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# Characterizing Quantum Properties of a Measurement Apparatus : Insights from the Retrodictive Approach

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Using the retrodictive approach of quantum physics, we show that the pre-measurement state retrodicted from the response of a measurement apparatus is a convenient tool to characterize its quantum properties. We propose a procedure for realizing the tomography of this state for any measurement apparatus. We then translate in terms of this state some interesting aspects of the quantum behavior of a detector, such as the projectivity, the ideality, the non-classicality or the non-gaussian character of its measurements. These properties are crucial in several quantum protocols, in particular in conditional preparation of non-classical states of light or measurement-driven quantum information processing.

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*Introduction.*—In the quantum world, the measurement process plays a central role, as it leads to an unavoidable and strong modification of the system which is measured. This singular feature has important consequences concerning the foundations of quantum theory [1, 2], but it also has many practical implications, because the information obtained through a measurement is often used for driving quantum information processing [3, 4] or preparing a target state conditioned on the result of this measurement [5, 6]. To master as much as possible these conditionings, it is therefore very important to characterize as precisely as possible the quantum properties of the measurement apparatus that one uses.

The first experimental quantum characterization of a detector has been achieved only very recently in quantum optics [7]. By using the quantum detector tomography technique (QDT), it is possible to realize the reconstruction of positive operator valued measures (POVMs) [8] characterizing any measuring device. This technique, by probing the behavior of its responses with a set of known states [9], gives a complete characterization of the detector seen as a black-box, i.e. only characterized by its responses and without any assumptions about its internal operation. QDT has opened the path of experimental study of novel concepts such as the non-classicality of detectors, with fundamental significance and particular relevance for experimental quantum state engineering.

The aim of this paper is to show that the retrodictive approach of quantum physics [10, 11], which is complementary to the usual predictive one, provides interesting physical insights on the behaviour of a measurement apparatus in terms of a quantum state: the *pre-measurement state*. The quantum properties of a measurement performed by a detector can then be associated with the properties of its pre-measurement state. We will first introduce the pre-measurement state from mathe-

matical foundations of quantum physics, and we will then propose a procedure for reconstructing this state from experimental data. Finally, we will examine several properties of a measurement apparatus from this perspective. In particular, we will provide estimators of certain properties, such as the ideality/projectivity of measurements or their fidelity with projective measurements. We will introduce a precise meaning for the non-classicality of measurements by illustrating its relevance for conditional preparation of non-classical states of light.

*States and Propositions.*—As a preliminary step, we recall a mathematical result demonstrated in [12], which is the recent generalization of the Gleason's theorem [13]. From very general requirements about probabilities and the mathematical structure of the Hilbert space, it provides the general expression of probabilities of checking any proposition about the system. First we remind that a proposition  $P_n$  is a property of the system corresponding to a *precise value for a given observable*. This one is represented in the Hilbert space by a *proposition operator*  $\hat{P}_n$ , which is in the simplest case a projector on the eigenstate corresponding to such a value. In the most general case, it can be represented only by a *hermitian and positive operator*.

One assumes that the probability  $\text{Pr}(n)$  of checking the proposition  $P_n$  on the physical system satisfies the three following conditions:

1.  $0 \leq \text{Pr}(n) \leq 1$  for any proposition  $P_n$ .
2.  $\sum_n \text{Pr}(n) = 1$  for any *exhaustive* set of propositions such that  $\sum_n \hat{P}_n = \hat{1}$ .
3.  $\text{Pr}(n_1 \text{ or } n_2 \text{ or } \dots) = \text{Pr}(n_1) + \text{Pr}(n_2) + \dots$  for any *non-exhaustive* set of propositions such that  $\hat{P}_{n_1} + \hat{P}_{n_2} + \dots \leq \hat{1}$ .

According to this theorem for a system needing predictions (i.e. with a Hilbert space of dimension  $D \geq 2$ ), this probability is given by  $\text{Pr}(n) = \text{Tr}\{\hat{\rho} \hat{P}_n\}$  in which  $\hat{\rho}$  is a hermitian, positive, and normalized operator. This operator allows us to make predictions about any properties

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of the system, and thus constitutes the most general form of its *quantum state*.

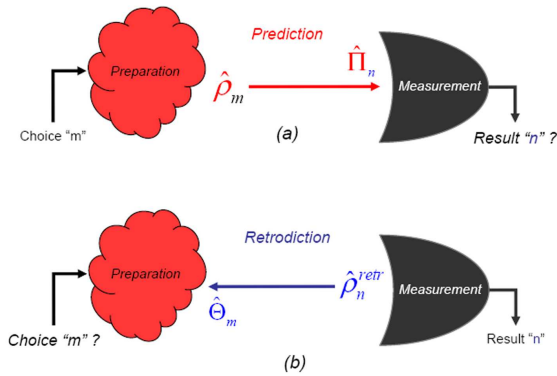


FIG. 1: (color online) (a) *Predictive approach* - From a preparation choice  $m$ , the prepared state  $\hat{\rho}_m$  resulting from this choice is used for making predictions about subsequent measurements performed on this preparation. (b) *Retrodictive approach* - From a measurement result  $n$ , the retrodicted state  $\hat{\rho}_n^{\text{retr}}$  assigned to the measured system allows retrodictions about state preparations leading to the result  $n$ .

*State preparations and Measurements.*— In quantum physics, any situation can be summarized by a preparation, an evolution and a measurement. One can make predictions about the preparation choices or about the measurement results. This corresponds to two approaches: the conventional predictive one, which provides predictions about measurement results, and the less usual retrodictive one, which provides predictions about preparation choices (called retrodictions). Each approach requires a quantum state and an exhaustive set of propositions about it, according to the previous section, as we will precise it in the following.

*Predictive approach.*— This is the usual approach of quantum physics: one prepares the system in a given state after some preparation process, which is generally a well-mastered physical phenomenon. Such a preparation of the system in a state  $\hat{\rho}_m$  can be associated with a piece of information that we call the *choice*  $m$ . With the prepared state  $\hat{\rho}_m$  resulting from this choice, we can make predictions about measurement results  $n$ , labeling the proposition operators which are simply the POVMs describing the behavior of responses of the apparatus performing their tests. Thus, from the previous theorem, the probability of obtaining the result  $n$  after the system was prepared in the state  $\hat{\rho}_m$  takes the form:

$$\Pr(n|m) = \text{Tr}\{\hat{\rho}_m \hat{\Pi}_n\}, \quad (1)$$

where  $\hat{\Pi}_n$  is the POVM element corresponding to the result  $n$ . This is in fact the Born's rule on which the conventional interpretation of quantum physics is based.

*Retrodictive approach.*— This approach [10, 11] is less popular in quantum physics. It has sometimes been used in quantum optics for instance to simplify the description

of protocols, like in state truncation [14, 15]. Let us start with an example in order to clarify this approach in a simple case: we consider a perfect photon counter able to discern the number of absorbed photons. If this detector displays the result  $n$  counts, the checked proposition is the projector  $\hat{P}_n = |n\rangle\langle n|$  on the photon-number state  $|n\rangle$ , and the pre-measurement state is precisely this photon-number state  $\hat{\rho}_n^{\text{retr}} = \hat{P}_n$ , with which we can make predictions about the preparations leading to such a result. Thus, for instance for preparations of the light in the photon-number states  $|m\rangle$ , the only possible preparation corresponds to  $m = n$  for which the retrodicted probability is equal to 1.

According to the generalization of Gleason's theorem, the retrodictive approach also needs a state and propositions. The state is assigned to the system on the basis of the measurement result  $n$ . This is the pre-measurement state  $\hat{\rho}_n^{\text{retr}}$  with which we make retrodictions about state preparations leading to this result. The propositions about this state simply correspond to the different preparations of the measured system before its interaction with the apparatus. We note  $\hat{\Theta}_m$  the proposition operator associated to each preparation choice  $m$ , and in order to have an *exhaustive set* of propositions, these operators  $\hat{\Theta}_m$  should be a resolution of the Hilbert space,  $\sum_m \hat{\Theta}_m = \hat{1}$ . Thus, the retrodictive probability of preparing the system in the state  $\hat{\rho}_m$  when the measurement gives the result  $n$  can be written as

$$\Pr(m|n) = \text{Tr}\{\hat{\rho}_n^{\text{retr}} \hat{\Theta}_m\}. \quad (2)$$

As previously derived by Barnett et al.[14] from an analogy with Born's rule, the expressions of the pre-measurement state  $\hat{\rho}_n^{\text{retr}}$  and proposition operators  $\hat{\Theta}_m$  can be derived from the predictive probabilities  $\Pr(n|m)$  from Bayes' theorem. When a measurement gives a result  $n$ , corresponding to a POVM element  $\hat{\Pi}_n$ , the pre-measurement state retrodicted from this result is simply given by

$$\hat{\rho}_n^{\text{retr}} = \hat{\Pi}_n / \text{Tr}\{\hat{\Pi}_n\}, \quad (3)$$

and the proposition operators are linked to the possible preparations by

$$\hat{\Theta}_m = D\Pr(m) \hat{\rho}_m \quad (4)$$

with  $\Pr(m)$  the probability of preparing the system in the state  $\hat{\rho}_m$ .

The strong condition that the operators  $\hat{\Theta}_m$  should be a resolution of the Hilbert space can also be viewed as a maximization requirement on the Von Neumann entropy,  $\mathcal{S}[\hat{\rho}] = -\text{Tr}\{\hat{\rho} \log \hat{\rho}\}$ , by the state  $\hat{\rho} = \sum_m \Pr(m) \hat{\rho}_m = \hat{1}/D$ . This state is then maximally mixed and probes all responses of the measurement apparatus. It also corresponds for example to an 'unread' measurement [16] in a conditional preparation scheme, i.e. a mixture of states conditioned on each result  $m$ , weighted by their respective success probabilities  $\Pr(m)$ . It can also be obtained

by a random preparation of the different states  $\hat{\rho}_m$ , as we will see below.

Finally, let us note that the retrodicted state can be propagated backward in time using the usual quantum evolution or propagation equations [15, 17]. This feature can be easily understood since the pre-measurement state is obtained from an observable (i.e. here a POVM) which evolves backward in time in the Heisenberg picture of the predictive approach.

*Quantum tomography of retrodicted states.*— We show now that the retrodicted states can be directly reconstructed from experimental data by using the same tools as the usual quantum state tomography. However, the tomography is here based on retrodictive approach and preparation choices contrary to the usual ones based on Born's rule and measurement results. At the fundamental level, this reconstruction shows the relevance of the retrodictive approach since the retrodicted state can be deduced from experimental data, in the same way as states of the predictive approach.

We probe the detector with a mixture of some states  $\{\hat{\rho}_m\}$ , characterized by the density matrix:

$$\hat{\rho}_{probe} = \sum_m \Pr(m) \hat{\rho}_m \quad (5)$$

The probabilities  $\Pr(m)$  of preparing the states  $\hat{\rho}_m$  are chosen in such a way that  $\det\{\hat{\rho}_{probe}\} \neq 0$  in order to generate the proposition operators:

$$\hat{\Lambda}_m = \Pr(m) (\hat{\sigma}^{-1})^\dagger \hat{\rho}_m \hat{\sigma}^{-1} \quad (6)$$

where we use the Cholesky decomposition of the mixture  $\hat{\rho}_{probe} = \hat{\sigma}^\dagger \hat{\sigma}$ .

Each time the result  $n$  occurs, we record the preparation choice  $m$  leading to this result. Thus, we directly measure the retrodictive probabilities  $\Pr(m|n)$  of preparing the light in the state resulting of the choice  $m$  and leading to the result  $n$ . Then, these experimental data are used for reconstructing the state  $\hat{\rho}_n$  giving the conditional probabilities  $\Pr(m|n) = \text{Tr}\{\hat{\rho}_n \hat{\Lambda}_m\}$  which are the closest to those measured. To this end, we replace POVMs corresponding to measurements in a reconstruction method, a Maximum-Likelihood iteration for instance, by the preparation operators  $\hat{\Lambda}_m$  which are also hermitian and positive operators resolving the Hilbert space. The pre-measurement state, retrodicted from the response  $n$  of the apparatus, is then given by

$$\hat{\rho}_n^{retr} = \hat{\sigma}^{-1} \hat{\rho}_n (\hat{\sigma}^{-1})^\dagger / \text{Tr}\{\hat{\sigma}^{-1} \hat{\rho}_n (\hat{\sigma}^{-1})^\dagger\}. \quad (7)$$

Moreover, we can also obtain the POVM from the reconstructed state  $\hat{\rho}_n$ :

$$\hat{\Pi}_n = \Pr(n) \hat{\sigma}^{-1} \hat{\rho}_n (\hat{\sigma}^{-1})^\dagger. \quad (8)$$

Note that we obtain the POVM element by focusing only on one response, contrary to a QDT which needs all responses of the device. This ability turns out to be convenient for a measurement apparatus with a continuous set of responses.

Finally, it is worth noting that the retrodictive probabilities  $\Pr(m|n)$  can also be obtained from data taken in QDT experiments [7], in which one directly measures the predictive probabilities (1) for each choice  $m$  and each result  $n$ . Indeed, if the preparation rate is the same for all the probe states  $\hat{\rho}_m$ , the probability of preparing the state  $\hat{\rho}_m$  is simply given by  $\Pr(m) = 1/M$ , where  $M$  is the number of different preparation choices. These retrodicted probabilities are then obtained by using Bayes' theorem:

$$\Pr(m|n) = \frac{\Pr(n|m)}{\sum_{m'=1}^M \Pr(n|m')}. \quad (9)$$

*Quantum properties of a measurement revealed by its pre-measurement state.*— We finally translate some quantum properties of measurements in terms of their pre-measurement states.

*1-Projectivity and Ideality.* An 'ideal' measurement checks a simple proposition corresponding to a projector  $\hat{\Pi}_n = |\psi_n\rangle\langle\psi_n|$  in the Hilbert space. However, in more realistic situations, a measuring device is characterized by POVM elements which are not at all projectors. An evaluation of the projectivity of a measurement is given by the purity  $\pi_n$  of its pre-measurement state:

$$\pi_n = \text{Tr} \left[ (\hat{\rho}_n^{retr})^2 \right]. \quad (10)$$

When the pre-measurement state is a pure quantum state with  $\pi_n = 1$ , the measurement performed by the apparatus is *projective* for the response  $n$ . However, it may be *non-ideal*. Indeed, the POVM element corresponding to such a projectivity ( $\pi_n = 1$ ) is in fact given by

$$\hat{\Pi}_n = \eta_n |\psi_n\rangle\langle\psi_n| \quad (11)$$

where  $\eta_n = \text{Tr}\{\hat{\Pi}_n\}$  can be viewed as the detection efficiency of the target state  $|\psi_n\rangle$ , by using the predictive approach in which the predictive probability  $\Pr(n|\psi_n) = \eta_n$ . The projective character of a measurement is revealed within the retrodictive approach, and not by the predictive one, for which the usual definition [1] of a projective measurement is the certainty of the result of two successive projective measurements. Such a measurement now corresponds to a very particular case: an ideal and projective measurement ( $\pi_n = \eta_n = 1$ ).

*2-Fidelity with a projective measurement.* We define this fidelity as the overlap between the pre-measurement state  $\hat{\rho}_n^{retr}$  retrodicted from a certain result  $n$  and a target state  $|\psi_{tar}\rangle$ , in which we would like checking the system before its interaction with the apparatus. Such a fidelity [18] can be written as

$$\mathcal{F}_n(\psi_{tar}) = \langle\psi_{tar}|\hat{\rho}_n^{retr}|\psi_{tar}\rangle. \quad (12)$$

Retrodictive approach provides an interesting interpretation for this overlap. This is the retrodictive probability of preparing the system in the target state  $|\psi_{tar}\rangle$ , before the measurement process giving the result  $n$ :

$$\mathcal{F}_n(\psi_{tar}) = \Pr(\psi_{tar}|n) = \text{Tr}\{\hat{\rho}_n^{retr} \hat{\Theta}_{tar}\}. \quad (13)$$

The proposition operator about the state of the system, just after its preparation, is  $\hat{\Theta}_{\text{tar}} = |\psi_{\text{tar}}\rangle\langle\psi_{\text{tar}}|$ . When the measurement giving the result  $n$  is sufficiently faithful  $\mathcal{F}_n(\psi_{\text{tar}}) \simeq 1$ , the most probable state in which the system was prepared before its interaction with the apparatus is this target state  $|\psi_{\text{tar}}\rangle$ .

*3-Non-classicality of a measurement.* There are different manifestations for the non-classicality of states. For instance in quantum optics, this property is revealed by different signatures [19]: variances below the standard quantum noise, upper bound for eigenvalues of the covariance matrix [20], negativity in particular quasi-probability distributions [21]. Whatever the signature that is used, the non-classicality of a measurement corresponds to the non-classicality of its pre-measurement state.

We illustrate the relevance of such a correspondance for optical detectors in the conditional preparation of non-classical states of light [22–24]. In such experiments, the Wigner representation of the conditional state can be written in a general way:

$$\mathcal{W}_n^{\text{cond}}(\alpha) = \frac{\int d^2\beta \mathcal{W}_{AB}(\alpha, \beta) \mathcal{W}_n^{\text{retr}}(\beta)}{\int d^2\alpha d^2\beta \mathcal{W}_{AB}(\alpha, \beta) \mathcal{W}_n^{\text{retr}}(\beta)} \quad (14)$$

where  $\mathcal{W}_{AB}$  and  $\mathcal{W}_n^{\text{retr}}$  are respectively the Wigner representations for the resource  $\hat{\rho}_{AB}$  and the pre-measurement state  $\hat{\rho}_n^{\text{retr}}$  retrodicted from the result  $n$ . When the resource has a non-negative representation, as gaussian states, the necessary condition for preparing a non-classical state  $\hat{\rho}_n^{\text{cond}}$  is to perform a non-classical measurement, in the sense of a non-positive Wigner representation  $\mathcal{W}_n^{\text{retr}}$ .

Finally, the non-Gaussian character of a measurement apparatus is also essential in many quantum information protocols such as entanglement purification [25]. This characteristic can be measured by the non-Gaussian character of its pre-measurement state. We can compute for instance its "non-Gaussianity" [26] defined by the relative Von Neumann entropy between this pre-measurement state and its reference gaussian state, which is the gaussian state with the same covariance matrix. The non-Gaussianity is equal to zero only for gaussian states. An interesting link can also be established between the Gaussian character of a measurement and its projectivity. The Hudson-Piquet's theorem [27] states that any pure state characterized by a non-negative Wigner representation has a Gaussian Wigner representation. We deduce from it that, when a measurement is projective and is characterized by a pre-measurement state having a non-negative Wigner representation, then the measurement is Gaussian.

*Conclusion.*— We have shown that the retrodictive approach of quantum physics provides an interesting way to characterize the quantum properties of a measurement apparatus: these properties are obtained from the ones of the pre-measurement state. This approach, significantly based on usual quantum state analysis, allows to define precise estimators for qualifying a measurement and, in particular, its non-classicality, which plays a central role in the context of measurement-based quantum protocols. Furthermore, the possibility to directly reconstruct pre-measurement states illustrates the experimental relevance of the retrodictive approach for the characterization of measurement apparatuses.

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