\mathcal{A}g$ and from our assumption there exists $h \in \text{Dom}(L)$ such that $(\mathcal{B} - \xi_j)h = -\mathcal{A}g$. As a consequence, we have $(\mathcal{B} - \xi_j)(g - h) = 0$ and from assumption **(H2)-(ii)** we obtain $g = h \in \text{Dom}(L)$ and $(L - \xi_j)g = 0$, so that $g \in N(L - \xi_j)$. \square

3. SEMIGROUP DECAY *versus* RESOLVENT ESTIMATES

Consider X a Banach space and $\Lambda \in \mathcal{C}(X)$. We shall make use of the following new assumptions:

(H1') **Localization of the spectrum:** $\Sigma(\Lambda) \cap \Delta_a = \{\xi_1, \dots, \xi_k\}$ for some $a \in \mathbb{R}$, $k \in \mathbb{N}$, and some *discrete* eigenvalues $\xi_1, \dots, \xi_k \in \Delta_a$ (isolated with finite dimension) with the convention $\{\xi_1, \dots, \xi_k\} = \emptyset$ if $k = 0$.

(H3) **Line control on the resolvent:** there exists some real numbers $a \in \mathbb{R}$, $K \in (0, \infty)$ such that

$$\sup_{y \in \mathbb{R}} \|R(a + iy)\|_{\mathcal{B}(X)} \leq K.$$

(H3') **Sectorial control on the resolvent:** there exists some real numbers $a \in \mathbb{R}$, $K, \theta \in (0, \infty)$ such that

$$\sup_{y = \pm\theta(x-a), x \leq a} \|R(x + iy)\|_{\mathcal{B}(X)} \leq K'.$$

(H4) **Mild control on the semigroup:** Λ generates a C_0 -semigroup of bounded operators $e^{t\Lambda}$ on X , and more precisely there exists $b \in \mathbb{R}$, $C_b \geq 0$ such that

$$\forall t \geq 0 \quad \|e^{t\Lambda}\|_{\mathcal{B}(X)} \leq C_b e^{bt}.$$

(H5) Accurate decomposition of the semigroup: there holds

$$(3.1) \quad \forall t \geq 0, \quad \left\| e^{t\Lambda} - \sum_{j=1}^k e^{tT_j} \Pi_j \right\|_{\mathcal{B}(X)} \leq C_{\tilde{a}} e^{\tilde{a}t},$$

for some $\tilde{a} \in \mathbb{R}$, $k \in \mathbb{N}$, some constant $C_{\tilde{a}} \geq 1$, some finite dimensional projection operators Π_1, \dots, Π_k such that $\Pi_i \Pi_j = 0$ for any $i \neq j$ and such that they all commute with the semigroup $e^{t\Lambda}$, some operators of the form $T_j = \xi_j Id_{X_j} + N_j$ with $\xi_j \in \Delta_{\tilde{a}}$, $X_j := \Pi_j X$, $N_j \in \mathcal{B}(X_j)$ is a nilpotent operator.

Theorem 3.1. *Consider a Banach space X and $\Lambda \in \mathcal{C}(X)$.*

- (1) **(General Banach case)** *Assume that **(H1')** and **(H3)** hold for the same real $a \in \mathbb{R}$ as well as **(H4)**. Then for any $\tilde{a} > a$ there exists a constant $C_{\tilde{a}} \geq 1$ (explicitly computable as a function of \tilde{a} , a , b , C_b and K) such that*

$$(3.2) \quad \forall t \geq 0, \quad \left\| e^{t\Lambda} - \sum_{j=1}^k e^{t\Lambda} \Pi_{\Lambda, \xi_j} \right\|_{\mathcal{B}(Y \rightarrow X)} \leq C_{\tilde{a}} e^{\tilde{a}t},$$

where $Y = D(\Lambda) \cap D(\Lambda^2)$ endowed with the norm $\|f\|_Y := \|f\|_X + \|\Lambda f\|_X + \|\Lambda^2 f\|_X$.

- (2) **(General Hilbert case)** *Assume that **(H1')** and **(H3)** hold for the same real $a \in \mathbb{R}$ as well as **(H4)**. Assume furthermore that X is a Hilbert space. Then **(H5)** holds for any $\tilde{a} > a$ and where $\Pi_j = \Pi_{\Lambda, \xi_j}$ is the spectral projector associated to ξ_j .*
- (3) **(Sectorial case)** *Assume that **(H1')** and **(H3')** hold for the same real a . Then **(H5)** holds for any $\tilde{a} > a$ and where again $\Pi_j = \Pi_{\Lambda, \xi_j}$ is the spectral projector associated to ξ_j .*
- (4) **(Partial reversed implication)** *Assume that **(H5)** holds. Then Λ satisfies **(H1')**, **(H3)** for any $a \in (\tilde{a}, \min_j \Re \xi_j)$, **(H4)** for any $b \geq \max_j \Re \xi_j$ and $\Pi_{\Lambda, \xi_j} = \Pi_j$ for any $1 \leq j \leq k$.*

Remarks 3.2. (1) *In a Hilbert space, we see that the assumptions **(H1)**, **(H3)**, **(H4)** on the one hand, and the assumption **(H5)** on the other hand are “almost” equivalent, in the sense that we must chose $\tilde{a} > a$ for the direct implication, or $a > \tilde{a}$ for the reversed one.*

- (2) *One could replace **(H3)** in this theorem by*

$$(3.3) \quad \int_0^\infty \|e^{t\Lambda}\|_{\mathcal{B}(X)} e^{-b't} dt \leq C_{b'}.$$

*Indeed, **(H3)** implies (3.3) for any $b' > b$, and in the proof we only use (3.3).*

- (3) *After finishing writing this paper, we were informed of the recent paper [17] which uses an idea similar to the proof below of point (2) of Theorem 3.1: use a Plancherel identity on the resolvent in Hilbert spaces in order to obtain explicit rates of decay on the semigroup in terms of bounds on the resolvent. Let us also add that this idea was more or less “well-known” in the spectral theory community (as is also acknowledged in [17]), even if it seems hard to*

find a reference where it is done in a quantitative way as here and in [17]. In the present paper however, the focus is on the combination of this method with our factorization theorem in order to modify the space for the decay estimate on the semigroup.

Proof of Theorem 3.1.

Proof of point (1). Assume first that $\{\xi_1, \dots, \xi_k\} = \emptyset$. Starting from the identity

$$\forall z \notin \Sigma(\Lambda), \quad R(z) = z^{-1} [-\text{Id} + R(z)\Lambda]$$

we deduce from **(H2)** that for any $f \in \text{Dom}(\Lambda)$

$$(3.4) \quad \|R(a + is)f\| \leq \frac{(1 + K)}{|a + is|} (\|f\|_X + \|\Lambda f\|_X) \rightarrow 0 \quad \text{as } |s| \rightarrow \infty$$

and it follows from the density of $\text{Dom}(\Lambda)$ in X that

$$\forall f \in X, \quad \|R(z)f\|_X \rightarrow 0, \quad z = a + is, \quad |s| \rightarrow \infty.$$

Then consider for $M > 0$

$$I_M(f) := \int_{a-iM}^{a+iM} e^{zt} R(z)f \, dz.$$

Since R is differentiable on the segment $[a - iM, a + iM]$ with $R'(z) = R(z)^2$, we compute by integration by parts

$$(3.5) \quad I_M(f) = \frac{e^{(a+iM)t}}{t} R(a + iM)f - \frac{e^{(a-iM)t}}{t} R(a - iM)f - \int_{a-iM}^{a+iM} \frac{e^{zt}}{t} R(z)^2 f \, dz.$$

We use (see for instance [26, Chapter 1, Section 1.7] and [31, Theorem 1.1]) the representation

$$e^{t\Lambda} f = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} e^{zt} R(z) f \, dz := \lim_{M \rightarrow \infty} \frac{1}{2i\pi} I_M(f).$$

From (3.4)-(3.5) this representation formula can be rewritten as

$$(3.6) \quad e^{t\Lambda} f = -\frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{t} R(z)^2 f \, dz := -\frac{1}{2i\pi} \lim_{M \rightarrow \infty} \int_{a-iM}^{a+iM} \frac{e^{zt}}{t} R(z)^2 f \, dz.$$

Then we use the identity

$$\forall z \notin \Sigma(\Lambda), \quad R(z) = z^{-1} [-\text{Id} + R(z)\Lambda]$$

to deduce

$$\|e^{t\Lambda} f\|_X \leq C \frac{e^{at}}{t} \frac{1}{2\pi} \left(\lim_{M \rightarrow \infty} \int_{a-iM}^{a+iM} \frac{dz}{|a + iz|^2} \right) (\|f\|_X + \|\Lambda f\|_X + \|\Lambda^2 f\|_X)$$

which concludes the proof.

Proof of (2). As in the proof of point 1, we start from the representation formula (3.6). Let $f \in \text{Dom}(\Lambda) \subset X$ and $\phi \in \text{Dom}(\Lambda^*) \subset X^*$ (where X^* denotes the dual space of X and Λ^* the adjoint operator of Λ). Let us estimate

$$\begin{aligned}
 \langle \phi, e^{t\Lambda} f \rangle &= -\frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{t} \langle \phi, R(z)^2 f \rangle dz, \\
 (3.7) \quad &:= -\frac{1}{2i\pi} \lim_{M \rightarrow \infty} \int_{a-iM}^{a+iM} \frac{e^{zt}}{t} \langle \phi, R(z)^2 f \rangle dz.
 \end{aligned}$$

Applying Cauchy-Schwartz inequality, assumption **(H3)** and the identity

$$R(a + is) = R(b + 1 + is) + (a - b - 1) R(a + is) R(b + 1 + is),$$

we get

$$\begin{aligned}
 \left| \int_{a-iM}^{a+iM} \frac{e^{zt}}{t} \langle \phi, R(z)^2 f \rangle dz \right| &\leq \frac{e^{at}}{t} \int_{-M}^M \|R(a + is)^* \phi\| \|R(a + is) f\| ds \\
 &\leq \left(1 + |b + 1 - a| K\right)^2 \frac{e^{at}}{t} \int_{-M}^M \|R(b + 1 + is)^* \phi\| \|R(b + 1 + is) f\| ds \\
 &\leq \left(1 + |b + 1 - a| K\right)^2 \frac{e^{at}}{t} \left(\int_{-M}^M \|R(b + 1 + is)^* \phi\|^2 ds \right)^{1/2} \\
 &\quad \times \left(\int_{-M}^M \|R(b + 1 + is) f\|^2 ds \right)^{1/2}.
 \end{aligned}$$

Plancherel's identity in X (this is where we need that X is a Hilbert space) and assumption **(H2)** imply:

$$\begin{aligned}
 \int_{\mathbb{R}} \|R(b + 1 + is) f\|^2 ds &= 2\pi \int_0^{+\infty} \|e^{-(b+1)t} e^{t\Lambda} f\|^2 dt \\
 &\leq \left(\int_0^{+\infty} \|e^{-(b+1)t} e^{t\Lambda}\|^2 dt \right) \|f\|^2 \\
 &\leq \left(\int_0^{+\infty} C_b e^{-t} dt \right) \|f\|^2 = C_b \|f\|_X^2.
 \end{aligned}$$

Since (Hilbert space) $X^* \sim X$ and $\|e^{t\Lambda}\| = \|(e^{t\Lambda})^*\|$, we have similarly

$$\int_{-M}^M \|R(b + 1 + is)^* \phi\|^2 ds \leq \int_0^{+\infty} \|e^{-(b+1)t} (e^{t\Lambda})^* \phi\|^2 dt \leq C_b \|\phi\|_{X^*}^2.$$

Gathering the estimates above, we get

$$|\langle \phi, e^{t\Lambda} f \rangle| \leq \underbrace{(1 + |b + 1 - a| K)^2 C_b}_{:=C} \frac{e^{at}}{t} \|\phi\|_{X^*} \|f\|_X$$

which implies

$$\|e^{t\Lambda}\| \leq \frac{C e^{at}}{t},$$

and concludes the proof (by combining this last estimate with **(H4)** for $t \leq 1$).

Suppose now that the set of isolated eigenvalues $\{\xi_1, \dots, \xi_k\}$ is not empty. In this case, the representation is

$$e^{t\Lambda} = \sum_{j=1}^k e^{\xi_j t} \Pi_j + \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{t} R(z)^2 dz,$$

and we proceed as above for the second term in the right-hand side.

Proof of point (3). The proof is classical and can be found in [26, Theorem 7.7 Chap 1] for instance.

Proof of point (4). On the one hand, define

$$\Pi_0 := \text{Id} - \Pi_1 - \dots - \Pi_k \quad \text{and} \quad S_0(t) := S(t) \Pi_0.$$

Its generator is $\Lambda \Pi_0$ and assumption **(H5)** together with the Hille-Yosida theorem (see [26, Theorem 5.3]) implies then $\Delta_a \cap \Sigma(\Lambda \Pi_0) = \emptyset$ and that the resolvent operator associate to $\Lambda \Pi_0$ satisfies **(H3)**. On the other hand, since $\Lambda \Pi_j$ acts on a finite dimensional space, by the Jordan decomposition, one can find $\eta_{j,1}, \dots, \eta_{j,k_j}$ and projectors $(\Pi_{j,i})$ such that

$$\Pi_j = \Pi_{j,1} + \dots + \Pi_{j,k_j}, \quad \Pi_{j,i} \Pi_{j,\ell} = 0 \quad \forall i \neq \ell,$$

where $\Lambda \Pi_{j,i}$ is a matrix with only eigenvalue $\eta_{j,i}$ (writing as a diagonal part plus some nilpotent part) for any $1 \leq i \leq k_j$ and $1 \leq j \leq k$. Using assumption **(H5)** again, we have by composing by Π_j

$$(3.8) \quad \left\| e^{t\Lambda \Pi_j} \Pi_j - e^{tT_j} \Pi_j \right\|_{\mathcal{B}(X)} \leq C_{\bar{a}} e^{\bar{a}t}$$

and next by composing by $\Pi_{j,i}$

$$(3.9) \quad \left\| e^{t\Lambda \Pi_{j,i}} \Pi_{j,i} - e^{tT_j} \Pi_{j,i} \right\|_{\mathcal{B}(X)} \leq C_{\bar{a}} e^{\bar{a}t}.$$

By comparing the large time behavior $t \rightarrow +\infty$ of the two semigroups we obtain that necessarily $\eta_{j,i} = \xi_j$ for any $1 \leq i \leq k_j$, so that we can take $k_j = 1$. In other words $\{\xi_1, \dots, \xi_k\}$ are discrete eigenvalues of Λ with associated eigenspaces $E_j := \Pi_j X$ which is nothing but **(H1')**.

Moreover by writing

$$e^{tT_j} = e^{t\xi_j} (P_{j,0} + t P_{j,1} + \dots + t^{k_j} P_{j,k_j})$$

for some new projectors $(P_{j,i})$ such that

$$P_j = P_{j,1} + \dots + P_{j,k_j}, \quad P_{j,i} P_{j,\ell} = 0 \quad \forall i \neq \ell,$$

and writing a similar decomposition of the semigroup generated by the matrix $\Lambda \Pi_j$, we easily deduce $\Lambda \Pi_j = T_j$. We conclude that **(H3)** and **(H4)** hold by gathering (through a direct sum) the informations obtained on each subspace $\Pi_j X$, $0 \leq j \leq k$.

Finally, define $\Pi := \Pi_1 - \dots - \Pi_k$ so that $\Pi_0 = \text{Id} - \Pi$, and $\tilde{\Pi} := \Pi_{\Lambda, \xi_1} + \dots + \Pi_{\Lambda, \xi_k}$ so that $\tilde{\Pi}_0 = \text{Id} - \tilde{\Pi}$. By applying point (1) to Λ , we have

$$\forall t \geq 0, \quad \left\| e^{t\Lambda} (\text{Id} - \tilde{\Pi}) \right\|_{\mathcal{B}(Y \rightarrow X)} \leq C_{\alpha} e^{\alpha t},$$

for any $\alpha > \tilde{a}$. Composing by Π_0 we deduce with the help of **(H5)**

$$\forall t \geq 0, \quad \|e^{t\Lambda} \tilde{\Pi} \Pi_0\|_{\mathcal{B}(Y \rightarrow X)} \leq \left\| e^{t\Lambda} \Pi_0 - e^{t\Lambda} \tilde{\Pi} \Pi_0 \right\|_{\mathcal{B}(Y \rightarrow X)} + \|e^{t\Lambda} \Pi_0\|_{\mathcal{B}(Y \rightarrow X)} \leq C_\alpha e^{\alpha t},$$

which implies (since the norms $\|\cdot\|_Y$ and $\|\cdot\|_X$ are equivalent on the finite dimensional subspace $\tilde{\Pi}X$, and Π_0 and $\tilde{\Pi}$ commute)

$$\forall t \geq 0, \quad \left\| e^{t\Lambda} \tilde{\Pi} \Pi_0 \right\|_{\mathcal{B}(X)} \leq C_\alpha e^{\alpha t}.$$

That last estimate together with the fact that $\Lambda \tilde{\Pi} \Pi_0$ is the restriction to the subspace $\tilde{\Pi} \Pi_0 X$ of the matrix $\Lambda \tilde{\Pi}$ with spectrum $\Sigma(\Lambda \tilde{\Pi}) = \{\xi_1, \dots, \xi_k\}$ yields $\tilde{\Pi} \Pi_0 = 0$. Similarly, we obtain $\Pi \tilde{\Pi}_0 = 0$ and then

$$\Pi = \Pi \tilde{\Pi}_0 + \Pi \tilde{\Pi} = \tilde{\Pi} \Pi_0 + \tilde{\Pi} \Pi = \tilde{\Pi}.$$

That last identity implies

$$\sum_{j=1}^k e^{\Lambda t} \Pi_{\Lambda, \xi_j} = e^{\Lambda t} \tilde{\Pi} = e^{\Lambda t} \Pi = \sum_{j=1}^k e^{T_j t} \Pi_j.$$

Coming back to the complex plane integral definition of Π_{Λ, ξ_j} we finally get that $e^{\Lambda t} \Pi_{\Lambda, \xi_j} = e^{T_j t} \Pi_j$ and therefore $\Pi_{\Lambda, \xi_j} = \Pi_j$. \square

We end this section by some recalls on the classical notion of *dissipative operators* together with the statement of a variant of the *Lumer-Phillips theorem*. The latter provides a very popular alternative method in order to prove assumption **(H5)** for a given operator. Its conclusion is stronger (the constant in front of the exponential is $C_{\tilde{a}} = 1$, that is the operator is a contraction up to a translation) but its assumptions are also more restrictive. Nevertheless, in all the applications we have in mind, it is the starting point from which we can develop our space enlargement method of the spectral gap decay estimate.

Definition 3.3. Consider X a Banach space, $\Lambda \in \mathcal{C}(X)$ and $a \in \mathbb{R}$. We say that $\Lambda - a$ is dissipative on a closed subspace $X_0 \subset X$ if

$$(3.10) \quad \forall f \in D(\Lambda) \cap X_0, \exists \varphi \in F(f) \quad \Re \langle \varphi, (\Lambda - a) f \rangle \leq 0,$$

where $F(f) \subset X'$ is the dual set of f defined by

$$F(f) := \{\varphi \in X'; \langle \varphi, f \rangle = \|f\|_X^2 = \|\varphi\|_{X'}^2\}.$$

In the case when X is a Hilbert space $F(f) = \{f\}$ and (3.10) writes

$$(3.11) \quad \forall f \in D(\Lambda) \cap X_0, \quad \Re \langle \Lambda f, f \rangle \leq a \|f\|^2.$$

We say that $\Lambda - a$ is m -dissipative is furthermore

$$R(\Lambda - a) = X.$$

Remark 3.4. Notice that Λ satisfies (3.10) if and only if

$$\forall f \in D(\Lambda), \forall \lambda > a \quad \|(\lambda - \Lambda) f\| \geq (\lambda - a) \|f\|.$$

We refer to [26, Chap 1, Theorem 4.2] for a proof of this claim.

Theorem 3.5. *Consider X a Banach space, $\Lambda \in \mathcal{C}(X)$. Assume that there exists $a, b \in \mathbb{R}$ and $\xi_1, \dots, \xi_k \in \mathbb{C}$ such that $a < \min \Re \xi_j \leq \max \Re \xi_j < b$. Assume that there exists a closed subspace X_0 and some finite dimension subspaces X_1, \dots, X_k such that $X_0 \oplus X_1 \oplus \dots \oplus X_k = X$ and*

- (i) $X_j \subset M(\Lambda - \xi_j)$ for any $j = 1, \dots, k$;
- (ii) X_0 is invariant under the action of Λ : $\Lambda f \in X_0$ for any $f \in X_0 \cap D(\Lambda)$;
- (iii) $\Lambda - a$ is dissipative on X_0 ;
- (iv) $\Lambda - b$ is m -dissipative on X_0 or on X .

Then Λ generates a C_0 -semigroup of bounded operators and **(H5)** with $\tilde{a} = a$ and $C_{\tilde{a}} = 1$.

Sketch of the proof of Theorem 3.5. The operator $\Lambda|_{X_j}$, $j = 1, \dots, k$, generates a semigroup which writes $S_j(t) := e^{T_j t}$ where $T_j = \xi_j \text{Id}_{X_j} + N_j$, N_j nilpotent operator on X_j . We remark that $\Lambda - b$ is m -dissipative on X_0 due to (iv) and $X_0 \oplus X_1 \oplus \dots \oplus X_k = X$. Then the classical Lumer-Phillips theorem (see [26, Chap 1, Theorem 4.3]) implies that $\Lambda|_{X_0}$ generates a semigroup $S_0(t)$. To conclude, we define

$$S(t) := S_0(t) P_0 + \dots + S_k(t) P_k,$$

where P_j is the projector on X_j . Using that each P_j commutes with Λ , it is easily checked that $S(t) = e^{t\Lambda}$ by decomposing $f \in D(\Lambda)$ as $f = f_0 + \dots + f_k$ with $f_j \in X_j \cap D(\Lambda)$ so that $f_0 \in D(\Lambda)$ and computing the time derivative of $S(t) f$. \square

4. SPECTRAL GAP ESTIMATES

In this section we combine the results obtained so far in order to prove that one can enlarge the functional space of decay estimates on semigroups.

Theorem 4.1. *Assume that L satisfies **(H1')** and*

$$\forall t \geq 0, \quad \left\| e^{tL} \left(\text{Id} - \sum_{j=1}^k \Pi_j \right) \right\|_{\mathcal{B}(E)} \leq C_a e^{at},$$

where $\Pi_j = \Pi_{L, \xi_j}$, $1 \leq j \leq k$ denotes the (finite dimensional) spectral projectors of the eigenvalue $\xi_j \in \Delta_a$, and \mathcal{L} satisfies **(H4)** as well as the decomposition assumption **(H2)-(ii)**, that is there exist two closed unbounded operators \mathcal{A} and \mathcal{B} (with domains containing $\text{Dom}(\mathcal{L})$) such that $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and $\mathcal{A} \in \mathcal{B}(\mathcal{E} \rightarrow E)$. Then we distinguish the following cases:

- (i) **(Sectorial framework)** \mathcal{B} satisfies an assumption of type **(H3')**: more precisely there exists some real numbers $a \in \mathbb{R}$ and $K', \theta \in (0, \infty)$ such that

$$\sup_{y = \pm \theta(x-a), x \leq a} \|(\mathcal{B} - (x + iy))^{-1}\|_{\mathcal{B}(X)} \leq K'.$$

Then \mathcal{L} is sectorial (with explicit estimates) and for any $\tilde{a} > a$ there exists an explicit constant $C_{\tilde{a}} \geq 1$ such that

$$(4.1) \quad \forall t \geq 0, \quad \left\| e^{t\mathcal{L}} \left(\text{Id} - \sum_{j=1}^k \Pi_j \right) \right\|_{\mathcal{B}(\mathcal{E})} \leq C_{\tilde{a}} e^{\tilde{a}t}.$$

- (ii) **(Hilbert space framework)** \mathcal{E} is an Hilbert space and \mathcal{B} satisfies an assumption of type **(H3)**: more precisely there exists some real numbers $a \in \mathbb{R}$ and $K \in (0, \infty)$ such that

$$(4.2) \quad \sup_{y \in \mathbb{R}} \|(\mathcal{B} - (a + iy))^{-1}\|_{\mathcal{B}(X)} \leq K.$$

Then \mathcal{L} satisfies **(H1)** and for any $\tilde{a} > a$ there exists an explicit constant $C_{\tilde{a}} \geq 1$ such that

$$\forall t \geq 0, \quad \left\| e^{t\mathcal{L}} \left(Id - \sum_{j=1}^k \Pi_j \right) \right\|_{\mathcal{B}(\mathcal{E})} \leq C_{\tilde{a}} e^{\tilde{a}t}.$$

- (iii) **(General Banach space framework)** \mathcal{E} is a Banach space and \mathcal{B} satisfies again an assumption of type **(H3)** in the form (4.2). Then \mathcal{L} satisfies **(H1)** and for any $\tilde{a} > a$ there exists an explicit constant $C_{\tilde{a}} \geq 1$ such that

$$\forall t \geq 0, \quad \left\| e^{t\mathcal{L}} \left(f - \sum_{j=1}^k \Pi_j f \right) \right\|_{\mathcal{E}} \leq C_{\tilde{a}} e^{\tilde{a}t} (\|f\|_{\mathcal{E}} + \|\mathcal{L}f\|_{\mathcal{E}} + \|\mathcal{L}^2 f\|_{\mathcal{E}}).$$

Proof of Theorem 4.1. The proof is immediate.

For the case (i) we combine uniform estimates on the resolvent of L for $\xi \in \{y = \pm\theta(x - a), x \leq a\}$ together with the sectorial estimate **(H3')** assumed on \mathcal{B} to deduce by Theorem (2.1) sectorial estimates on \mathcal{L} for $\xi \in \{y = \pm\theta(x - a), x \leq a\}$. We conclude by applying Theorem 3.1, point (3).

For the case (ii) we combine uniform estimates on the resolvent of L for $\{\Re \xi = a\}$ together with the estimate **(H3)** assumed on \mathcal{B} to deduce by Theorem (2.1) uniform estimates of type **(H3)** on the resolvent of \mathcal{L} for $\{\Re \xi = a\}$. We conclude by applying Theorem 3.1, point (2).

For the case (iii) we combine uniform estimates on the resolvent of L for $\{\Re \xi = a\}$ together with the estimate **(H3)** assumed on \mathcal{B} to deduce by Theorem (2.1) uniform estimates of type **(H3)** on the resolvent of \mathcal{L} for $\{\Re \xi = a\}$. We conclude by applying Theorem 3.1, point (1). \square

In the following corollary we give a kind of “practical” case of application for PDEs, where the small space E is a Hilbert space where the operator is self-adjoint and dissipative (typically $L^2(\mu^{-1})$ where μ is the stationary distribution), and where one has some conservation law in the larger Banach space \mathcal{E} in which we want to apply our space enlargement method.

Corollary 4.2 (Application to PDEs).

Consider a Hilbert space E and a Banach space \mathcal{E} such that $E \subset \mathcal{E}$ and E is dense in \mathcal{E} . Consider two unbounded closed operators with dense domain L on E and \mathcal{L} on \mathcal{E} such that $\mathcal{L}|_E = L$.

- Assume in E :

- (i) There are $G_1, \dots, G_n \in E$ linearly independent, and $\lambda_1, \dots, \lambda_n \in \Delta_\alpha$ for some $\alpha \in \mathbb{R}$, such that $G_j \in M(L - \lambda_j)$ and $\|G_j\|_E = 1$ for any $1 \leq j \leq n$ (note that the λ_j are not necessarily distinct).

(ii) Defining $\psi_j(f) := \langle f, G_j \rangle_E G_j$ for any $1 \leq j \leq n$, the space

$$E_0 := \{f \in E; \forall j = 1, \dots, n, \psi_j(f) = 0\}$$

is invariant under the action of L , and $E = E_0 \oplus \text{Span}\{G_1, \dots, G_n\}$.

(iii) $L - \alpha$ is dissipative on E_0 for some scalar product $\langle \langle \cdot, \cdot \rangle \rangle_E$ on E equivalent to $\langle \cdot, \cdot \rangle_E$:

$$\forall f \in D(L) \cap E_0, \quad \langle \langle (L - \alpha)f, f \rangle \rangle_E \leq 0.$$

(iv) $L - b$ is m -dissipative in E_0 or E for some $b \geq \alpha$.

• Assume in \mathcal{E} :

(v) \mathcal{L} decomposes as $\mathcal{L} = \mathcal{A} + \mathcal{B}$, $\mathcal{A}, \mathcal{B} \in \mathcal{C}(\mathcal{E})$, with \mathcal{A} bounded from \mathcal{E} to E and $\mathcal{B} - \alpha$ is dissipative.

(vi) There exists $\Psi_j \in \mathcal{B}(\mathcal{E} \rightarrow E)$ such that $\Psi_j|_E = \psi_j$.

Then, $\{\lambda_1, \dots, \lambda_n\} = \Sigma(\mathcal{L}) \cap \Delta_a$, G_1, \dots, G_n is a base of the algebraic eigenspaces of \mathcal{L} associated to the eigenvalues $\lambda_1, \dots, \lambda_n$, and, introducing the notation ξ_1, \dots, ξ_k for the distinct eigenvalues, that is $\{\xi_1, \dots, \xi_k\} = \{\lambda_1, \dots, \lambda_n\}$ and $\xi_j \neq \xi_i$ for $i \neq j$, we have $\Pi_{\mathcal{L}, \xi_j|_E} = \Pi_{L, \xi_j}$ for any $1 \leq j \leq k$. Moreover, the space

$$\mathcal{E}_0 := \{f \in \mathcal{E}; \forall j = 1, \dots, n, \Psi_j(f) = 0\}$$

is invariant under the action of \mathcal{L} and for any $a > \alpha$ there exists $C_a \geq 1$ such that for any $f \in X \cap \mathcal{E}_0$ there holds:

$$\forall t \geq 0 \quad \|e^{t\mathcal{L}} f\|_{\mathcal{E}} \leq C_a e^{at} \|f\|_X,$$

where $X = \mathcal{E}$ if \mathcal{E} is a Hilbert space and $X = D(\mathcal{L}^2) \cap D(\mathcal{L})$ endowed with the norm of the graphs of \mathcal{L} and \mathcal{L}^2 if \mathcal{E} is a Banach space.

Remark 4.3. Note that in the case where L is self-adjoint in the small space E (which however is not always satisfied in the following applications, in particular for inhomogeneous kinetic models), then there can be no nilpotent part in the action of the operator in the eigenspaces.

Proof of Corollary 4.2. First from (i), (ii), (iii), (iv) and Theorem 3.5, the operator L satisfies **(H1')**, generates a semigroup e^{tL} on E and more precisely satisfies the bound

$$\forall t \geq 0, \quad \left\| e^{tL} \left(\text{Id} - \sum_{j=1}^k \Pi_j \right) \right\|_{\mathcal{B}(E)} \leq C_a e^{at}.$$

Next, because $A|_E \in \mathcal{B}(E)$, the operator $\mathcal{B}|_E = L - A|_E$ also generates a semigroup on E . We claim that

$$(4.3) \quad \forall f \in E, \forall t \geq 0 \quad \|e^{t\mathcal{B}} f\|_{\mathcal{E}} \leq e^{Ct} \|f\|_{\mathcal{E}},$$

for some constant $C \in \mathbb{R}$. Indeed, for any $f \in D(L)$ we may compute (in the case where \mathcal{E} is a Hilbert space)

$$\frac{d}{dt} \|e^{t\mathcal{B}} f\|_{\mathcal{E}}^2 = 2 \Re \langle \mathcal{B} e^{t\mathcal{B}} f, e^{t\mathcal{B}} f \rangle \leq 2\alpha \|e^{t\mathcal{B}} f\|_{\mathcal{E}}^2,$$

and we deduce (4.3) with $C = \alpha$. In the general Banach case, we can argue as follows. For any $\varepsilon > 0$ we introduce the norm $\|f\|_{\varepsilon} := \|f\|_{\mathcal{E}} + \varepsilon \|f\|_E$ so that $\|\cdot\|_{\varepsilon}$ is

equivalent to $\|\cdot\|_E$ (for any $\epsilon > 0$). Define $C := \alpha + \|\mathcal{A}\|_{\mathcal{B}(\mathcal{E} \rightarrow E)}$. Using the second definition recalled in Remark 3.4 and the fact that $L - \alpha$ is dissipative in E , it is clear that $\mathcal{B} - C$ is m -dissipative in the Banach space $(E, \|\cdot\|_\epsilon)$. The Lumer-Phillips theorem says that $\mathcal{B} - C$ generates a semigroups of contractions on $(E, \|\cdot\|_\epsilon)$, in particular for any $f \in E$, $t \geq 0$,

$$\|e^{t(\mathcal{B}-C)} f\|_\epsilon + \epsilon \|e^{t(\mathcal{B}-C)} f\|_E \leq \|f\|_\epsilon + \epsilon \|f\|_E.$$

Letting ϵ goes to 0, we obtain (4.3). Together with a density argument, this implies that \mathcal{B} generates a C_0 -semigroup on \mathcal{E} and more precisely $\mathcal{B} - C$ is m -dissipative on \mathcal{E} . Using again that $\mathcal{A} \in \mathcal{B}(\mathcal{E})$ we deduce then that $\mathcal{L} = \mathcal{A} + \mathcal{B}$ generates a C_0 -semigroup and more precisely that \mathcal{L} satisfies **(H4)** with a growth rate smaller than $b = C + \|\mathcal{A}\|_{\mathcal{B}(\mathcal{E})}$.

Because of Remark 3.4, the line control (4.2) on the resolvent of \mathcal{B} on any line $\{\xi = a + iy, y \in \mathbb{R}\}$, $a > \alpha$, is consequence of the fact $\mathcal{B} - \alpha$ is dissipative. The assumptions of Theorem 4.1 being fulfilled, we deduce that for any $f \in X$

$$\forall t \geq 0 \quad \left\| e^{t\mathcal{L}} f - \sum_{j=1}^k e^{\mathcal{L}t} \Pi_{\mathcal{L}, \xi_j} f \right\|_\mathcal{E} \leq C_a e^{at} \|f\|_X.$$

where by definition $\Pi_{\mathcal{L}, \xi_j}$ is the spectral projection on the eigenspace associated to the eigenvalue ξ_j in \mathcal{E} through the integral formula (2.1).

In order to conclude we have to prove that $\Pi_{\mathcal{L}, \xi_j} f = \Pi_j f$ for any $f \in \mathcal{E}$, where we define

$$\Pi_j := \sum_{\ell} \Pi_{j,\ell} \quad \Pi_{j,\ell} := \sum_{i, \lambda_i = \xi_j, d(G_i) = \ell} \Psi_i$$

and $d(G_i)$ stands for the smallest $q \in \mathbb{N}$ such that $(L - \lambda_i)^q G_i = 0$. First, for $f \in E \cap Y$, we write for some polynomial function $P_{j,\ell}(t)$

$$\begin{aligned} & \left\| \sum_{j=1}^k e^{\mathcal{L}t} \Pi_{\mathcal{L}, \xi_j} f - \sum_{j=1}^k \sum_{\ell} e^{\xi_j t} P_{j,\ell}(t) \Pi_{j,\ell} f \right\|_\mathcal{E} \leq \\ & \leq \left\| \sum_{j=1}^k e^{\mathcal{L}t} \Pi_{\mathcal{L}, \xi_j} f - e^{t\mathcal{L}} f \right\|_\mathcal{E} + \left\| e^{t\mathcal{L}} f - \sum_{j=1}^k \sum_{\ell} e^{\xi_j t} P_{j,\ell}(t) \Pi_{j,\ell} f \right\|_\mathcal{E} \leq C_a e^{at}, \end{aligned}$$

because $\Pi_{\mathcal{L}, \xi_j} = \Pi_j$ as a consequence of the first step and point (4) in Theorem 3.1. When the ξ_j are real numbers, the fact that $P_{j,\ell}$ has exactly degree ℓ implies that all the the functions $e^{\xi_j t} P_{j,\ell}(t)$ have different growth when $t \rightarrow \infty$. By induction, we deduce that $\Pi_{\mathcal{L}, \xi_j} f = \Pi_j f$ for any $1 \leq j \leq k$. With similar arguments (and taking into account the frequency of oscillation) we have the same conclusion in the general case where the eigenvalues are not purely real. Because both semigroups are continuous on \mathcal{E} the same identity holds on the whole space \mathcal{E} . \square

Remarks 4.4. (1) *The assumptions made on E in Corollary 4.2 are nothing but those made on E in Theorem 3.5 and are then equivalent (up to a slight translation of value of $a \in \mathbb{R}$) to know an accurate decomposition **(H5)** of the semigroup e^{tL} .*

- (2) In section 5 and 6 below, we apply our space enlargement of the decay of semigroups from E to $\mathcal{E} \supset E$. Since the conditions on E are already known results and condition (ii)-(vi) is a straightforward consequence of the conservation laws of the equation in \mathcal{E} , the only thing we have to do in the following application is to exhibit a suitable decomposition $\mathcal{L} = \mathcal{A} + \mathcal{B}$ for which (v) holds.

5. APPLICATION TO FOKKER-PLANCK EQUATIONS

5.1. **The space homogeneous Fokker-Planck equation.** Consider the equation

$$(5.1) \quad \partial_t f = Lf := \nabla_v \cdot (\nabla_v f + F f), \quad f(\cdot, 0) = f_{in}(\cdot),$$

for $f = f(t, v)$, $t \geq 0$, $v \in \mathbb{R}^d$ and $F = F(v) \in \mathbb{R}^d$ which takes the form

$$(5.2) \quad F = \nabla_v \phi + U.$$

Assume

(FP1) The potential $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies: $\mu(dv) := e^{-\phi(v)} dv$ is a probability measure and (strong version of the Poincaré inequality) for some $\nu > 0$ and for any f such that $\int f dv = 0$ one has

$$(5.3) \quad \int \left| \nabla \left(\frac{f}{\mu} \right) \right|^2 \mu(dv) \geq \nu \int f^2 (1 + |\nabla \phi|^2) \mu^{-1}(dv).$$

(FP2) The additional force field U satisfies

$$\forall v \in \mathbb{R}^d, \quad \nabla \cdot U = 0, \quad \nabla_v \phi \cdot U = 0, \quad |U| \leq C (1 + |\nabla \phi|).$$

It is immediate to check that $L(\mu) = 0$ and

$$\langle Lf, f \rangle_{L^2(\mu^{-1})} := \int Lf f \mu^{-1}(dv) = - \int |\nabla(f/\mu)|^2 \mu(dv) \leq 0.$$

Let us first recall more or less well-known results about the spectrum and spectral gap properties of the Fokker-Planck equation in the space $L^2(\mu^{-1})$.

Theorem 5.1. *Assume that F satisfies **(FP1)**-**(FP2)**. Then in the space $E = L^2(\mu^{-1})$, the operator L satisfies: there exists $\lambda > 0$ such that $\Sigma(L) \subset \{z \in \mathbb{C} \mid \operatorname{Re}(z) \leq -\lambda\} \cup \{0\}$, the null space of L is exactly $\mathbb{R}\mu$, and any solution to the initial value problem (5.1-5.2) for any initial datum $f_{in} \in E$ satisfies*

$$\forall t \geq 0, \quad \|f(t) - \mu \langle f_{in} \rangle\|_E \leq e^{-\lambda t} \|f_{in} - \langle f_{in} \rangle\|_E$$

where we denote

$$\langle f_{in} \rangle := \int f_{in} dv.$$

Remark 5.2. Equation (5.3) is a strengthened version of the classical Poincaré inequality

$$\int \left| \nabla \left(\frac{f}{\mu} \right) \right|^2 \mu(dv) \geq \nu \int f^2 \mu^{-1}(dv).$$

The latter is satisfied for instance under the so-called Bakry-Emery condition $\operatorname{Hess}(\phi) \geq \nu \operatorname{Id}$ (see [4, 28, 3]).

A (more or less well-known) sufficient condition for the strenghtened form (5.3) is for instance $\alpha|\nabla\phi|^2 - 2\Delta\phi \geq c$ for $\alpha \in (0, 1)$ and $c \in \mathbb{R}$ and for any $|v| \geq R$ for some $R > 0$ (see [25] for a quantitative proof).

Let us now consider a weight function $m(v) := e^{-\theta(\phi(v))}$ with $\theta \in C^2$ and the associated Hilbert space $\mathcal{E} = L^2(m^{-1})$ such that $E \subset \mathcal{E}$. The new result is a spectral gap estimate in this larger space under some specific assumptions on m . Let us denote by \mathcal{L} the operator (5.1-5.2) in the space \mathcal{E} .

Assume that m satisfies:

(FP3) The function

$$\psi(v) := \frac{1}{2m} \nabla_v \cdot \left(\mu \nabla_v \left(\frac{m}{\mu} \right) \right)$$

satisfies

$$\psi(v) \xrightarrow{|v| \rightarrow \infty} -\infty.$$

Remarks 5.3. (1) *Other technical assumptions could have been chosen for the function m in the proof below, however the formulation **(FP3)** seems to us the most natural one since it is based on the comparison of the Fokker-Planck operators for two different force field. In any case, the core idea in the decomposition in the proof below is that a coercive \mathcal{B} in \mathcal{E} is obtained by a negative local perturbation of the whole operator.*

(2) *Remark that by mollification the smoothness of m could be relaxed: if m is not smooth but \tilde{m} is smooth, satisfies **(FP3)** and is such that $c_1 m < \tilde{m} \leq c_2 m$, then*

$$\|f_t - \mu\|_{L^2(m)} \leq c_2 \|f_t - \mu\|_{L^2(\tilde{m})} \leq C e^{-\lambda_1 t} \|f_{in} - \mu\|_{L^2(\tilde{m})} \leq C' e^{-\lambda_1 t} \|f_{in} - \mu\|_{L^2(m)}.$$

Many interesting ϕ and m satisfy our assumptions (we omit the proof which is straightforward calculations):

Proposition 5.4. *One has*

$$\psi(v) = \frac{1}{2} \left[|\nabla_v \phi|^2 (\theta'(\phi)^2 - \theta'(\phi) - \theta''(\phi)) + \Delta_v \phi (1 - \theta'(\phi)) \right]$$

and admissible potentials and weights for **(FP1)**-**(FP3)** are for instance

- (i) $\phi(v) = C \langle v \rangle^s$, $s \geq 1$ and $\theta(z) = z^\alpha$, $\alpha \in (0, 1)$ with $s(1 + \alpha) > 2$;
- (ii) $\phi(v) = C \langle v \rangle^s$, $s \geq 1$ and $\theta(z) = k \log z$ with $s > 2$ and $ks > d + s - 2$.

Theorem 5.5. *Assume that F satisfies **(FP1)**-**(FP2)** and m satisfies **(FP3)**. Then for any λ_1 with $0 < \lambda_1 < \lambda$ (where λ is given in Theorem 5.1) such $\Sigma(\mathcal{L}) \subset \{z \in \mathbb{C} \mid \Re(z) \leq -\lambda_1\} \cup \{0\}$ and there exists $C_{\lambda_1} \in [1, \infty)$ such that, for any initial data $f_{in} \in \mathcal{E}$, equation (5.1-5.2) has a solution $f \in \mathcal{E}$ such that*

$$\forall t \geq 0 \quad \|f(t) - \mu \langle f_{in} \rangle\|_{\mathcal{E}} \leq C_{\lambda_1} e^{-\lambda_1 t} \|f_{in} - \mu \langle f_{in} \rangle\|_{\mathcal{E}}$$

where we denote

$$\langle f_{in} \rangle := \int f_{in} dv.$$

- Remarks 5.6.** (1) *The case of equality $\lambda_1 = \lambda$ could easily be obtained by considering a more detailed expansion of the semigroup for the first eigenvalues in the use of Theorem 5.1, and showing that the eigenspaces are the same in E and \mathcal{E} .*
- (2) *One could prove a variant of this theorem for some Banach large space \mathcal{E} such as $L^1(m^{-1})$ (under some different conditions on m).*
- (3) *Let us make a last important remark. One could have thought of another natural method of proof by interpolation: using the exponential relaxation in E together with a uniform bound in L^1 (by mass conservation and preservation of nonnegativity) one could (hopefully) try to obtain some rate of decay in some intermediate spaces. This method would yield probably quite complicated rates. Moreover it would not be satisfactory in at least two aspects: first it would not recover optimal rate of decay, and second, most importantly, **it would not apply to semigroups which do not preserve sign and therefore the L^1 norm, such as those obtained by linearization** (e.g. the linearized Boltzmann equation in section 7).*

Proof of Theorem 5.5. We prove that one can apply Corollary 4.2 with $n = 1$ (1-dimensional null space spanned by μ), $E = L^2(\mu^{-1})$, $\mathcal{E} = L^2(m^{-1})$ and $\psi(f) = \langle f, \mu \rangle_E \mu$. Because theorem 5.1 implies that assumptions (i), (ii), (iii) and (iv) hold and the mass conservation along the flow implies that assumption (vi) holds, the only thing to do is to verify the operator splitting assumption (v).

Let us write the following splitting:

$$L = L^s + L^{as}, \quad L^s = \nabla_v \cdot (\nabla_v f + f \nabla_v \phi), \quad L^{as} = \nabla_v (U f)$$

on E and similarly $\mathcal{L} = \mathcal{L}^s + \mathcal{L}^{as}$ on the space \mathcal{E} . The operator L^s is symmetric in E , since

$$\langle L^s f, g \rangle_E = - \int \nabla \left(\frac{f}{\mu} \right) \cdot \nabla \left(\frac{g}{\mu} \right) \mu dv$$

and the operators L^{as} and \mathcal{L}^{as} are both anti-symmetric in resp. E and \mathcal{E} by immediate computation (using **(FP2)** and that m is a function of ϕ).

One can compute

$$\langle \mathcal{L}^s f, f \rangle_{\mathcal{E}} = - \int \left| \nabla_v \left(\frac{f}{m} \right) \right|^2 m dv + \int f^2 \psi(v) m^{-1}(dv).$$

Consider $\chi \in C_c^\infty(\mathbb{R}^d)$, $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $B(0, 1)$, $\|\chi\|_\infty \leq 1$ and define $\chi_R(v) = \chi(R^{-1}v)$, $R \geq 1$. Consider R such that $\psi(v) \leq 0$ for $|v| \geq R$ and M such that $|\psi(v)| \leq M/2$ for $|v| \leq M$. We then define the decompositions $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with

$$\mathcal{A}f := \chi_R M f \quad \text{and} \quad \mathcal{B}f := \mathcal{L}f - \chi_R M f.$$

The operator \mathcal{A} is clearly bounded from \mathcal{E} into E thanks to the cutoff function, and for ξ with real part a :

$$\begin{aligned} \langle (\mathcal{B} - \xi)f, f \rangle_E &= \langle (\mathcal{L} - \xi)f, f \rangle_{\mathcal{E}} - \int \chi_R M f^2 m^{-1}(dv) \\ &= \langle (\mathcal{L}^s - \xi)f, f \rangle_{\mathcal{E}} - \int \chi_R M f^2 m^{-1}(dv) \\ &= - \int \left| \nabla_v \left(\frac{f}{m} \right) \right|^2 m dv + \int f^2 (\psi(v) - M \chi_R - \Re e \xi) m^{-1}(dv). \end{aligned}$$

We then use that $\psi(v) - M \chi_R \leq 0$ and $\psi(v) - M \chi_R \rightarrow -\infty$ as $|v| \rightarrow \infty$ to deduce that (taking maybe larger M and R)

$$(\psi(v) - M \chi_R - \Re e \xi) \leq -C < 0$$

which proves that $\mathcal{B} - a$ is dissipative. \square

5.2. The kinetic Fokker-Planck equation. Consider the equation

$$(5.4) \quad \partial_t f = Lf := \nabla_v \cdot (\nabla_v f + v f) - v \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_v f, \quad f(\cdot, 0) = f_{in}(\cdot),$$

for $f = f(t, x, v)$, $t \geq 0$, $x, v \in \mathbb{R}^d$ and $\Phi = \Phi(x) \in \mathbb{R}^d$.

Assume

(FP1') The potential $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies: $\pi(dx) := e^{-\Phi(x)} dx$ is a probability measure and (strong version of the Poincaré inequality) for some $\zeta > 0$ and for any $f = f(x) \in L^2(\pi^{-1})$ such that $\int f dx = 0$

$$(5.5) \quad \int \left| \nabla \left(\frac{f}{\pi} \right) \right|^2 \pi(dx) \geq \zeta \int f^2 (1 + |\nabla \Phi|^2) \pi^{-1}(dx).$$

Let us denote

$$\mu(v) := \frac{1}{(2\pi)^{-d/2}} e^{-|v|^2/2},$$

and

$$\gamma(x, v) := \pi \mu = \frac{1}{(2\pi)^{-d/2}} e^{-|v|^2/2 - \Phi(x)}$$

which is a probability measure on $\mathbb{R}^d \times \mathbb{R}^d$.

Let us first recall a recent result proved in [14, 13] for the kinetic Fokker-Planck equation in the space $L^2(\gamma^{-1})$.

Theorem 5.7. *Assume that Φ satisfies (FP1'). Then in the space $E = L^2(\gamma^{-1})$, the operator L satisfies: there exists $C > 0$ and $\lambda > 0$ such that $\Sigma(L) \subset \{z \in \mathbb{C} \mid \Re e(z) \leq -\lambda\} \cup \{0\}$, the null space of L is exactly $\text{Span}(\{\gamma\})$, and any solution to the initial value problem (5.4) for any initial datum $f_{in} \in E$ satisfies*

$$\forall t \geq 0, \quad \|f(t) - \gamma \langle f_{in} \rangle_E\|_E \leq C e^{-\lambda t} \|f_{in} - \gamma \langle f_{in} \rangle_E\|_E$$

where we denote

$$\langle f_{in} \rangle := \int f_{in} dx dv.$$

Let us denote $e(x, v) = |v|^2/2 + \Phi(x)$ and consider some weight $m = e^{-\theta(e)}$ with $\theta \in C^2$ and the associated Hilbert space $\mathcal{E} = L^2(m^{-1})$ such that $E \subset \mathcal{E}$. The new result is a spectral gap estimate in this larger space under some specific assumptions on m . Let us denote by \mathcal{L} the operator (5.4) in the space \mathcal{E} .

Assume that m satisfies:

(FP3') The function

$$\Psi(x, v) := \frac{1}{2m} \nabla_v \cdot \left(\gamma \nabla_v \left(\frac{m}{\gamma} \right) \right)$$

satisfies

$$\Psi(x, v) \xrightarrow{|e(x, v)| \rightarrow \infty} -\infty.$$

Theorem 5.8. *Assume that Φ satisfies **(FP1')** and m satisfies **(FP3')**. Then for any λ_1 with $0 < \lambda_1 < \lambda$ (where λ is given in Theorem 5.7) such $\Sigma(\mathcal{L}) \subset \{z \in \mathbb{C} \mid \Re(z) \leq -\lambda_1\} \cup \{0\}$ and there exists $C_{\lambda_1} \in [1, \infty)$ such that, for any initial data $f_{in} \in \mathcal{E}$, equation (5.4) has a solution $f \in \mathcal{E}$ such that*

$$\forall t \geq 0 \quad \|f(t) - \gamma \langle f_{in} \rangle\|_{\mathcal{E}} \leq C_{\lambda_1} e^{-\lambda_1 t} \|f_{in} - \gamma \langle f_{in} \rangle\|_{\mathcal{E}}$$

where we denote

$$\langle f_{in} \rangle := \int f_{in} dx dv.$$

Remarks 5.9. (1) *Again one could obtain the case of equality $\lambda_1 = \lambda$ for the rate of convergence by considering a more detailed expansion of the semigroup for the first eigenvalue.*

(2) *It would be straightforward (and simpler) to adapt the previous theorem in the case of the kinetic Fokker-Planck equation in the periodic torus $x \in T^d$.*

(3) *Another possible modification could be to extend the decay estimate to larger Banach L^1 -type weighted spaces but it would require specific additional work, since the operator \mathcal{A} in the decomposition below does not map L^1 into L^2 whatever the weights. We refer to section 7 where it is done for the linearized Boltzmann equation.*

Proof of Theorem 5.8. Again we prove that one can apply Corollary 4.2 with $n = 1$ (1-dimensional null space spanned by γ), $E = L^2(\gamma^{-1})$, $\mathcal{E} = L^2(m^{-1})$ and $\psi(f) = \langle f, \gamma \rangle_E \gamma$.

Let us write the following splitting:

$$L = L^s + L^{as}, \quad L^s = \nabla_v \cdot (\nabla_v f + v f), \quad L^{as} = -v \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_v f$$

on E and similarly $\mathcal{L} = \mathcal{L}^s + \mathcal{L}^{as}$ on the space \mathcal{E} . The operator L^s is symmetric in E , since

$$\langle L^s f, g \rangle_E = - \int \nabla \left(\frac{f}{\gamma} \right) \cdot \nabla \left(\frac{g}{\gamma} \right) \gamma dx dv$$

and the operators L^{as} and \mathcal{L}^{as} are both anti-symmetric in resp. E and \mathcal{E} by immediate computation (using that $-v \cdot \nabla_x e + \nabla_x \Phi \cdot \nabla_v e = 0$ and that m is a function of e).

One can compute

$$\langle \mathcal{L}^s f, f \rangle_{\mathcal{E}} = - \int \left| \nabla_v \left(\frac{f}{m} \right) \right|^2 m dx dv + \int f^2 \psi(v) m^{-1}(dx dv).$$

Consider $\chi \in C_c^\infty(\mathbb{R}^d)$, $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $B(0, 1)$, $\|\chi\|_\infty \leq 1$ and define $\chi_R(v) = \chi(R^{-1}v)$, $R \geq 1$. Consider R such that $\Psi(x, v) \leq 0$ for $|e(x, v)| \geq R$ and M such that $|\Psi| \leq M/2$ for $|e(x, v)| \leq M$. We then define the decomposition $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with

$$\mathcal{A}f := \chi_R(e) M f \quad \text{and} \quad \mathcal{B}f := \mathcal{L}f - \chi_R(e) M f.$$

The operator \mathcal{A} is clearly bounded from E into \mathcal{E} thanks to the cutoff function. For the invertibility of $\mathcal{B} - \xi$ we compute

$$\begin{aligned} \langle (\mathcal{B} - \xi)f, f \rangle_E &= \langle (\mathcal{L} - \xi)f, f \rangle_{\mathcal{E}} - \int \chi_R M f^2 m^{-1}(dx dv) \\ &= \langle (\mathcal{L}^s - \xi)f, f \rangle_{\mathcal{E}} - \int \chi_R(e) M f^2 m^{-1}(dx dv) \\ &= - \int \left| \nabla_v \left(\frac{f}{m} \right) \right|^2 m dv + \int f^2 (\Psi(e) - M \chi_R(e) - \Re e \xi) m^{-1}(dv). \end{aligned}$$

We then use that $\Psi(e) - M \chi_R(e) \leq 0$ and $\psi(e) - M \chi_R(e) \rightarrow -\infty$ as $|e| \rightarrow \infty$ to deduce that (taking maybe larger M and R)

$$(\Psi(e) - M \chi_R(e) - \Re e \xi) \leq -C < 0$$

which proves that $\mathcal{B} - a$ is dissipative. \square

6. APPLICATION TO THE LINEAR BOLTZMANN EQUATION

6.1. The space homogeneous linear Boltzmann equation. Let us consider as a first example the spatially homogeneous linear Boltzmann equation

$$(6.1) \quad \partial_t f = Lf := \int_{\mathbb{R}^d} (k(v, v_*) f(v_*) - k(v_*, v) f(v)) dv_*$$

for $f = f(t, v)$, $t \geq 0$, $v \in \mathbb{R}^d$ and $k(v, v_*) \geq 0$.

Assume

(LBE1) There is $\mu = \mu(v)$ probability measure such that

$$k(v, v_*) = b(v, v_*) \mu(v) \quad \text{and} \quad 0 < b_* \leq b(v, v_*) = b(v_*, v) \leq b^* < \infty.$$

Remark 6.1. *More general setting without detailed balance could be considered within framework. We do not try to optimize the assumptions here but rather show how our method apply to Boltzmann equations.*

It is well-known that

$$\langle Lf, f \rangle_E = -\frac{1}{2} \int b(v, v_*) \left(\frac{f(v)}{\mu(v)} - \frac{f(v_*)}{\mu(v_*)} \right)^2 dv dv_*$$

and that the homogeneous linear Boltzmann equation with such a kernel has a spectral gap in the Hilbert space $E = L^2(\mu^{-1})$:

Theorem 6.2. *Assume that k satisfies **(LB1)**. Then in the space $E = L^2(\mu^{-1})$, the operator L satisfies: there exists $\lambda > 0$ such that $\Sigma(L) \subset \{z \in \mathbb{C} \mid \Re(z) \leq -\lambda\} \cup \{0\}$, the null space of L is exactly $\mathbb{R}\mu$, and any solution to the initial value problem (6.1) for any initial datum $f_{in} \in E$ satisfies*

$$\forall t \geq 0, \quad \|f(t) - \mu \langle f_{in} \rangle\|_E \leq e^{-\lambda t} \|f_{in} - \mu \langle f_{in} \rangle\|_E$$

where we denote

$$\langle f_{in} \rangle := \int f_{in} dv.$$

Let us now consider a measurable weight function $m(v) > 0$ and the associated Hilbert space $\mathcal{E} = L^2(m^{-1})$ such that $E \subset \mathcal{E}$. The new result is a spectral gap estimate in this larger space under some specific assumptions on m . Let us denote by \mathcal{L} the operator (6.1) in the space \mathcal{E} .

Assume that m satisfies:

(LB2) The L^1 norm is bounded from above by the norm \mathcal{E} times some constant and the norm \mathcal{E} is controlled from above by the norm E times some constant.

Theorem 6.3. *Assume that k satisfies **(LB1)** and m satisfies **(LB2)**. Then there exists a positive constant λ_1 with $0 < \lambda_1 \leq \lambda$ (where λ is given in Theorem 6.2) such $\Sigma(\mathcal{L}) \subset \{z \in \mathbb{C} \mid \Re(z) \leq -\lambda_1\} \cup \{0\}$ and there exists $C_{\lambda_1} \in [1, \infty)$ such that, for any initial data $f_{in} \in \mathcal{E}$, equation (6.1) has a solution $f \in \mathcal{E}$ such that*

$$\forall t \geq 0 \quad \|f(t) - \mu \langle f_{in} \rangle\|_{\mathcal{E}} \leq C_{\lambda_1} e^{-\lambda_1 t} \|f_{in} - \mu \langle f_{in} \rangle\|_{\mathcal{E}}$$

where we denote

$$\langle f_{in} \rangle := \int f_{in} dv.$$

Proof of Theorem 6.3. We only sketch the proof which is similar to the one of Theorem 5.5 for the spatially homogeneous Fokker-Planck equation.

We use the decomposition $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with

$$\mathcal{A}f := \int k(v, v_*) f(v_*) dv_* \quad \text{and} \quad \mathcal{B}f := \int k(v, v_*) f(v) dv_*.$$

The fact that \mathcal{A} is bounded from \mathcal{E} to E is clear:

$$\begin{aligned} \|\mathcal{A}f\|_E^2 &= \int (\mathcal{A}f)^2 \mu^{-1} \leq \int \left(\int b^* |f(v_*)| dv_* \right)^2 \mu(v) dv \\ &\leq (b^*)^2 \|f\|_{L^1}^2 \leq C \|f\|_{\mathcal{E}}^2 \end{aligned}$$

Finally dissipativity of \mathcal{B} is clear from the fact that $\mathcal{B}f = \nu(v) f$ with $\nu \geq \nu_0 > 0$. \square

6.2. The space inhomogeneous linear Boltzmann equation in the torus.

Consider

$$(6.2) \quad \partial_t f = Lf := \int_{\mathbb{R}^d} (k(v, v_*) f(v_*) - k(v_*, v) f(v)) dv_* - v \cdot \nabla_x f$$

for $f = f(t, x, v)$, $t \geq 0$, $x \in \mathbb{T}^d$ (d -dimensional torus), $v \in \mathbb{R}^d$ and $k(v, v_*) \geq 0$.

Assume **(LB1)** on k again.

Remark 6.4. *Again more general settings could be considered: most importantly the case of a confining potential in $x \in \mathbb{R}^d$. We leave this interesting question for a follow-up paper.*

Let us denote

$$\mu(v) := \frac{1}{(2\pi)^{-d/2}} e^{-|v|^2/2},$$

and assume without loss of generality that the torus has unit measure. Then μ is a probability measure on $\mathbb{R}^d \times \mathbb{R}^d$.

Let us first recall a recent result proved in [14, 13] for the kinetic Fokker-Planck equation in the space $L^2(\mu^{-1})$.

Theorem 6.5. *Assume that k satisfies **(LB1)** and Φ satisfies **(LB3)**. Then in the space $E = L^2(\gamma^{-1})$, the operator L satisfies: there exists $\lambda > 0$ and $C > 0$ such that $\Sigma(L) \subset \{z \in \mathbb{C} \mid \operatorname{Re}(z) \leq -\lambda\} \cup \{0\}$, the null space of L is exactly $\mathbb{R}\mu$, and any solution to the initial value problem (6.2) for any initial datum $f_{in} \in E$ satisfies*

$$(6.3) \quad \forall t \geq 0, \quad \|f(t) - \mu \langle f_{in} \rangle\|_E \leq C e^{-\lambda t} \|f_{in} - \mu \langle f_{in} \rangle\|_E$$

where we denote

$$\langle f_{in} \rangle := \int f_{in} dx dv.$$

Consider some weight $m = m(v)$ and the associated Hilbert space $\mathcal{E} = L^2(m^{-1})$ such that $E \subset \mathcal{E}$. The new result is a spectral gap estimate in this larger space under some specific assumptions on m . Let us denote by \mathcal{L} the operator (6.2) in the space \mathcal{E} .

Assume that m satisfies:

(LB2') The L^1 norm (in x and v) is bounded from above by the norm \mathcal{E} times some constant and the norm \mathcal{E} (in x and v) is controlled from above by the norm E (in x and v) times some constant.

Theorem 6.6. *Assume that k satisfies **(LB1)**, Φ satisfies **(LB3)** and m satisfies **(LB2')**. Then for any λ_1 with $0 < \lambda_1 < \lambda$ (where λ is given in Theorem 6.5) such $\Sigma(\mathcal{L}) \subset \{z \in \mathbb{C} \mid \operatorname{Re}(z) \leq -\lambda_1\} \cup \{0\}$ and there exists $C_{\lambda_1} \in [1, \infty)$ such that, for any initial data $f_{in} \in \mathcal{E}$, equation (5.4) has a solution $f \in \mathcal{E}$ such that*

$$\forall t \geq 0 \quad \|f(t) - \mu \langle f_{in} \rangle\|_{\mathcal{E}} \leq C_{\lambda_1} e^{-\lambda_1 t} \|f_{in} - \mu \langle f_{in} \rangle\|_{\mathcal{E}}$$

where we denote

$$\langle f_{in} \rangle := \int f_{in} dx dv.$$

Remarks 6.7. (1) *As before the case of equality $\lambda_1 = \lambda$ for the rate of convergence could be obtained by refining the expansions on the semigroup.*

(2) *This would be again possible to extend the decay estimate to larger Banach L^1 -type weighted spaces, see section 7 where it is done for the linearized Boltzmann equation.*

Proof of Theorem 6.3. We only sketch the proof which is exactly similar to the one of Theorem 5.8 for the spatially inhomogeneous Fokker-Planck equation (it is in fact

even simpler since the confinement is ensured by the periodic domain and not by some external potential).

Define the decomposition $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with

$$\mathcal{A}f := \int k(v, v_*) f(v_*) dv_* \quad \text{and} \quad \mathcal{B}f := \int k(v, v_*) f(v) dv_* - v \cdot \nabla_x f.$$

Then the operator \mathcal{A} is again clearly bounded from \mathcal{E} into E , and the dissipativity of \mathcal{B} is clear from

$$\langle \mathcal{B}f, f \rangle_{\mathcal{E}} = \nu(v) \|f\|_{\mathcal{E}}^2$$

since $v \cdot \nabla_x$ is anti-symmetric in \mathcal{E} as m is a function of v . \square

7. APPLICATION TO THE NONLINEAR BOLTZMANN EQUATION

7.1. The Boltzmann equation. Consider the Boltzmann equation for hard potentials with cutoff (including hard spheres), which writes in the spatially homogeneous case

$$(7.4) \quad \partial_t f = Q(f, f) - v \cdot \nabla_x f$$

for $f = f(t, x, v)$, $x \in \mathbb{T}^d$ (d -dimensional torus), $v \in \mathbb{R}^d$ and

$$(7.5) \quad Q(f, f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \left[f(v'_*) f(v') - f(v) f(v_*) \right] b(\cos \theta) |v - v_*|^\zeta dv_* d\sigma$$

with

$$v' = \frac{v + v_*}{2} + \sigma \frac{|v - v_*|}{2}, \quad v'_* = \frac{v + v_*}{2} - \sigma \frac{|v - v_*|}{2},$$

and where b is a positive integrable function on the sphere \mathbb{S}^{d-1} with $\cos \theta = \sigma \cdot (v - v_*)/|v - v_*|$, and $\zeta \in (0, 1]$.

Assume without loss of generality that the torus has volume one. Then the unique global equilibrium (see [12] for instance) is

$$\mu(v) := \frac{1}{(2\pi)^{-d/2}} e^{-|v|^2/2}.$$

The linearized equation around this equilibrium is

$$(7.6) \quad \partial_t f = L(f) = C(f) - v \cdot \nabla_x f$$

for $f = f(t, x, v)$, $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$ and

$$C(f) := \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \left[\mu(v'_*) f(v') + \mu(v') f(v'_*) - \mu(v_*) f(v) - \mu(v) f(v_*) \right] b(\cos \theta) |v - v_*|^\zeta dv_* d\sigma.$$

It is well-known that L is self-adjoint non-positive and has a spectral gap in the Hilbert space $\bar{E} = L^2(\mu^{-1} dv)$ (see [7, 15, 16] and then [5] for explicit estimates). This result would be easily extended to $H^1(\mu^{-1} dv)$.

In [22], this spectral gap estimate on the semigroup was extended to the space $\bar{\mathcal{E}} := L^1(e^{a|v|^s} dv)$ for $a > 0$ and $s \in (0, 2)$. Let us denote \mathcal{C} and \mathcal{L} the operators C and L when considered resp. in $L^1(e^{a|v|^s} dv)$ and $\mathcal{E}_0 := L^1(e^{a|v|^s} dx dv)$. Let us denote $E := H^1(\mu^{-1} dx dv)$.

We shall extend the estimate of [22] to the full inhomogeneous case in the torus. First we shall recall the decomposition of L devised in [22], then recall an hypocoercivity property in the torus proved in [24], then introduce a diffusive approximation and finally combine these preliminary steps with Corollary 4.2 (in the Banach space version) in order to extend decay estimates in $L^1(e^{a|v|^s} dx dv)$ in the torus, first for the diffusive approximation, and second for the original equation by relaxing the diffusion parameter (a key point is that the estimates are made uniform in terms of the diffusion with the help of the hypocoercivity theory). In the last subsection we shall turn to the nonlinear Boltzmann equation.

7.2. The decomposition of [22]. Let us state the key decomposition result which was proved in [22, Propositions 2.1-2.3-2.4-2.5-2.6]:

Proposition 7.8. *There exists a decomposition*

$$(7.7) \quad \begin{cases} Cf = \bar{A}f + \bar{B}f, \\ Cf = \bar{\mathcal{A}}f + \bar{\mathcal{B}}f \end{cases}$$

such that \bar{A} and $\bar{\mathcal{A}}$ (resp. \bar{B} and $\bar{\mathcal{B}}$) coincide on $\bar{E} \cap \bar{\mathcal{E}}$ and such that $\bar{\mathcal{A}}$ is bounded from $\bar{\mathcal{E}}$ into \bar{E} and also from $\bar{\mathcal{E}}$ into $H^1(\mu^{-1} dv)$ and $\bar{B} - \xi$ and $\bar{\mathcal{B}} - \xi$ are dissipative and invertible resp. in \bar{E} (or $H^1(\mu^{-1} dv)$) and $\bar{\mathcal{E}}$, for any $\Re \xi = a$ for some $a < 0$.

7.3. Hypocoercivity in the small Hilbert space. We have the following theorem from [24, Theorem 1.1]:

Theorem 7.9. *In the space $E = H^1(\mu^{-1} dx dv)$, the operator L satisfies: there exists $\lambda > 0$ such that $\Sigma(L) \subset \{z \in \mathbb{C} \mid \Re(z) \leq -\lambda\} \cup \{0\}$, the null space of L is spanned by μ , $v_i \mu$, $1 \leq i \leq d$ and $|v|^2 \mu$, and any solution to the initial value problem (7.6) for any initial datum $f_{in} \in E$ satisfies*

$$\forall t \geq 0, \quad \|f(t) - \Pi f_{in}\|_E \leq C e^{-\lambda t} \|f_{in} - \Pi f_{in}\|_E$$

for some $C \geq 1$, where we denote Πf_{in} the orthogonal projection in $L^2(\mu^{-1} dx dv)$ onto the null space of L .

7.4. A diffusive approximation. Let us introduce the following approximate equation:

$$(7.8) \quad \partial_t f = L_\eta(f) = C(f) - v \cdot \nabla_x f + \eta \Delta_x f$$

for any $\eta > 0$ and the corresponding approximate operator \mathcal{L}_η in \mathcal{E} . Then Theorem 7.9 can be extended in the following way to this approximate equation:

Theorem 7.10. *In the space $E = H^1(\mu^{-1} dx dv)$, there exists $\lambda > 0$ such that for any $\eta \in (0, 1]$, the operator L_η satisfies: $\Sigma(L) \subset \{z \in \mathbb{C} \mid \Re(z) \leq -\lambda\} \cup \{0\}$, the null space of L is spanned by μ , $v_i \mu$, $1 \leq i \leq d$ and $|v|^2 \mu$, and any solution to the initial value problem (7.8) for any initial datum $f_{in} \in E$ satisfies*

$$\forall t \geq 0, \quad \|f(t) - \Pi f_{in}\|_E \leq C e^{-\lambda t} \|f_{in} - \Pi f_{in}\|_E$$

for some $C \geq 1$, where we denote Πf_{in} the orthogonal projection in $L^2(\mu^{-1} dx dv)$ onto the null space of L .

Remark 7.11. *Note the crucial point: the estimate from below λ on the spectral gap is independent of the viscosity parameter η .*

Proof of Theorem 7.10. Here we need to go back to the proof of [24, Theorem 1.1] in [24, Section 2]. The core of the proof there is the construction of a norm equivalent to H^1 in the form

$$\begin{aligned} \|f\|_*^2 &= A \|f\|_{L^2(\mu^{-1})}^2 + \alpha \|\nabla_x f\|_{L^2(\mu^{-1})}^2 \\ &\quad + \beta \|(\nabla_v + v/2)f\|_{L^2(\mu^{-1})}^2 + \gamma \langle \nabla_x f, (\nabla_v + v/2)f \rangle_{L^2(\mu^{-1})} \end{aligned}$$

by choosing $\beta > 0$ big enough, then $A > 0$ big enough, then $\gamma > 0$ enough, then $\alpha > 0$ big enough (in the paper [24] all computations are done for $h = \mu^{-1/2}f$ in L^2 without weight, and when translating these computations in $L^2(\mu^{-1})$ one should be careful with $\nabla_v h = \mu^{-1/2}(\nabla_v f + v/2)f$).

Then let us compute the action of the term $\eta \Delta_x f$ in the time-derivative of this norm:

$$\begin{aligned} \eta \langle \Delta_x f, f \rangle_* &= -A \eta \|\nabla_x f\|_{L^2(\mu^{-1})}^2 - \alpha \eta \|\nabla_x^2 f\|_{L^2(\mu^{-1})}^2 \\ &\quad - \beta \eta \|\nabla_x(\nabla_v + v/2)f\|_{L^2(\mu^{-1})}^2 - \gamma \eta \langle \nabla_x^2 f, \nabla_x(\nabla_v + v/2)f \rangle_{L^2(\mu^{-1})}^2. \end{aligned}$$

This only term of the right-hand side which is not non-positive is the fourth one, and by taking α big enough (possibly taking an α slightly larger than in the original proof in [24]), it can be controlled by the second and third terms of the right-hand side, *uniformly in terms of η* . Finally the choice of α big enough is also uniform in η . This concludes the proof. \square

7.5. Localization of the spectrum in a larger Banach space for the approximate operator. Let us consider for $\ell > 1 + d/2$ the Banach space

$$\mathcal{E}_\ell := W_x^{\ell,1} L_v^1(e^{a|v|^s} dx dv)$$

defined by the norm

$$\|f\|_{\mathcal{E}_\ell} := \sum_{i=0}^{\ell} \int_{\mathbb{T}^d \times \mathbb{R}^d} |\nabla_x^i f| e^{a|v|^s} dx dv.$$

In this subsection we shall prove that the spectrum is unchanged from E to \mathcal{E}_ℓ . This will allow to justify the estimates on the resolvent *with loss of derivatives* which will be done in the next subsection.

Proposition 7.12. *In the space \mathcal{E}_ℓ , for any $\eta \in (0, 1]$, the operator \mathcal{L}_η satisfies: $\Sigma(\mathcal{L}_\eta) \subset \{\xi \in \mathbb{C} \mid \Re e(\xi) \leq -\lambda\} \cup \{0\}$ (where λ is given by Theorem 7.10) and the null space of L_η is spanned by $\{\mu, v_1 \mu, \dots, v_d \mu, |v|^2 \mu\}$.*

Proof of Theorem 7.12. We start from the decompositions

$$L_\eta = A + B_\eta, \quad \mathcal{L}_\eta = \mathcal{A} + \mathcal{B}_\eta$$

with

$$\begin{cases} Af := \bar{A}f \quad \text{and} \quad B_\eta f := \bar{B}f - v \cdot \nabla_x f + \eta \Delta_x f, \\ \mathcal{A}f := \bar{\mathcal{A}}f \quad \text{and} \quad \mathcal{B}_\eta f := \bar{\mathcal{B}}f - v \cdot \nabla_x f + \eta \Delta_x f. \end{cases}$$

Then first we prove that \mathcal{A} is compact relatively to \mathcal{B}_η . This is a trivial consequence of the following facts: (1) the operator \bar{A} regularizes in v and cuts large velocities $v \in \mathbb{R}^d$, (2) the Dirichlet form of \mathcal{B}_η controls the L^1 norm of the gradient in $x \in \mathbb{T}^d$.

Proceeding as in the proof of [22, Proposition 3.4] (variant of Weyl's theorem [18, Chapter 4, Section 5] in Banach spaces based on the classification of the spectrum by the Fredholm theory), we deduce that the essential spectrum of \mathcal{L}_η is the same as the one of \mathcal{B}_η . Moreover \mathcal{B}_η is dissipative with constant λ independent of η since

$$\langle (\mathcal{B}_\eta - \xi) f, f \rangle_{\mathcal{E}} \leq \sum_{i=0}^{\ell} \int \left(\int (\bar{\mathcal{B}} - \xi) (\nabla_x^i f) \operatorname{sign} (\nabla_x^i f) e^{a|v|^s} dv \right) dx$$

and for each i and any fixed $x \in \mathbb{T}^d$ we use the dissipativity of $\bar{\mathcal{B}}$ in $L^1(e^{a|v|^s} dv)$. Hence the essential spectrum of \mathcal{L}_η is included in $\{z \in \mathbb{C} \mid \Re(z) \leq -\lambda\}$.

Then let us show that the discrete spectrum of L_η and \mathcal{L}_η are the same in the region $\{\xi \in \mathbb{C} \mid \Re(\xi) > -\lambda\}$: consider an isolated eigenvalue ξ of \mathcal{L}_η in \mathcal{E}_ℓ with multiplicity 1 (in case of higher multiplicity a similar argument can be performed on each Jordan block) and write the eigenvalue equation

$$\mathcal{A}f = \xi f - \mathcal{B}_\eta f$$

with $\Re(\xi) > -\lambda$. Then by using the dissipativity of \mathcal{B}_η in \mathcal{E}_ℓ and E and the fact that \mathcal{A} is bounded from \mathcal{E}_ℓ to E (it is immediate since \mathcal{A}_0 is bounded from $L^1(e^{a|v|^s} dv)$ into $H_v^1(\mu^{-1})$ and by Sobolev embedding $W_x^{\ell,1} \rightarrow H_x^1$ in the torus as $\ell > 1 + d/2$), we deduce that $f \in E$ and therefore ξ is an eigenvalue of L_η in E . The converse implication (that the eigenvalues of L_η are eigenvalues of \mathcal{L}_η) is trivial from the fact that these eigenvalues are known in an explicit ways (note that a more general argument could be the use of the regularizing properties in v of A , and in x of $\eta \Delta_x$). \square

7.6. Hypocoercivity in a larger Banach space.

Theorem 7.13. *In the space \mathcal{E}_ℓ with $\ell > d/2 + 3$, the semigroup of the operator \mathcal{L} satisfies: for any $\lambda_1 \in (0, \lambda)$, any solution to the initial value problem (7.6) for any initial datum $f_{in} \in \mathcal{E}_\ell$ satisfies*

$$\forall t \geq 0, \quad \|f(t) - \Pi f_{in}\|_{\mathcal{E}_0} \leq C_1 e^{-\lambda_1 t} \|\langle v \rangle^2 (f_{in} - \Pi f_{in})\|_{\mathcal{E}_\ell}$$

for some $C_1 \geq 1$, where we denote Π the orthogonal projection in $L^2(\mu^{-1})$ onto the null space of L .

Remark 7.14. *As before the optimal rate $\lambda_1 = \lambda$ is reachable by refining the expansion of the semigroup.*

Proof of Theorem 7.13. We first work at the level of the approximate operator \mathcal{L}_η , and we start from the already used representation (see for instance [26, Chapter 1,

Section 1.7] and [31, Theorem 1.1])

$$(7.9) \quad e^{t\mathcal{L}_\eta} (f - \tilde{\Pi}f) = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} e^{zt} \mathcal{R}_\eta(z) f dz$$

$$= \lim_{M \rightarrow \infty} \frac{1}{2i\pi} \int_{\lambda_1 - iM}^{\lambda_1 + iM} e^{zt} \mathcal{R}_\eta(z) f dz := \lim_{M \rightarrow \infty} \frac{1}{2i\pi} I_{M,\eta}(f)$$

for any $\lambda_1 > \lambda$ and where \mathcal{R}_η is the resolvent of \mathcal{L}_η in \mathcal{E}_ℓ and $\tilde{\Pi}$ the spectral projector of \mathcal{L}_η for the eigenvalue 0 in \mathcal{E}_ℓ ; this is justified in \mathcal{E}_ℓ by the localization of the spectrum of \mathcal{L}_η in Proposition 7.12.

Consider again the decompositions $L_\eta = A + B_\eta$ and $\mathcal{L}_\eta = \mathcal{A} + \mathcal{B}_\eta$ with

$$\begin{cases} Af := \bar{A}f & \text{and} & B_\eta f := \bar{B}f - v \cdot \nabla_x f + \eta \Delta_x f, \\ \mathcal{A}f := \bar{\mathcal{A}}f & \text{and} & \mathcal{B}_\eta f := \bar{\mathcal{B}}f - v \cdot \nabla_x f + \eta \Delta_x f. \end{cases}$$

We know that for $\lambda_1 \in (0, \lambda)$, the resolvent $\mathcal{R}_\eta(\xi)$ of \mathcal{L}_η with $\Re \xi = \lambda_1$ is well-defined in \mathcal{E}_ℓ , with the representation (7.9). Moreover we have seen in the proof of Proposition 7.12 that \mathcal{A} is bounded from \mathcal{E}_ℓ into E and $\mathcal{B}_\eta - \xi$ is dissipative for $\Re \xi = \lambda_1$ *uniformly in terms of η* . Therefore we can write the factorization formula as in Theorem 2.1:

$$\mathcal{R}_\eta(\xi) = -(\mathcal{B}_\eta - \xi)^{-1} + R_\eta(\xi) \mathcal{A} (\mathcal{B}_\eta - \xi)^{-1}$$

where $R_\eta(\xi)$ is the resolvent of L_η in E .

We now estimate $I_{M,\eta}$ using that

$$\begin{aligned} \|\mathcal{R}_\eta(\xi)\|_{\mathcal{E}_\ell \rightarrow \mathcal{E}_0} &\leq \|(\mathcal{B}_\eta - \xi)^{-1}\|_{\mathcal{E}_\ell \rightarrow \mathcal{E}_\ell} \\ &\quad + \|R_\eta(\xi)\|_{E \rightarrow \mathcal{E}_0} \|\mathcal{A}\|_{\mathcal{E}_{\ell-2} \rightarrow E} \|(\mathcal{B}_\eta - \xi)^{-1}\|_{\mathcal{E}_\ell \rightarrow \mathcal{E}_{\ell-2}} \end{aligned}$$

where we have used that the norm of \mathcal{E}_0 is controlled by the norm of E and the fact that for because $g \in \mathcal{E}_{\ell-2}$ we have $\mathcal{A}g \in W_x^{\ell-2,1}(H_v^1) \subset H_x^{[\ell-d/2]}(H_v^1) \subset E$ by $\ell - d/2 > 1$.

Finally by using

$$\|(\mathcal{B}_\eta - \xi) f\|_{\mathcal{E}_{\ell-2}} \leq \frac{C}{|\Im m \xi|^2} \|\langle v \rangle^2 f\|_{\mathcal{E}_\ell}$$

(loss of twice the graph norm) for some constant C *independent of η* , we deduce that for $\ell > d/2 + 3$:

$$\|\mathcal{R}_\eta(\xi) f\|_{\mathcal{E}_0} \leq \frac{C}{|\Im m \xi|^2} \|\langle v \rangle^2 f\|_{\mathcal{E}_\ell}$$

for some constant C *independent of η* and therefore we obtain the decay of the semigroup of \mathcal{L}_η by the representation formula (7.9).

Finally it is clear (from a Gronwall estimate for instance) that the semigroup of \mathcal{L}_η pointwise converges to the semigroup of \mathcal{L} in \mathcal{E}_ℓ as $\eta \rightarrow 0$. Therefore we can let the parameter η goes to 0 since the estimate does not depend on η , and we obtain the desired estimate for the semigroup of L , which concludes the proof. \square

7.7. Relaxation for the nonlinear Boltzmann equation. Let us first give a theorem combining existing results:

Theorem 7.15 ([12]). *Let us $(f_t)_{t \geq 0}$ be a nonnegative nonzero smooth solution of (7.4) such that for k, s big enough*

$$\sup_{t \geq 0} (\|f\|_{H^k(\mathbb{T}^d \times \mathbb{R}^d)} + \|f\|_{L^1(1+|v|^s)}) \leq C < +\infty.$$

Then ([21, Theorem 1.1]) for some $\tau, K, A, q > 0$

$$\inf_{t \geq \tau, x \in \mathbb{T}^d} f_t(x, v) \geq K e^{A|v|^s}$$

and there exists an explicit function $\varphi = \varphi(t)$ which goes to 0 as t goes to infinity such that

$$\forall t \geq 0, \quad \|f_t - M\|_{W_x^{k,1}(L^1(e^{a|v|^s}))} \leq \varphi(t)$$

where $M = \Pi f$ is the Gaussian equilibrium associated with f .

Proof of Theorem [21]. This theorem is an immediate consequence of [12, Theorem 2] together with lower bounds proved in [21, Theorem 1.1] and the proof that the moments $L^1(e^{a|v|^s})$, $s \in (0, 1)$ appears and are then uniformly bounded in [22, Lemma 4.7] and [20, Proposition 3.2] (in the latter paper the proof was done in the homogeneous case, however one can integrate in x these moment estimates by using the uniform bounds in x from above and below on f). \square

Then we write the nonlinear equation for the fluctuation $f = M + h$ as

$$\partial_t h = \mathcal{L}h + Q(h, h)$$

and we use that (for some k big enough in terms of the dimension)

$$\|Q(h, h)\|_{W_x^{k,1}(L^1(e^{a|v|^s}))} \leq C \|h\|_{W_x^{k,1}(L^1(e^{a|v|^s}))} \|h(1 + |v|^\gamma)\|_{W_x^{k,1}(L^1(e^{a|v|^s}))}.$$

The latter inequality is a consequence of the L^1 theory:

$$\|Q(h, h)\|_{L^1(e^{a|v|^s})} \leq C \|h\|_{L^1(e^{a|v|^s})} \|h(1 + |v|^\gamma)\|_{L^1(e^{a|v|^s})}$$

(see for instance [20]) together with the Leibniz formula for the derivatives in x , and Sobolev embeddings.

Finally by elementary Gronwall estimates we obtain the following theorem:

Theorem 7.16. *Let us $(f_t)_{t \geq 0}$ be a nonnegative nonzero smooth solution of (7.4) such that for k, s big enough*

$$\sup_{t \geq 0} (\|f\|_{H^k(\mathbb{T}^d \times \mathbb{R}^d)} + \|f\|_{L^1(1+|v|^s)}) \leq C < +\infty.$$

Then for some $C > 0$ and $\lambda_1 \in (0, \lambda)$ (where λ is given by the rate of decay of the linearized semigroup in $W_x^{k,1}(L^1(e^{a|v|^s}))$)

$$\forall t \geq 0, \quad \|f_t - M\|_{W_x^{k,1}(L^1(e^{a|v|^s}))} \leq C e^{-\lambda t}$$

where $M = \Pi f$ is the Gaussian equilibrium associated with f .

In particular we deduce a proof of the exponential decay of the relative entropy:

$$\forall t \geq 0, \quad \int f_t \log \frac{f_t}{M} dx dv \leq C e^{-\lambda t}.$$

7.8. **Other applications.** Here are some natural questions left open by this study:

- The extension of Theorem 7.16 to the Boltzmann equation for non-cutoff hard and moderately soft potentials (in the case of long-range interactions)
- The extension of Theorem 7.16 in the case of a confining potential as in Theorem 5.8 for the kinetic Fokker-Planck equation.

Moreover two other interesting applications of our new estimates should be looked in the futur:

- To develop a complete theory of nonlinear stability around the Maxwellian equilibrium in $W_x^{k,1}(L^1(e^{a|v|^s}))$.
- To develop a quantitative close-to-homogeneous theory for the Boltzmann equation, improving on the non-constructive results [1]. This is allowed by our estimates since the functional space of the nonlinear stability (1) is compatible with the nonlinear flow of the spatially homogeneous Boltzmann equation.

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