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► **To cite this version:**

Daniel Palacin, Frank Olaf Wagner. Elimination of Hyperimaginaries and Stable Independence in simple CM-trivial theories. 2011. hal-00495655v2

HAL Id: hal-00495655

<https://hal.science/hal-00495655v2>

Preprint submitted on 12 Dec 2011 (v2), last revised 19 Mar 2012 (v5)

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ELIMINATION OF HYPERIMAGINARIES AND STABLE INDEPENDENCE IN SIMPLE CM-TRIVIAL THEORIES

D. PALACÍN AND F. O. WAGNER

ABSTRACT. In a simple CM-trivial theory every hyperimaginary is interbounded with a sequence of finitary hyperimaginaries. Moreover, such a theory eliminates hyperimaginaries whenever it eliminates finitary hyperimaginaries. In a supersimple CM-trivial theory, the independence relation is stable.

1. INTRODUCTION

An important notion introduced by Shelah for a first-order theory is that of an *imaginary* element: the class of a finite tuple by a \emptyset -definable equivalence relation. The construction obtained by adding all imaginary elements to a structure does not change its basic model-theoretic properties, but introduces a convenient context and language to talk about quotients (by definable equivalence relations) and *canonical parameters* of definable sets. In the context of a stable theory it also ensures the existence of *canonical bases* for arbitrary complete types, generalizing the notion of a field of definition of an algebraic variety.

The generalization of stability theory to the wider class of simple theories necessitated the introduction of hyperimaginaries, classes of countable tuples modulo \emptyset -type-definable equivalence relations. Although the relevant model-theory for hyperimaginaries has been reasonably well understood [4], they cannot simply be added as extra sorts to the underlying structure, since inequality of two hyperimaginaries is equivalent to non-equivalence, and thus *a priori* is an open, but not a closed condition. While hyperimaginary elements are needed for the

The first author was partially supported by research project MTM 2008-01545 of the Spanish government and research project 2009SGR 00187 of the Catalan government. The second author was partially supported by the research project ANR-09-BLAN-0047 of the Agence Nationale de la Recherche.

This work was partially done while the first author was visiting the Institut Camille Jordan of Lyon. The first author wishes to express his gratitude to the members of Lyon logic group for their hospitality. He also would like to thank Enrique Casanovas for his valuable comments.

general theory, all known examples of a simple theory *eliminate* them in the sense that they are interdefinable (or at least interbounded) with a sequence of ordinary imaginaries; the latter condition is called *weak elimination*. It has thus been asked (and even been conjectured):

Question. Do all simple theories eliminate hyperimaginaries?

The answer is positive for stable theories [11], and for supersimple theories [1]. Among non-simple theories, the relation of being infinitely close in a non-standard real-closed field gives rise to non-eliminable hyperimaginaries; Casanovas and the second author have constructed non-eliminable hyperimaginaries in a theory without the strict order property [3].

A hyperimaginary is *finitary* if it is the class of a finite tuple modulo a type-definable equivalence relation. Kim [5] has shown that small theories eliminate finitary hyperimaginaries, and a result of Lascar and Pillay [8] states that bounded hyperimaginaries can be eliminated in favour of finitary bounded ones. We shall show that in a CM-trivial simple theory all hyperimaginaries are interbounded with sequences of finitary hyperimaginaries. We shall deduce that in such a theory hyperimaginaries can be eliminated in favour of finitary ones. In particular, a small CM-trivial simple theory eliminates hyperimaginaries. However, even the question whether all one-based simple theories eliminate hyperimaginaries is still open.

Elimination of hyperimaginaries is closely related to another question, the *stable forking conjecture*:

Question. In a simple theory, if $a \not\downarrow_B M$ for some model M containing B , is there a stable formula in $\text{tp}(a/M)$ which forks over B ?

If we do not require M to be a model, nor to contain B , this is called *strong stable forking*. Every known simple theory has stable forking; Kim [6] has shown that one-based simple theories with elimination of hyperimaginaries have stable forking. Kim and Pillay [7] have strengthened this to show that one-based simple theories with weak elimination of imaginaries hyperimaginaries have strong stable forking; on the other hand pseudofinite fields (which are supersimple of SU-rank 1) do not. Conversely, stable forking implies weak elimination of hyperimaginaries (Adler).

While we shall not attack the stable forking conjecture as such, we shall show in the last section that the independence relation $x \downarrow_{y_1} y_2$ is stable, meaning that it cannot order an infinite indiscernible sequence.

2. PRELIMINARIES

As usual, we shall work in the monster model \mathfrak{C} of a complete first-order theory (with infinite models), and all sets of parameters and all sequences of elements will live in \mathfrak{C}^{eq} . Given any sequences a, b and any set of parameters A , we write $a \equiv_A b$ whenever a and b have the same type over A . We shall write $a \equiv_A^s b$ if in addition a and b lie in the same class modulo all A -definable finite equivalence relations (i.e. if a and b have the same *strong type* over A), and $a \equiv_A^{Ls} b$ if they lie in the same class modulo all A -invariant bounded equivalence relations (i.e. if a and b have the same *Lascar strong type* over A). Recall that a theory is G -compact over a set A iff \equiv_A^{Ls} is type-definable over A (in which case it is the finest bounded equivalence relation type-definable over A). A theory T is G -compact whenever it is G -compact over any A . In particular, simple theories are G -compact [5].

Definition 2.1. A hyperimaginary h is *bounded* if $h \in \text{bdd}(\emptyset)$; it is *finitary* if $h \in \text{dcl}^{heq}(a)$ for some finite tuple a of imaginaries, and *quasi-finitary* if $h \in \text{bdd}(a)$ for some finite tuple a of imaginaries.

Definition 2.2. A hyperimaginary h is *eliminable* if it is interdefinable with (equivalent to) a sequence $e = (e_i : i \in I)$ of imaginaries. T eliminates (finitary/quasi-finitary) hyperimaginaries if all (finitary/quasi-finitary) hyperimaginaries are eliminable.

Remark 2.3. [8, Corollary 1.5] A hyperimaginary is finitary iff it is equivalent to the class of a finite tuple modulo some type-definable equivalence relation.

Lemma 2.4. *Let e be a finitary hyperimaginary. If T eliminates finitary hyperimaginaries, then $T(e)$ eliminates finitary hyperimaginaries.*

Proof. Let a be a finite tuple with $e \in \text{dcl}^{heq}(a)$, and h a finitary hyperimaginary over e . So there is a finite tuple b with $h \in \text{dcl}^{heq}(eb) \subseteq \text{dcl}^{heq}(ab)$. Then there is a type-definable equivalence relation E over \emptyset with $e = a_E$, and a type-definable equivalence relation F_a over a with $h = b_{F_a}$. Moreover, F_a only depends on the E -class of a , that is, if $a' E a$, then $F_{a'} = F_a$.

Type-define an equivalence relation by

$$xy\bar{E}uv \iff xEu \wedge yF_x v.$$

It is easy to see that h is equivalent to $(ab)_{\bar{E}}$ over e . Moreover, $(ab)_{\bar{E}}$ is clearly finitary, and hence eliminable in T . So h is eliminable in $T(e)$. \square

Fact 2.5. [8, Proof of Proposition 2.2], [1, Lemma 2.17] Let h be a hyperimaginary and let a be a sequence of imaginaries such that $a \in \text{bdd}(h)$ and $h \in \text{dcl}^{heq}(a)$. Then, h is eliminable.

This is also Lemma 18.6 of [2] or Lemma 3.6.3 of [12] (the global assumption of simplicity is not used for this lemma).

Fact 2.6. [1, Lemma 2.18] Let h, e be hyperimaginaries with $h \in \text{bdd}(e)$. Then the set of e -conjugates of h is equivalent to a hyperimaginary h' .

Fact 2.7. [8, Theorem 4.15] A bounded hyperimaginary is equivalent to a sequence of finitary hyperimaginaries.

Proposition 2.8. *If T eliminates finitary hyperimaginaries, then T eliminates quasi-finitary hyperimaginaries.*

Proof. Let h be a quasi-finitary hyperimaginary and let a be a finite tuple of imaginaries such that $h \in \text{bdd}(a)$. Consider $a' \equiv_h a$ with $\text{bdd}(a) \cap \text{bdd}(a') = \text{bdd}(h)$. Let h' be the hyperimaginary corresponding to the set of aa' -conjugates of h . Then h' is aa' -invariant, and hence finitary. It is thus equivalent to a sequence e of imaginaries.

On the other hand, $h \in \text{bdd}(a) \cap \text{bdd}(a')$, as are all its aa' -conjugates. Thus $h' \in \text{bdd}(a) \cap \text{bdd}(a') = \text{bdd}(h)$. Hence $e \in \text{acl}^{eq}(h)$ and $h \in \text{bdd}(h') = \text{bdd}(e)$. By Fact 2.7, there is a sequence h'' of finitary hyperimaginaries equivalent to h over e . By Lemma 2.4 and elimination of finitary hyperimaginaries h'' is equivalent over e to a sequence e' of imaginaries. So $h \in \text{dcl}^{heq}(ee')$ and $e' \in \text{dcl}^{eq}(eh)$. Moreover, $ee' \in \text{acl}^{eq}(h)$ since $e \in \text{acl}^{eq}(h)$. Hence h is eliminable by Fact 2.5. \square

The following remarks and lemmata will need G -compactness.

Remark 2.9. Let T be G -compact over a set A . The following are equivalent:

- (1) $a \equiv_A^{Ls} b$ iff $a \equiv_A^s b$ for all sequences a, b .
- (2) $\text{Aut}(\mathfrak{C}/\text{bdd}(A)) = \text{Aut}(\mathfrak{C}/\text{acl}^{eq}(A))$.
- (3) $\text{bdd}(A) = \text{dcl}^{heq}(\text{acl}^{eq}(A))$.

Proof. Easy exercise. \square

Remark 2.10. Let T be a G -compact theory and assume further that $a \equiv_A^{Ls} b \Leftrightarrow a \equiv_A^s b$ for all sequences a, b and for any set A . Let now h be a hyperimaginary and let e be a sequence of imaginaries such that h and e are interbounded. Then h is eliminable.

Proof. It follows from Remark 2.9 that $\text{bdd}(e) = \text{dcl}^{heq}(\text{acl}^{eq}(e))$. Fix an enumeration \bar{e} of $\text{acl}^{eq}(e)$ and observe that $h \in \text{dcl}^{heq}(\bar{e})$ and $\bar{e} \in \text{bdd}(h)$. Then apply Fact 2.5 to eliminate h . \square

It turns out for G -compact theories that elimination of hyperimaginaries can be decomposed as weak elimination of hyperimaginaries plus the equality between Lascar strong types and strong types over parameter sets.

Fact 2.11. [2, Proposition 18.27] Assume that T is G -compact. Then T eliminates all bounded hyperimaginaries iff $a \equiv^{Ls} b \Leftrightarrow a \equiv^s b$ for all sequences a, b .

Proof. The proof in [2] is nice and intuitive; however, we will give another one using Remark 2.10. If T eliminates bounded hyperimaginaries, then $\text{Aut}(\mathfrak{C}/\text{bdd}(\emptyset)) = \text{Aut}(\mathfrak{C}/\text{acl}^{eq}(\emptyset))$. By Remark 2.9 we get $\text{Lstp} = \text{stp}$. For the other direction, let $e \in \text{bdd}(\emptyset)$ and let \bar{a} be an enumeration of $\text{acl}^{eq}(\emptyset)$. It is clear that e and \bar{a} are interbounded. By Remark 2.10, e is eliminable. \square

Lemma 2.12. *Suppose T is G -compact and assume further that T eliminates finitary hyperimaginaries. Then $a \equiv_A^{Ls} b$ iff $a \equiv_A^s b$ for all sequences a, b and for any set A .*

Proof. Since T is G -compact, it is enough to check the condition for finite A . But then $T(A)$ eliminates finitary hyperimaginaries by Remark 2.4, and hence all bounded hyperimaginaries by Fact 2.7. Now applying Fact 2.11 we obtain that $a \equiv^{Ls} b$ iff $a \equiv^s b$ in $T(A)$. \square

3. ELIMINATION OF HYPERIMAGINARIES IN SIMPLE THEORIES

In this section T will be a simple theory.

Remark 3.1. Let a be an imaginary finite tuple and let h be a quasi-finitary hyperimaginary, say $h \in \text{bdd}(b)$. Then $\text{Cb}(a/h)$ is finitary, as $\text{Cb}(a/h) \in \text{bdd}(h) \cap \text{dcl}(ab)$.

Recall the definition of CM-triviality.

Definition 3.2. A simple theory T is *CM-trivial* if for every tuple a and for any sets $A \subseteq B$ with $\text{bdd}(aA) \cap \text{bdd}(B) = \text{bdd}(A)$ we have $\text{Cb}(a/A) \subseteq \text{bdd}(\text{Cb}(a/B))$.

Remark 3.3. As in [10, Corollary 2.5], in the definition of CM-triviality we may take $A \subseteq B$ to be models of the ambient theory and a to be a tuple from the home sort. Therefore, it makes no difference in the definition of CM-triviality whether we consider hyperimaginaries or just imaginaries.

Now we characterize canonical bases in simple CM-trivial theories in terms of finitary hyperimaginaries.

Lemma 3.4. *If $c \in \text{Cb}(a/e)$, then $\text{dcl}(ac) \cap \text{bdd}(c) \subseteq \text{Cb}(a/e)$. If moreover $e \in \text{dcl}(a)$, then $\text{Cb}(a/e) = \text{dcl}(a) \cap \text{bdd}(e)$.*

Proof. The second statement is essentially [1, Remark 3.8], see also [2, Proposition 18.7]. Put $c' = \text{dcl}(ac) \cap \text{bdd}(c)$. Since $c' \in \text{dcl}(ac)$ there is an equivalence relation E on $\text{tp}(a/c)$ type-definable over c such that c' is interdefinable over c with a_E . As $c' \in \text{bdd}(c)$ and $c \in \text{Cb}(a/e)$, the E -class of a over $\text{Cb}(a/e)$ is bounded and hence definable over $\text{Cb}(a/e)$, as $\text{tp}(a/\text{Cb}(a/e))$ is Lascar-strong.

Now if $e \in \text{dcl}(a)$ then clearly $e \in \text{Cb}(a/e)$. By the first part, $\text{dcl}(a) \cap \text{bdd}(e) \subseteq \text{Cb}(a/e)$. Conversely, $\text{dcl}(a) \cap \text{bdd}(e)$ contains e as well as the class of a modulo every type-definable equivalence relation which is bounded over e , so $\text{Cb}(a/e) \subseteq \text{dcl}(a) \cap \text{bdd}(e)$. \square

Proposition 3.5. *Assume the theory is simple CM-trivial. If a is a finite imaginary tuple, then*

$$\text{dcl}(\text{Cb}(a/b) : b \in X) \subseteq \text{dcl}(\text{Cb}(a/B)) \subseteq \text{bdd}(\text{Cb}(a/b) : b \in X),$$

where X is the set of all finitary $b \in \text{dcl}(\text{Cb}(a/B))$.

Proof. As $\text{Cb}(a/b) \subseteq \text{dcl}(ab) \cap \text{bdd}(b)$, we have $\text{Cb}(a/b) \subseteq \text{Cb}(a/B)$ for all $b \in X$ by Lemma 3.4. For the second inclusion, let \hat{b} be a representative of $\text{Cb}(a/b)$ for every $b \in X$ such that

$$(\hat{b} : b \in X) \downarrow_{(\text{Cb}(a/b):b \in X)} aB,$$

whence $(\hat{b} : b \in X) \downarrow_B a$. Now, if $a \not\downarrow_{(\hat{b}:b \in X)} B$ then there is a finite tuple $b' \in B \cup \{\hat{b} : b \in X\}$ and a formula $\varphi(x, b') \in \text{tp}(a/B, \hat{b} : b \in X)$ which divides over $(\hat{b} : b \in X)$. Put $\bar{b} = \text{bdd}(ab') \cap \text{bdd}(B, \hat{b} : b \in X)$. Then \bar{b} is a quasi-finitary hyperimaginary, and by CM-triviality

$$\text{Cb}(a/\bar{b}) \subseteq \text{bdd}(\text{Cb}(a/B, \hat{b} : b \in X)) = \text{bdd}(\text{Cb}(a/B)).$$

Since $\text{Cb}(a/\bar{b})$ is finitary by Remark 3.1, it belongs to X . Note that $b' \in \text{dcl}(\bar{b})$; but $a \downarrow_{\text{Cb}(a/\bar{b})} \bar{b}$, so $\varphi(x, b')$ cannot divide over $\text{Cb}(a/\bar{b})$, and even less over $(\hat{b} : b \in X)$. Thus, $a \downarrow_{(\hat{b}:b \in X)} B$, whence $a \downarrow_{(\text{Cb}(a/b):b \in X)} B$ by transitivity. Therefore

$$\text{Cb}(a/B) \subseteq \text{bdd}(\text{Cb}(a/b) : b \in X).$$

\square

Question. The same proof will work without assuming CM-triviality if for every finite tuple $b \in B$ there is some quasi-finitary hyperimaginary $\bar{b} \in \text{bdd}(B)$ with $b \in \text{dcl}(\bar{b})$ such that $\text{Cb}(a/\bar{b}) \subseteq \text{bdd}(\text{Cb}(a/B))$. Is this true in general?

We can now state (and prove) the main result.

Theorem 3.6. *Let T be a simple CM-trivial theory. Then every hyperimaginary is interbounded with a sequence of finitary hyperimaginaries.*

Proof. By Lemma 3.4 every hyperimaginary is interbounded with a canonical base. Since $\text{Cb}(A/B)$ is interdefinable with $\bigcup\{\text{Cb}(\bar{a}/B) : \bar{a} \in A \text{ finite}\}$, it is enough to show that canonical bases of types of finite tuples are interbounded with sequences of finitary hyperimaginaries. This is precisely Proposition 3.5. \square

Corollary 3.7. *A simple CM-trivial theory eliminates hyperimaginaries whenever it eliminates finitary ones.*

Proof. By Theorem 3.6 every hyperimaginary is interbounded with a sequence of finitary hyperimaginaries and so with a sequence of imaginaries. Since T is simple, it is G -compact, whence $\text{Lstp} = \text{stp}$ over any set by Lemma 2.12. We conclude that every hyperimaginary is eliminable by Remark 2.10. \square

Corollary 3.8. *Every small simple CM-trivial theory eliminates hyperimaginaries.*

Proof. A small simple theory eliminates finitary hyperimaginaries by [5]. Now apply Corollary 3.7. \square

4. STABLE INDEPENDENCE FOR CM-TRIVIAL THEORIES

Recall that an \emptyset -invariant relation $R(x, y)$ is *stable* if there is no infinite indiscernible sequence $(a_i, b_i : i < \omega)$ such that $R(a_i, b_j)$ holds if and only if $i < j$. In this section, we shall show that independence is a stable relation, even with varying base set. We hope that this will help elucidate the stable forking problem.

Theorem 4.1. *In a supersimple CM-trivial theory, the relation $R(x; y_1 y_2)$ given by $x \downarrow_{y_1} y_2$ is stable.*

Proof. Suppose not. Then there is an indiscernible sequence $I = (a_i : i \in \mathbb{Q})$ and tuples b, c such that

- $I^+ = (a_i : i > 0)$ is indiscernible over I^-bc ,
- $I^- = (a_i : i < 0)$ is indiscernible over I^+bc , and
- $a_i \downarrow_c b$ if and only if $i > 0$.

We consider limit types with respect to the cut at 0. Let $p = \lim(I/I)$, $p^+ = \lim(I^+/Ibc)$ and $p^- = \lim(I^-/Ibc)$. By finite satisfiability, p^+ and p^- are both non-forking extensions of p , which is Lascar-strong. Let

$$A = \text{Cb}(p) = \text{Cb}(p^+) = \text{Cb}(p^-) \in \text{bdd}(I^+) \cap \text{bdd}(I^-).$$

We consider first $e_0 = \text{bdd}(a_1c) \cap \text{bdd}(Ac)$. Then

$$\text{bdd}(a_1e_0) \cap \text{bdd}(Ae_0) = e_0,$$

whence by CM-triviality

$$A_0 = \text{Cb}(a_1/e_0) \in e_0 \cap \text{bdd}(\text{Cb}(a_1/Ae_0)) \subseteq \text{bdd}(a_1c) \cap \text{bdd}(A).$$

Note that $a_1 \downarrow_c b$ yields $a_1 \downarrow_{A_0c} b$. Moreover $a_1 \downarrow_{A_0} e_0$ implies $a_1 \downarrow_{A_0} c$, whence $a_1 \downarrow_{A_0} cb$. On the other hand, $b \downarrow_c A_0$; since $b \not\downarrow_c a_{-1}$ we must have $bc \not\downarrow_{A_0} a_{-1}$. Since I remains indiscernible over A_0 , we may add A_0 to the parameters and suppose $c = \emptyset$ (replacing b by bc).

Fact 4.2. [12, Theorem 5.2.18] In a supersimple theory, for any finitary a there is some $B \downarrow a$ and hyperimaginary finite tuple \bar{a} of independent realizations of regular types over B , such that a and \bar{a} are domination-equivalent over B .

By Fact 4.2 there is $B \downarrow a_1$ and \bar{a}_1 , an independent tuple of realizations of regular types, such that a_1 and \bar{a}_1 are domination-equivalent over B . Since $B \downarrow a_1$ and I is indiscernible, we may assume by [12, Theorem 2.5.4] that $Ba_i \equiv Ba_1$ for all $i \in \mathbb{Q}$, and $B \downarrow I$. So there are \bar{a}_i for $i \in \mathbb{Q}$ with $Ba_i\bar{a}_i \equiv Ba_1\bar{a}_1$. We can also assume $B \downarrow_I b$, whence $B \downarrow Ib$. In particular, $b \downarrow_{a_i} B$ for $i > 0$, whence $b \downarrow Ba_i$ and thus $b \downarrow_B a_i$, while for $i < 0$ we have $b \not\downarrow a_i B$ and $b \downarrow B$, whence $b \not\downarrow_B a_i$. Similarly, there is $C \downarrow_B Ib$ such that there is a tuple \bar{b} of realizations of regular types domination-equivalent with b over BC . Then $b \downarrow_{a_i B} C$ and $b \downarrow_B a_i$ imply $b \downarrow_B a_i C$ and $b \downarrow_{BC} a_i$ for $i > 0$, whence $a_i \downarrow_B bC$. For $i < 0$ on the other hand, $b \not\downarrow_B a_i$ and $b \downarrow_B C$ yield $b \not\downarrow_{BC} a_i$.

By compactness and Ramsey we may suppose in addition that $\bar{I} = (\bar{a}_i : i \in \mathbb{Q})$ is indiscernible over B , $\bar{I}^+ = (\bar{a}_i : i > 0)$ is indiscernible over $BbC\bar{I}^-$ and $\bar{I}^- = (\bar{a}_i : i < 0)$ is indiscernible over $BbC\bar{I}^+$. We shall add B to the parameters and suppress it from the notation.

Since $a_i \downarrow C$ for all $i \in \mathbb{Q}$ and $b \downarrow C$, the domination-equivalence of a_i and \bar{a}_i is preserved over C . Thus $\bar{a}_i \downarrow_C \bar{b}$ for $i > 0$, whereas $\bar{a}_i \not\downarrow_C \bar{b}$ for $i < 0$. We may assume that $\bar{a}'_i \downarrow_C \bar{b}'$ for any subtuples $\bar{a}'_i \subseteq \bar{a}_i$ and $\bar{b}' \subseteq \bar{b}$ such that at least one inclusion is proper. It follows that all the regular types in $\bar{a}_i\bar{b}$ are non-orthogonal. Let $w_{\mathcal{P}}(\cdot)$ denote the weight with respect to that non-orthogonality class \mathcal{P} of regular types.

We again consider limit types with respect to the cut at 0. Let $\bar{p} = \lim(\bar{I}/\bar{I})$, $\bar{p}^+ = \lim(\bar{I}^+/\bar{I}bC)$ and $\bar{p}^- = \lim(\bar{I}^-/\bar{I}bC)$. Again, \bar{p} is Lascar-strong and \bar{p}^+ and \bar{p}^- are non-forking extensions of \bar{p} by finite

satisfiability; let

$$\bar{A} = \text{Cb}(\bar{p}) = \text{Cb}(\bar{p}^+) = \text{Cb}(\bar{p}^-) \in \text{bdd}(\bar{I}^+) \cap \text{bdd}(\bar{I}^-).$$

Note that \bar{A} is \mathcal{P} -simple of finite \mathcal{P} -weight $w_{\mathcal{P}}(\bar{A})$.

Put $e_1 = \text{bdd}(\bar{a}_{-1}\bar{b}C) \cap \text{bdd}(\bar{A}\bar{b}C)$. Then

$$\text{bdd}(\bar{a}_{-1}e_1C) \cap \text{bdd}(\bar{A}e_1C) = e_1,$$

whence by CM-triviality

$$A_1 = \text{Cb}(\bar{a}_{-1}/e_1) \in e_1 \cap \text{bdd}(\text{Cb}(\bar{a}_{-1}/\bar{A}e_1C)) \subseteq \text{bdd}(\bar{a}_{-1}\bar{b}C) \cap \text{bdd}(\bar{A}).$$

Then $\bar{a}_{-1} \downarrow_{A_1} C\bar{b}$.

If $w_{\mathcal{P}}(\bar{b}/A_1C\bar{a}_1) < w_{\mathcal{P}}(\bar{b}/A_1C)$, we can add A_1 to the parameters, and replace the \bar{a}_i and \bar{b} by maximal subtuples independent of A_1 over C . Exchanging the rôles of \mathbb{Q}^+ and \mathbb{Q}^- , we obtain a counterexample of reduced \mathcal{P} -weight, since

$$w_{\mathcal{P}}(\bar{a}_i/A_1) = w_{\mathcal{P}}(\bar{a}_{-1}/A_1C\bar{b}) \leq w_{\mathcal{P}}(\bar{a}_{-1}/C\bar{b}) < w_{\mathcal{P}}(\bar{a}_{-1}/C) = w_{\mathcal{P}}(\bar{a}_i).$$

So we may assume $w_{\mathcal{P}}(\bar{b}/A_1C\bar{a}_1) = w_{\mathcal{P}}(\bar{b}/A_1C)$. Now $\bar{a}_{-1} \downarrow_{A_1} C\bar{b}$ implies $w_{\mathcal{P}}(\bar{a}_{-1}/A_1C\bar{b}) = w_{\mathcal{P}}(\bar{a}_{-1}/A_1)$, whence

$$\begin{aligned} w_{\mathcal{P}}(\bar{a}_{-1}\bar{b}/C) &= w_{\mathcal{P}}(\bar{a}_{-1}/A_1C\bar{b}) + w_{\mathcal{P}}(\bar{b}/A_1C) + w_{\mathcal{P}}(A_1/C) \\ &= w_{\mathcal{P}}(\bar{a}_{-1}/A_1) + w_{\mathcal{P}}(\bar{b}/A_1C) + w_{\mathcal{P}}(A_1/C) \\ &= w_{\mathcal{P}}(\bar{a}_1/A_1) + w_{\mathcal{P}}(\bar{b}/A_1C) + w_{\mathcal{P}}(A_1/C) \\ &\geq w_{\mathcal{P}}(\bar{a}_1/A_1C) + w_{\mathcal{P}}(\bar{b}/A_1C\bar{a}_1) + w_{\mathcal{P}}(A_1/C) \\ &= w_{\mathcal{P}}(\bar{a}_1\bar{b}/A_1C) + w_{\mathcal{P}}(A_1/C) \\ &= w_{\mathcal{P}}(\bar{a}_1\bar{b}A_1/C) \\ &= w_{\mathcal{P}}(A_1/\bar{a}_1\bar{b}C) + w_{\mathcal{P}}(\bar{a}_1\bar{b}/C) \\ &\geq w_{\mathcal{P}}(\bar{a}_1\bar{b}/C) > w_{\mathcal{P}}(\bar{a}_{-1}\bar{b}/C), \end{aligned}$$

since $\bar{a}_{-1} \not\downarrow_C \bar{b}$ and $\bar{a}_1 \downarrow_C \bar{b}$. This final contradiction proves the theorem. \square

Remark 4.3. Note that the proof only uses the conclusion of Fact 4.2. The theorem thus still holds for simple CM-trivial theories with finite weights (strongly simple theories) and enough regular types, for instance CM-trivial simple theories without dense forking chains.

Question. By [9, Theorem 4.20] it is sufficient to assume that every regular type is CM-trivial, as this implies global CM-triviality. However, for a regular type p a more general notion of CM-triviality is often more appropriate, namely

$$\text{cl}_p(aA) \cap \text{cl}_p(B) = \text{cl}_p(A) \quad \Rightarrow \quad \text{Cb}(a/\text{cl}_p(A)) \subseteq \text{cl}_p(\text{Cb}(a/\text{cl}_p(B))).$$

If this holds for all regular types p , is independence still stable?

Corollary 4.4. *An ω -categorical supersimple CM-trivial theory has stable independence.*

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