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HYPERIMAGINARIES IN SIMPLE CM -TRIVIAL THEORIES

D. PALACÍN AND F. O. WAGNER

ABSTRACT. A simple CM -trivial theory eliminates hyperimaginaries whenever it eliminates finitary hyperimaginaries.

1. INTRODUCTION

An important notion introduced by Shelah for a first-order theory is that of an *imaginary* element: the class of a finite tuple by a \emptyset -definable equivalence relation. The construction obtained by adding all imaginary elements to a structure does not change its basic model-theoretic properties, but introduces a convenient context and language to talk about quotients (by definable equivalence relations) and *canonical parameters* of definable sets. In the context of a stable theory it also ensures the existence of *canonical bases* for arbitrary complete types, generalizing the notion of a field of definition of an algebraic variety.

The generalization of stability theory to the wider class of simple theories necessitated the introduction of hyperimaginaries, classes of countable tuples modulo \emptyset -type-definable equivalence relations. Although the relevant model-theory for hyperimaginaries has been reasonably well understood [4], they cannot simply be added as extra sorts to the underlying structure, since inequality of two hyperimaginaries is equivalent to non-equivalence, and thus *a priori* is an open, but not a closed condition. While hyperimaginary elements are needed for the general theory, all known examples of a simple theory *eliminate* them in the sense that they are interdefinable with a sequence of ordinary imaginaries. It has thus been asked (and even been conjectured):

Question. Do all simple theories eliminate hyperimaginaries?

The answer is positive for stable theories [7], and for supersimple theories [1]. Among non-simple theories, the relation of being infinitely close in a non-standard real-closed field gives rise to non-eliminable hyperimaginaries; Casanovas and the second author have constructed non-eliminable hyperimaginaries in a theory without the strict order property [3].

A hyperimaginary is *finitary* if it is the class of a finite tuple modulo a type-definable equivalence relation. Kim [5] has shown that small theories eliminate finitary hyperimaginaries, and a result of Lascar and Pillay [6] states that bounded hyperimaginaries can be eliminated in favour of finitary bounded ones. We shall show that in a CM -trivial simple theory hyperimaginaries can be eliminated in favour of finitary ones. In particular, a small CM -trivial simple theory eliminates hyperimaginaries.

2. PRELIMINARIES

As usual, we shall work in the monster model \mathfrak{C} of a complete first-order theory (with infinite models), and all sets of parameters and all sequences of elements will live in \mathfrak{C}^{eq} . Given any sequences a, b and any set of parameters A , we write $a \equiv_A b$ whenever a and b have

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the same type over A . We shall write $a \equiv_A^s b$ if in addition a and b lie in the same class modulo all A -definable finite equivalence relations (i.e. if a and b have the same *strong type* over A), and $a \equiv_A^{Ls} b$ if they lie in the same class modulo all A -invariant bounded equivalence relations (i.e. if a and b have the same *Lascaz strong type* over A). Recall that a theory is G -compact over a set A iff \equiv_A^{Ls} is type-definable over A (in which case it is the finest bounded equivalence relation type-definable over A). A theory T is G -compact whenever it is G -compact over any A . In particular, simple theories are G -compact [5].

Definition 2.1. A hyperimaginary h is *bounded* if $h \in \text{bdd}(\emptyset)$; it is *finitary* if $h \in \text{dcl}^{heq}(a)$ for some finite tuple a of imaginaries, and *quasi-finitary* if $h \in \text{bdd}(a)$ for some finite tuple a of imaginaries.

Definition 2.2. A hyperimaginary h is *eliminable* if it is interdefinable with (equivalent to) a sequence $e = (e_i : i \in I)$ of imaginaries. T eliminates (finitary/quasi-finitary) hyperimaginaries if all (finitary/quasi-finitary) hyperimaginaries are eliminable.

Remark 2.3. [6, Corollary 1.5] A hyperimaginary is finitary iff it is equivalent to the class of a finite tuple modulo some type-definable equivalence relation.

Lemma 2.4. *Let e be a finitary hyperimaginary. If T eliminates finitary hyperimaginaries, then $T(e)$ eliminates finitary hyperimaginaries.*

Proof. Let a be a finite tuple with $e \in \text{dcl}^{heq}(a)$, and h a finitary hyperimaginary over e . So there is a finite tuple b with $h \in \text{dcl}^{heq}(eb) \subseteq \text{dcl}^{heq}(ab)$. Then there is a type-definable equivalence relation E over \emptyset with $e = a_E$, and a type-definable equivalence relation F_a over a with $h = b_{F_a}$. Moreover, F_a only depends on the E -class of a , that is, if $a'Ea$, then $F_{a'} = F_a$.

Type-define an equivalence relation by

$$xy\bar{E}uv \Leftrightarrow xEu \wedge yF_xv.$$

It is easy to see that h is equivalent to $(ab)_E$ over e . Moreover, $(ab)_E$ is clearly finitary, and hence eliminable in T . So h is eliminable in $T(e)$. \square

Fact 2.5. [6, Proof of Proposition 2.2], [1, Lemma 2.17] Let h be a hyperimaginary and let a be a sequence of imaginaries such that $a \in \text{bdd}(h)$ and $h \in \text{dcl}^{heq}(a)$. Then, h is eliminable.

This is also Lemma 18.6 of [2] or Lemma 3.6.3 of [8] (the global assumption of simplicity is not used for this lemma).

Fact 2.6. [1, Lemma 2.18] Let h, e be hyperimaginaries with $h \in \text{bdd}(e)$. Then the set of e -conjugates of h is equivalent to a hyperimaginary h' .

Fact 2.7. [6, Theorem 4.15] A bounded hyperimaginary is equivalent to a sequence of finitary hyperimaginaries.

Proposition 2.8. *If T eliminates finitary hyperimaginaries, then T eliminates quasi-finitary hyperimaginaries.*

Proof. Let h be a quasi-finitary hyperimaginary and let a be a finite tuple of imaginaries such that $h \in \text{bdd}(a)$. Consider $a' \equiv_h a$ with $\text{bdd}(a) \cap \text{bdd}(a') = \text{bdd}(h)$. Let h' be the hyperimaginary corresponding to the set of aa' -conjugates of h . Then h' is aa' -invariant, and hence finitary. It is thus equivalent to a sequence e of imaginaries.

On the other hand, $h \in \text{bdd}(a) \cap \text{bdd}(a')$, as are all its aa' -conjugates. Thus $h' \in \text{bdd}(a) \cap \text{bdd}(a') = \text{bdd}(h)$. Hence $e \in \text{acl}^{eq}(h)$ and $h \in \text{bdd}(h') = \text{bdd}(e)$. By Fact 2.7, there is a sequence h'' of finitary hyperimaginaries equivalent to h over e . By Lemma 2.4 and elimination of finitary hyperimaginaries h'' is equivalent over e to a sequence e' of imaginaries. So $h \in \text{dcl}^{heq}(ee')$ and $e' \in \text{dcl}^{eq}(eh)$. Moreover, $ee' \in \text{acl}^{eq}(h)$ since $e \in \text{acl}^{eq}(h)$. Hence h is eliminable by Fact 2.5. \square

The following remarks and lemmata will need G -compactness.

Remark 2.9. Let T be G -compact over a set A . The following are equivalent:

- (1) $a \equiv_A^{Ls} b$ iff $a \equiv_A^s b$ for all sequences a, b .
- (2) $Aut(\mathfrak{C}/bdd(A)) = Aut(\mathfrak{C}/acl^{eq}(A))$.
- (3) $bdd(A) = dcl^{heq}(acl^{eq}(A))$.

Proof. Easy exercise. \square

Remark 2.10. Let T be a G -compact theory and assume further that $a \equiv_A^{Ls} b \Leftrightarrow a \equiv_A^s b$ for all sequences a, b and for any set A . Let now h be a hyperimaginary and let e be a sequence of imaginaries such that h and e are interbounded. Then h is eliminable.

Proof. It follows from Remark 2.9 that $bdd(e) = dcl^{heq}(acl^{eq}(e))$. Fix an enumeration \bar{e} of $acl^{eq}(e)$ and observe that $h \in dcl^{heq}(\bar{e})$ and $\bar{e} \in bdd(h)$. Then apply Fact 2.5 to eliminate h . \square

Fact 2.11. [2, Proposition 18.27] Assume that T is G -compact. Then T eliminates all bounded hyperimaginaries iff $a \equiv_A^{Ls} b \Leftrightarrow a \equiv_A^s b$ for all sequences a, b .

Proof. The proof in [2] is nice and intuitive; however, we will give another one using Remark 2.10. If T eliminates bounded hyperimaginaries, then $Aut(\mathfrak{C}/bdd(\emptyset)) = Aut(\mathfrak{C}/acl^{eq}(\emptyset))$. By Remark 2.9 we get $Lstp = stp$. For the other direction, let $e \in bdd(\emptyset)$ and let \bar{a} be an enumeration of $acl^{eq}(\emptyset)$. It is clear that e and \bar{a} are interbounded. By Remark 2.10, e is eliminable. \square

Lemma 2.12. *Suppose T is G -compact and assume further that T eliminates finitary hyperimaginaries. Then $a \equiv_A^{Ls} b$ iff $a \equiv_A^s b$ for all sequences a, b and for any set A .*

Proof. Since T is G -compact, it is enough to check the condition for finite A . But then $T(A)$ eliminates finitary hyperimaginaries by Remark 2.4, and hence all bounded hyperimaginaries by Fact 2.7. Now applying Fact 2.11 we obtain that $a \equiv_A^{Ls} b$ iff $a \equiv_A^s b$ in $T(A)$. \square

3. ELIMINATION OF HYPERIMAGINARIES IN SIMPLE THEORIES

In this section T will be a simple theory.

Fact 3.1. [6, Proposition 2.2] T eliminates hyperimaginaries iff the canonical base of every Lascar strong type is equivalent to a sequence of imaginaries.

Proof. Let $h = a_E$ be a hyperimaginary. Then, $h \in dcl^{heq}(Cb(a/h))$ and $Cb(a/h) \in bdd(h)$. So h is eliminable by Fact 2.5. \square

Lemma 3.2. *Let a be an imaginary finite tuple and let h be a quasi-finitary hyperimaginary. Then $Cb(a/h)$ is finitary.*

Proof. Suppose $h \in bdd(b)$ for some finite tuple b ; we may choose $b \downarrow_h a$. We claim that $Cb(a/h) \in dcl^{heq}(ba)$. Indeed, every bounded h -type-definable equivalence relation E has a bounded refinement E' type-definable over b (namely the intersection of the b -conjugates of E), and the class of a modulo E' is definable over ab . \square

Recall the definition of CM -triviality.

Definition 3.3. A simple theory T is CM -trivial if for every tuple a and for any sets $A \subseteq B$ with $bdd(aA) \cap bdd(B) = bdd(A)$ we have $Cb(a/A) \subseteq bdd(Cb(a/B))$.

We can now state (and prove) the main result. In fact, it also holds if the condition of CM -triviality is only verified for imaginary a , A and B .

Theorem 3.4. *Let T be a simple CM -trivial theory. If T eliminates finitary hyperimaginaries, then T eliminates hyperimaginaries.*

Proof. By Fact 3.1 it is enough to show that canonical bases are eliminable. Since $\text{Cb}(A/B)$ is interdefinable with $\bigcup\{\text{Cb}(\bar{a}/B) : \bar{a} \in A \text{ finite}\}$, it is enough to show that canonical bases of types of finite tuples are eliminable.

Given a finite tuple a and some set B , let X be the set of all quasi-finitary hyperimaginaries $b \in \text{bdd}(B)$ such that $\text{bdd}(ab) \cap \text{bdd}(B) = \text{bdd}(b)$. We will show that

Claim. $\text{bdd}(\text{Cb}(a/B)) = \text{bdd}(\bigcup_{b \in X} \text{Cb}(a/b))$.

Proof. By CM -triviality $\text{Cb}(a/b) \subseteq \text{bdd}(\text{Cb}(a/B))$ for every $b \in X$, so $\text{bdd}(\text{Cb}(a/B)) \supseteq \bigcup_{b \in X} \text{Cb}(a/b)$. (If we only have CM -triviality for imaginaries, note that any $b \in X$ is equivalent to a sequence of imaginaries by Proposition 2.8.)

To prove the other inclusion suppose that $a \not\downarrow_{\bigcup_{b \in X} \text{Cb}(a/b)} B$. Hence there is a finite tuple $b' \in B$ and a formula $\varphi(x, b') \in \text{tp}(a/B)$ which divides over $\bigcup_{b \in X} \text{Cb}(a/b)$. Put $b'' = \text{bdd}(ab') \cap \text{bdd}(B)$. Then b'' is quasi-finitary and $\text{bdd}(ab'') \cap \text{bdd}(B) = \text{bdd}(b'')$, whence $b'' \in X$. Note that $b' \in \text{dcl}^{eq}(b'')$. But $a \downarrow_{\text{Cb}(a/b'')} b''$, so $\varphi(x, b')$ cannot divide over $\text{Cb}(a/b'')$, and even less over $\bigcup_{b \in X} \text{Cb}(a/b)$, a contradiction. Thus $a \downarrow_{\bigcup_{b \in X} \text{Cb}(a/b)} B$, and $\text{Cb}(a/B) \subseteq \text{bdd}(\bigcup_{b \in X} \text{Cb}(a/b))$. \square

Since $\text{Cb}(a/b)$ is eliminable for every $b \in X$ by Lemma 3.2, there is a sequence e of imaginaries with $\text{Cb}(a/B) \in \text{bdd}(e)$ and $e \in \text{acl}^{eq}(\text{Cb}(a/B))$. Since T is simple, it is G -compact, whence $\text{Lstp} = \text{stp}$ over any set by Lemma 2.12. We conclude that $\text{Cb}(a/B)$ is eliminable by Remark 2.10. \square

Question. The same proof will work without assuming CM -triviality if for every $b \in B$ there is some $b' \in B$ with $b \in \text{dcl}^{eq}(b')$ and $\text{Cb}(a/b') \in \text{bdd}(\text{Cb}(a/B))$. Is this true in general?

Corollary 3.5. *Every small simple CM -trivial theory eliminates hyperimaginaries.*

Proof. A small simple theory eliminates finitary hyperimaginaries by [5]. Now apply Theorem 3.4. \square

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