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**SEMIPARAMETRIC ESTIMATION OF A BINARY RESPONSE
MODEL WITH A CHANGE-POINT DUE TO A COVARIATE
THRESHOLD***

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February 2008

Abstract

This paper is concerned with semiparametric estimation of a threshold binary response model. The estimation method considered in the paper is semiparametric since the parameters for a regression function are finite-dimensional, while allowing for heteroskedasticity of unknown form. In particular, the paper considers Manski (1975, 1985)'s maximum score estimator. The model in this paper is irregular because of a change-point due to an unknown threshold in a covariate. This irregularity coupled with the discontinuity of the objective function of the maximum score estimator complicates the analysis of the asymptotic behavior of the estimator. Sufficient conditions for the identification of parameters are given and the consistency of the estimator is obtained. It is shown that the estimator of the threshold parameter, γ_0 , is n^{-1} -consistent and the estimator of the remaining regression parameters, θ_0 , is $n^{-1/3}$ -consistent. Furthermore, we obtain the asymptotic distribution of the estimator. It turns out that both estimators $\hat{\gamma}$ and $\hat{\theta}$ are oracle-efficient in that $n(\hat{\gamma}_n - \gamma_0)$ and $n^{1/3}(\hat{\theta}_n - \theta_0)$ converge weakly to the distributions to which they would converge weakly if the other parameter(s) were known.

Key Words: Binary response model, maximum score estimation, nonlinear random utility models, semiparametric estimation, threshold regression.

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1 Introduction

A binary response model is very commonly used in a number of fields. In this model, an observable binary outcome Y is modelled typically as $Y = 1(Y^* \geq 0)$, where $1(A)$ denotes the indicator function, i.e., $1(A) = 1$ if A is true and zero otherwise, and Y^* is an unobservable continuous variable that determines the binary outcome Y (see e.g. Manski, 1988). In most applications, Y^* has the following form:

$$(1.1) \quad Y^* = X' \beta_0 + U,$$

where X is a vector of observed random variables including an intercept term, β_0 is a vector of unknown parameters, and U is an unobserved random variable.

In this paper, we consider a threshold regression model for Y^* instead of the linear regression model (1.1). In particular, the model is

$$(1.2) \quad Y^* = \begin{cases} W_1' \beta_{10} + W_2' \beta_{20} + U, & \text{if } D \leq \gamma_0, \\ W_1' \beta_{10} + W_2' \beta_{30} + U, & \text{if } D > \gamma_0, \end{cases}$$

or equivalently

$$(1.3) \quad Y^* = W' \beta_0 + Z' \delta_0 1 \{D > \gamma_0\} + U,$$

where $W = (W_1', W_2)'$, $Z = W_2$, $\beta_0 = (\beta_{10}', \beta_{20}')'$, and $\delta_0 = \beta_{30} - \beta_{20}$. In particular, D is the observed threshold variable and γ_0 is the threshold parameter. We denote by X the set of regressors $(W', D)'$ and by $\theta_0 = (\beta_0', \delta_0)'$ the set of regression parameters excluding γ_0 . The threshold variable D can be an element of W . The literature on the threshold model (also called change-point model, two-phase regression, or sample splitting) is vast. It has been studied for autoregressive models (e.g. Tong, 1990; Chan, 1993), for linear regression models (e.g. Hansen, 2000; Koul and Qian, 2002), for nonparametric models (e.g. Delgado and Hidalgo, 2000), and more recently for transformation models (e.g. Pons, 2003; Kosorok and Song, 2007).

Threshold binary regression models have a wide variety of applications. For example, in economics, finance, and management, random utility models that are nonlinear in income

and/or price are commonly employed (see, e.g. Herriges and Kling, 1999; Dagsvik and Karlstrom, 2005). Terui and Dahana (2006) adopted an asymmetric market response model with the threshold due to price based on consumer behavior theory. The effect of financial contagion (see, e.g. Forbes and Rigobon, 2002) can also be modelled as a discontinuous threshold effect as in Pesaran and Pick (2007). In biostatistics, dose-response models are modelled with threshold parameters (see, e.g. Cox, 1987; Schwartz et al., 1995). In epidemiology, logistic regressions with change-points are used to model the relationship between the continuous exposure variable and disease risk (see Pastor and Guallar, 1998; Pastor-Barriuso et al., 2003). While the threshold model is easier to interpret than more complex nonlinear models, the irregular feature of the model makes estimation of the unknown change-point complicated. To our best knowledge, all existing methods for estimating a change-point according to a covariate threshold in binary response models assumes that the distribution of U in (1.2) belongs to a family of parametric distributions.

The purpose of this paper is to develop a method for estimating (θ_0, γ_0) in (1.2) without imposing the parametric distribution on U . First, we establish identification of (θ_0, γ_0) , allowing for conditional heteroskedasticity of unknown form, under the conditional median independence assumption. This is substantially weaker than the typical assumption such that the distribution of U is parametric and U and X are independent. The conditional heteroskedasticity is particularly important in random utility models as shown in Brown and Walker (1989). Manski (1988) has shown that if the distribution of X has sufficiently rich support, the finite-dimensional parameters in a binary regression model with (1.1) are identified up to scale. As in Manski (1988), (θ_0, γ_0) in (1.2) is identified (up to scale with respect to θ_0) under some regularity conditions, which will be given in Section 2. Once (θ_0, γ_0) is identified via the conditional median independence assumption, a natural estimation strategy for (θ_0, γ_0) in (1.2) is to apply Manski (1975, 1985)'s maximum score estimator to the threshold model (1.2). Section 3 describes the corresponding maximum score estimator and section 4 establishes the consistency of the estimator.

We also develop convergence rates and asymptotic distributions, which are highly non-standard due to mixed irregularities of the model and maximum score estimation. In Section 5, it is shown that $\hat{\gamma}_n$ is n^{-1} -consistent and $\hat{\theta}_n$ is $n^{-1/3}$ -consistent. It is interesting to find that the fast rate of convergence for the threshold estimator is preserved for the maximum score estimation. Section 6 gives the asymptotic distributions of $n(\hat{\gamma}_n - \gamma_0)$ and $n^{1/3}(\hat{\theta}_n - \theta_0)$. It turns out that both estimators $\hat{\gamma}$ and $\hat{\theta}$ are oracle-efficient in that $n(\hat{\gamma}_n - \gamma_0)$ and $n^{1/3}(\hat{\theta}_n - \theta_0)$ converge weakly to the distributions to which they would converge weakly if the other parameter(s) were known. Section 7 discusses the subsampling inference that enables us to carry out inference, although the distributions of $n(\hat{\gamma}_n - \gamma_0)$ and $n^{1/3}(\hat{\theta}_n - \theta_0)$ are nonstandard and cannot be tabulated. Section 8 reports some simulation results that investigate the finite sample performance of the subsampling confidence intervals for the regression parameters. Concluding remarks are given in Section 9. The proofs of theorems are given in the Appendix.

2 Identification of θ_0 and γ_0

This section provides regularity conditions under which θ_0 and γ_0 are identified. Let $F_{U|X}(u|x)$ and $f_{U|X}(u|x)$, respectively, denote the cumulative distribution function and probability density function of U conditional on $X = x$. We make the following assumptions.

Assumption 1. *Assume that $\|\theta_0\| = 1$ and $\delta_0 \neq 0$.*

Assumption 2. *The distribution of U conditional on $X = x$ is absolutely continuous with respect to Lebesgue measure and the corresponding conditional density is uniformly continuous and positive everywhere with probability one. In addition, $F_{U|X}(0|x) = 0.5$ for almost every x .*

Assumption 3. *The support of $W \in \mathbb{R}^q$ is not contained in any proper linear subspace of \mathbb{R}^q under the conditional distribution of W given $D = d$ for almost every d .*

Assumption 4. *There exists at least one $k \in [1, \dots, q]$ such that the k -th element of β_0 is non-zero and such that, for almost every value of $\tilde{w} \equiv (w_1, \dots, w_{k-1}, w_{k+1}, \dots, w_q)$ and d , $\Pr[a_1 < W^{(k)} < a_2 | \tilde{W} = \tilde{w}, D = d] > 0$ for all open intervals $(a_1, a_2) \in \mathbb{R}$, where W is decomposed into the k -th element $W^{(k)}$ and other elements \tilde{W} .*

Assumption 5. *D is continuously distributed with support containing γ_0 .*

Assumption 1 imposes restrictions on parameters. As in the linear binary response model (1.1), the scale of θ_0 is not identified since the distribution of U conditional on $X = x$ is unknown. In this paper, the scale normalization on θ_0 is given by $\|\theta_0\| = 1$. If δ_0 is zero, then γ_0 is unidentified. Assumption 2 allows for an arbitrary form of dependence between U and X as long as the conditional median independence assumption is satisfied. This can be extended easily to any other quantile independence assumption as well (see, Assumption QI in Manski, 1988). Assumptions 3 and 4 are simply the restatement of Conditions X1 and X3 of Manski (1988), respectively. Assumption 3 imposes a usual full rank condition on the distribution of X and Assumption 4 requires a rich support of one of elements of W with a non-zero coefficient conditional on other regressors. Assumption 5 ensures the identification of γ_0 . If γ_0 were not contained in the support of D , then the threshold model (1.2) would not have a two-phase feature in observed data.

The following theorem shows that θ_0 and γ_0 are identified.

Theorem 2.1. *Let Assumptions 1-5 hold. Then θ_0 and γ_0 are identified.*

3 The Maximum Score Estimator

We consider estimation with an independent and identically distributed sample $\{(Y_i, X_i) : i = 1, \dots, n\}$ of (Y, X) . To describe the maximum score estimator, for any $\theta = (\beta', \delta')'$, γ and $x = (w, z, d)$, define

$$(3.1) \quad S_n(\theta, \gamma) = \sum_{i=1}^n (2Y_i - 1) 1 \{G(X_i, \theta, \gamma) \geq 0\},$$

where

$$G(x, \theta, \gamma) = w'\beta + z'\delta 1(d > \gamma).$$

Then Manski (1975, 1985)'s maximum score estimator $(\hat{\theta}_n, \hat{\gamma}_n)$ of (θ_0, γ_0) can be obtained in two steps. In the first stage, for fixed γ , obtain

$$\hat{\theta}_n(\gamma) = \operatorname{argmax}_{\theta} S_n(\theta, \gamma)$$

with the scale normalization on θ with $\|\theta\| = 1$. In the second stage, let $\hat{S}_n(\gamma) = S_n(\hat{\theta}_n(\gamma), \gamma)$. Obtain $\hat{\gamma}_n = \inf\{\gamma_n : \gamma_n = \operatorname{argmax}_{\gamma} \hat{S}_n(\gamma)\}$. Then the maximum score estimator of θ_0 is $\hat{\theta}_n = \hat{\theta}_n(\hat{\gamma}_n)$. Since there could be infinitely many γ 's at which $\hat{S}_n(\gamma)$ is maximized, $\hat{\gamma}_n$ is defined as the infimum of those.

The discontinuity of S_n with respect to (θ, γ) complicates the analysis of the asymptotic behavior of the maximum score estimator. There exist two distinct sources of discontinuity: on the one hand, the discontinuity arises because of the presence of the indicator function $1\{G(X_i, \theta, \gamma) \geq 0\}$ in (3.1); on the other hand, it may occur because of a possible change-point in (1.2) due to an unknown threshold γ_0 . We now turn to the asymptotic properties of the maximum score estimator.

4 Consistency of the Estimator

This section gives conditions under which the maximum score estimator is consistent.

Assumption 6. *Assume that θ_0 and γ_0 are in a compact subset of \mathbb{R}^p , where p is the dimension of (θ_0, γ_0) .*

Assumption 7. *Assume that $\{(Y_i, X_i) : i = 1, \dots, n\}$ is independent and identically distributed.*

These are standard assumptions in the literature. The following theorem gives the consistency result.

Theorem 4.1. *Let Assumptions 1-7 hold. Then $(\hat{\theta}_n, \hat{\gamma}_n) \rightarrow_p (\theta_0, \gamma_0)$.*

5 Rates of Convergence

This section establishes the rates of convergence in probability of $(\hat{\theta}_n, \hat{\gamma}_n)$ to (θ_0, γ_0) . First, we show that $\hat{\gamma}_n$ is n^{-1} -consistent for γ_0 . We make the following additional assumptions.

Assumption 8. *Assume that D is continuously distributed with full support on \mathbb{R} and its probability density function is strictly positive, bounded and continuous in a neighborhood of γ_0 .*

Assumption 9. *With positive probability, $Z'\delta_0 \neq 0$ at $D = \gamma_0$.*

Assumption 8 is needed to prove a useful lemma (see Lemma A.1 in the Appendix). Roughly speaking, this condition ensures that there exist enough data points around γ_0 for sufficiently large n . Assumption 9 imposes that the regression function in (1.2) is discontinuous. When the regression function is continuous, then we would have different rates of convergence and asymptotic distributions. We will make a brief comment on this at the end of this section.

The following theorem gives the n^{-1} -consistency of $\hat{\gamma}_n$ to γ_0 .

Theorem 5.1. *Let Assumptions 1-9 hold. Then, $\hat{\gamma}_n = \gamma_0 + O_p(n^{-1})$.*

We now turn to the rates of convergence of $\hat{\theta}_n$ to θ_0 . Given the result of Theorem 5.1 and Kim and Pollard (1990), it is not surprising that we establish the $n^{-1/3}$ -consistency for $\hat{\theta}_n$.

Assumption 10. *As a function of θ , $E[(2Y - 1)1\{G(X, \theta, \gamma_0) \geq 0\}]$ has a strictly negative definite second derivative matrix at θ_0 .*

This assumption is necessary to apply Corollary 4.2 of Kim and Pollard (1990), who also give sufficient conditions for the maximum score estimator of the binary response model with (1.1) (see Section 6.4 in Kim and Pollard, 1990). Assumption 10 imposes a local identification at θ_0 and is a weak smoothness condition. The following theorem gives the $n^{-1/3}$ -consistency of $\hat{\theta}_n$ to θ_0 .

Theorem 5.2. *Let Assumptions 1-10 hold. Then, $\hat{\theta}_n = \theta_0 + O_p(n^{-1/3})$.*

We end this section by making a comment on the case when the model (1.2) is a continuous threshold model, in which $z'\delta$ becomes zero when $d = \gamma$ and $G(x, \theta, \gamma) = w'\beta + \delta(d - \gamma)1\{d > \gamma\}$. This violates Assumption 9, invalidating Theorem 5.1. Instead, Kim and Pollard (1990) applies to both parameters θ and γ , since the limit objective function can now be assumed to be twice differentiable in γ as well as in θ .¹

6 The Asymptotic Distribution of the Estimator

To obtain the asymptotic distribution of the maximum score estimator, write

$$(6.1) \quad S_n(\theta, \gamma) = S_{n1}(\gamma) + S_{n2}(\theta) + S_{n3}(\theta, \gamma),$$

where $S_{n1}(\gamma) = S_n(\theta_0, \gamma) - S_n(\theta_0, \gamma_0)$, $S_{n2}(\theta) = S_n(\theta, \gamma_0) - S_n(\theta_0, \gamma_0)$, and

$$S_{n3}(\theta, \gamma) = [S_n(\theta, \gamma) - S_n(\theta, \gamma_0)] - [S_n(\theta_0, \gamma) - S_n(\theta_0, \gamma_0)].$$

Since $\hat{\gamma}$ is n^{-1} -consistent and $\hat{\theta}$ is $n^{-1/3}$ -consistent, it suffices to consider a n^{-1} -neighborhood of γ_0 and a $n^{-1/3}$ -neighborhood of θ_0 . Then we obtain the following lemma.

Lemma 6.1. *For every $A > 0$, as $n \rightarrow \infty$:*

$$\sup_{\theta, \gamma} \left\{ |S_{n3}(\theta, \gamma)| : n^{1/3} \|\theta - \theta_0\| \leq A, n|\gamma - \gamma_0| \leq A \right\} = O_p(n^{-1/6}).$$

To establish the asymptotic distribution of $n(\hat{\gamma} - \gamma_0)$, we now consider the weak convergence of S_{n1} as a random variable on the space of cadlag functions, denoted by \mathcal{D} , equipped with the Skorohod topology and on its restrictions to the space of cadlag functions on $[-A, A]$, denoted by \mathcal{D}_A , for any $A > 0$. This approach is similar to that taken in the recent

¹A sufficient condition for this would be the conditional density $f_{D|W}$ of D given W is differentiable twice. Then, assuming $c = -w'\beta/\delta$ and $\delta > 0$, we note that

$$\Pr\{G \geq 0|W = w\} = \Pr\{(D - \gamma)1\{D > \gamma\} \geq c|W = w\} = \int_{c \vee 0}^{\infty} f_{D|W=w}(x + \gamma) dx$$

has second derivative.

literature (for example, Chan, 1993; Koul and Qian, 2002; Pons, 2003; Kosorok and Song, 2007).

Now define $\nu = n(\gamma - \gamma_0)$. Using arguments identical to prove (A.1) in the Appendix, write a rescaled version of $S_{n1}(\gamma)$ as

$$\tilde{S}_{n1}(\nu) = \tilde{S}_{n1}^+(\nu)1(\nu > 0) + \tilde{S}_{n1}^-(\nu)1(\nu < 0),$$

where

$$\begin{aligned}\tilde{S}_{n1}^+(\nu) &= \sum_{i=1}^n 1\{\gamma_0 < D_i \leq \gamma_0 + n^{-1}\nu\} \zeta_i^+(\theta_0), \\ \tilde{S}_{n1}^-(\nu) &= \sum_{i=1}^n 1\{\gamma_0 + n^{-1}\nu < D_i \leq \gamma_0\} \zeta_i^-(\theta_0),\end{aligned}$$

$\zeta_i^+(\theta) = [I_{1i}^+(\theta) - I_{2i}^+(\theta) + I_{3i}^+(\theta) - I_{4i}^+(\theta)]$ with

$$\begin{aligned}I_{1i}^+(\theta) &= 1\{W_i'\beta \geq 0, W_i'\beta + Z_i'\delta < 0, W_i'\beta_0 + Z_i'\delta_0 \geq -U_i\} \\ I_{2i}^+(\theta) &= 1\{W_i'\beta \geq 0, W_i'\beta + Z_i'\delta < 0, W_i'\beta_0 + Z_i'\delta_0 < -U_i\} \\ I_{3i}^+(\theta) &= 1\{W_i'\beta < 0, W_i'\beta + Z_i'\delta \geq 0, W_i'\beta_0 + Z_i'\delta_0 < -U_i\} \\ I_{4i}^+(\theta) &= 1\{W_i'\beta < 0, W_i'\beta + Z_i'\delta \geq 0, W_i'\beta_0 + Z_i'\delta_0 \geq -U_i\},\end{aligned}$$

and $\zeta_i^-(\theta) = [I_{1i}^-(\theta) - I_{2i}^-(\theta) + I_{3i}^-(\theta) - I_{4i}^-(\theta)]$ with

$$\begin{aligned}I_{1i}^-(\theta) &= 1\{W_i'\beta \geq 0, W_i'\beta + Z_i'\delta < 0, W_i'\beta_0 < -U_i\} \\ I_{2i}^-(\theta) &= 1\{W_i'\beta \geq 0, W_i'\beta + Z_i'\delta < 0, W_i'\beta_0 \geq -U_i\} \\ I_{3i}^-(\theta) &= 1\{W_i'\beta < 0, W_i'\beta + Z_i'\delta \geq 0, W_i'\beta_0 \geq -U_i\} \\ I_{4i}^-(\theta) &= 1\{W_i'\beta < 0, W_i'\beta + Z_i'\delta \geq 0, W_i'\beta_0 < -U_i\}.\end{aligned}$$

To describe the asymptotic behavior of $\tilde{S}_{n1}(\nu)$, let $f_D(d)$ denote the probability density function of D , let $\tilde{\nu}^+$ and $\tilde{\nu}^-$ denote two independent jump processes on \mathbb{R} such that $\tilde{\nu}^+(s)$ is a Poisson random variable with parameter $sf_D(\gamma_0)$ for $s > 0$ and $\tilde{\nu}^+(s) = 0$ for $s \leq 0$ and $\tilde{\nu}^-(s)$ is a Poisson random variable with parameter $-sf_D(\gamma_0)$ for $s < 0$ and $\tilde{\nu}^-(s) = 0$ for

$s \geq 0$. In addition, let $\{\tilde{\zeta}_k^+ : k = 0, 1, \dots\}$ and $\{\tilde{\zeta}_k^- : k = 0, 1, \dots\}$ be independent sequences of i.i.d. random variables with characteristic functions

$$\varphi^+(t) = E[\exp(it\zeta^+) | D = \gamma_0^+] \quad \text{and} \quad \varphi^-(t) = E[\exp(it\zeta^-) | D = \gamma_0^-],$$

respectively, and let $\tilde{\zeta}_0^+ = \tilde{\zeta}_0^- = 0$. Let $\tilde{S}_1(s) = \tilde{S}_1^+(s)1\{s > 0\} + \tilde{S}_1^-(s)1\{s < 0\}$ be a right-continuous jump process on \mathbb{R} , where

$$\tilde{S}_1^+(s) = \sum_{0 \leq k \leq \tilde{\nu}^+(s)} \tilde{\zeta}_k^+ \quad \text{and} \quad \tilde{S}_1^-(s) = \sum_{0 \leq k \leq \tilde{\nu}^-(s)} \tilde{\zeta}_k^-.$$

The following lemma establishes the weak convergence of \tilde{S}_{n1} .

Lemma 6.2. *As $n \rightarrow \infty$, \tilde{S}_{n1} converges weakly to \tilde{S}_1 in \mathcal{D}_A for every $A > 0$.*

Let $\nu_{\tilde{S}_1}^* = \inf\{\nu^* : \nu^* = \operatorname{argmax}_{\nu} \tilde{S}_1(\nu)\}$. As in the literature (see, e.g., Chan, 1993; Koul and Qian, 2002; Pons, 2003; Kosorok and Song, 2007), \tilde{S}_1 tends to $-\infty$ almost surely as $|\nu| \rightarrow \infty$, thereby implying that the distribution of $\nu_{\tilde{S}_1}^*$ is tight. An application of the argmax continuous mapping theorem of van der Vaart and Wellner (1996, Theorem 3.2.2, p.286), combined with Lemmas 6.1 and 6.2, gives the main theorem of this paper.

Theorem 6.1. *Let Assumptions 1-10 hold. Then, as $n \rightarrow \infty$, $n(\hat{\gamma} - \gamma_0)$ converges weakly to $\nu_{\tilde{S}_1}^*$. In addition, $n^{1/3}(\hat{\theta} - \theta_0)$ and $\operatorname{argmax}_{\mu} n^{-1/3} S_{n2}(\theta_0 + n^{-1/3}\mu)$ converge weakly to the same limiting distribution.*

Theorem 6.1 implies that the limiting distributions of $n(\hat{\gamma} - \gamma_0)$ and $n^{1/3}(\hat{\theta} - \theta_0)$ are the same as those they would be if the other parameters were known. The limiting distribution of $n^{1/3}(\hat{\theta} - \theta_0)$ is not shown here since it is already given in Kim and Pollard (1990, Example 6.4).

7 Inference

The asymptotic distributions obtained in this paper are nonstandard and cannot be tabulated as they depend on nuisance parameters in complex manners. It is known from

Abrevaya and Huang (2005) that the bootstrap does not estimate the asymptotic distribution of $\hat{\theta}$ consistently. While the m out of n bootstrap in Lee and Pun (2006) is more general and allows for the standard maximum score estimator, their regularity conditions do not permit threshold models. However, subsampling provides a consistent inferential method for the asymptotic distributions of both $\hat{\theta}$ and $\hat{\gamma}$, as in Delgado et al. (2001) and Gonzalo and Wolf (2005). Confidence intervals can be constructed following the standard subsampling procedure, see e.g. Politis et al. (1999).

As our asymptotic development is based on the discontinuity of the regression function (Assumption 9), the above asymptotics breaks down when the model comes close to a continuous one. In this case, the subsampling confidence intervals may be problematic as their convergence is not uniform around the continuity point, see e.g. Andrews and Guggenberger (2006).

Another interesting inferential approach can be based on the smoothing of the objective function. This can be done by replacing the indicator functions in (3.1) with integrated kernels with smoothing parameters that converge to zero. Horowitz (1992) and Seo and Linton (2007) have developed the smoothing approach for the maximum score estimation and threshold estimation, respectively. This method enables the standard normal inference for both γ_0 and θ_0 while it causes the convergence rate of $\hat{\theta}$ to increase and that of $\hat{\gamma}$ to decrease. The smoothing, however, demands more stringent smoothness conditions than the ones in this paper as demonstrated in Horowitz (1993).

8 Monte Carlo Experiments

We examine the finite sample performance of the subsampling inference for the regression parameters $\theta_0 = (\beta_0, \delta_0)$ and the threshold parameter γ_0 . The samples of size $n = 500$ are generated independently from

$$\begin{aligned} Y^* &= \beta_0 X_1 + \delta_0 X_2 1\{D > \gamma_0\} + U \\ Y &= 1\{Y^* > 0\}, \end{aligned}$$

where $\beta_0 = \delta_0 = 1/\sqrt{2}$, $\gamma_0 = 2$, X_1 , X_2 , and D are independently distributed as $N(0, 1)$, $2N(0, 1)$, and $N(0, 1) + 2$, respectively. U is independent of all the other variables and generated from either logistic distribution (Dist 1) with median 0 and variance 0.1 or student's t distribution (Dist 2) with 3 degrees of freedom and with variance 0.1. These two classes of error distributions are considered by Horowitz (1992).

The model parameters β_0 , δ_0 and γ_0 are estimated by the grid search.² The grid for δ is set as $\{0, .01, .02, \dots, 1\}$ and that for γ as the data points of D after trimming at lower and upper 10 percentiles. We focus on δ and γ as the normalization restriction yields $\beta^2 + \delta^2 = 1$. Let $\hat{\delta}$ and $\hat{\gamma}$ denote the estimates.

We describe subsampling hypothesis tests for θ_0 . Let the null hypothesis of interest be $H_0 : g(\theta_0) = 0$ against the alternative hypothesis $H_1 : g(\theta_0) \neq 0$, where g is a smooth function whose first derivative is of full rank at $\theta = \theta_0$. We may consider a test statistic $T_n = n^{1/3} \left\| g(\hat{\theta}) \right\|$, where $\|\cdot\|$ is the Euclidean norm. Now, let $m < n$ be the block length for the subsampling and randomly draw m samples with replacement from a given data set and estimate δ_0 and γ_0 from this subsample. Let $\hat{\delta}^*$ and $\hat{\gamma}^*$ denote subsample estimates and construct $T_n^* = m^{1/3} \left\| g(\hat{\theta}^*) - g(\hat{\theta}) \right\|$. Repeat this many times, say B times, and obtain empirical distributions of T_n^* . Denote the $1 - \alpha$ quantile of the distribution as $c_{1-\alpha}$. Then, the nominal level α test rejects H_0 if and only if $T_n > c_{1-\alpha}$. In case of the hypothesis test for γ_0 , the rate of convergence $n^{1/3}$ and $m^{1/3}$ above need to be replaced by n and m , respectively.

We examine the rejection rates of the test for the hypothesis $\delta_0 = 1/\sqrt{2}$ with the two nominal levels $\alpha = 0.05$ and 0.1 . Also considered is the test for the hypothesis $\gamma_0 = 2$. All the simulations are based on 500 repetitions with the subsampling repetition $B = 500$. As we do not pursue an automatic selection rule for the block length m ,³ we experimented with several values of m to see how the choice affects the performance of subsampling. The

²Alternatively, one could use a random search algorithm based on Manski and Thompson (1986) rather than the simple grid search.

³See Delgado et al (2001) for an automatic selection rule, while it is computationally demanding to perform a Monte Carlo experiment.

results are reported in Table 1.

The results look promising. The rejection frequencies of the subsampling test for the threshold parameter γ_0 are close to their nominal levels with a broad range of block size m . The subsampling test for the slope parameter δ_0 appears rather conservative for larger values of m and liberal for smaller values of m . Furthermore, we note that the block length m needs to be different for the threshold and slope parameters for best results.

Table 1: Rejection rates of subsampling hypothesis tests

Threshold parameter γ_0					
	Nominal Level	Block Size (m)			
		30	40	50	60
Dist 1	0.05	0.064	0.042	0.030	0.028
	0.1	0.126	0.106	0.096	0.082
Dist 2	0.05	0.080	0.058	0.050	0.050
	0.1	0.134	0.104	0.116	0.100
Slope parameter δ_0					
	Nominal Level	Block Size (m)			
		10	15	20	30
Dist 1	0.05	0.090	0.046	0.030	0.014
	0.1	0.152	0.114	0.094	0.052
Dist 2	0.05	0.070	0.030	0.022	0.006
	0.1	0.138	0.080	0.058	0.048

We also experimented the subsampling test for the hypothesis for δ_0 with γ fixed at the true value 2. The results are quite similar to the one reported in Table 1 and thus omitted. This reflects a finding in Theorem 6.1, which is that the asymptotic distribution of $\hat{\delta}$ is not affected by the estimation of γ_0 .

9 Conclusions

This paper has considered the maximum score estimator of a binary response model with a change-point according to the unknown threshold of a covariate, allowing for an arbitrary form of heteroskedasticity. We have obtained the rates of convergence and asymptotic

distribution of the estimator. It turns out that the estimator of the threshold parameter is n^{-1} -consistent, the estimator of the remaining regression parameters is $n^{-1/3}$ -consistent, and the limiting distributions of the estimators normalized by the rates of convergence are the same as if the other parameters were known. Therefore, an important practical implication of this paper is that the unknown threshold parameter can be estimated precisely with a small or moderate sample size and that there is no loss of efficiency of not knowing other parameters, which are difficult to estimate precisely without a large sample. For future research, it is an interesting but challenging research topic to develop a linearity test of the maximum score model against threshold alternatives.

A Appendix: Proofs of Theorems

Proof of Theorem 2.1. Note that by Assumptions 2, 3 and 4, for each fixed $D = d$, (β_{10}, β_{20}) or (β_{10}, β_{30}) is identified by Corollary 2 of Manski (1988). Then, given Assumption 5, there is a unique value of d at which the slope coefficient changes. This change point is identified as γ_0 . ■

Proof of Theorem 4.1. Define

$$H^*(x) = E[1\{G(X, \theta_0, \gamma_0) + U \geq 0\} - 1\{G(X, \theta_0, \gamma_0) + U < 0\} | X = x]$$

and

$$\Delta^*(\theta, \gamma) = n^{-1}E[S_n(\theta, \gamma) - S_n(\theta_0, \gamma_0)].$$

Then it can be shown that

$$\Delta^*(\theta, \gamma) = E\left[H^*(X)\left(1\{G(X, \theta, \gamma) \geq 0 > G(X, \theta_0, \gamma_0)\} - 1\{G(X, \theta_0, \gamma_0) \geq 0 > G(X, \theta, \gamma)\}\right)\right].$$

Note that

$$H^*(x) = 1 - 2F_{U|X}[-G(x, \theta_0, \gamma_0)|x].$$

By the assumption that $F_{U|X}[0|x] = 0.5$, note that $H^*(x) \geq 0$ when $G(x, \theta_0, \gamma_0) \geq 0$ and that $H^*(x) < 0$ when $G(x, \theta_0, \gamma_0) < 0$. Define

$$\mathcal{Q}(\theta, \gamma) = \left[x \in \text{supp}(X) : \{G(x, \theta, \gamma) \geq 0 > G(x, \theta_0, \gamma_0)\} \cup \{G(x, \theta_0, \gamma_0) \geq 0 > G(x, \theta, \gamma)\} \right].$$

By arguments identical to those used to prove Proposition 2 of Manski (1988), (θ_0, γ_0) are identified if and only if $\Pr(X \in \mathcal{Q}(\theta, \gamma)) > 0$ for any $(\theta, \gamma) \neq (\theta_0, \gamma_0)$. Therefore, $\Delta^*(\theta, \gamma)$ is non-positive everywhere and is equal to zero only when $(\theta, \gamma) = (\theta_0, \gamma_0)$. Also, notice that $G(S, \theta, \gamma)$ is continuous at each θ and γ with probability one. Thus, $S_n(\theta, \gamma)$ converges in probability to $E[S_n(\theta, \gamma)]$ uniformly over (θ, γ) by Lemma 2.4 of Newey and McFadden (1994). Then the consistency of the estimator follows from Theorem 2.1 of Newey and McFadden (1994). ■

To obtain the converge rates for $\hat{\gamma}_n$, we begin with the following lemma, which is similar to Claim 2 in the proof of Proposition 1 of Chan (1993) and Lemma 3.2 of Koul and Qian (2002). The proof of the lemma is given in Appendix B.

Lemma A.1. *For any random variables (V, D) satisfying $E[V|D] = 0$ and $E[V^2|D] < C < \infty$ almost surely for some constant C , assume that $\{(V_i, D_i) : i = 1, \dots, n\}$ is a random sample of (V, D) and that D is continuously distributed and has a bounded, continuous, positive density in a neighborhood of $x \in \mathbb{R}$. Then, for each $\delta > 0$ and $\varepsilon > 0$, there exists a positive constant $B < \infty$ such that for all $0 < \Delta < 1$ and for all $n > B/\Delta$,*

$$(a) \Pr \left\{ \sup_{\frac{B}{n} < r < \Delta} \left| \frac{1}{n} \sum_{i=1}^n \frac{1 \{x < D_i < x + r\}}{\Pr \{x < D < x + r\}} - 1 \right| > \delta \right\} < \varepsilon$$

$$(b) \Pr \left\{ \sup_{\frac{B}{n} < r < \Delta} \left| \frac{1}{n} \sum_{i=1}^n V_i \frac{1 \{x < D_i < x + r\}}{\Pr \{x < D_i < x + r\}} \right| > \delta \right\} < \varepsilon.$$

Proof of Theorem 5.1. Define

$$\begin{aligned} \Delta_{S_n}(\theta, \gamma) &= S_n(\theta, \gamma) - S_n(\theta, \gamma_0) \\ &= \sum_{i=1}^n (2Y_i - 1) (1 \{G(X_i, \theta, \gamma) \geq 0\} - 1 \{G(X_i, \theta, \gamma_0) \geq 0\}). \end{aligned}$$

Suppose that $\gamma > \gamma_0$. Then if $D_i > \gamma$ or $D_i \leq \gamma_0$,

$$1 \{G(X_i, \theta, \gamma) \geq 0\} - 1 \{G(X_i, \theta, \gamma_0) \geq 0\} = 0.$$

Suppose now that $\gamma < \gamma_0$. Then if $D_i > \gamma_0$ or $D_i \leq \gamma$,

$$1 \{G(X_i, \theta, \gamma) \geq 0\} - 1 \{G(X_i, \theta, \gamma_0) \geq 0\} = 0.$$

Using these, we may write

$$(A.1) \quad \Delta_{S_n}(\theta, \gamma) = \sum_{i=1}^n 1 \{\gamma_0 < D_i \leq \gamma\} \zeta_i^+(\theta) + \sum_{i=1}^n 1 \{\gamma < D_i \leq \gamma_0\} \zeta_i^-(\theta),$$

where $\zeta_i^+(\theta)$ and $\zeta_i^-(\theta)$ are defined in the main text.

The following lemma, which is proved in Appendix B, is useful to prove the theorem.

Lemma A.2. *For any $\Delta > 0$, define $\Theta_\Delta = \{\theta : \|\theta - \theta_0\| < \Delta\}$. There exists a sufficiently small $\Delta > 0$ such that*

$$\sup_{\theta \in \Theta_\Delta} E[\zeta_i^+(\theta)|D_i] < 0 \quad \text{and} \quad \sup_{\theta \in \Theta_\Delta} E[\zeta_i^-(\theta)|D_i] < 0$$

almost surely.

Now write

$$\begin{aligned} \Delta_{S_n}(\theta, \gamma) &= \sum_{i=1}^n 1 \{\gamma_0 < D_i \leq \gamma\} E[\zeta_i^+(\theta)|D_i] + \sum_{i=1}^n 1 \{\gamma < D_i \leq \gamma_0\} E[\zeta_i^-(\theta)|D_i] \\ &\quad + \sum_{i=1}^n 1 \{\gamma_0 < D_i \leq \gamma\} \{\zeta_i^+(\theta) - E[\zeta_i^+(\theta)|D_i]\} \\ &\quad + \sum_{i=1}^n 1 \{\gamma < D_i \leq \gamma_0\} \{\zeta_i^-(\theta) - E[\zeta_i^-(\theta)|D_i]\}. \end{aligned}$$

Then apply Lemmas A.1 and A.2 to obtain $\Delta_{S_n}(\theta, \gamma) < 0$ with probability approaching one for all $|\gamma - \gamma_0| > B/n$ and $\theta \in \Theta_\Delta$ and with some sufficiently large B and sufficiently small Δ . As $(\hat{\theta}, \hat{\gamma})$ is the maximizer of $S_n(\theta, \gamma)$, we conclude that $\hat{\gamma} = \gamma_0 + O_p(1/n)$. ■

To obtain the converge rates for $\hat{\theta}_n$, we first give the following lemma, which is proved in Appendix B.

Lemma A.3. For every $A > 0$, as $n \rightarrow \infty$:

$$\sup_{\theta, \gamma} \{|S_{n3}(\theta, \gamma)| : n|\gamma - \gamma_0| \leq A\} = O_p(1).$$

Proof of Theorem 5.2. In view of Corollary 4.2 of Kim and Pollard (1990), it suffices to show that

$$n^{-1}S_n(\hat{\theta}) \geq \sup_{\theta} n^{-1}S_n(\theta) - O_p(n^{-2/3}),$$

which in turn follows from the same Corollary, Lemma A.3, and the fact that S_{n1} does not depend on θ . ■

Proof of Theorem 6.1. Note that $n(\hat{\gamma} - \gamma_0)$ is uniformly tight by Theorem 5.1. Since S_{n2} depends on only θ , Lemmas 6.1 and 6.2 ensure that conditions of the argmax continuous mapping theorem of van der Vaart and Wellner (1996, Theorem 3.2.2, p.286) are satisfied. Then the first conclusion follows from the argmax continuous mapping theorem. Similarly, $n^{1/3}(\hat{\theta} - \theta_0)$ is uniformly tight by Theorem 5.2. Since S_{n1} depends on only γ , by Lemma 6.2 and again by the argmax continuous mapping theorem, the second conclusion follows. ■

B Appendix: Proofs of Lemmas

Proof of Lemma A.1. For any $r \in (0, \Delta]$, there exist $0 < m < M < \infty$ such that

$$(B.1) \quad mr < \Pr\{x < D < x + r\} < Mr.$$

Consider a partition of the interval $(B/n, \Delta) \equiv \cup_{k=0}^{K-1} J_k$, where K is the largest integer such that $b^{K-1}B/n < \Delta$ for some $b > 1$, and $J_k = (b^k B/n, b^{k+1} B/n]$, $k = 0, \dots, K-2$ and $J_{K-1} = (b^{K-1} B/n, \Delta)$. Here we prove part (b) only as part (a) can be proved in the same but simpler way. For a $r \in J_k$, we have

$$\begin{aligned} \left| \frac{1}{n} \sum_{i=1}^n V_i 1\{x < D_i < x + r\} \right| &\leq \frac{1}{n} \sum_{i=1}^n |V_i| 1\{x + b^k B/n < D_i \leq x + b^{k+1} B/n\} \\ &\quad + \left| \frac{1}{n} \sum_{i=1}^n V_i 1\{x < D_i \leq x + b^k B/n\} \right| \end{aligned}$$

and

$$\Pr \{x < D < x + r\} \geq \Pr \left\{ x < D < x + b^k B/n \right\}.$$

Thus,

(B.2)

$$\begin{aligned} \sup_{\frac{B}{n} < r < \Delta} \left| \frac{1}{n} \sum_{i=1}^n V_i \frac{1 \{x < D_i < x + r\}}{\Pr \{x < D < x + r\}} \right| &\leq \max_{0 \leq k < K} \frac{1}{n} \sum_{i=1}^n \frac{|V_i| 1 \{x + b^k B/n < D_i \leq x + b^{k+1} B/n\}}{\Pr \{x < D_i < x + b^k B/n\}} \\ &+ \max_{0 \leq k < K} \left| \frac{1}{n} \sum_{i=1}^n \frac{V_i 1 \{x < D_i \leq x + b^k B/n\}}{\Pr \{x < D \leq x + b^k B/n\}} \right|. \end{aligned}$$

First, consider the second term in (B.2). It follows from the Markov and Cauchy-Schwarz inequalities that for any $\delta_1 > 0$,

$$\begin{aligned} (B.3) \quad &\Pr \left\{ \max_{0 \leq k < K} \left| \frac{1}{n} \sum_{i=1}^n \frac{V_i 1 \{x < D_i \leq x + b^k B/n\}}{\Pr \{x < D \leq x + b^k B/n\}} \right| > \delta_1 \right\} \\ &\leq \delta_1^{-2} \sum_{k=0}^{K-1} E \left| \frac{1}{n} \sum_{i=1}^n \frac{V_i 1 \{x < D_i \leq x + b^k B/n\}}{\Pr \{x < D < x + b^k B/n\}} \right|^2 \\ &\leq \delta_1^{-2} \sum_{k=0}^{K-1} \frac{E[V^2]}{n \cdot \Pr \{x < D \leq x + b^k B/n\}} \\ &\leq \frac{E[V^2]}{m \delta_1^2 B (1 - b^{-1})}. \end{aligned}$$

Next, note that due to (B.1),

$$\text{var} \left(|V| 1 \left\{ x + b^k B/n < D \leq x + b^{k+1} B/n \right\} \right) \leq C b^k (b - 1) B/n,$$

for some constant C . Thus, for the first term in (B.2), we have that

$$\begin{aligned} &\Pr \left\{ \max_{0 \leq k < K} \frac{1}{n} \sum_{i=1}^n \frac{|V_i| 1 \{x + b^k B/n < D_i \leq x + b^{k+1} B/n\}}{\Pr \{x < D < b^k B/n\}} > \delta_2 \right\} \\ &\leq \sum_{k=0}^{K-1} \frac{1}{n} \text{var} \left(\frac{|V_i| 1 \{x + b^k B/n < D_i \leq x + b^{k+1} B/n\}}{\Pr \{x < D < x + b^k B/n\}} \delta_2 \right) \\ &+ \max_{0 \leq k < K} E \left(\frac{|V| 1 \{x + b^k B/n < D \leq x + b^{k+1} B/n\}}{\Pr \{x < D < x + b^k B/n\}} \delta_2 \right) \\ &\leq \sum_{k=0}^{K-1} \frac{1}{n} \frac{C b^k (b - 1) B/n}{(\delta_2 m b^k B/n)^2} + E[V^2] (b - 1) / \delta_2 m \\ &= \frac{C (b - 1)}{(\delta_2 m)^2 (1 - b^{-1}) B} + E[V^2] (b - 1) / \delta_2 m. \end{aligned}$$

Therefore, we can choose b close enough to 1 and then B large enough so that these two quantities and (B.3) can be made arbitrarily small for any $\delta_1, \delta_2 > 0$. ■

Proof of Lemma A.2. We first show that $E \sup_{\theta \in \Theta_\Delta} [I_{1i}^+(\theta) - I_{2i}^+(\theta) | D_i] < 0$ for Δ sufficiently small. Let $\varepsilon > 0, 1/2 < \xi < 1$, and $0 < \eta < (4\varepsilon\xi - 2\varepsilon) / (1 + 4\varepsilon\xi - 2\varepsilon)$. In view of Assumption 2, Choose Δ to be small enough so that $|W_i'(\beta_0 - \beta) + Z_i'(\delta_0 - \delta)| \leq a < \infty$ for all $\theta \in \Theta_\Delta$ with probability $1 - \eta$, for $a > 0$ such that $\Pr\{U_i < a | X_i\} \leq 1/2 + \varepsilon$ and $\Pr\{U_i > -a | X_i\} \leq 1/2 - \varepsilon$ almost surely and that $\Pr\{A_{i,a} | A_i, D_i\} = \xi > 1/2$, where

$$\begin{aligned} A_i &= 1 \{W_i'\beta > 0, W_i'\beta + Z_i'\delta \leq 0\} \\ A_{i,a} &= 1 \{W_i'\beta > 0, W_i'\beta + Z_i'\delta \leq -2a < 0\}. \end{aligned}$$

Then for all $\theta \in \Theta_\Delta$, we have

$$\begin{aligned} E(I_{1i}^+(\theta) | A_i, D_i) &\leq \eta + (1 - \eta)((1/2 - \varepsilon)\xi + (1/2 + \varepsilon)(1 - \xi)) \\ E(I_{2i}^-(\theta) | A_i, D_i) &\geq (1 - \eta)((1/2 + \varepsilon)\xi + (1/2 - \varepsilon)(1 - \xi)), \end{aligned}$$

almost surely, which in turn yields that

$$\begin{aligned} E[I_{1i}^+(\theta) - I_{2i}^-(\theta) | A_i, D_i] &\leq \eta + (1 - \eta)(-2\varepsilon\xi + 2\varepsilon(1 - \xi)) \\ &= \eta + (1 - \eta)(2\varepsilon - 4\varepsilon\xi) \\ &< 0 \end{aligned}$$

almost surely. Similarly, $E \sup_{\theta \in \Theta_\Delta} [I_{3i}^+(\theta) - I_{4i}^+(\theta) | D_i] < 0$ almost surely for Δ sufficiently small. Therefore, $\sup_{\theta \in \Theta_\Delta} E[\zeta_i^+(\theta) | D_i] < 0$ almost surely for Δ sufficiently small. The proof for $\zeta_i^-(\theta)$ is similar. ■

Proof of Lemma A.3. It follows from (A.1) that

$$S_{n3}(\theta, \gamma) = \sum_{i=1}^n 1\{\gamma_0 < D_i \leq \gamma\} [\zeta_i^+(\theta) - \zeta_i^+(\theta_0)] + \sum_{i=1}^n 1\{\gamma < D_i \leq \gamma_0\} [\zeta_i^-(\theta) - \zeta_i^-(\theta_0)].$$

Since $\zeta_i^+(\theta)$ and $\zeta_i^-(\theta)$ consists of just several indicator functions, there exists a universal constant $C < \infty$ such that

$$|S_{n3}(\theta, \gamma)| \leq C \left[\sum_{i=1}^n 1 \{\gamma_0 < D_i \leq \gamma\} + \sum_{i=1}^n 1 \{\gamma < D_i \leq \gamma_0\} \right]$$

uniformly over θ . Thus, it suffices to show that for every $A > 0$,

$$(B.4) \quad \sup_{\gamma: n|\gamma - \gamma_0| \leq A} \sum_{i=1}^n 1 \{\gamma_0 < D_i \leq \gamma\} = O_p(1) \quad \text{and} \quad \sup_{\gamma: n|\gamma - \gamma_0| \leq A} \sum_{i=1}^n 1 \{\gamma < D_i \leq \gamma_0\} = O_p(1).$$

To show the first claim of (B.4), consider a class of functions indexed by γ , $\mathcal{M}_A = \{1 \{\gamma_0 < D \leq \gamma\} : n|\gamma - \gamma_0| \leq A\}$. Then \mathcal{M}_A is a VC class of functions with an envelope function $M_A = 1 \{\gamma_0 < D \leq \gamma_0 + A/n\}$. By Theorem 2.14.1 of van der Vaart and Wellner (1996, p.239), there exists a universal constant $C < \infty$ such that

$$E \left[\sup_{\gamma: n|\gamma - \gamma_0| \leq A} \left| \sum_{i=1}^n [1 \{\gamma_0 < D_i \leq \gamma\} - E 1 \{\gamma_0 < D \leq \gamma\}] \right| \right] \leq C n^{1/2} J(1, \mathcal{M}_A) (EM_A^2)^{1/2},$$

where $J(1, \mathcal{M}_A)$ is the uniform entropy integral defined in van der Vaart and Wellner (1996, p.239). Since $J(1, \mathcal{M}_A)$ is bounded for a VC class and $(EM_A^2)^{1/2} = O(n^{-1/2})$,

$$\sup_{\gamma: n|\gamma - \gamma_0| \leq A} \left| \sum_{i=1}^n [1 \{\gamma_0 < D_i \leq \gamma\} - E 1 \{\gamma_0 < D \leq \gamma\}] \right| = O_p(1).$$

The desired result follows since

$$(B.5) \quad \sup_{\gamma: n|\gamma - \gamma_0| \leq A} E 1 \{\gamma_0 < D \leq \gamma\} = O(n^{-1}).$$

The proof of the second claim of (B.4) is similar. ■

Proof of Lemma 6.1. The proof of this lemma is similar to that of Lemma A.3. It suffices to show that

$$(B.6) \quad \sum_{i=1}^n 1 \{\gamma_0 < D_i \leq \gamma\} [\zeta^+(\theta) - \zeta^+(\theta_0)] = O_p(n^{-1/6})$$

and

$$(B.7) \quad \sum_{i=1}^n 1 \{\gamma < D_i \leq \gamma_0\} [\zeta^-(\theta) - \zeta^-(\theta_0)] = O_p(n^{-1/6}).$$

uniformly over (θ, γ) satisfying $n^{1/3}|\theta - \theta_0| \leq A$ and $n|\gamma - \gamma_0| \leq A$. We will prove only (B.6) since the proof of (B.7) is similar. Consider a class of functions indexed by (θ, γ) , $\mathcal{L}_A = \{1\{\gamma_0 < D \leq \gamma\}[\zeta^+(\theta) - \zeta^+(\theta_0)] : n^{1/3}|\theta - \theta_0| \leq A \text{ and } n|\gamma - \gamma_0| \leq A\}$. Then \mathcal{L}_A is a VC class of functions with an envelope function $L_A = \sup_{\theta, \gamma} \{1\{\gamma_0 < D \leq \gamma\}[\zeta^+(\theta) - \zeta^+(\theta_0)] : n^{1/3}|\theta - \theta_0| \leq A \text{ and } n|\gamma - \gamma_0| \leq A\}$. Then as in the proof of Lemma A.3, (B.6) follows from Theorem 2.14.1 of Van der Vaart and Wellner (1996, p.239), the fact that $(EL_A^2)^{1/2} = O(n^{-1/2}n^{-1/6})$, (B.5), and

$$\sup_{\theta: n^{1/3}|\theta - \theta_0| \leq A} E[\zeta^+(\theta) - \zeta^+(\theta_0)] = O(n^{-1/6}).$$

■

Proof of Lemma 6.2. Since \tilde{S}_{n1}^+ and \tilde{S}_{n1}^- are independent and similarly defined, we prove the convergence of \tilde{S}_{n1}^+ . Define $\tilde{S}_{n1}^+(\nu) = S_{n1}(\gamma_0 + n^{-1}\nu)$. We first prove the tightness of $\tilde{S}_{n1}^+(\nu)$. By the D -tightness criterion of Billingsley (1968, equation (15.21)), it suffices to show that there exists a finite, universal constant $C < \infty$ such that

$$(B.8) \quad E \left| \tilde{S}_{n1}^+(\nu) - \tilde{S}_{n1}^+(\nu_1) \right| \left| \tilde{S}_{n1}^+(\nu_2) - \tilde{S}_{n1}^+(\nu) \right| \leq C(\nu_2 - \nu_1)^2$$

for any ν, ν_1 and ν_2 satisfying $0 < \nu_1 < \nu < \nu_2$. Note that

$$\begin{aligned} & E \left| \tilde{S}_{n1}^+(\nu) - \tilde{S}_{n1}^+(\nu_1) \right| \left| \tilde{S}_{n1}^+(\nu_2) - \tilde{S}_{n1}^+(\nu) \right| \\ & \leq 4E \sum_{i=1}^n \sum_{j=1}^n 1 \left\{ \gamma_0 + \frac{\nu_1}{n} < D_i \leq \gamma_0 + \frac{\nu}{n} \right\} 1 \left\{ \gamma_0 + \frac{\nu}{n} < D_j \leq \gamma_0 + \frac{\nu_2}{n} \right\} \\ & \leq 4E \sum_{i=1}^n \sum_{j=1, j \neq i}^n 1 \left\{ \gamma_0 + \frac{\nu_1}{n} < D_i \leq \gamma_0 + \frac{\nu_2}{n} \right\} 1 \left\{ \gamma_0 + \frac{\nu_1}{n} < D_j \leq \gamma_0 + \frac{\nu_2}{n} \right\} \\ & \leq 4n(n-1) \left[F_D \left(\gamma_0 + \frac{\nu_2}{n} \right) - F_D \left(\gamma_0 + \frac{\nu_1}{n} \right) \right]^2 \\ & \leq 4C_{F_D} \frac{n(n-1)}{n^2} (\nu_2 - \nu_1)^2, \end{aligned}$$

where the first inequality comes from the fact $|\zeta_i^+(\theta_0)| \leq 2$ for all i , the second inequality follows since $1\{\gamma_1 < D_i \leq \gamma\}$ and $1\{\gamma < D_j \leq \gamma_2\}$ are disjoint, and $1\{\gamma_1 < D_i \leq \gamma_2\}$ is nondecreasing as γ_1 decreases or as γ_2 increases, the third inequality is obvious since

D_i are i.i.d. with the common distribution F_D , and the fourth inequality follows from the assumption that F_D is Lipschitz continuous with the constant C_{F_D} . Then (B.8) follows immediately by taking $C = 4C_{F_D}$.

We now consider the weak convergence of the finite-dimensional distributions of $\tilde{S}_{n1}^+(\nu)$. To use the Cramér-Wold device, let $0 < \nu_1 < \dots < \nu_J$ for $J \in \mathbb{N}$ and q_1, \dots, q_J be constants. Instead of obtaining the weak convergence of $\sum_{j=1}^J q_j \tilde{S}_{n1}^+(\nu_j)$, we will show below the weak convergence of \bar{S}_{n1} , where \bar{S}_{n1} is defined as

$$\bar{S}_{n1} = \sum_{j=1}^J q_j \left[\tilde{S}_{n1}^+(\nu_j) - \tilde{S}_{n1}^+(\nu_{j-1}) \right].$$

As shown in the proof of Theorem 4 of Pons (2003), it is easier to deal with \bar{S}_{n1} . Specifically, since \bar{S}_{n1} is a linear combination of $\{\tilde{S}_{n1}^+(\nu_1), \dots, \tilde{S}_{n1}^+(\nu_J)\}$, the weak convergence of the finite-dimensional distributions of $\tilde{S}_{n1}^+(\nu)$ follows if we show that its characteristic function converges to the characteristic function of

$$\sum_{j=1}^J q_j \left[\tilde{S}_1^+(\nu_j) - \tilde{S}_1^+(\nu_{j-1}) \right].$$

We now consider the characteristic function of \bar{S}_{n1} , which has the form

$$\varphi_{\bar{S}_{n1}}(t) = \left\{ E \left[\exp \left(it \sum_{j=1}^J q_j I_{nj}(D) \zeta(\theta_0) \right) \right] \right\}^n.$$

Note that as in the proof of Theorem 4 of Pons (2003), using the fact that for each i , there is at most one index j such that $I_{nj}(D_i) \neq 0$, write

$$\exp \left(it \sum_{j=1}^J q_j I_{nj}(D) \zeta(\theta_0) \right) = 1 + \sum_{j=1}^J [\exp(itq_j I_{nj}(D) \zeta(\theta_0)) - 1],$$

so that

$$\begin{aligned} & E \left[\exp \left(it \sum_{j=1}^J q_j I_{nj}(D) \zeta(\theta_0) \right) \right] \\ &= 1 + \sum_{j=1}^J n^{-1} [(\nu_j - \nu_{j-1}) f_D(\gamma_0) \{ E [\exp(itq_j \zeta(\theta_0)) - 1 | D = \gamma_0^+] \} + o(1)]. \end{aligned}$$

Then using the fact that $A^n = \exp(n \log A)$ for any $A > 0$ and that $\log(1 + a_n) = a_n + o(a_n)$ for any $a_n \rightarrow 0$, we have

$$\lim_{n \rightarrow \infty} \varphi_{\tilde{S}_{n1}}(t) = \sum_{j=1}^J (\nu_j - \nu_{j-1}) f_D(\gamma_0) \{E[\exp(itq_j \zeta(\theta_0)) - 1 | D = \gamma_0^+]\}.$$

Then the lemma follows from the fact that the characteristic function $\varphi_{\tilde{S}_1(s)}(t)$ of $\tilde{S}_1(s)$ is $\varphi_{\tilde{S}_1(s)}(t) = \exp[sf_D(\gamma_0)\{\varphi^+(t) - 1\}]$, as in Lemma 5 of Pons (2003). ■

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