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COMPOSITION OPERATORS ON BERGMAN-ORLICZ AND HARDY-ORLICZ SPACES OF \mathbb{B}_N

STÉPHANE CHARPENTIER

ABSTRACT. We give embedding theorems for Bergman-Orlicz and Hardy-Orlicz spaces of the ball and then apply our results to the study of the boundedness and the compactness of composition operators in this context. As one of the motivations of this work, we show that there exist some Bergman-Orlicz or Hardy-Orlicz spaces, different from H^∞ , on which every composition operator is bounded.

1. INTRODUCTION AND PRELIMINARIES

1.1. Introduction. The continuity and the compactness of composition operators C_ϕ , defined by $C_\phi(f) = f \circ \phi$, on usual analytic functions spaces have been studied in different ways. In one variable, the Littlewood subordination principle is the main tool to show that every composition operator is bounded on spaces such as weighted Bergman spaces, Hardy spaces (see [20]) or Bergman-Orlicz ([15]) and Hardy-Orlicz ([13]) spaces of the disk \mathbb{D} . The compactness of composition operators in one variable can be characterized in terms of Nevanlinna counting function. For a complete study of these points of view in the one dimensional case, we refer to [20] in particular for Bergman and Hardy spaces. For the Bergman-Orlicz and Hardy-Orlicz spaces, we refer to [13, 15, 16]. An other approach for these questions in a general framework is to notice that composition operators can be interpreted as embedding operators of usual spaces into $L^p(\mu)$ or $L^\psi(\mu)$ spaces for some convenient measure μ on the ball (or the closed ball) and then to apply results about continuity or compactness of such embedding operators to composition operators. One of the motivation to do this is that methods using Littlewood subordination principle or Nevanlinna counting function fail to be true in the several variables context. An other interest of this is that it provides geometric understandings of the phenomena and especially crucial properties of the defining symbol ϕ . Moreover, a lot of embedding theorems have been given in the classical spaces. All these results involve Carleson measures. The first Carleson embedding theorem has been given by L. Carleson in 1962 ([3],) as a part of his work on the corona problem. He gave a characterization of measures μ such that inclusion $H^p(\mathbb{D}) \hookrightarrow L^p(\mu)$ holds (the continuity is automatic because of closed graph theorem.) In [9], L. Hormander generalized this latter to the strictly pseudo-convex domains of \mathbb{C}^n in 1967, while S. Power simplified his proof for the unit ball in 1985 ([17].) Similar results have been obtained in the context of non-weighted Bergman or weighted Bergman spaces by several authors such as W. Hasting, D. Stegenga or J. Cima and W. Wogen. We refer to [8, 21, 4, 5]. Characterization theorems about compactness of such embedding operators also exist for all these spaces and all these results have then be applied to composition operators. For classical Hardy and Bergman spaces, these kinds of statement can be found in [7] as well as a lot of geometric interpretations.

In 2009 and 2010, P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza gave similar characterizations in Hardy-Orlicz and Bergman-Orlicz spaces of the disk in respectively [13, 15]. As a motivation for these two papers, the authors recalled that there is a brutal change regarding compactness of composition operators when we pass from finite values of p to the value $p = \infty$: C_ϕ is compact on $H^\infty(\mathbb{D})$ if and only if $\|\phi\|_\infty < 1$ whereas for any $1 \leq p < \infty$, one may find $\phi : \mathbb{D} \rightarrow \mathbb{D}$ such that $\overline{\phi(\mathbb{D})} \cap \mathbb{T}$ is not empty, yet C_ϕ is compact on $H^p(\mathbb{D})$. B. MacCluer and J. Shapiro even gave an example of a surjective holomorphic ϕ of \mathbb{D} such that C_ϕ is compact on

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$H^p(\mathbb{D})$ ([12, Example 3.12].) The same ideas also have an interest in the context of Bergman spaces, and it may appear legitimate to introduce the intermediate Bergman-Orlicz spaces A^ψ and the Hardy-Orlicz spaces H^ψ between respectively H^∞ and A^p or H^∞ and H^p , in order to understand where this change takes place -if it exists- and eventually to find some class of Orlicz functions ψ such that composition operators act on the associated A^ψ or H^ψ somehow as they act on H^∞ . However, [16, Theorem 4.1] gives a precise answer to the question of compactness of C_ϕ in $H^\psi(\mathbb{B}_N)$ by proving that the previous result by MacCluer and Shapiro holds in this context. Moreover, [13, Theorem 5.7] implicitly says that, as soon as ψ grows very fast, if C_ϕ is bounded from $H^\psi(\mathbb{D})$ into itself, then it is bounded as an operator from $A^\psi(\mathbb{D})$ into itself, what let us think, in a not very satisfactory way, that H^∞ also appears really as a singular case in the context of Bergman-orlicz spaces.

In the several-variables context, this motivation appears to be even more important, since it concerns continuity and not only compactness. Indeed it is well-known that there even exist some symbol ϕ such that C_ϕ is not bounded on the classical Bergman $A^p(\mathbb{B}_N)$ or Hardy $H^p(\mathbb{B}_N)$ spaces, although every C_ϕ is bounded on H^∞ . Precisely, in addition to give necessary and sufficient conditions for the continuity and the compactness of composition operators on $A_\alpha^\psi(\mathbb{B}_N)$ or $H^\psi(\mathbb{B}_N)$ spaces in terms of generalized Carleson measures, it seems to us be interesting to find a precise growth or regular condition on ψ in order that every composition operator on such $A_\alpha^\psi(\mathbb{B}_N)$ or $H^\psi(\mathbb{B}_N)$ is bounded.

Moreover, up to now, in the specific context of Hardy spaces, it remains possible to consider composition operators as embedding operators; in one variable, it was made possible due to a so-called Lindelöf Theorem, while in the classical several variables analytic functions spaces it was due to the fact that the ball algebra $A(\mathbb{B}_N)$, consisting of holomorphic functions in \mathbb{B}_N continuous up to the boundary, is dense in $H^p(\mathbb{B}_N)$. Nevertheless, it is known that the density of $A(\mathbb{B}_N)$ is more or less specific to the classical Hardy spaces in the sense that this property fails to be true in $H^\psi(\mathbb{B}_N)$ Hardy-Orlicz spaces which differs from H^p spaces, in a way which can be precised: $A(\mathbb{B}_N)$ is dense in $H^\psi(\mathbb{B}_N)$ if and only if ψ satisfies the Δ_2 -Condition (see Definition 1.4.) This reason underlines the fact that studying composition operators in terms of embedding theorem is maybe not so well adapted when we consider spaces smaller than the classical Hardy spaces. Anyway, even in this context, we can wonder how much useful remains the methods involved in the embedding theorems to deduce satisfying necessary or sufficient condition for the boundedness or the compactness of composition operators on Hardy-Orlicz spaces. We notice that all these questions do not concern Bergman-Orlicz spaces.

Yet, this Δ_2 -Condition allows to establish embedding theorems and to study composition operators on Hardy-Orlicz or Bergman-Orlicz spaces, by using methods which are quite similar to that used in the classical Hardy and Bergman spaces framework. In this way, Z. J. Jiang gave embedding theorems and characterizations of the boundedness and the compactness of composition operators on Bergman-Orlicz spaces $A^\psi(\mathbb{B}_N)$ when ψ satisfies Δ_2 -Condition ([10].) As we could guess, these characterizations are the same than that known for Bergman spaces, and their applications to composition operators do not provide different results from that obtained in the classical framework; especially, they give no information for “small” Bergman-Orlicz spaces.

Section 2 and Section 3 are devoted to these questions. The first one deals with general Bergman-Orlicz spaces. Given two arbitrary Orlicz functions ψ_1 and ψ_2 , we exhibit in Theorem 2.8 and Theorem 2.12 necessary and sufficient conditions on a measure μ on the ball under which the canonical embedding $A_{\alpha}^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$ holds (and then is bounded) or is compact. In general, we do not get characterizations, yet we see that we do when $\psi_1 = \psi_2$ satisfies some convenient regular conditions. Applications are given to composition operators following the ideas described above, which allow us to positively answer the question: does there exist some Orlicz function ψ defining Bergman-Orlicz space $A_{\alpha}^{\psi}(\mathbb{B}_N)$ between H^∞ and $A_{\alpha}^p(\mathbb{B}_N)$ on which every composition operator is bounded?

In Section 3, we get in a first part results analogous to that obtained in the previous section for the embedding $H^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$. This is the purpose of both Theorem 3.14 and Theorem 3.19.

After we explain why these results can not be directly used to get conditions for the boundedness or the compactness of composition operators, we see how their proofs can be slightly adapted to composition operators and then we state characterizations of continuity and compactness of such operators, if we make some usual regular assumptions on ψ . As for Bergman-Orlicz case, we apply our results to show that there are some Hardy-Orlicz space $H^\psi(\mathbb{B}_N)$ on which every composition operator is bounded.

In a last section, we intend to give a brief comparison between the classical framework and the Orlicz corresponding one.

1.2. Orlicz spaces - Preliminaries. We recall the definition of Orlicz functions and of the associated Orlicz spaces.

Definition 1.1. A strictly convex function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called an Orlicz function if it satisfies the following properties:

- (1) $\psi(0) = 0$ and ψ is continuous at 0;
- (2) $\frac{\psi(x)}{x} \xrightarrow{x \rightarrow \infty} +\infty$.

In particular, an orlicz function is non-decreasing.

Let (Ω, \mathbb{P}) be a probability space; throughout this paper, Ω will be a ball or a sphere in \mathbb{C}^N and \mathbb{P} the normalized Lebesgue measure on it.

Definition 1.2. Let ψ be an Orlicz function. The Orlicz spaces $L^\psi(\Omega)$ is the space of all (equivalence classes of) measurable complex functions f on Ω for which there is a constant $C > 0$ such that

$$\int_{\Omega} \psi\left(\frac{|f|}{C}\right) d\mathbb{P} < \infty.$$

This space may be normalized by the Luxemburg norm

$$\|f\|_{\psi} = \inf \left\{ C > 0, \int_{\Omega} \psi\left(\frac{|f|}{C}\right) d\mathbb{P} \leq 1 \right\},$$

which makes $(L^\psi(\Omega), \|\cdot\|_{\psi})$ a Banach space satisfying:

$$L^\infty(\Omega) \subset L^\psi(\Omega) \subset L^1(\Omega).$$

Note that if ψ grows sufficiently fast in order that

$$\frac{\psi(x)}{x^p} \xrightarrow{x \rightarrow \infty} \infty$$

for every $p < \infty$ then we have

$$L^\infty(\Omega) \subset L^\psi(\Omega) \subset \bigcap_{0 < p < \infty} L^p(\Omega).$$

Finally, if $\psi(x) = x^p$ for every x , then it is easy to verify that

$$L^\psi(\Omega) = L^p(\Omega).$$

We also introduce $M^\psi(\Omega)$ as the subspace of $L^\psi(\Omega)$ generated by $L^\infty(\Omega)$.

In order to give some words about the duality in Orlicz spaces framework, we need to introduce the complementary function of an Orlicz function. Then let ψ be an Orlicz function and define $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$\Phi(y) = \sup_{x \in \mathbb{R}_+} \{xy - \psi(x)\}.$$

We may verify that Φ is also an Orlicz function (see [18], Section 1.3.) In the same book (Theorem 6) it is shown the following general result concerning duality:

Theorem 1.3. *Let (Ω, \mathbb{P}) be a probability space and let ψ be an Orlicz function and Φ its complementary Orlicz function. We have $(M^\psi(\Omega))^* = L^\Phi(\Omega)$ in the sense that for every continuous linear functional $x^* \in (M^\psi(\Omega))^*$, there exists a unique function $f \in L^\Phi(\Omega)$ such that for every $g \in M^\psi(\Omega)$ we have*

$$x^*(g) = \int_{\Omega} fg d\mathbb{P}.$$

Moreover $\|x^*\| = \|f\|_{\Phi}$.

This result will be essentially useful to describe the weak-star topology induced on the unit ball of the upcoming Bergman-Orlicz and Hardy-Orlicz spaces.

We now introduce four classes of Orlicz functions which will appear several times in this paper. The first one is the so-called Δ_2 -Condition which is a regular condition satisfied by Orlicz functions which do not grow faster than x^p for x large enough and for some $p > 1$.

Definition 1.4. Let ψ be an Orlicz function. We say that ψ satisfies the Δ_2 -Condition if there exist $x_0 > 0$ and a constant $K > 1$, such that

$$\psi(2x) \leq K\psi(x)$$

for any $x \geq x_0$.

For example, $x \mapsto x^p(1 + \log(x))$, $p > 1$, satisfies the Δ_2 -Condition. Corollary 5, Chapter II of [18] formalizes what we announced above the last definition:

Proposition 1.5. *Let ψ be an Orlicz function satisfying the Δ_2 -Condition, then there are some $p > 1$ and $C > 0$ such that $\psi(x) \leq Cx^p$, for x large enough.*

The ∇_2 -class contains the Orlicz functions whose complementary ones satisfy the Δ_2 -condition. It is a condition of fast growing:

Definition 1.6. Let ψ be an Orlicz function. ψ belongs to the ∇_2 -class if its complementary function belongs to the Δ_2 -class.

The two last conditions are regular conditions which are satisfied by most of the Orlicz functions which are interesting for us.

Definition 1.7. Let ψ be an Orlicz function. We say that ψ satisfies the ∇_0 -Condition if there exist some $x_0 > 0$ and some constant $C \geq 1$, such that for every $x_0 \leq x \leq y$ we have

$$\frac{\psi(2x)}{\psi(x)} \leq \frac{\psi(2Cy)}{\psi(y)}.$$

We refer to Proposition 4.6 of [13] to verify that we have the following:

Proposition 1.8. *Let ψ be an Orlicz function. Then ψ satisfies the ∇_0 -Condition if and only if there exists $x_0 > 0$ such that for every $\beta > 1$, there exists a constant $C_\beta \geq 1$ such that*

$$\frac{\psi(\beta x)}{\psi(x)} \leq \frac{\psi(\beta C_\beta y)}{\psi(y)}$$

for every $x_0 \leq x \leq y$.

The uniform ∇_0 -Condition is defined as follows:

Definition 1.9. Let ψ be an Orlicz function. We say that ψ satisfies the *uniform* ∇_0 -Condition if and only if there exist $x_0 > 0$ and a constant $C \geq 1$ such that for every $\beta > 1$ we have

$$\frac{\psi(\beta x)}{\psi(x)} \leq \frac{\psi(\beta Cy)}{\psi(y)}$$

for every $x_0 \leq x \leq y$.

We recall Proposition 4.7 (2) of [13]:

Proposition 1.10. *Let ψ be an Orlicz function. If ψ satisfies uniform ∇_0 -Condition, then it satisfies ∇_2 -Condition.*

Let us notice that for any $1 < p < \infty$ and any $a > 0$, every function $x \mapsto x^p$ or $x \mapsto e^{ax} - 1$ is an Orlicz function which satisfies uniform ∇_0 , and then ∇_2 and ∇_0 -conditions, according to the previous proposition. $x \mapsto x^p$ belongs to the Δ_2 -class whereas $x \mapsto e^{ax} - 1$ does not. For a complete study of Orlicz spaces, we refer to [18]. We can also find precise information in context of composition operators, such as other classes of Orlicz functions and their link together with, in [13].

2. WEIGHTED BERGMAN-ORLICZ SPACES OF \mathbb{B}_N

The purpose of this section is to extend Carleson Theorem for Bergman spaces to the Bergman-Orlicz framework. Applications will be given to the characterization of continuity and compactness of composition operators on Bergman-Orlicz spaces in terms of adapted Carleson measures.

2.1. Definitions and classical results. Let $\alpha > -1$ and let dv_α be the normalized weighted Lebesgue measure on \mathbb{B}_N

$$dv_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha dv(z)$$

where dv is the normalized volume Lebesgue measure on \mathbb{B}_N . The constant c_α is equal to

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)}.$$

The weighted Bergman-Orlicz spaces of the ball, $A_\alpha^\psi(\mathbb{B}_N)$, is defined from the ψ -Orlicz space associated to the probabilistic measure dv_α on \mathbb{B}_N , $L^\psi(\mathbb{B}_N, v)$.

Definition 2.1. The weighted Bergman-Orlicz space $A_\alpha^\psi(\mathbb{B}_N)$ of the ball is the set of holomorphic functions f on \mathbb{B}_N such that there exists a constant $0 < C < \infty$ such that the integral

$$\int_{\mathbb{B}_N} \psi\left(\frac{|f|}{C}\right) dv_\alpha$$

is finite.

Endowed with the norm

$$\|f\|_{A_\alpha^\psi} = \inf \left\{ C, \int_{\mathbb{B}_N} \psi\left(\frac{|f|}{C}\right) dv_\alpha \leq 1 \right\},$$

$A_\alpha^\psi(\mathbb{B}_N)$ is a Banach space.

In fact,

$$A_\alpha^\psi(\mathbb{B}_N) = L_\alpha^\psi(\mathbb{B}_N) \cap H(\mathbb{B}_N) \subset A_\alpha^1(\mathbb{B}_N).$$

For $a \in \mathbb{B}_N$, we denote by δ_a the point evaluation functional at point a . The following proposition infers that δ_a is bounded on every $A_\alpha^\psi(\mathbb{B}_N)$ for any a in the ball.

Proposition 2.2. *Let $\alpha > -1$ and let ψ be an Orlicz function. Let also $a \in \mathbb{B}_N$. Then the point evaluation functional δ_a at a is bounded on $A_\alpha^\psi(\mathbb{B}_N)$; more precisely, we have*

$$\frac{1}{4^{N+1+\alpha}} \psi^{-1} \left(\left(\frac{1+|a|}{1-|a|} \right)^{N+1+\alpha} \right) \leq \|\delta_a\| \leq \psi^{-1} \left(\left(\frac{1+|a|}{1-|a|} \right)^{N+1+\alpha} \right).$$

Proof. First, let φ_a an automorphism of \mathbb{B}_N such that $\varphi(0) = a$. Fix $f \in A_\alpha^\psi(\mathbb{B}_N)$ and set $C = \|f\|_{A_\alpha^\psi}$. By the change of variables formula (see for example [22], Proposition 1.13), we get

$$\int_{\mathbb{B}_N} \psi\left(\frac{|f \circ \varphi_a|}{C}\right) dv_\alpha = \int_{\mathbb{B}_N} \psi\left(\frac{|f(z)|}{C}\right) \left(\frac{1-|a|^2}{|1-\langle z, a \rangle|^2}\right)^{N+1+\alpha} dv_\alpha(z).$$

The subharmonicity of $\psi\left(\frac{|f \circ \varphi_a|}{C}\right)$ yields

$$\begin{aligned} \psi\left(\frac{|f(a)|}{C}\right) &\leq \int_{\mathbb{B}_N} \psi\left(\frac{|f(z)|}{C}\right) \left(\frac{1-|a|^2}{|1-\langle z, a \rangle|^2}\right)^{N+1+\alpha} dv_\alpha(z) \\ &\leq \left(\frac{1+|a|}{1-|a|}\right)^{N+1+\alpha}. \end{aligned}$$

Since ψ^{-1} is non-decreasing, we have

$$|f(a)| \leq C\psi^{-1}\left(\left(\frac{1+|a|}{1-|a|}\right)^{N+1+\alpha}\right).$$

In particular, we have

$$\|\delta_a\| \leq \psi^{-1}\left(\left(\frac{1+|a|}{1-|a|}\right)^{N+1+\alpha}\right).$$

Conversely, let us compute $\delta_a(H_a)$ where

$$H_a(z) = \left(\frac{1-|a|^2}{|1-\langle z, a \rangle|^2}\right)^{N+1+\alpha}$$

is the Berezin kernel at $a \in \mathbb{B}_N$. It is not hard to check that $\|H_a\|_\infty = \left(\frac{1+|a|}{1-|a|}\right)^{N+1+\alpha}$ and that $\|H_a\|_{L^1} = 1$ (this is a direct application of the change of variables formula,) hence

$$\begin{aligned} \|\delta_a\| &\geq \frac{|H_a(a)|}{\|H_a\|_{A_\alpha^\psi}} \\ &\geq \frac{1}{(1-|a|^2)^{N+1+\alpha}} \frac{\psi^{-1}(\|H_a\|_\infty)}{\|H_a\|_\infty} \quad (\text{by Lemma 3.9 from [13]}) \\ &\geq \frac{1}{(1+|a|)^{2(N+1+\alpha)}} \psi^{-1}\left(\left(\frac{1+|a|}{1-|a|}\right)^{N+1+\alpha}\right) \\ (2.1) \quad &\geq \frac{1}{4^{N+1+\alpha}} \psi^{-1}\left(\left(\frac{1+|a|}{1-|a|}\right)^{N+1+\alpha}\right). \end{aligned}$$

□

We will need to describe the weak-star convergence on the unit ball of Bergman-Orlicz spaces in terms of uniform convergence on every compact subset of \mathbb{B}_N :

Proposition 2.3. *Let $\alpha > -1$, let ψ be an Orlicz function and let Φ be its complementary Orlicz function. On the unit ball of $A_\alpha^\psi(\mathbb{B}_N)$, the induced weak-star topology*

$$\sigma\left(L^\psi(\mathbb{B}_N, dv_\alpha), M^\Phi(\mathbb{B}_N, dv_\alpha)\right)$$

coincides with the uniform convergence on every compact subset of \mathbb{B}_N .

Proof. First, we observe that both above topologies are metrizable. Indeed, this is well-know for the topology of uniform convergence on compacta and, for the weak-star topology, this follows from the fact the Morse-Transue space $M^\Phi(\mathbb{B}_N)$ is separable. Thus we can deal with convergent sequences. Now, let $(f_n)_n$ be a sequence in the unit ball B_α^ψ of $A_\alpha^\psi(\mathbb{B}_N)$ which converges to

$f \in B_\alpha^\psi$ for the weak-star topology. For $a \in \mathbb{B}_N$, let $H_a(z) = \left(\frac{1-|a|^2}{|1-\langle z, a \rangle|^2}\right)^{N+1+\alpha}$; $\|H_a\|_\infty =$

$\left(\frac{1+|a|}{1-|a|}\right)^{N+1+\alpha}$ so that $H_a \in M^\Phi$ and therefore

$$(f_n - f)(a) = \int_{\mathbb{B}_N} (f_n - f) H_a dv_\alpha \xrightarrow{n \rightarrow \infty} 0.$$

Moreover, since $(f_n)_n$ is in the unit ball of $A_\alpha^\psi(\mathbb{B}_N)$, thanks to Proposition 2.2 for example, $(f_n)_n$ is bounded on every compact subsets of \mathbb{B}_N . By Montel Theorem and a compactness argument, we get that $(f_n)_n$ converges uniformly on every compact subsets of \mathbb{B}_N to f .

The proof of the converse is identical to the proof of the second part of (ii), Proposition 5.3, of [13]. \square

2.2. Embedding Theorems for Bergman-Orlicz spaces - Adapted Bergman-Carleson measures. We will need a version of Carleson theorem for Bergman spaces slightly different from the traditional one. This is inspired from [15]. Anyway, as for the study of continuity and compactness of composition operators on Bergman spaces or Hardy spaces of the ball in terms of Carleson measure, we will need to introduce the objects and notions involved. We first recall the definition of the non-isotropic distance on the sphere \mathbb{S}_N , which we denote by d . For $(\zeta, \xi) \in \mathbb{S}_N^2$, it is given by

$$d(\zeta, \xi) = \sqrt{|1 - \langle \zeta, \xi \rangle|}.$$

We may verify that the map d is a distance on \mathbb{S}_N and that it satisfies the triangular inequality on $\overline{\mathbb{B}_N}$. For $\zeta \in \overline{\mathbb{B}_N}$ and $h \in]0, 1]$, we define the non-isotropic ball of \mathbb{B}_N by

$$S(\zeta, h) = \left\{ z \in \mathbb{B}_N, d(\zeta, z)^2 < h \right\}.$$

and its analogue in $\overline{\mathbb{B}_N}$ by

$$S_f(\zeta, h) = \left\{ z \in \overline{\mathbb{B}_N}, d(\zeta, z)^2 < h \right\}.$$

Let also us denote

$$Q_f = S_f(\zeta, h) \cap \mathbb{S}_N.$$

Next, for $\zeta \in \mathbb{S}_N$ and $h \in]0, 1]$, we define

$$W(\zeta, h) = \left\{ z \in \mathbb{B}_N, 1 - |z| < h, \frac{z}{|z|} \in Q_f(\zeta, h) \right\}.$$

$W(\zeta, h)$ is called a Carleson window.

We introduce the following two functions:

$$\varrho_\mu(h) = \sup_{\xi \in \mathbb{S}_N} \mu(W(\xi, h))$$

where μ is positive Borel measure on \mathbb{B}_N . We now set

$$K_{\mu, \alpha}(h) = \sup_{0 < t \leq h} \frac{\varrho_\mu(t)}{t^{N+1+\alpha}}.$$

μ is said to be a α -Bergman-Carleson measure if $K_{\mu, \alpha}$ is bounded. As

$$(2.2) \quad t^{N+1+\alpha} \sim v_\alpha(W(\xi, t))$$

for every $\xi \in \mathbb{S}_N$, μ is a α -Bergman-Carleson measure if and only if there exists a constant C such that

$$\mu(W(\xi, h)) \leq C v_\alpha(W(\xi, h))$$

for any $\xi \in \mathbb{S}_N$ and any $h \in (0, 1)$ (or equivalently any $h \in (0, h_A)$ for some $0 < h_A \leq 1$). Let us remark that, in the definition of ϱ_μ and $K_{\mu, \alpha}$, we may have taken $S(\xi, h)$ instead of $W(\xi, h)$, since these two sets are equivalent in the sense that there exist two constants $C_1 > 0$ and $C_2 > 0$ such that

$$S(\xi, C_1 h) \subset W(\xi, h) \subset S(\xi, C_2 h).$$

Next we may work indifferently with non-isotropic balls or Carleson windows if there is no possible confusion.

We have the following covering lemma which will be useful for our version of Carleson theorem:

Proposition 2.4. *There exists an integer $M > 0$ such that for any $0 < r < 1$, we can find a finite sequence $\{\xi_k\}_{k=1}^m$ (m depending on r) in \mathbb{S}_N with the following properties:*

- (1) $\mathbb{S}_N = \bigcup_k Q_f(\xi_k, r)$.
- (2) The sets $Q_f(\xi_k, r/4)$ are mutually disjoint.
- (3) Each point of \mathbb{S}_N belongs to at most M of the sets $Q_f(\xi_k, 4r)$.

Proof. The proof, using a variant of [22, Lemma 2.22] for the non-isotropic distance at the boundary is quite identical to the one of [22, Theorem 2.23]. That we can take a finite union follows from a compactness argument. \square

From now on, M will always stand for the constant involved in Theorem 2.4. We will now define a maximal operator associated to a covering of the ball with convenient subsets. Let $n \geq 0$ be an integer and denote by C_n the corona

$$C_n = \left\{ z \in \mathbb{B}_N, 1 - \frac{1}{2^n} \leq |z| < 1 - \frac{1}{2^{n+1}} \right\}.$$

For any $n \geq 0$, let $(\xi_{n,k})_k \subset \mathbb{S}_N$ be given by Theorem 2.4 putting $r = \frac{1}{2^n}$. For $k \geq 0$, we set

$$T_{0,k} = \left\{ z \in \mathbb{B}_N \setminus \{0\}, \frac{z}{|z|} \in Q_f(\xi_{0,k}, 1) \right\} \cup \{0\}.$$

Then let us define the sets $T_{n,k}$, for $n \geq 1$ and $k \geq 0$, by

$$T_{n,k} = \left\{ z \in \mathbb{B}_N \setminus \{0\}, \frac{z}{|z|} \in Q_f\left(\xi_{n,k}, \frac{1}{2^n}\right) \right\}.$$

We have both

$$\bigcup_{n \geq 0} C_n = \mathbb{B}_N$$

and

$$\begin{aligned} \bigcup_{k \geq 0} T_{0,k} &= \mathbb{B}_N; \\ \bigcup_{k \geq 0} T_{n,k} &= \mathbb{B}_N \setminus \{0\}, \quad n \geq 1. \end{aligned}$$

For $(n, k) \in \mathbb{N}^2$, we finally define the subset $\Delta_{(n,k)}$ of \mathbb{B}_N by

$$\Delta_{(n,k)} = C_n \cap T_{n,k}.$$

We have

$$\begin{aligned} \Delta_{(0,k)} &= (W(\xi_{0,k}, 1) \cap C_0) \cup \{0\}; \\ \Delta_{(n,k)} &= W\left(\xi_{n,k}, \frac{1}{2^n}\right) \cap C_n, \quad n \geq 1. \end{aligned}$$

The $\Delta_{(n,k)}$'s satisfy the following properties:

$$(1) \bigcup_{(n,k) \in \mathbb{N}^2} \Delta_{(n,k)} = \mathbb{B}_N.$$

- (2) For every (n, k) , $\Delta_{(n,k)}$ is a subset of the closed Carleson window $\overline{W\left(\xi_{n,k}, \frac{1}{2^n}\right)}$ and by construction, we can find a constant $\tilde{C} > 0$, independent of (n, k) such that

$$v_\alpha \left(W\left(\xi_{n,k}, \frac{1}{2^n}\right) \right) \leq \tilde{C} v_\alpha(\Delta_{(n,k)}).$$

(3) Given $0 < \varepsilon < 1/2$, if C_n^ε denotes the corona defined by

$$C_n^\varepsilon = \left\{ z \in \mathbb{B}_N, (1 + \varepsilon) \left(1 - \frac{1}{2^n} \right) \leq |z| < (1 + \varepsilon) \left(1 - \frac{1}{2^{n+1}} \right) \right\},$$

then each point of \mathbb{B}_N belongs to at most M of the sets $\Delta_{(n,k)}^\varepsilon$'s defined by

$$\begin{aligned} \Delta_{(0,k)}^\varepsilon &= (W(\xi_{0,k}, 1 + \varepsilon) \cap C_0^\varepsilon) \cup \{0\}; \\ \Delta_{(n,k)}^\varepsilon &= W\left(\xi_{n,k}, (1 + \varepsilon) \frac{1}{2^n}\right) \cap C_n^\varepsilon, \quad n \geq 1. \end{aligned}$$

This comes from the construction and the previous covering proposition. In particular, we have

$$\sum_{(n,k) \in \mathbb{N}^2} v_\alpha(\Delta_{(n,k)}^\varepsilon) \leq M v_\alpha(\mathbb{B}_N) = M.$$

For any $f \in A_\alpha^\psi(\mathbb{B}_N)$, we define the following maximal function Λ_f :

$$(2.3) \quad \Lambda_f = \sum_{n,k \geq 0} \sup_{\Delta_{(n,k)}} (|f(z)|) \chi_{\Delta_{(n,k)}}$$

where $\chi_{\Delta_{(n,k)}}$ is the characteristic function of $\Delta_{(n,k)}$. The next proposition says that the maximal operator $\Lambda : f \mapsto \Lambda_f$ is bounded from $A_\alpha^\psi(\mathbb{B}_N)$ to $L_\alpha^\psi(\mathbb{B}_N, v_\alpha)$.

Proposition 2.5. *Let ψ be an Orlicz function and let $\alpha > -1$. Then the maximal operator Λ defined above is bounded from $A_\alpha^\psi(\mathbb{B}_N)$ to $L_\alpha^\psi(\mathbb{B}_N, v_\alpha)$. More precisely there exists $B \geq 1$ such that for every $f \in A_\alpha^\psi(\mathbb{B}_N)$, we have*

$$\|\Lambda_f\|_{L_\alpha^\psi} \leq 2B \|f\|_{A_\alpha^\psi}.$$

Proof. Fix $f \in A_\alpha^\psi(\mathbb{B}_N)$ and set $C = \|f\|_{A_\alpha^\psi}$. We denote by $c_{(n,k)} = \sup_{\Delta_{(n,k)}} (|f|)$ and let $\tau_{(n,k)} \in \Delta_{(n,k)}$ be such that $|f(\tau_{(n,k)})| \geq \frac{c_{(n,k)}}{2}$. Since $\frac{\psi \circ |f|}{C}$ is subharmonic, and by the mean value property, we have

$$\begin{aligned} \int_{\mathbb{B}_N} \psi\left(\frac{\Lambda_f}{2C}\right) dv_\alpha &\leq \sum_{n,k \geq 0} \psi\left(\frac{|f(\tau_{(n,k)})|}{C}\right) v_\alpha(\Delta_{(n,k)}) \\ &\leq \sum_{n,k \geq 0} \frac{v_\alpha(\Delta_{(n,k)})}{v_\alpha(\Delta_{(n,k)}^\varepsilon)} \int_{\Delta_{(n,k)}^\varepsilon} \psi\left(\frac{|f|}{C}\right) dv_\alpha. \end{aligned}$$

Of course,

$$\frac{v_\alpha(\Delta_{(n,k)})}{v_\alpha(\Delta_{(n,k)}^\varepsilon)} \leq D_\varepsilon$$

where D_ε is a positive constant which only depends on ε . Therefore we get,

$$\int_{\mathbb{B}_N} \psi\left(\frac{\Lambda_f}{2C}\right) dv_\alpha \leq D_\varepsilon \sum_{n,k \geq 0} \int_{\Delta_{(n,k)}^\varepsilon} \psi\left(\frac{|f|}{C}\right) dv_\alpha.$$

Now, we have $C_n^\varepsilon = \cup_{k \geq 0} \Delta_{(n,k)}^\varepsilon$ and, because of the third property following the construction of the $\Delta_{(n,k)}$'s, for every n , each point of C_n^ε belongs to at most M of the sets $\Delta_{(n,k)}^\varepsilon$. Then, for n fixed,

$$\sum_{k \geq 0} \int_{\Delta_{(n,k)}^\varepsilon} \psi\left(\frac{|f|}{C}\right) dv_\alpha \leq M \int_{C_n^\varepsilon} \psi\left(\frac{|f|}{C}\right) dv_\alpha.$$

Next, we of course have $\mathbb{B}_N \subset \cup_{n \geq 0} C_n^\varepsilon$ and each point of \mathbb{B}_N belongs to at most 3 of the C_n^ε 's. It follows that

$$\begin{aligned} \int_{\mathbb{B}_N} \psi\left(\frac{\Lambda_f}{2C}\right) dv_\alpha &\leq D_\varepsilon M \sum_{n \geq 0} \int_{C_n^\varepsilon} \psi\left(\frac{|f|}{C}\right) dv_\alpha \\ &\leq B \int_{\mathbb{B}_N} \psi\left(\frac{|f|}{C}\right) dv_\alpha \end{aligned}$$

for some constant $B \geq 1$. Now, by convexity, we get

$$\int_{\mathbb{B}_N} \psi\left(\frac{\Lambda_f}{2BC}\right) dv_\alpha \leq 1,$$

hence

$$\|\Lambda_f\|_{L_\alpha^\psi} \leq 2B \|f\|_{A_\alpha^\psi}.$$

□

We state our version of Carleson theorem as follows:

Theorem 2.6. *There exists a constants $\tilde{C} > 0$ such that, for every $f \in A_\alpha^1(\mathbb{B}_N)$ and every positive finite Borel measure μ on \mathbb{B}_N , we have*

$$\mu(\{z \in \mathbb{B}_N, |z| > 1 - h \text{ and } |f(z)| > t\}) \leq \tilde{C} K_{\mu, \alpha}(2h) v_\alpha(\{\Lambda_f > t\})$$

for every $h \in (0, 1/2)$ and every $t > 0$.

Proof. The proof is quite identical to the one of [15, Lemma 2.3]. Anyway, we prefer to give the details. Fix $0 < h < 1$ and $t > 0$. We identify $i \in \mathbb{N}$ and $(n, k) \in \mathbb{N}^2$ thanks to an arbitrary bijection from \mathbb{N}^2 onto \mathbb{N} . We will write $i \longleftrightarrow (n, k)$ without possible confusion. Define

$$I = \left\{ i \longleftrightarrow (n, k), \sup_{\Delta_i} |f| > t \right\}$$

and

$$I_h = \left\{ i \longleftrightarrow (n, k), h > \frac{1}{2^{n+1}} \text{ and } \sup_{\Delta_i} |f| > t \right\}.$$

Denoting by W_i the smallest Carleson window containing Δ_i , and by the three remarks done above about the Δ_i 's, we can find some constants $C > 0$ and $\tilde{C} > 0$ such that

$$\begin{aligned} \mu(\{z \in \mathbb{B}_N, |z| > 1 - h \text{ and } |f(z)| > t\}) &\leq \sum_{i \in I_h} \mu(\Delta_i) \\ &\leq \sum_{i \in I_h} \mu(W_i) \\ &\leq C \sum_{i \in I_h} K_{\mu, \alpha}(2h) v_\alpha(W_i) \\ &\leq C \tilde{C} K_{\mu, \alpha}(2h) \sum_{i \in I} v_\alpha(\Delta_i). \end{aligned}$$

The third inequality comes from (2.2) and from the fact that, for every $i \in I_h$, as the radius of W_i is smaller than $\frac{1}{2^n}$, it is then smaller than $2h$. Now, as each point of \mathbb{B}_N belongs to at most M of the Δ_i 's, we have

$$\sum_{i \in I} v_\alpha(\Delta_i) \leq M v_\alpha\left(\bigcup_{i \in I} \Delta_i\right) \leq M v_\alpha(\{\Lambda_f > t\}).$$

and

$$\mu(\{z \in \mathbb{B}_N, |z| > 1 - h \text{ and } |f(z)| > t\}) \lesssim K_{\mu, \alpha}(2h) v_\alpha(\{\Lambda_f > t\}).$$

□

The last lemma gives the following technical result.

Lemma 2.7. *Let μ be a finite positive Borel measure on \mathbb{B}_N and let ψ_1 and ψ_2 be two Orlicz functions. Assume that there exist $A > 0$, $\eta > 0$ and $h_A \in (0, 1/2)$ such that*

$$K_{\mu, \alpha}(h) \leq \eta \frac{1/h^{N+1+\alpha}}{\psi_2(A\psi_1^{-1}(1/h^{N+1+\alpha}))}$$

for every $h \in (0, h_A)$. Then, there exist three constants $B > 0$, $x_A > 0$ and C_1 (this latter does not depend on A , η and h_A) such that, for every $f \in A_\alpha^{\psi_1}(\mathbb{B}_N)$ such that $\|f\|_{A_\alpha^{\psi_1}} \leq 1$ and every Borel subset E of \mathbb{B}_N , we have

$$\int_E \psi_2\left(\frac{|f|}{B}\right) d\mu \leq \mu(E) \psi_2(x_A) + C_1 \eta \int_{\mathbb{B}_N} \psi_1(\Lambda_f) dv_\alpha.$$

Proof. Let A , h_A and η satisfying the assumptions of the lemma. For $f \in A_\alpha^{\psi_1}(\mathbb{B}_N)$, $\|f\|_{A_\alpha^{\psi_1}} \leq 1$, and E a Borel subset of \mathbb{B}_N , we begin by writing the following formula, based on Fubini's integration:

$$(2.4) \quad \int_E \psi_2(|f|) d\mu = \int_0^\infty \psi_2'(t) \mu(\{|f| > t\} \cap E) dt.$$

We concentrate our attention on the expression $\mu(\{|f| > t\})$. We use the upper estimate of the point evaluation functional obtained in Proposition 2.2 to get that if $|f(z)| > t$, then since $\|f\|_{A_\alpha^{\psi_1}} \leq 1$ we have

$$(2.5) \quad \begin{aligned} t &< \psi_1^{-1}\left(\left(\frac{1+|z|}{1-|z|}\right)^{N+1+\alpha}\right) \\ &\leq 2^{N+1+\alpha} \psi_1^{-1}\left(\left(\frac{1}{1-|z|}\right)^{N+1+\alpha}\right) \end{aligned}$$

because ψ is a convex function. Inequality (2.5) is now equivalent to the following one:

$$|z| > 1 - \left(\frac{1}{\psi_1\left(\frac{t}{2^{N+1+\alpha}}\right)}\right)^{1/(N+1+\alpha)}.$$

Carleson Theorem (Theorem 2.6) then yields that

$$(2.6) \quad \begin{aligned} \mu(\{|f| > t\}) &= \mu\left(\{|f| > t\} \cap \left\{|z| > 1 - \left(\frac{1}{\psi_1\left(\frac{t}{2^{N+1+\alpha}}\right)}\right)^{1/(N+1+\alpha)}\right\}\right) \\ &\leq \tilde{C} K_{\mu, \alpha} \left(2 \left(\frac{1}{\psi_1\left(\frac{t}{2^{N+1+\alpha}}\right)}\right)^{1/(N+1+\alpha)}\right) v_\alpha(\{\Lambda_f > t\}). \end{aligned}$$

Now, the hypothesis of the lemma ensures that, if

$$\frac{1}{2^{N+1+\alpha}} \psi_1\left(\frac{3.2^{N+\alpha}}{A} s\right) > 1/h_A^{N+1+\alpha}$$

i. e. $s \geq x_A := \frac{A}{3.2^{N+\alpha}} \psi_1^{-1}\left((2/h_A)^{N+1+\alpha}\right)$, then

$$(2.7) \quad K_{\mu, \alpha} \left(2 \left(\frac{1}{\psi_1\left(\frac{3.2^{N+\alpha}}{A} s\right)}\right)^{1/(N+1+\alpha)}\right) \leq \frac{\eta}{2^{N+1+\alpha}} \frac{\psi_1\left(\frac{3.2^{N+\alpha}}{A} s\right)}{\psi_2\left(\frac{3}{2} s\right)}.$$

Hence, applying (2.4) to $\frac{A}{6.4^{N+\alpha}} |f|$, together with (2.6) and (2.7), and putting $t = \frac{6.4^{N+\alpha}}{A} s$ in (2.6), we get

$$(2.8) \quad \int_E \psi_2 \left(\frac{A}{6.4^{N+\alpha}} |f| \right) d\mu \leq \int_0^{x_A} \psi_2'(s) \mu(E) ds \\ + \frac{\eta \tilde{C}}{2^{N+1+\alpha}} \int_{x_A}^\infty \psi_2'(s) \frac{\psi_1 \left(\frac{3.2^{N+\alpha}}{A} s \right)}{\psi_2 \left(\frac{3}{2} s \right)} v_\alpha \left(\left\{ \Lambda_f > \frac{6.4^{N+\alpha}}{A} s \right\} \right) ds.$$

For the second integral of the right hand side, notice that for an Orlicz function ψ , we have

$$x \psi'(x) \leq C \psi \left(\frac{(C+1)x}{C} \right)$$

for any $C > 0$ and any $x \geq 0$. Indeed, as $\psi'(t)$ is non-decreasing, we have

$$\frac{x}{C} \psi'(x) \leq \int_x^{\frac{C+1}{C}x} \psi'(t) dt \leq \psi \left(\frac{C+1}{C} x \right).$$

Therefore

$$\frac{\psi_2'(s)}{\psi_2 \left(\frac{3}{2} s \right)} \leq \frac{2}{s}$$

and (2.8) yields

$$\int_E \psi_2 \left(\frac{A}{6.4^{N+\alpha}} |f| \right) d\mu \leq \psi_2(x_A) \mu(E) \\ + \frac{\eta \tilde{C}}{2^{N+\alpha}} \int_{x_A}^\infty \frac{1}{s} \psi_1 \left(\frac{3.2^{N+\alpha}}{A} s \right) v_\alpha \left(\left\{ \Lambda_f > \frac{6.4^{N+\alpha}}{A} s \right\} \right) ds.$$

Using the convexity of the function ψ_1 , we get

$$\int_E \psi_2 \left(\frac{A}{6.4^{N+\alpha}} |f| \right) d\mu \leq \psi_2(x_A) \mu(E) \\ + \frac{\eta \tilde{C}}{2^{N+\alpha}} \frac{3.2^{N+\alpha}}{A} \int_0^\infty \psi_1' \left(\frac{3.2^{N+\alpha}}{A} s \right) v_\alpha \left(\left\{ \Lambda_f > \frac{6.4^{N+\alpha}}{A} s \right\} \right) ds$$

i.e.

$$\int_E \psi_2 \left(\frac{A}{6.4^{N+\alpha}} |f| \right) d\mu \leq \psi_2(x_A) \mu(E) + \frac{\eta \tilde{C}}{2^{N+\alpha}} \int_0^\infty \psi_1'(u) v_\alpha(\{\Lambda_f > 2^{N+1+\alpha} u\}) du \\ \leq \psi_2(x_A) \mu(E) + \frac{\eta \tilde{C}}{2^{N+\alpha}} \int_{\mathbb{B}_N} \psi_1 \left(\frac{\Lambda_f}{2^{N+1+\alpha}} \right) dv_\alpha \\ \leq \psi_2(x_A) \mu(E) + \frac{\eta \tilde{C}}{2.4^{N+\alpha}} \int_{\mathbb{B}_N} \psi_1(\Lambda_f) dv_\alpha$$

and the proof of the lemma is complete. \square

In the usual context of Bergman spaces (respectively Hardy spaces,) Carleson's Theorem (resp. the adapted one to Hardy spaces) ensures that, given a positive finite Borel measure μ on \mathbb{B}_N (resp. $\overline{\mathbb{B}_N}$), the inclusion $A_\alpha^p(\mathbb{B}_N) \subset L^p(\mu)$ (resp. $H^p(\mathbb{B}_N) \subset L^p(\mu)$) holds (and is continuous) if and only if μ is an α -Bergman-Carleson measure (resp. a Carleson measure.) The compactness is characterized in terms of vanishing α -Bergman-Carleson measure (resp. vanishing Carleson measure.) It first reveals that these characterizations do not depend on the exponent p . In the context of Bergman-Orlicz (respectively Hardy-Orlicz) spaces, the previous formulation of Carleson theorem and the technical lemma (see Section 3 for the statement of these results in the context of Hardy-Orlicz spaces) is sufficiently general to permit us to deduce two conditions,

respectively sufficient and necessary, for a measure μ to get continuous or compact inclusion of a Bergman-Orlicz space $A_\alpha^{\psi_1}(\mathbb{B}_N)$ (resp. Hardy-Orlicz $H^{\psi_1}(\mathbb{B}_N)$) into an Orlicz space $L^{\psi_2}(\mathbb{B}_N, \mu)$ (resp. $L^{\psi_2}(\mathbb{S}_N, \mu)$).

Recently, Z. J. Jiang showed in [10] that these characterizations in Bergman-Orlicz spaces are the same as that in Bergman spaces, whenever $\psi_1 = \psi_2 = \psi$ satisfies the Δ_2 -Condition. As Orlicz functions which satisfies Δ_2 -Condition are nearly $x \mapsto x^p$ functions, his proof and the applications of his results are very similar to the usual ones in the classical Bergman spaces.

In the sequel, we deal with embedding theorems in the general case of two arbitrary Orlicz functions ψ_1 and ψ_2 , and applying our results to composition operators, we point out original behaviors of them on $A_\alpha^{\psi_1}(\mathbb{B}_N)$ (resp. $H^{\psi_1}(\mathbb{B}_N)$) when $\psi_1 = \psi_2 = \psi$ grows fast enough, and which does not occur when ψ satisfies the Δ_2 -Condition.

We first deal with the existence of such an embedding.

2.2.1. *The canonical embedding $A_\alpha^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$.* Using Lemma 2.7, we get the following embedding boundedness theorem in the Bergman-Orlicz spaces framework.

Theorem 2.8. *Let μ be a finite positive Borel measure on \mathbb{B}_N and let ψ_1 and ψ_2 be two Orlicz functions. Then:*

- (1) *If inclusion $A_\alpha^{\psi_1}(\mathbb{B}_N) \subset L^{\psi_2}(\mu)$ holds and is continuous, then there exists some $A > 0$ such that*

$$(2.9) \quad \varrho_\mu(h) = O_{h \rightarrow 0} \left(\frac{1}{\psi_2(A\psi_1^{-1}(1/h^{N+1+\alpha}))} \right).$$

- (2) *If there exists some $A > 0$ such that*

$$(2.10) \quad K_{\mu, \alpha}(h) = O_{h \rightarrow 0} \left(\frac{1/h^{N+1+\alpha}}{\psi_2(A\psi_1^{-1}(1/h^{N+1+\alpha}))} \right)$$

then inclusion $A_\alpha^{\psi_1}(\mathbb{B}_N) \subset L^{\psi_2}(\mu)$ holds and is continuous.

- (3) *If in addition $\psi_1 = \psi_2 = \psi$ satisfies the uniform ∇_0 -Condition (see Definition 1.9,) then (2.9) and (2.10) are equivalent.*

As we already said in the introduction, embedding $A_\alpha^{\psi_1}(\mathbb{B}_N) \subset L^{\psi_2}(\mu)$ is continuous as soon as it holds. It is just an application of the closed graph theorem.

Proof of Theorem 2.8. 1) For the first part, let us denote by C the norm of the canonical embedding $j_\alpha : A_\alpha^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$. Let $a \in \mathbb{B}_N$, $|a| = 1 - h$ and $\xi \in \mathbb{S}_N$ be such that $a = (1 - h)\xi$. Let us consider the map

$$\begin{aligned} f_a &= \frac{1}{2^{N+1+\alpha}} \frac{\psi_1^{-1}(1/h^{N+1+\alpha})}{1/h^{N+1+\alpha}} H_a(z) \\ &= \frac{1}{2^{N+1+\alpha}} \frac{\psi_1^{-1}(1/h^{N+1+\alpha})}{1/h^{N+1+\alpha}} \left(\frac{h(2-h)}{|1 - (1-h)\langle z, \xi \rangle|^2} \right)^{N+1+\alpha} \end{aligned}$$

Recall that H_a is the Berezin kernel introduced in Proposition 2.2. As we saw in the proof of Proposition 2.2, f is in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$ and our assumption ensures that

$$\|j_\alpha(f)\|_{L^{\psi_2}(\mu)} = \|f\|_{L^{\psi_2}(\mu)} \leq C$$

so that

$$(2.11) \quad 1 \geq \int_{\mathbb{B}_N} \psi_2 \left(\frac{|f|}{C} \right) d\mu.$$

Let us minorize the right hand side of (2.11). We just get a minorization of $|f|$ on the non-isotropic “ball” $S(\xi, h)$. If $z \in S(\xi, h)$, then we have

$$\begin{aligned} |1 - \langle z, a \rangle| &\leq |1 - \langle \xi, a \rangle| + |\langle \xi, a \rangle - \langle z, a \rangle| \\ &\leq h + (1 - h) |\langle \xi, \xi \rangle - \langle z, \xi \rangle| \\ &\leq h + (1 - h) |1 - \langle z, \xi \rangle| \\ &\leq h + (1 - h) h \\ &\leq 2h. \end{aligned}$$

Hence, for any $z \in S(a, h)$,

$$\begin{aligned} |f_a(z)| &\geq \frac{1}{2^{N+1+\alpha}} \frac{\psi_1^{-1}(1/h^{N+1+\alpha})}{1/h^{N+1+\alpha}} \left(\frac{1}{4h}\right)^{N+1+\alpha} \\ &= \frac{\psi_1^{-1}(1/h^{N+1+\alpha})}{8^{N+1+\alpha}}. \end{aligned}$$

Therefore

$$\begin{aligned} 1 &\geq \int_{\mathbb{B}_N} \psi_2 \left(\frac{|f|}{C} \right) d\mu \\ &\geq \int_{S(\xi, h)} \psi_2 \left(\frac{\psi_1^{-1}(1/h^{N+1+\alpha})}{8^{N+1+\alpha} C} \right) d\mu \\ &\geq \psi_2 \left(\frac{\psi_1^{-1}(1/h^{N+1+\alpha})}{8^{N+1+\alpha} C} \right) \mu(S_f(a, h)) \end{aligned}$$

which is Condition (2.9) and the first part of the theorem.

2) The second part will need Lemma 2.7. First of all, we know (Proposition 2.5) that there exists a constant $C_M \geq 1$ such that, for every $f \in A_\alpha^{\psi_1}(\mathbb{B}_N)$, $\|\Lambda_f\|_{L_\alpha^{\psi_1}(\mathbb{B}_N)} \leq C_M \|f\|_{A_\alpha^{\psi_1}(\mathbb{B}_N)}$ where Λ_f is the maximal function introduced above (To see that $C_M \geq 1$, it suffices to check with $f = 1$.) Let now f be in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$; it suffices to show that $\|f\|_{L^{\psi_2}(\mu)} \leq C_0$ for some constant $C_0 > 0$ which does not depend on f . Let $\tilde{C} > 1$ be a constant whose value will be precised later. Condition (2.10) is supposed to be realized, that is there exists some constants $A > 0$, $h_A \in (0, 1/2]$ and $\eta > 0$ such that

$$(2.12) \quad K_{\mu, \alpha}(h) \leq \eta \frac{1/h^{N+1+\alpha}}{\psi_2(A\psi_1^{-1}(1/h^{N+1+\alpha}))}$$

for any $h \in (0, h_A)$. By using convexity of ψ_2 and Lemma 2.7 for f/C_M (which of course still satisfies $\|f/C_M\|_{A_\alpha^{\psi_1}} \leq 1$), $E = \mathbb{B}_N$, η and h_A as in (2.12), there are constants $B > 0$, x_A and $C_1 > 0$, all independent of f , such that

$$\begin{aligned} \int_{\mathbb{B}_N} \psi_2 \left(\frac{|f|}{BC_M \tilde{C}} \right) d\mu &\leq \frac{1}{\tilde{C}} \int_{\mathbb{B}_N} \psi_2 \left(\frac{|f|}{BC_M} \right) d\mu \\ &\leq \frac{1}{\tilde{C}} \left(\mu(\mathbb{B}_N) \psi_2(x_A) + C_1 \eta \int_{\mathbb{B}_N} \psi_1 \left(\frac{\Lambda_f}{C_M} \right) dv_\alpha \right) \\ &\leq \frac{1}{\tilde{C}} (\mu(\mathbb{B}_N) \psi_2(x_A) + C_1 \eta). \end{aligned}$$

Of course, C_1 may be supposed to be large enough so that $C_1 \eta \geq 1$ and up to fix $\tilde{C} = \mu(\mathbb{B}_N) \psi_2(x_A) + C_1 \eta \geq 1$, we get that $\|f\|_{L^{\psi_2}(\mu)} \leq C_0 := BC_M \tilde{C}$ which completes the proof of (2) of Theorem 2.8.

3) The proof of this part is based on the following claim:

Claim. Under the notations of the theorem, if inclusion $A\psi_1^{\psi_1}(\mathbb{B}_N) \subset L^{\psi_2}(\mu)$ holds and is continuous, then there exist some A as large as we want and $\eta > 0$ such that

$$(2.13) \quad \varrho_\mu(h) \leq \eta \frac{1}{\psi_2(A\psi_1^{-1}(h_A/h^{N+1+\alpha}))}$$

for some h_A , $0 < h_A \leq 1$ and for any $0 < h < h_A$.

Proof of the claim. Thanks to the first point of the current proof, inequality

$$(2.14) \quad \varrho_\mu(h) \leq \eta \frac{1}{\psi_2(\tilde{A}\psi_1^{-1}(1/h^{N+1+\alpha}))}$$

holds for some $\tilde{A} \geq 0$, \tilde{h}_A , $0 < \tilde{h}_A \leq 1$, $\eta > 0$ and any $0 < h < \tilde{h}_A$. We fix $A > 1$ and we look for some constant $h_{\tilde{A},A} \leq 1$ such that

$$(2.15) \quad \frac{1}{\psi_2(\tilde{A}\psi_1^{-1}(1/h^{N+1+\alpha}))} \leq \frac{1}{\psi_2\left(A\psi_1^{-1}\left(\left(h_{\tilde{A},A}/h\right)^{N+1+\alpha}\right)\right)}$$

for $0 < h < h_{\tilde{A},A}$. Now it is easy to verify that Inequality (2.15) is equivalent to

$$\frac{A}{\tilde{A}} \leq \frac{\psi_1^{-1}(1/h^{N+1+\alpha})}{\psi_1^{-1}\left(\left(h_{\tilde{A},A}/h\right)^{N+1+\alpha}\right)} \leq \frac{1}{h_{\tilde{A},A}^{N+1+\alpha}}$$

by concavity of ψ^{-1} . Then the claim follows by choosing $h_{\tilde{A},A}$ small enough. \square

We come back to the proof of the third point. First, it is clear that Condition (2.10) implies Condition (2.9). The converse is also true. Indeed, let suppose that ψ belongs to the uniform ∇_0 -class of Orlicz functions and let $A > 0$, $h_A \in (0, 1]$ and $\eta > 0$ be such that

$$\varrho_\mu(h) \leq \eta \frac{1}{\psi(A\psi^{-1}(1/h^{N+1+\alpha}))}$$

for every $h \in (0, h_A)$. The previous claim says that we can find $B \geq 1$ and $0 < K = K_{B,A} \leq 1$ such that

$$\varrho_\mu(h) \leq \eta \frac{1}{\psi\left(B\psi^{-1}\left((K/h)^{N+1+\alpha}\right)\right)}$$

for every $0 < h < K$. Therefore, we have

$$\begin{aligned} K_{\mu,\alpha}(h) &= \sup_{0 < t \leq h} \frac{\varrho_\mu(t)}{t^{N+1+\alpha}} \leq \eta \sup_{0 < t \leq h} \frac{1/t^{N+1+\alpha}}{\psi\left(B\psi^{-1}\left((K/t)^{N+1+\alpha}\right)\right)} \\ &= \eta \sup_{x \geq \psi^{-1}\left((K/h)^{N+1+\alpha}\right)} \frac{1}{K^{N+1+\alpha}} \frac{\psi(x)}{\psi(Bx)} \end{aligned}$$

for any $0 < h \leq K$. Let C be the constant induced by the uniform ∇_0 -Condition satisfied by ψ and let β be such that $B = \beta C$. The claim allows us to take B large enough and therefore to assume that $\beta > 1$. We then have, since ψ satisfies the uniform ∇_0 -Condition,

$$\frac{\psi\left(\beta\psi^{-1}\left((K/h)^{N+1+\alpha}\right)\right)}{(K/h)^{N+1+\alpha}} \leq \frac{\psi(Bx)}{\psi(x)}$$

for any $x \geq \psi^{-1}\left((K/h)^{N+1+\alpha}\right)$. Hence, for every $0 < h \leq K$,

$$K_{\mu,\alpha}(h) \leq \eta \frac{1/h^{N+1+\alpha}}{\psi\left(\beta\psi^{-1}\left((K/h)^{N+1+\alpha}\right)\right)} \leq \eta \frac{1/h^{N+1+\alpha}}{\psi\left(\beta K^{N+1+\alpha}\psi^{-1}(1/h^{N+1+\alpha})\right)}$$

by concavity of ψ^{-1} , and Condition (2.10) is satisfied. \square

The third point of the previous theorem leads us to define (ψ, α) -Bergman-Carleson measures on the ball:

Definition 2.9. Let μ be a positive Borel measure on \mathbb{B}_N and let ψ be an Orlicz function. We say that μ is a (ψ, α) -Bergman-Carleson measure if there exists some $A > 0$, such that

$$(2.16) \quad \mu(W(\xi, h)) = O_{h \rightarrow 0} \left(\frac{1}{\psi(A\psi^{-1}(1/h^{N+1+\alpha}))} \right)$$

for any $\xi \in \mathbb{S}_N$.

We notice that (2.16) is equivalent to (2.9). Therefore, it seems to us to be relevant to state the next corollary:

Corollary 2.10. Let μ be a finite positive Borel measure on \mathbb{B}_N and let ψ be an Orlicz function satisfying the uniform ∇_0 -Condition. Inclusion $A_\alpha^\psi(\mathbb{B}_N) \hookrightarrow L^\psi(\mu)$ holds and is continuous if and only if μ is a (ψ, α) -Bergman-Carleson measure.

2.2.2. Compactness of the canonical embedding $A_\alpha^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$. For the study of compactness, we usually need some compactness criterion.

Proposition 2.11. Let μ be a finite positive measure on \mathbb{B}_N and let ψ_1 and ψ_2 be two Orlicz functions. We suppose that the canonical embedding $j_{\mu, \alpha} : A_\alpha^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$ holds and is bounded. The three following assertions are equivalent:

- (1) $j_{\mu, \alpha} : A_\alpha^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$ is compact;
- (2) Every sequence in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$, which is convergent to 0 uniformly on every compact subset of \mathbb{B}_N , is strongly convergent to 0 in $L^{\psi_2}(\mu)$.
- (3) $\lim_{r \rightarrow 1^-} \|I_r\| = 0$, where $I_r(f) = f \cdot \chi_{\mathbb{B}_N \setminus r\overline{\mathbb{B}_N}}$.

Proof. 1 \Rightarrow 2) We first assume that $j_{\mu, \alpha}$ is compact. Let $(f_n)_n$ be a sequence in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$, which is convergent to 0 uniformly on every compact subset of \mathbb{B}_N . Of course, $j_{\mu, \alpha}(f_n)$ converges to 0 everywhere. By contradiction, suppose up to extract a subsequence that $\liminf_n \|j_{\mu, \alpha}(f_n)\|_{L^{\psi_2}(\mu)} > 0$. By compactness of $j_{\mu, \alpha}$, up to an other extraction, we may assume that $(j_{\mu, \alpha}(f_n))_n$ strongly converges to some $g \in L^{\psi_2}(\mu)$ and we must have $\|g\|_{L^{\psi_2}(\mu)} > 0$. As convergence in norm in $L^{\psi_2}(\mu)$ entails μ -almost everywhere convergence, we get a contradiction.

2 \Rightarrow 1) Conversely, let $(f_n)_n$ be a sequence in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$. In particular, $(f_n)_n$ is in the unit ball of $A_\alpha^1(\mathbb{B}_N)$ and the Cauchy's formula ensures that $(f_n)_n$ is uniformly bounded on every compact subset of \mathbb{B}_N , so that, up to an extraction, we may suppose that $(f_n)_n$ is uniformly convergent on compact subsets of \mathbb{B}_N to f holomorphic in \mathbb{B}_N , by Montel's theorem. Now, Lebesgue's theorem ensures that $f \in A_\alpha^{\psi_1}(\mathbb{B}_N)$ and, up to divide by a constant large enough, we may assume that $f_n - f$, which converges to 0 on every compact subset of \mathbb{B}_N , is in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$. Therefore, our assumption implies that $(j_{\mu, \alpha}(f_n) - j_{\mu, \alpha}(f))_n$ converges to 0 in the norm of $L^{\psi_2}(\mu)$ and $j_{\mu, \alpha}$ is compact as expected.

3 \Rightarrow 2) Let $(f_n)_n$ be in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$ converging to 0 uniformly on every compact subset of \mathbb{B}_N . We have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|f_n\|_{L^{\psi_2}(\mu)} &= \limsup_{r \rightarrow 1^-} \limsup_{n \rightarrow \infty} \left\| I_r(f_n) + f_n \cdot \chi_{r\overline{\mathbb{B}_N}} \right\|_{L^{\psi_2}(\mu)} \\ &\lesssim \limsup_{r \rightarrow 1^-} \|I_r\| + \limsup_{r \rightarrow 1^-} \limsup_{n \rightarrow \infty} \left\| f_n \cdot \chi_{r\overline{\mathbb{B}_N}} \right\|_\infty \\ &= 0. \end{aligned}$$

2 \Rightarrow 3) By contradiction suppose that (3) is not satisfied so that there exist a constant $\delta > 0$ and a sequence $(f_n)_n$ in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$ such that $\left\| I_{(1-\frac{1}{n})}(f_n) \right\|_{L^{\psi_2}} \geq \delta$ for every $n \geq 0$.

Up to an extraction, we may suppose that $(f_n)_n$ converges uniformly on compact subsets of \mathbb{B}_N to $f \in A_\alpha^{\psi_1}(\mathbb{B}_N)$. By Lebesgue's theorem, $\lim_{n \rightarrow \infty} \left\| I_{(1-\frac{1}{n})}(f) \right\|_{L^{\psi_2}} = 0$; thus, for n large enough,

$$\|f_n - f\|_{L^{\psi_2}} \geq \left\| I_{(1-\frac{1}{n})}(f_n - f) \right\|_{L^{\psi_2}} \geq \delta/2$$

which contradicts (2). \square

As for the boundedness, Proposition 2.11 and Lemma 2.7 yield the following embedding compactness theorem for Bergman-Orlicz spaces.

Theorem 2.12. *Let μ be a finite positive Borel measure on \mathbb{B}_N , and let ψ_1 and ψ_2 be two Orlicz functions. Then:*

(1) *If embedding $A_\alpha^{\psi_1}(\mathbb{B}_N) \subset L^{\psi_2}(\mu)$ holds and is compact, then for every $A > 0$ we have*

$$(2.17) \quad \varrho_\mu(h) = o_{h \rightarrow 0} \left(\frac{1}{\psi_2(A\psi_1^{-1}(1/h^{N+1+\alpha}))} \right).$$

(2) *If*

$$(2.18) \quad K_{\mu,\alpha}(h) = o_{h \rightarrow 0} \left(\frac{1/h^{N+1+\alpha}}{\psi_2(A\psi_1^{-1}(1/h^{N+1+\alpha}))} \right)$$

for every $A > 0$, then $A_\alpha^{\psi_1}(\mathbb{B}_N)$ embeds compactly in $L^{\psi_2}(\mu)$.

(3) *If in addition $\psi_1 = \psi_2 = \psi$ satisfies the ∇_0 -Condition (see Definition 1.7,) then Conditions (2.17) and (2.18) are equivalent.*

Proof. 1) We suppose that the canonical embedding is compact but that Condition (2.17) failed to be satisfied. This means that there exist some $\varepsilon_0 \in (0, 1)$ and $A > 0$, some sequences $(h_n)_n \subset (0, 1)$ decreasing to 0 and $(\xi_n)_n \subset \mathbb{S}_N$, such that

$$\mu(S(\xi_n, h_n)) \geq \frac{\varepsilon_0}{\psi_2(A\psi_1^{-1}(1/h_n^{N+1+\alpha}))}.$$

Let $a_n := (1 - h_n)\xi_n$ and consider the functions

$$(2.19) \quad \begin{aligned} f_n(z) := f_{a_n}(z) &:= \frac{1}{2^{N+1+\alpha}} \frac{\psi_1^{-1}(1/h_n^{N+1+\alpha})}{1/h_n^{N+1+\alpha}} H_{a_n}(z) \\ &= \frac{1}{2^{N+1+\alpha}} \frac{\psi_1^{-1}(1/h_n^{N+1+\alpha})}{1/h_n^{N+1+\alpha}} \left(\frac{h_n(2-h_n)}{|1 - (1-h_n)\langle z, \xi_n \rangle|^2} \right)^{N+1+\alpha} \end{aligned}$$

where H_{a_n} is the Berezin kernel (see Proposition 2.2.) By the second part of Proposition 2.2 (Inequalities (2.1),) every f_n lays in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$. By (2.19), as ψ_1 is an Orlicz function, $(f_n)_n$ converges to 0 uniformly on every compact subset of \mathbb{B}_N . So Proposition 2.11 ensures that $(f_n)_n$ converges to 0 in norm of $L^{\psi_2}(\mu)$.

Now, by the proof of the first part of Theorem 2.8, the following estimation holds:

$$|H_{a_n}(z)| \geq \left(\frac{1}{4h_n} \right)^{N+1+\alpha}$$

for any $z \in S(\xi_n, h_n)$, which gives

$$|f_n(z)| \geq \frac{\psi_1^{-1}(1/h_n^{N+1+\alpha})}{8^{N+1+\alpha}}$$

for any $z \in S(\xi_n, h_n)$; therefore

$$\begin{aligned} \int_{\mathbb{B}_N} \psi_2 \left(\frac{8^{N+1+\alpha} A}{\varepsilon_0} |f_n| \right) d\mu &\geq \psi_2 \left(\frac{A}{\varepsilon_0} \psi_1^{-1} \left(\frac{1}{h_n^{N+1+\alpha}} \right) \right) \mu(S_f(\xi_n, h_n)) \\ &\geq \psi_2 \left(\frac{A}{\varepsilon_0} \psi_1^{-1} \left(\frac{1}{h_n^{N+1+\alpha}} \right) \right) \frac{\varepsilon_0}{\psi_2 \left(A \psi_1^{-1} \left(1/h_n^{N+1+\alpha} \right) \right)} \\ &\geq 1 \end{aligned}$$

by the convexity of ψ_2 . This yields $\|f_n\|_{L^{\psi_2}(\mu)} \geq \frac{\varepsilon_0}{8^{N+1+\alpha} A}$ for every n , which is a contradiction and completes the proof of the first part.

2) We now suppose that Condition (2.18) is satisfied. Thanks to the second point of Proposition 2.11, it is sufficient to prove that, for every $\varepsilon > 0$, the norm of the embedding

$$I_r : A_\alpha^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mathbb{B}_N \setminus r\overline{\mathbb{B}_N}, \mu)$$

is smaller than ε for some $r_0(\varepsilon)$ and every r such that $r_0(\varepsilon) \leq r < 1$. Let $\eta \in (0, 1)$ and let $A := A(\varepsilon) = \frac{6 \cdot 4^{N+\alpha}}{\varepsilon} > 0$; Condition (2.18) ensures that there exists $h_A \in (0, 1/2)$ such that

$$K_{\mu, \alpha}(h) \leq \eta \frac{1/h^{N+1+\alpha}}{\psi_2(A \psi_1^{-1}(1/h^{N+1+\alpha}))}$$

for $h \leq h_A$. Let now f be in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$ and $r \in (0, 1)$. By the proof of Lemma 2.7, applied to $E = \mathbb{B}_N \setminus r\overline{\mathbb{B}_N}$ and f , there exist a constant $B > 0$ given by $B = \frac{6 \cdot 4^{N+\alpha}}{A} = \varepsilon$, and some constants $x_A > 0$ and $C_1 > 0$, independent of f , such that

$$\begin{aligned} \int_{\mathbb{B}_N \setminus r\overline{\mathbb{B}_N}} \psi_2 \left(\frac{|f|}{\varepsilon} \right) d\mu &= \int_{\mathbb{B}_N \setminus r\overline{\mathbb{B}_N}} \psi_2 \left(\frac{|f|}{B} \right) d\mu \\ &\leq \mu(\mathbb{B}_N \setminus r\overline{\mathbb{B}_N}) \psi_2(x_A) + C_1 \eta \int_{\mathbb{B}_N} \psi_1(\Lambda_f) dv_\alpha, \end{aligned}$$

where Λ_f denotes the maximal function of f , defined by (2.3). Now, we choose η such that $C_1 \eta \int_{\mathbb{B}_N} \psi_1(\Lambda_f) dv_\alpha \leq \frac{1}{2}$ (which is possible thanks to the fact that ψ_1 satisfies ∇_2 -Condition, and thanks to Corollary 3.9) and we take $r_0 \in (0, 1)$ such that $\mu(\mathbb{B}_N \setminus r\overline{\mathbb{B}_N}) \psi_2(x_A) \leq \frac{1}{2}$ for every $r \in (r_0, 1)$ (note that r_0 depends on ε because A depends on ε). We get $\|I_r(f)\|_{L^{\psi_2}(\mu)} \leq \varepsilon$ as soon as $r_0 < r < 1$, what completes the proof.

3) The proof of the third point is essentially contained in that of the third part of Theorem 4.11 of [13]. \square

This leads us to the definition of *vanishing* (ψ, α) -Bergman-Carleson measure on the ball:

Definition 2.13. Let ψ be an Orlicz function and let μ be a Borel positive measure on \mathbb{B}_N . We say that μ is a vanishing (ψ, α) -Bergman-Carleson measure if, for every $A > 0$,

$$\mu(W(\xi, h)) = o_{h \rightarrow 0} \left(\frac{1}{\psi(A \psi^{-1}(1/h^{N+1+\alpha}))} \right)$$

for any $\xi \in \mathbb{S}_N$.

We have the next corollary:

Corollary 2.14. Let ψ be an Orlicz function satisfying the ∇_0 -Condition and let μ be a Borel positive measure on \mathbb{B}_N . Then $A_\alpha^\psi(\mathbb{B}_N)$ embeds compactly into $L^\psi(\mu)$ if and only if μ is a vanishing (ψ, α) -Bergman-Carleson measure.

2.3. Application to composition operators on weighted Bergman-Orlicz spaces. Theorem 2.8 and Theorem 2.12 allow us to give the following characterization with some constraints on the Orlicz function ψ :

Theorem 2.15. *Let ψ be an Orlicz function and $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be holomorphic. We denote by μ_ϕ^α the pull-back measure of ϕ for the weighted Lebesgue measure v_α on \mathbb{B}_N , namely $\mu_\phi^\alpha(E) = v_\alpha(\phi^{-1}(E))$ for every Borel subset E of \mathbb{B}_N . Then*

- (1) *If ψ satisfies the uniform ∇_0 -Condition, then C_ϕ is bounded from $A_\alpha^\psi(\mathbb{B}_N)$ into itself if and only if μ_ϕ^α is a (ψ, α) -Bergman-Carleson measure.*
- (2) *If ψ satisfies the ∇_0 -Condition, then C_ϕ is compact from $A_\alpha^\psi(\mathbb{B}_N)$ into itself if and only if μ_ϕ^α is a vanishing (ψ, α) -Bergman-Carleson measure.*

Proof. Thanks to Corollary 2.10 and Corollary 2.14, it suffices to notice that the continuity (resp. compactness) of the canonical embedding $I_{\mu_\phi^\alpha} : A_\alpha^\psi(\mathbb{B}_N) \hookrightarrow L^\psi(\mu_\phi^\alpha)$ is equivalent to the boundedness (resp. compactness) of $C_\phi : A_\alpha^\psi(\mathbb{B}_N) \rightarrow A_\alpha^\psi(\mathbb{B}_N)$. This just proceeds from the fact that

$$\begin{aligned} \|C_\phi(f)\|_{A_\alpha^\psi(\mathbb{B}_N)} &= \inf \left\{ C > 0, \int_{\mathbb{B}_N} \psi \left(\frac{|f \circ \phi|}{C} \right) d\sigma \leq 1 \right\} \\ &= \inf \left\{ C > 0, \int_{\mathbb{B}_N} \psi \left(\frac{|f|}{C} \right) d\mu_\phi^\alpha \leq 1 \right\} \\ &= \left\| I_{\mu_\phi^\alpha}(f) \right\|_{L^\psi(\mu_\phi^\alpha)}, \end{aligned}$$

for any $f \in A_\alpha^\psi(\mathbb{B}_N)$. □

As a particular case of the previous theorem, we state and verify [10, Theorem 3.6 and Theorem 4.3]:

Theorem 2.16. *Let ψ be an Orlicz function which satisfies Δ_2 -Conditions and let $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be holomorphic. Then*

- (1) *C_ϕ is bounded from $A_\alpha^\psi(\mathbb{B}_N)$ into itself if and only if μ_ϕ^α is an α -Bergman-Carleson measure.*
- (2) *C_ϕ is compact from $A_\alpha^\psi(\mathbb{B}_N)$ into itself if and only if μ_ϕ^α is a vanishing α -Bergman-Carleson measure.*

Proof. It suffices to observe that

$$\frac{1}{\psi(A\psi^{-1}(1/h^{N+1+\alpha}))} \approx h^{N+1+\alpha}$$

for every $A > 0$, whenever ψ is an Orlicz function which satisfies the Δ_2 -Condition (see Remark 2 (a) following Theorem 4.11 in [13].) □

As we said, we are interested in finding where the break of condition for boundedness of C_ϕ happens between $H^\infty(\mathbb{B}_N)$ and $A_\alpha^p(\mathbb{B}_N)$. More precisely, we can wonder if there are some spaces different from $H^\infty(\mathbb{B}_N)$ and smaller than some $A_\alpha^p(\mathbb{B}_N)$ on which every composition operator C_ϕ is bounded. In [11], the authors show the following proposition:

Proposition 2.17. *Let $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be analytic. Then*

$$(2.20) \quad \mu_\phi^\alpha(S(\xi, h)) = O_{h \rightarrow 0}(h^{\alpha+2})$$

for every $\xi \in \mathbb{S}_N$.

In fact, this result is stated for general strongly pseudo-convex domains instead of \mathbb{B}_N ([11, Proposition 4].) A comparison of Condition (2.20) and the definition (ψ, α) -Bergman-Carleson measure with ψ satisfying the uniform ∇_0 -Condition leads us to define an other class of Orlicz functions:

Definition 2.18. Let ψ be an Orlicz function. We say that ψ belongs to the Δ_α^N -Condition if there exist two constants $K > 0$ and $C > 1$ such that, for every $x > 0$ large enough, we have

$$(2.21) \quad \psi(x) \frac{N+1+\alpha}{\alpha+2} \leq K\psi(Cx).$$

This condition is clearly a condition of fast enough growth. For instance, $\psi(x) = e^{ax} - 1$ with $a > 0$ satisfies this Δ_α^N -Condition. A straightforward computation entails:

Proposition 2.19. *Let ψ be an Orlicz function satisfying the Δ_α^N -Condition. Then there exist $A > 0$ and $C > 1$ such that*

$$(2.22) \quad Ch^{\alpha+2} \leq \frac{1}{\psi(A\psi^{-1}(1/h^{N+1+\alpha}))}$$

for every h small enough.

In fact, Inequality (2.22) is equivalent to the fact that ψ satisfies the Δ_α^N -Condition. Now, Corollary 2.10, Proposition 2.17 and Proposition 2.19 give the following theorem, which answers to the question of the existence of some Bergman-Orlicz space, on which every composition operator is bounded:

Theorem 2.20. *Let ψ be an Orlicz function satisfying the $\Delta_\alpha^N \cap \nabla_0^{\text{unif}}$ -Condition. Then every composition operator is bounded from $A_\alpha^\psi(\mathbb{B}_N)$ into itself.*

We remark that every $\psi(x) = e^{ax} - 1$ for $a > 0$ belongs to the $\nabla_0^{\text{unif}} \cap \Delta_\alpha^N$ -Class, so that our result is not empty.

Let us notice that when $N = 1$, Theorem 4.6 (adapted to the α -order weighted area measure on the disk) holds ([15, Theorem 3.1], up to the order α), then uniform ∇_0 -Condition is not necessary and Δ_α^1 is satisfied by every Orlicz function, and so is Inequality (2.22). Therefore we deduce from Theorem 2.20 that every composition operator is bounded on every Bergman-Orlicz space of the unit disk \mathbb{D} , which is already known ([13, Proposition 5.4]; note that our proof is different from that of the latter, which is based on the Littlewood subordination principle.)

3. HARDY-ORLICZ SPACES OF \mathbb{B}_N

Concerning the study of composition operators, as it is usual for Hardy spaces, working with Hardy-Orlicz spaces need more precautions than working with Bergman-Orlicz spaces; this is essentially because of the fact that their definition takes more into account the boundary behavior of holomorphic functions.

3.1. Definitions and classical results. As for the Hardy spaces, we define the Hardy-Orlicz spaces of the ball from the Orlicz spaces of the sphere:

Definition 3.1. Let ψ be an Orlicz function. The Hardy-Orlicz space $H^\psi(\mathbb{B}_N)$ of the ball of \mathbb{C}^N is the space of analytic functions $f : \mathbb{B}_N \rightarrow \mathbb{C}$ such that

$$\sup_{0 < r < 1} \|f_r\|_\psi < \infty,$$

where $f_r \in L^\psi(\mathbb{S}_N)$ is defined by $f_r(z) = f(rz)$.

Since $L^\psi(\mathbb{B}_N) \subset L^1(\mathbb{B}_N)$, $H^\psi(\mathbb{B}_N)$ is contained in $H^1(\mathbb{B}_N)$. In particular, any $f \in H^\psi(\mathbb{B}_N)$, admits a radial limit f^* almost everywhere on \mathbb{S}_N , and we have the following theorem, which can be generalized from the well-known case $\psi(x) = x^p$:

Theorem 3.2. *Let $f \in H^\psi(\mathbb{B}_N)$ and let f^* be its almost everywhere boundary limit. Then $f^* \in L^\psi(\mathbb{S}_N)$ and*

$$\|f^*\|_\psi = \sup_{0 < r < 1} \|f_r\|_\psi.$$

If we denote by $\|f\|_{H^\psi} := \|f^*\|_\psi$, then $H^\psi(\mathbb{B}_N)$ endowed with the norm $\|\cdot\|_{H^\psi}$ is a Banach space.

The proof of this result is quite identical to the one variable case which is contained in the proof of Proposition 3.1 in [13]. When there is no possible confusion, we will indifferently denote by $\|\cdot\|_\psi$ the norm $\|\cdot\|_{H^\psi}$ on $H^\psi(\mathbb{B}_N)$. Moreover, $H^\psi(\mathbb{B}_N)$ can be seen as a subspace of $L^\psi(\mathbb{S}_N)$.

Contrary to what happens for the Hardy space, the ball algebra $A(\mathbb{B}_N)$ is not always dense in $H^\psi(\mathbb{B}_N)$, as the following result indicates (from now on, $HM^\psi(\mathbb{B}_N)$ denotes $H^\psi(\mathbb{B}_N) \cap M^\psi(\mathbb{B}_N)$.)

Theorem 3.3 (See Theorem 4, Chapter IX of [18] for $\Omega = \mathbb{B}_N$). *Let ψ be an Orlicz function. $A(\mathbb{B}_N)$ is dense in $H^\psi(\mathbb{B}_N)$ if and only if $H^\psi(\mathbb{B}_N)$ is separable, which in turn is equivalent to the fact that ψ satisfies the Δ_2 -Condition. However, $HM^\psi(\mathbb{B}_N)$ is always separable.*

The non-separability of $H^\psi(\mathbb{B}_N)$ will cause some problems when we will try to deduce results on composition operators from embedding theorems.

As for Bergman-Orlicz spaces, the following theorem will be useful to get embedding theorems in the context of Hardy-Orlicz spaces. Its proof is very similar to that of Proposition 2.3.

Theorem 3.4. *On the unit ball of $H^\psi(\mathbb{B}_N)$, the weak-star topology coincides with the uniform convergence on every compact subset of \mathbb{B}_N .*

We finish this section by recalling an interpolation theorem which is not general, but which will be sufficient for our purpose. It is nothing but [13, Proposition 3.5].

Proposition 3.5. *Let ψ be an Orlicz function which satisfies the ∇_2 -Condition (see Definition 1.6.) Then every linear, or sub-linear, operator which is of weak-type $(1, 1)$ and strong type (∞, ∞) is bounded from L^ψ into L^ψ .*

For more complete results concerning interpolation of linear operators between Orlicz spaces, we refer to Chapter VI of [18].

3.2. Embedding Theorems for Hardy-Orlicz spaces. We begin with an estimation of the norm of point evaluation linear functional maps acting on $H^\psi(\mathbb{B}_N)$. We denote by δ_a the point evaluation functional at $a \in \mathbb{B}_N$, so that, for f defined on \mathbb{B}_N , we have $\delta_a(f) = f(a)$.

First, for $a \in \mathbb{S}_N$ and $0 < r < 1$, let us define the function $u_{a,r} \in H^\infty(\mathbb{B}_N)$ by

$$u_{a,r}(z) = \left(\frac{1-r}{1-r\langle z, a \rangle} \right)^{2N}$$

for $z \in \mathbb{B}_N$. It is indeed clear that $\|u_{a,r}\|_\infty = 1$ since $|u_{a,r}(z)| \leq 1$ for every $z \in \mathbb{B}_N$ and $u_{a,r}(z) \xrightarrow{z \rightarrow a} 1$. Moreover, let notice that for every $z \in \mathbb{B}_N$

$$|u_{a,r}(z)| = \left(\frac{1-r}{1+r} \right)^N P(ra, z),$$

where

$$P_w(z) = P(w, z) = \left(\frac{1-|w|^2}{|1-\langle w, z \rangle|^2} \right)^N$$

for $w, z \in \mathbb{B}_N$, is the invariant Poisson Kernel. Therefore, it follows that

$$\|u_{a,r}\|_1 = \left(\frac{(1-r)^2}{1-r^2} \right)^N \|P_w\|_1 = \left(\frac{1-r}{1+r} \right)^N.$$

The last inequality is nothing but Cauchy's formula and the positivity of P (see [22, Subsection 4.1].)

The following proposition says that the point evaluation functional is bounded on $H^\psi(\mathbb{B}_N)$ and yields lower and upper estimations of its norm.

Proposition 3.6. *If $a \in \mathbb{B}_N$ then*

$$\frac{1}{4^N} \psi^{-1} \left(\left(\frac{1+|z|}{1-|z|} \right)^N \right) \leq \|\delta_a\|_{(H^\psi(\mathbb{B}_N))^*} \leq \psi^{-1} \left(\left(\frac{1+|z|}{1-|z|} \right)^N \right).$$

Moreover,

$$(3.1) \quad \|\delta_a\|_{(HM^\psi(\mathbb{B}_N))^*} = \|\delta_a\|_{(H^\psi(\mathbb{B}_N))^*} := \|\delta_a\|.$$

Proof. First, since $f_r \in HM^\psi(\mathbb{B}_N)$ for every $f \in H^\psi(\mathbb{B}_N)$ and since $f_r(z) \xrightarrow[r \rightarrow 1]{} f(z)$, $z \in \mathbb{B}_N$, we have

$$(3.2) \quad \|\delta_z\|_{(HM^\psi(\mathbb{B}_N))^*} \geq \|\delta_z\|_{(H^\psi(\mathbb{B}_N))^*}$$

so that (3.1) is true, as the converse inequality in (3.2) is trivial.

Now, for every $f \in H^\psi$, we have

$$f(z) = \int_{\mathbb{S}_N} P_z(\zeta) f(\zeta) d\sigma(\zeta),$$

with $P_z \in L^\infty(\mathbb{S}_N)$ and $\|P_z\|_\infty = \left(\frac{1+|z|}{1-|z|}\right)^N$. Set $C = \|f\|_\psi$; Jensen's Inequality gives

$$\begin{aligned} \psi\left(\left|\frac{f(z)}{C}\right|\right) &\leq \int_{\mathbb{S}_N} P_z(\zeta) \psi\left(\frac{|f(\zeta)|}{C}\right) d\sigma(\zeta) \\ &\leq \left(\frac{1+|z|}{1-|z|}\right)^N. \end{aligned}$$

Hence

$$|f(z)| \leq \|f\|_\psi \psi^{-1}\left(\left(\frac{1+|z|}{1-|z|}\right)^N\right).$$

For the lower bound, it is sufficient to write

$$\|\delta_z\| \geq \frac{|u_{a,r}(z)|}{\|u_{a,r}\|_\psi}$$

with $r = |z|$ and $\bar{a}z = r$, and to argue as in the proof of Proposition 2.2. \square

Embedding theorems will involve some geometric conditions. We recall the definition of the non-isotropic distance on the sphere \mathbb{S}_N , which we denote by d . For $(\zeta, \xi) \in \mathbb{S}_N$, it is given by

$$d(\zeta, \xi) = \sqrt{|1 - \langle \zeta, \xi \rangle|}.$$

As in Subsection 2.2, we will use non-isotropic ‘‘balls’’ $S(\zeta, h)$ and $S_f(\zeta, h)$, the associated ‘‘true’’ balls $Q_f(\zeta, h)$ in \mathbb{S}_N , extending their definition for ζ in the whole closed ball $\overline{\mathbb{B}_N}$, and Carleson windows $W(\zeta, h)$ and $W_f(\zeta, h)$ ($\zeta \in \mathbb{S}_N$ this time,) the latter being defined by

$$W_f(\zeta, h) = \left\{ z \in \overline{\mathbb{B}_N}, 1 - |z| < h, \frac{z}{|z|} \in Q_f(\zeta, h) \right\}.$$

Finally, we define the Korányi approach region $D(\eta)$ for $\eta \in \mathbb{S}_N$ by

$$D(\eta) = \left\{ z \in \mathbb{B}_N, d(z, \eta)^2 < 1 - |z|^2 \right\}.$$

Given f continuous on \mathbb{B}_N and $\xi \in \mathbb{S}_N$, the maximal function N_f of f associated to the Korányi approach regions is defined as follows:

$$N_f(\xi) = \sup_{w \in D(\xi)} \{|f(w)|\}.$$

The Hardy-Littlewood maximal function M_f of $f \in L^1(\mathbb{S}_N)$ is given by

$$M_f(\xi) = \sup_{\delta > 0} \frac{1}{\sigma(Q_f(\xi, \delta))} \int_{Q_f(\xi, \delta)} |f| d\sigma,$$

for $\xi \in \mathbb{S}_N$. It is well-known that the sub-linear maximal operator $M : f \mapsto M_f$ is of weak type $(1, 1)$ on \mathbb{S}_N (see [22], Lemma 4.8) and that it maps $L^\infty(\mathbb{S}_N)$ into itself boundedly. Therefore,

Proposition 3.5 yields the following result concerning Hardy-Littlewood maximal function on Hardy-Orlicz spaces, under the ∇_2 -condition (see Definition 1.6):

Theorem 3.7. *Let ψ be an Orlicz function satisfying the ∇_2 -condition. Then, the Hardy-Littlewood maximal operator M maps $L^\psi(\mathbb{S}_N)$ into itself boundedly. More precisely, there exists a constant $C_\psi > 0$ such that*

$$\|Mf\|_\psi \leq C_\psi \|f\|_\psi,$$

for every $f \in L^\psi(\mathbb{S}_N)$.

Moreover, N_f is dominated by M_f in the ‘‘Hardy-Orlicz following sense’’:

Theorem 3.8. *Let ψ be an Orlicz function. Then there exists a constant $C > 0$ such that*

$$N_f(\xi) \leq CM_{f^*}(\xi)$$

for every $f \in H^\psi(\mathbb{B}_N)$ and for every $\xi \in \mathbb{S}_N$, where $f^* \in L^\psi(\mathbb{S}_N)$ is the radial limit of f .

Proof. First, $f^* \in L^1(\mathbb{S}_N)$ since $f \in H^\psi(\mathbb{B}_N)$, so $f^*d\sigma$ is a finite complex Borel measure on \mathbb{S}_N and we can use Theorem 4.10 of [22] to get

$$N_{P[f^*]} \leq CM_{f^*}$$

for some constant $C > 0$, where

$$P[f^*](\xi) = \int_{\mathbb{S}_N} P(\xi, z) f^*(z) d\sigma(z).$$

We finish the proof by noticing that

$$P[f^*] = f,$$

for example because $f \in H^1(\mathbb{B}_N)$. □

We get a result similar to Theorem 3.7 for the maximal operator N associated to Korányi approach regions:

Corollary 3.9. *Let ψ be an Orlicz function satisfying the ∇_2 -condition. Then, the maximal operator N associated to Korányi approach regions maps $H^\psi(\mathbb{B}_N)$ into $L^\psi(\mathbb{S}_N)$ boundedly. More precisely, there exists a constant $C_\psi > 0$ such that*

$$\|Nf\|_{L^\psi(\mathbb{S}_N)} \leq C_\psi \|f\|_{H^\psi(\mathbb{B}_N)}$$

for every $f \in H^\psi(\mathbb{B}_N)$.

Proof. It is an immediate consequence of Theorem 3.7 and Theorem 3.8, with the fact that $\|f^*\|_{L^\psi(\mathbb{S}_N)} = \|f\|_{H^\psi(\mathbb{B}_N)}$ for every $f \in H^\psi(\mathbb{B}_N)$. □

As in the Bergman-Orlicz case, we will need a specific formulation of Carleson’s theorem, but here for the closed ball. In the case of the disk, we refer to [13, Theorem 4.13]. As usual, we define two functions involved in this context:

Definition 3.10. For any positive finite Borel measure μ on $\overline{\mathbb{B}_N}$, we set

$$\varrho_\mu(h) = \sup_{\xi \in \mathbb{S}_N} \mu(W_f(\xi, h))$$

and

$$K_\mu(h) = \sup_{0 < t < h} \frac{\mu(W_f(\xi, t))}{t^N}$$

for $h \in (0, 1)$.

We think that there is no possible confusion to denote in the same way the function ϱ_μ in Bergman-Orlicz and Hardy-Orlicz contexts. Usually, we say that μ is, by definition, a Carleson measure (in the classical Hardy spaces framework) if $K_\mu(h)$ is finite for some $h \in (0, 1)$. If $f \in H^1(\mathbb{B}_N)$, we will identify f^* , the function equal to f in \mathbb{B}_N and to the boundary radial limits of f on the full Lebesgue measure subset of \mathbb{S}_N where these limits exist, with f itself. In the statement of the following Carleson Theorem, we will not precise on which subset of $\overline{\mathbb{B}_N}$ the function f^* is well-defined and we will write $f(z)$ for $z \in \overline{\mathbb{B}_N}$, by keeping in mind that this has sense only for $z \in \mathbb{B}_N \cup (\mathbb{S}_N \cap E_f)$ where E_f is some subset of \mathbb{S}_N such that $\sigma(\mathbb{S}_N \setminus E_f) = 0$. We think that there is still no possible confusion.

Theorem 3.11. *There exist two constants $\tilde{C} > 0$ and $C > 1$ such that, for every $f \in H^1(\mathbb{B}_N)$ and every positive finite Borel measure μ on $\overline{\mathbb{B}_N}$, we have*

$$\mu(\{z \in \overline{\mathbb{B}_N}, |z| > 1 - h \text{ and } |f(z)| > t\}) \leq \tilde{C}K_\mu(Ch)\sigma(\{N_f > t\})$$

for every $h \in (0, 1/C)$ and every $t > 0$.

We will need a covering lemma:

Lemma 3.12. *Let g be a continuous function on \mathbb{B}_N , $a > 0$ and $h \in (0, 1)$. Then, either $|g(w)| < a$ in $\mathbb{B}_N \setminus (1 - h)\mathbb{B}_N$ or there exist w_1, w_2, \dots in $\mathbb{B}_N \setminus (1 - h)\mathbb{B}_N$ such that:*

- (1) $|g(w_i)| \geq a$ for every $i \geq 1$,
- (2) The set

$$\{w \in \mathbb{B}_N, |g(w)| \geq a\} \cap (\mathbb{B}_N \setminus (1 - h)\mathbb{B}_N)$$

is contained in

$$\bigcup_{i \geq 1} S(w_i, 4(1 - |w_i|^2)).$$

- (3) The sets $Q_f(w_i, (1 - |w_i|^2))$, $i \geq 1$, are pairwise disjoint.

Proof. The proof of this lemma can be found in [17] in the case $h = 1/2$. There is nothing to do to adapt this proof and get the result for any $h \in (0, 1)$. \square

Proof of Theorem 3.11. We fix $t > 0$. We may suppose that there exists $a \in \mathbb{B}_N \setminus (1 - h)\mathbb{B}_N$ such that $|f(a)| > t$, with $|a| > 1 - h$. Then the previous lemma ensures that there exists $(w_i)_{i \geq 1} \subset \mathbb{B}_N \setminus (1 - h)\mathbb{B}_N$ such that

$$(3.3) \quad \mu(\{z \in \mathbb{B}_N, |z| > 1 - h \text{ and } |f(z)| > t\}) \leq \sum_{i \geq 1} \mu(S(w_i, 4(1 - |w_i|^2))).$$

We must also have

$$\mu(\{z \in \mathbb{S}_N, |f(z)| > t\}) \leq \sum_{i \geq 1} \mu(\overline{Q_f(w_i, 4(1 - |w_i|^2))}),$$

since

$$\{z \in \mathbb{S}_N, |f(z)| > t\} \subset \bigcup_{i \geq 1} \overline{Q_f(w_i, 4(1 - |w_i|^2))}.$$

The last inclusion comes from the fact that if $z \in \mathbb{S}_N$ satisfies $|f(z)| > t$, then we can find $r \in (0, 1)$ as close as possible to 1 such that there is at least one $i \geq 1$ such that $rz \in S(w_i, 4(1 - |w_i|^2))$; letting r tends to 1, we get $z \in \overline{Q_f(w_i, 4(1 - |w_i|^2))}$. So we can rewrite inequality (3.3) for μ as follows:

$$(3.4) \quad \mu(\{z \in \overline{\mathbb{B}_N}, |z| > 1 - h \text{ and } |f(z)| > t\}) \leq \sum_{i \geq 1} \mu(\overline{S_f(w_i, 4(1 - |w_i|^2))}).$$

Moreover, the definitions of the Korányi approach region $D(\eta)$, $\eta \in \mathbb{S}_N$, and of Q_f yields

$$\{\eta \in \mathbb{S}_N, w_i \in D(\eta)\} = Q_f(w_i, 1 - |w_i|^2),$$

so $N_f(\eta) \geq t$ whenever $\eta \in Q_f(w_i, 1 - |w_i|^2)$. Therefore and since the sets $Q_f(w_i, 1 - |w_i|^2)$ are pairwise disjoint, we have

$$(3.5) \quad \sum_{i \geq 1} \sigma(Q_f(w_i, 1 - |w_i|^2)) \leq \sigma(\{N_f \geq t\}).$$

Now, the triangle inequality ensures that if we set $r = 9(1 - |w_i|^2)$, then

$$\mu\left(\overline{S_f(w_i, 4(1 - |w_i|^2))}\right) \leq \mu\left(S_f\left(\frac{w_i}{|w_i|}, r\right)\right) \leq \mu\left(W_f\left(\frac{w_i}{|w_i|}, C_0 r\right)\right)$$

for some $C_0 > 1$. By definition of K_μ and as $r \leq 2h$, we can find some absolute constant $C > 1$ (in fact, we can take $C = 2C_0$) such that

$$(3.6) \quad \mu\left(\overline{S_f(w_i, 4(1 - |w_i|^2))}\right) \leq C_0^N r^N \frac{\mu\left(W_f\left(\frac{w_i}{|w_i|}, C_0 r\right)\right)}{C_0^N r^N} \leq C_0^N r^N K_\mu(Ch).$$

Now, by using [22, Lemma 4.6] and by homogeneity of the Lebesgue measure on \mathbb{S}_N , we get

$$(3.7) \quad r^N \lesssim \sigma\left(Q_f\left(\frac{w_i}{|w_i|}, r\right)\right) \lesssim \sigma\left(Q_f(w_i, 1 - |w_i|^2)\right).$$

Hence, inequalities (3.4), (3.5), (3.6) and (3.7) give that there exist two constants $C > 1$ and $\tilde{C} > 0$ such that

$$\mu(\{z \in \overline{\mathbb{B}_N}, |z| > 1 - h \text{ and } |f(z)| > t\}) \leq \tilde{C} K_\mu(Ch) \sigma(\{N_f \geq t\}).$$

□

As for the Bergman-Orlicz case, we will need a lemma which is an application of Theorem 3.11. Its proof follows exactly the same ideas than that of Lemma 2.7; we do not go into details.

Lemma 3.13. *Let μ be a finite positive Borel measure on $\overline{\mathbb{B}_N}$ and let ψ_1 and ψ_2 be two Orlicz functions. Let $C \geq 1$ be the constant appearing in Theorem 3.11. Assume that there exist $A > 0$, $\eta > 0$ and $h_A \in (0, 1/C)$ such that*

$$K_\mu(h) \leq \eta \frac{1/h^N}{\psi_2(A\psi_1^{-1}(1/h^N))}$$

for every $h \in (0, h_A)$. Then, there exist three constants $B > 0$, $x_A > 0$ and $C_1 > 0$ (this latter does not depend on A , η and h_A), such that for every $f \in H^{\psi_1}(\mathbb{B}_N)$ such that $\|f\|_{\psi_1} \leq 1$ and every Borel subset E of $\overline{\mathbb{B}_N}$

$$\int_E \psi_2\left(\frac{|f|}{B}\right) d\mu \leq \mu(E) \psi_2(x_A) + C_1 \eta \int_{\mathbb{S}_N} \psi_1(N_f) d\sigma.$$

From this point on, we are able to give the results analogous to Theorem 2.8 and Theorem 3.19, which give a necessary condition and a sufficient condition for the existence (and then the boundedness) and actually the compactness of the embedding $j_\mu : H^{\psi_1}(\mathbb{B}_N) \rightarrow L^{\psi_2}(\mu)$ for some finite positive Borel measure μ on $\overline{\mathbb{B}_N}$.

3.2.1. The canonical embedding $H^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$. The main theorem of this section is the following:

Theorem 3.14. *Let μ be a finite positive Borel measure on $\overline{\mathbb{B}_N}$ and let ψ_1 and ψ_2 be two Orlicz functions; we suppose that ψ_1 satisfies the ∇_2 -condition. Then:*

- (1) *If inclusion $H^{\psi_1}(\mathbb{B}_N) \subset L^{\psi_2}(\mu)$ holds and is continuous, then there exists some $A > 0$ such that*

$$(3.8) \quad \varrho_\mu(h) = O_{h \rightarrow 0} \left(\frac{1}{\psi_2(A\psi_1^{-1}(1/h^N))} \right).$$

(2) If there exists some $A > 0$ such that

$$(3.9) \quad K_\mu(h) = O_{h \rightarrow 0} \left(\frac{1/h^N}{\psi_2(A\psi_1^{-1}(1/h^N))} \right)$$

then inclusion $H^{\psi_1}(\mathbb{B}_N) \subset L^{\psi_2}(\mu)$ holds and is continuous.

(3) If in addition $\psi_1 = \psi_2 = \psi$ satisfies the uniform ∇_0 -Condition, then (3.8) and (3.9) are equivalent.

Proof. We just give the sketch of the proof which is quite similar to that of Theorem 2.8.

1) We may consider the map

$$f = \psi^{-1} \left(\frac{1}{h^N} \right) u_{a,1-h},$$

where $u_{a,1-h}(z) = \left(\frac{h}{1 - (1-h)\langle z, a \rangle} \right)^{2N}$ for $0 < h < 1$ and then proceed as in the proof of the first part of Theorem 2.8

2) This is an application of Lemma 3.13 and Corollary 3.9, as in Theorem 2.8.

3) The proof of the third part is quite identical to the one of 3) of Theorem 2.8. \square

The third part of the previous theorem leads to introduce the ψ -Hardy-Carleson measures:

Definition 3.15. Let μ be a finite positive Borel measure on $\overline{\mathbb{B}_N}$ and ψ be an Orlicz function. We say that μ is a ψ -Carleson measure if there exists $A > 0$, such that

$$(3.10) \quad \mu(S_f(\xi, h)) = O_{h \rightarrow 0} \left(\frac{1}{\psi(A\psi^{-1}(1/h^N))} \right)$$

for every $\xi \in \mathbb{S}_N$.

We remark that (3.10) is equivalent to (3.8), and we have the following corollary:

Corollary 3.16. Let μ be a finite positive Borel measure on $\overline{\mathbb{B}_N}$ and let ψ be an Orlicz function. Then, if ψ satisfies the uniform ∇_0 -Condition, then inclusion $H^\psi(\mathbb{B}_N) \subset L^\psi(\mu)$ holds and is continuous if and only if μ is a ψ -Hardy-Carleson measure.

Proof. We just have to notice that ∇_0 -Condition entails ∇_2 -Condition, which is contained in Proposition 1.10. \square

3.2.2. Compactness of the canonical embedding $H^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$. For this purpose, we need some general facts about compactness of operators from $H^{\psi_1}(\mathbb{B}_N)$ into $L^{\psi_2}(\mu)$. Most of the criteria have already been given in Paragraph 2.2.2. Anyway, some difficulties appear as we have to take into account the boundary of \mathbb{B}_N . In the one-dimensional case, this is a natural necessary condition that the measure μ must be null on $\partial\mathbb{D}$ as soon as the canonical embedding $H^{\psi_1}(\mathbb{D}) \hookrightarrow L^{\psi_2}(\mu)$ is compact. However we do not know if a similar result is true in the several variables case. It seems to be interesting to state this question in the following conjecture:

Conjecture 3.17. Let μ be a finite positive measure on $\overline{\mathbb{B}_N}$, ψ_1 and ψ_2 be two Orlicz functions. If the canonical embedding $H^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$ is compact, then $\mu(\mathbb{S}_N) = 0$.

We give a necessary and sufficient condition for the canonical embedding $H^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$ to be compact under the hypothesis $\mu(\mathbb{S}_N) = 0$.

Proposition 3.18. Let μ be a finite positive measure on $\overline{\mathbb{B}_N}$ and let ψ_1 and ψ_2 be two Orlicz functions. We suppose that the canonical embedding $H^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$ holds and is bounded.

(1) Consider the two following assertions:

(a) The canonical embedding $H^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$ is compact;

(b) Every sequence in the unit ball of $H^{\psi_1}(\mathbb{B}_N)$, which is convergent to 0 uniformly on every compact subset of \mathbb{B}_N , is convergent to 0 in $L^{\psi_2}(\mu)$.

Then, if (a) holds and $\mu(\mathbb{S}_N) = 0$, then (b) holds. Conversely, if (b) holds, then (a) holds.

- (2) If $H^{\psi_1}(\mathbb{B}_N)$ is continuously embedded in $L^{\psi_2}(\mu)$ and if $\lim_{r \rightarrow 1^-} \|I_r\| = 0$, where $I_r(f) = f \cdot \chi_{\overline{\mathbb{B}_N} \setminus r\mathbb{B}_N}$, then the canonical embedding $H^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$ is compact.

Proof. This is the same than that of Proposition 2.11, except that the measure μ is defined on the closed ball $\overline{\mathbb{B}_N}$; actually, this difficulty is canceled by the assumption $\mu(\mathbb{S}_N) = 0$. \square

Thanks to Lemma 3.13 and to Proposition 3.18, we get our main theorem about compactness of the canonical embedding $H^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$.

Theorem 3.19. *Let μ be a finite positive Borel measure on $\overline{\mathbb{B}_N}$ such that $\mu(\mathbb{S}_N) = 0$, and let ψ_1 and ψ_2 be two Orlicz functions. We assume that ψ_1 satisfies the ∇_2 -condition. Then:*

- (1) *If inclusion $H^{\psi_1}(\mathbb{B}_N) \subset L^{\psi_2}(\mu)$ holds and is compact, then for every $A > 0$ we have*

$$(3.11) \quad \varrho_\mu(h) = o_{h \rightarrow 0} \left(\frac{1}{\psi_2(A\psi_1^{-1}(1/h^N))} \right).$$

- (2) *If*

$$(3.12) \quad K_\mu(h) = o_{h \rightarrow 0} \left(\frac{1/h^N}{\psi_2(A\psi_1^{-1}(1/h^N))} \right)$$

for every $A > 0$, then inclusion $H^{\psi_1}(\mathbb{B}_N) \subset L^{\psi_2}(\mu)$ holds and is compact.

- (3) *If in addition $\psi_1 = \psi_2 = \psi$ satisfies both ∇_0 and ∇_2 -Conditions then (3.11) and (3.12) are equivalent.*

Remark 3.20. It is not necessary to assume that $\mu(\mathbb{S}_N) = 0$ in (2). Indeed, for $h \in (0, 1)$, if $C(h)$ denotes the minimal number of non-isotropic balls $Q_f(\zeta, h)$ which are needed to cover the corona $\overline{\mathbb{B}_N} \setminus (1-h)\mathbb{B}_N$, then we know that there is a constant C independent of h , such that

$$C(h) \leq \frac{C}{h^N}.$$

(This is essentially contained in [19, Lemma 5.2.3].) Therefore, if Condition (3.12) holds, then

$$\mu(\overline{\mathbb{B}_N} \setminus (1-h)\mathbb{B}_N) \leq C \frac{\varrho_\mu(h)}{h^N} \leq CK_\mu(h) \xrightarrow{h \rightarrow 0} 0.$$

Proof. As for Theorem 3.14, we just give a sketch of the proof which is similar to that of Theorem 2.12.

- 1) We consider the functions

$$f_n(z) = \psi_1^{-1} \left(\frac{1}{h_n^N} \right) u_{\xi_n, 1-h_n}(z) = \left(\frac{h_n}{1 - (1-h_n)\langle z, \xi_n \rangle} \right)^{2N},$$

and then follow the proof of 1) of Theorem 2.12.

- 2) It is a consequence of Lemma 3.13, Corollary 3.9 and the assumption $\mu(\mathbb{S}_N) = 0$.

3) The proof of the third point is essentially contained in the proof of the third part of Theorem 4.11 of [13]. \square

The third point of the last theorem leads us to define what one calls *vanishing ψ -Carleson measures*:

Definition 3.21. Let μ be a finite positive Borel measure on $\overline{\mathbb{B}_N}$ and let ψ be an Orlicz function. We say that μ is a *vanishing ψ -Carleson measure* if, for every $A > 0$,

$$(3.13) \quad \mu(S_f(\xi, h)) = o_{h \rightarrow 0} \left(\frac{1}{\psi(A\psi^{-1}(1/h^N))} \right)$$

for every $\xi \in \mathbb{S}_N$.

We now state the following corollary:

Corollary 3.22. *Let ψ be an Orlicz function and let μ be a finite positive Borel measure on $\overline{\mathbb{B}_N}$ such that $\mu(\mathbb{S}_N) = 0$. If ψ satisfies the $\nabla_0 \cap \nabla_2$ -Condition, then inclusion $H^\psi(\mathbb{B}_N) \subset L^\psi(\mu)$ holds and is compact if and only if μ is a vanishing ψ -Carleson measure.*

3.3. Application to Composition operators on Hardy-Orlicz spaces. In this section, we intend to apply the previous results (or at least the ideas) to composition operators. Let $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be holomorphic. We denote by $\phi^* : \overline{\mathbb{B}_N} \rightarrow \overline{\mathbb{B}_N}$ the map which is equal to ϕ on \mathbb{B}_N and which is equal to the boundary limit of ϕ on \mathbb{S}_N , i.e. for all $\zeta \in \mathbb{B}_N$ and σ -almost every $\zeta \in \mathbb{S}_N$, $\phi^*(\zeta) = \lim_{r \rightarrow 1} \phi(r\zeta)$; we define the *pull-back measure* μ_ϕ on $\overline{\mathbb{B}_N}$ induced by ϕ as the image of the normalized Lebesgue measure σ on \mathbb{S}_N :

$$\mu_\phi(E) = \sigma\left(\phi^{*-1}(E) \cap \mathbb{S}_N\right)$$

for every Borel subset $E \subset \overline{\mathbb{B}_N}$.

We also denote by C_ϕ the composition operator associated to ϕ and defined by $C_\phi(f) = f \circ \phi$ for $f \in H^\psi(\mathbb{B}_N)$, with ψ an Orlicz function. In the classical study of composition operators on Hardy spaces, it is possible, and usually done, to make a link between the embedding operator j_{μ_ϕ} associated to the pull-back measure μ_ϕ and the composition operator C_ϕ . Just first look at these two expressions:

$$\begin{aligned} \|j_{\mu_\phi}(f)\|_{L^\psi(\mu)} &= \inf\left(C > 0, \int_{\overline{\mathbb{B}_N}} \psi\left(\frac{|f|}{C}\right) d\mu_\phi \leq 1\right) \\ (3.14) \qquad \qquad \qquad &= \inf\left(C > 0, \int_{\mathbb{S}_N} \psi\left(\frac{|f^* \circ \phi^*|}{C}\right) d\sigma \leq 1\right) \end{aligned}$$

and

$$(3.15) \qquad \|C_\phi(f)\|_\psi = \inf\left(C > 0, \int_{\mathbb{S}_N} \psi\left(\frac{|(f \circ \phi)^*|}{C}\right) d\sigma \leq 1\right).$$

Because of Carleson Theorem for Hardy-Orlicz spaces, it is quite natural to compare (3.14) and (3.15) and to wonder if these two expressions are equal in order to appeal to the results known concerning embedding operators to solve the questions of continuity and compactness of composition operators. For that purpose, we can ask the questions:

- (1) Is it true that $f^* \circ \phi^* = (f \circ \phi)^*$ as soon as these two expressions have sense?

In the one variable case, Lindelöf Theorem gives a positive answer when $f \in H^p(\mathbb{D})$ for any $1 \leq p \leq \infty$ (and so when $f \in H^\psi(\mathbb{D})$), and therefore it suffices to state the problem of continuity and compactness of composition operators in terms of embedding theorems for convenient measures. Nevertheless, it fails to be true in general, whenever the dimension is greater than 1. There exists a weakened several-variables version of Lindelöf Theorem, the so-called Cirka's theorem ([19, Theorem 8.4.4]), yet it is not sufficient for our purpose. So it leads to ask the following question:

- (2) Is inequality $\int_{\mathbb{S}_N} \psi\left(\frac{|(f \circ \phi)^*|}{C}\right) d\sigma \leq \int_{\mathbb{S}_N} \psi\left(\frac{|f^* \circ \phi^*|}{C}\right) d\sigma$ true?

A positive answer should still be sufficient to conclude that the boundedness (resp. compactness) of j_μ entails the boundedness (resp. compactness) of C_ϕ . And in an easier way than answering the first question, it reveals that the answer is yes if f is $H^p(\mathbb{B}_N)$, thanks to the fact that the ball algebra $A(\mathbb{B}_N)$ is dense in $H^p(\mathbb{B}_N)$. And in fact this is still true if f is in any $H^\psi(\mathbb{B}_N)$ in which $A(\mathbb{B}_N)$ is dense. As already mentioned, this is precisely characteristic of the Orlicz function ψ which satisfies the Δ_2 -Condition (Theorem 3.3.) Recall that Δ_2 -Condition constrains the function ψ to verify $\psi(x) \leq Cx^p$ for some $p > 1$ and $C > 0$ (Proposition 1.5.) Thus $H^\psi(\mathbb{B}_N)$ has to contain at most one $H^p(\mathbb{B}_N)$ and then this does not give any result for Hardy-Orlicz spaces “close” to $H^\infty(\mathbb{B}_N)$.

For these reasons, we may see the limit of the possibility to somehow consider composition operators as embedding operators. Nevertheless, we will see how the proofs of Theorem 3.11 and Lemma

3.13 permit to give sufficient conditions, stronger than that obtained for embedding operators, for composition operators to be bounded or compact.

First of all, according to the previous point (2), as a corollary of both Theorem 3.14 and Theorem 3.19, we have the following result, concerning continuity and compactness of composition operators, when the Orlicz function ψ satisfied the Δ_2 -Condition.

Theorem 3.23. *Let $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be an analytic map and let μ_ϕ be its pull-back measure on $\overline{\mathbb{B}_N}$. Let also ψ be an Orlicz functions satisfying the $\Delta_2 \cap \nabla_2$ -Condition. Then:*

(1) C_ϕ is continuous on $H^\psi(\mathbb{B}_N)$ if and only if

$$(3.16) \quad \varrho_\mu(h) = O_{h \rightarrow 0}(h^N).$$

(2) C_ϕ is compact on $H^\psi(\mathbb{B}_N)$ if and only if

$$(3.17) \quad \varrho_\mu(h) = o_{h \rightarrow 0}(h^N).$$

Proof. We recall that, for every Orlicz function ψ satisfying the Δ_2 -Condition,

$$\frac{1}{\psi(A\psi^{-1}(x))} \approx x$$

for every $A > 0$ and every $x > 0$. □

Remark 3.24. In the statement of (2) of the last theorem, we do not assume that $\mu_\phi(\mathbb{S}_N) = 0$, although it is an assumption of Theorem 3.19 (1). In fact, we can proceed as in the proof of Theorem 3.35 (2) of [7] and notice that this hypothesis is not necessary. More precisely, the proof of this part lays on the following corollary of the first part of the previous theorem:

Corollary 3.25. *Let ψ be an Orlicz function which satisfies the Δ_2 -Condition. Let $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be analytic. If C_ϕ is bounded on $H^\psi(\mathbb{B}_N)$ into itself, then ϕ^* cannot carry a set of positive Lebesgue measure in \mathbb{S}_N into a set of Lebesgue measure 0.*

Proof. It is similar to that of Corollary 3.38 of [7]; it is not difficult to see where we may use the Δ_2 -Condition. □

It is interesting to see that, if ψ satisfies the Δ_2 -Condition, then this corollary together with the density of $A(\mathbb{B}_N)$ in $H^\psi(\mathbb{B}_N)$ entails in fact that $(f \circ \phi)^* = f^* \circ \phi^*$ almost everywhere on $\overline{\mathbb{B}_N}$, for every function $f \in H^\psi(\mathbb{B}_N)$, whenever C_ϕ is bounded from $H^\psi(\mathbb{B}_N)$ into itself.

We now state our main general result about composition operators on Hardy-Orlicz spaces of the ball.

Theorem 3.26. *Let $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be an analytic map and μ_ϕ its induced pull-back measure on $\overline{\mathbb{B}_N}$. Let also ψ be an Orlicz function satisfying the ∇_2 -Condition. Then*

(1) *If C_ϕ is bounded from $H^\psi(\mathbb{B}_N)$ into itself then μ_ϕ is a ψ -Carleson measure.*

(2) *For $0 < r < 1$, let us denote by ϕ_r the analytic function on $\overline{\mathbb{B}_N}$ defined by $\phi_r(z) = \phi(rz)$. If there exists some $A > 0$ such that*

$$(3.18) \quad \sup_{0 < r < 1} K_{\mu_{\phi_r}}(h) = O_{h \rightarrow 0} \left(\frac{1/h^N}{\psi(A\psi^{-1}(1/h^N))} \right),$$

then C_ϕ is bounded on $H^\psi(\mathbb{B}_N)$.

(3) *If C_ϕ is compact from $H^\psi(\mathbb{B}_N)$ into itself then μ_ϕ is a vanishing ψ -Carleson measure.*

(4) *If for every $A > 0$, we have*

$$(3.19) \quad \sup_{0 < r < 1} K_{\mu_{\phi_r}}(h) = o_{h \rightarrow 0} \left(\frac{1}{\psi(A\psi^{-1}(1/h^N))} \right),$$

then C_ϕ is compact on $H^\psi(\mathbb{B}_N)$.

Proof. We will refer to the proofs of both Theorem 3.14 and Theorem 3.19 which have just been sketched. By the way, they are similar to that of both Theorem 2.8 and Theorem 2.12 which have been detailed.

(1) This is an easy adaptation of the first point of the proof of Theorem 3.14. We suppose that C_ϕ is continuous with norm equal to C . Let us consider the function already introduced in the proof of Theorem 3.14,

$$f = \psi^{-1} \left(\frac{1}{h^N} \right) u_{a,1-h},$$

for $a \in \mathbb{S}_N$ and $0 < h < 1$. One can show that $f \in H^\psi(\mathbb{B}_N)$, $\|f\|_\psi \leq 1$ and

$$|f(z)| \geq \frac{1}{4^N} \psi^{-1} \left(\frac{1}{h^N} \right)$$

for any $z \in S_f(a, h)$. Moreover, f is continuous up to the boundary. Consequently

$$\begin{aligned} 1 &\geq \int_{\mathbb{S}_N} \psi \left(\frac{|(f \circ \phi)^*|}{C} \right) d\sigma = \int_{\mathbb{S}_N} \psi \left(\frac{|f \circ \phi^*|}{C} \right) d\sigma \\ &= \int_{\mathbb{B}_N} \psi \left(\frac{|f|}{C} \right) d\mu \\ &\geq \int_{S_f(a, h)} \psi \left(\frac{1}{4^N} \psi^{-1} \left(\frac{1}{h^N} \right) \right) d\mu \\ (3.20) \quad &= \mu(S_f(a, h)) \psi \left(\frac{1}{4^N} \psi^{-1} \left(\frac{1}{h^N} \right) \right). \end{aligned}$$

Hence the first point.

(3) The third point proceeds from the same kind of argument and its proof is somehow contained in that of the first point of Theorem 3.19 by observing that the functions f_n introduced in the latter proof are continuous up to the boundary.

(2) and (4) We need to pay a little more attention to the proof of the second part of respectively Theorem 3.14 and Theorem 3.19.

First, we deal with (2); let f be in the unit ball of $H^\psi(\mathbb{B}_N)$, and let us observe that, for every $C > 0$,

$$\begin{aligned} \int_{\mathbb{S}_N} \psi \left(\frac{|(f \circ \phi)^*|}{C} \right) d\sigma &= \sup_{0 < r < 1} \int_{\mathbb{S}_N} \psi \left(\frac{|f \circ \phi_r|}{C} \right) d\sigma \\ (3.21) \quad &= \sup_{0 < r < 1} \int_{\mathbb{B}_N} \psi \left(\frac{|f|}{C} \right) d\mu_{\phi_r}. \end{aligned}$$

In order to avoid any confusion, remark that there is nothing wrong to write f instead of f^* in the right hand side of (3.21) since $\mu_{\phi_r}(\mathbb{S}_N) = 0$. Moreover, if we look at the proof of (2) of Theorem 3.14, we notice that Condition (3.18) and an application of Lemma 3.13 give a majorization of $\|f\|_{L^\psi(\mu_{\phi_r})}$ by some constant $C_0 > 0$ which does not depend on r (but just on N, A, η, C and \tilde{C} , the two last constants being the constants involved in our Carleson theorem, Theorem 3.11.) Then, setting $C = C_0$ in (3.21), we get

$$\int_{\mathbb{S}_N} \psi \left(\frac{|(f \circ \phi)^*|}{C_0} \right) d\sigma \leq \sup_{0 < r < 1} \int_{\mathbb{B}_N} \psi \left(\frac{|f|}{C_0} \right) d\mu_{\phi_r} \leq 1$$

which gives the result.

We turn on (4). It can be treated in a similar way. Nevertheless, we need a specific criterion of compactness of composition operators on $H^\psi(\mathbb{B}_N)$. Indeed, for the usual reasons we can not directly make use of Proposition 3.18 for every Orlicz function ψ . Anyway, the following criterion is a generalization of the classical one on $H^p(\mathbb{B}_N)$, $1 \leq p < \infty$, and its proof is an easy adaptation of the one of Proposition 3.18.

Proposition 3.27. *Let ψ be an Orlicz function and ϕ an analytic map from \mathbb{B}_N into itself. C_ϕ is compact on $H^\psi(\mathbb{B}_N)$ iff, for every sequence $(f_n)_n$ in the unit ball of $H^\psi(\mathbb{B}_N)$ converging to 0 uniformly on every compact subset of \mathbb{B}_N , $f_n \circ \phi$ converges to 0 in $H^\psi(\mathbb{B}_N)$.*

We come back to the proof of (4). Let $(f_n)_n$ be a sequence in the unit ball of $H^\psi(\mathbb{B}_N)$ which converges uniformly on every compact subset of \mathbb{B}_N to 0. Thanks to Proposition 3.27, it suffices to show that $\|f_n \circ \phi\|_\psi$ tends to 0 as n tends to infinity. Set $\varepsilon > 0$.

$$\int_{\mathbb{S}_N} \psi\left(\frac{|(f_n \circ \phi)^*|}{\varepsilon}\right) d\sigma = \sup_{0 < r < 1} \int_{\mathbb{B}_N} \psi\left(\frac{|f_n|}{\varepsilon}\right) d\mu_{\phi_r}.$$

Now, we fix $0 < r < 1$ and $0 < r' < 1$; the value of r' will be chosen independently of r later.

$$\int_{\mathbb{B}_N} \psi\left(\frac{|f_n|}{\varepsilon}\right) d\mu_{\phi_r} = \int_{\mathbb{B}_N \setminus r'\mathbb{B}_N} \psi\left(\frac{|f_n|}{\varepsilon}\right) d\mu_{\phi_r} + \int_{r'\mathbb{B}_N} \psi\left(\frac{|f_n|}{\varepsilon}\right) d\mu_{\phi_r}.$$

Similarly to the proof of (2) of Theorem 2.12, by using Lemma 2.7 and Condition (3.19), we can find some constants $x_A > 0$ and C_1 , both independent of r and r' , such that

$$\sup_{0 < r < 1} \int_{\mathbb{B}_N \setminus r'\mathbb{B}_N} \psi\left(\frac{|f_n|}{\varepsilon}\right) d\mu_{\phi_r} \leq \psi_2(x_A) \sup_{0 < r < 1} \left(\mu_{\phi_r}(\overline{\mathbb{B}_N} \setminus r'\mathbb{B}_N)\right) + \frac{1}{4}.$$

Let us come back to Condition (3.19); it implies that

$$\sup_{0 < r < 1} \sup_{\xi \in \mathbb{S}_N} (\mu_{\phi_r}(W_f(\xi, h))) \leq \eta(h) \cdot h^N$$

for some function η which tends to 0 when h tends to 0. If $C(h)$ denotes the minimal number of non-isotropic balls $Q_f(\xi, h)$ which are needed to cover \mathbb{S}_N , then we know that there is a constant C independent of h such that

$$C(h) \leq \frac{C}{h^N}.$$

(We already use this in Remark 3.20.) Therefore

$$\sup_{0 < r < 1} (\mu_{\phi_r}(\overline{\mathbb{B}_N} \setminus h\mathbb{B}_N)) \leq \frac{C}{h^N} \cdot \eta(h) \cdot h^N = C\eta(h) \xrightarrow{h \rightarrow 0} \infty,$$

and we can find some r' such that

$$\psi_2(x_A) \sup_{0 < r < 1} \left(\mu_{\phi_r}(\overline{\mathbb{B}_N} \setminus r'\mathbb{B}_N)\right) \leq \frac{1}{4}.$$

Next, since $(f_n)_n$ tends to 0 uniformly on every compact subset of \mathbb{B}_N , there exists some n_0 , depending only on r' (which has been fixed before,) such that for any $n \geq n_0$

$$\int_{r'\mathbb{B}_N} \psi\left(\frac{|f_n|}{\varepsilon}\right) d\mu_{\phi_r} \leq \frac{1}{2}$$

for every $0 < r < 1$. Hence, for any $n \geq n_0$,

$$\int_{\mathbb{S}_N} \psi\left(\frac{|(f_n \circ \phi)^*|}{\varepsilon}\right) d\sigma = \sup_{0 < r < 1} \int_{\mathbb{B}_N} \psi\left(\frac{|f_n|}{\varepsilon}\right) d\mu_{\phi_r} \leq \frac{1}{2} + \frac{1}{2} = 1$$

which completes the proof. \square

We make a remark which will be useful for the next purpose of this section.

Remark 3.28. 1) Let us have a look on the proof of the first part of the previous theorem. We can modify the chain rule of inequalities (3.20) in the following way:

$$\begin{aligned}
1 &\geq \int_{\mathbb{S}_N} \psi \left(\frac{|(f \circ \phi)^*|}{C} \right) d\sigma = \sup_{0 < r < 1} \int_{\mathbb{S}_N} \psi \left(\frac{|f \circ \phi_r|}{C} \right) d\sigma \\
&= \sup_{0 < r < 1} \int_{\mathbb{B}_N} \psi \left(\frac{|f|}{C} \right) d\mu_{\phi_r} \\
&\geq \sup_{0 < r < 1} \int_{S_f(a, h)} \psi \left(\frac{1}{4^N} \psi^{-1} \left(\frac{1}{h^N} \right) \right) d\mu_{\phi_r} \\
&= \sup_{0 < r < 1} \mu_{\phi_r} (S_f(a, h)) \psi \left(\frac{1}{4^N} \psi^{-1} \left(\frac{1}{h^N} \right) \right)
\end{aligned}$$

and then we obtain that the continuity of C_ϕ on $H^\psi(\mathbb{B}_N)$ not only implies that μ_ϕ is a ψ -Carleson measure but also that there exists some $A > 0$ such that

$$(3.22) \quad \sup_{0 < r < 1} \mu_{\phi_r} (S_f(\xi, h)) \leq \frac{1}{\psi(A\psi^{-1}(1/h^N))}$$

for every $\xi \in \mathbb{S}_N$ and any $0 < h < 1$.

Let us observe that this fact may have been proved directly by using an easy adaptation of [6, Lemma 3.2] to the Hardy-Orlicz context (this adaptation just consists in writing the above chain rule for arbitrary μ_{ϕ_β} , $\beta \in I$, such that C_{ϕ_β} are all norm-bounded by a constant independent of β .) Indeed, it is plain that $\|C_{\phi_r}\| \leq \|C_\phi\|$ for any $0 < r < 1$.

It seems interesting to notice that it is not clear that being a ψ -Carleson measure implies Condition (3.22).

This remark together with the proof of the third point of Theorem 2.8 shows that if ψ satisfies the uniform ∇_0 -Condition, then Condition (3.22) and Condition (3.18) are equivalent. We can then state the following characterization of the boundedness of composition operators on $H^\psi(\mathbb{B}_N)$ for ψ satisfying the uniform ∇_0 -Condition (which implies ∇_2 -Condition.)

Theorem 3.29. *Let ψ be an Orlicz function satisfying the uniform ∇_0 -Condition. Let also $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be analytic. C_ϕ is bounded from $H^\psi(\mathbb{B}_N)$ into itself if and only if there exists a constant $A > 0$ such that*

$$(3.23) \quad \sup_{0 < r < 1} \mu_{\phi_r} (S_f(\xi, h)) = O_{h \rightarrow 0} \left(\frac{1}{\psi(A\psi^{-1}(1/h^N))} \right)$$

for every $\xi \in \mathbb{S}_N$.

Remark 3.30. As

$$\frac{1}{\psi(A\psi^{-1}(1/h^N))} \approx h^N$$

for every $A > 0$ and every $h \in (0, 1)$, whenever the Orlicz function ψ satisfies the Δ_2 -Condition, we have the following corollary:

Corollary 3.31. *Let ψ be an Orlicz function satisfying $\Delta_2 \cap \nabla_2$ -Condition, and let $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be holomorphic. C_ϕ is bounded from $H^\psi(\mathbb{B}_N)$ into itself if and only if*

$$(3.24) \quad \sup_{0 < r < 1} \mu_{\phi_r} (S_f(\xi, h)) = O_{h \rightarrow 0} (h^N)$$

for every $\xi \in \mathbb{S}_N$.

Note that this corollary holds in particular for $x \mapsto x^p$, $1 < p < \infty$ (Conditions Δ_2 is satisfied by $x \mapsto x^p$, $1 < p < \infty$.) yet it is plain that all the previous studies work if $\psi(x) = x$ and we get the result for the value $p = 1$.

We also observe that the characterization of the boundedness of C_ϕ has already been obtained in Theorem 3.23, in a different way. This proves in particular that

$$\sup_{0 < r < 1} \mu_{\phi_r} (S_f(\xi, h)) = O_{h \rightarrow 0} (h^N)$$

for every $\xi \in \mathbb{S}_N$ holds if and only if

$$\mu_\phi(S_f(\xi, h)) = O_{h \rightarrow 0}(h^N)$$

for every $\xi \in \mathbb{S}_N$ holds. Surprisingly, we do not know if, given an arbitrary Orlicz function ψ ,

$$\sup_{0 < r < 1} \mu_{\phi_r}(S_f(\xi, h)) = O_{h \rightarrow 0}\left(\frac{1}{\psi(A\psi^{-1}(1/h^N))}\right)$$

for some $A > 0$ and for every $\xi \in \mathbb{S}_N$ holds if and only if

$$\mu_\phi(S_f(\xi, h)) = O_{h \rightarrow 0}\left(\frac{1}{\psi(A\psi^{-1}(1/h^N))}\right)$$

for some $A > 0$ and for every $\xi \in \mathbb{S}_N$ holds.

This corollary, together with Theorem 3.26, yields the following proposition:

Proposition 3.32. *Let ψ and ν be two Orlicz functions; assume that ψ satisfies the ∇_2 -Condition and that ν satisfies the $\Delta_2 \cap \nabla_2$ -Condition. Let also $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be holomorphic. If C_ϕ is bounded from $H^\nu(\mathbb{B}_N)$ into itself, then C_ϕ is bounded as an operator from $H^\psi(\mathbb{B}_N)$ into itself.*

Proof. It is sufficient to check that if the inequality

$$\sup_{0 < r < 1} K_{\mu_{\phi_r}}(h) \leq C$$

holds for some $C \geq 1$ and any h small, then the inequality

$$\sup_{0 < r < 1} K_{\mu_{\phi_r}}(h) \leq \frac{1/h^N}{\psi(A\psi^{-1}(1/h^N))}$$

holds for $A = 1/C$ and any h small. Now, the convexity of ψ entails

$$\psi\left(\frac{1}{C}\psi^{-1}(1/h^N)\right) \leq \frac{1}{Ch^N}$$

for every $h > 0$. □

As in the Bergman-Orlicz case, we wonder if there are some Hardy-Orlicz spaces $H^\psi(\mathbb{B}_N)$, different from $H^\infty(\mathbb{B}_N)$ and smaller than some $H^p(\mathbb{B}_N)$, on which every composition operator C_ϕ is bounded. In [11], the authors explain that Proposition 2.17 can be adapted to the case of measures μ_ϕ on the closed ball $\overline{\mathbb{B}_N}$, and then to the Hardy spaces indirectly.

Proposition 3.33. *If $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ is analytic, then there exists a constant B_ϕ such that*

$$(3.25) \quad \mu_\phi(S_f(\xi, h)) \leq B_\phi \cdot h$$

for every $\xi \in \mathbb{S}_N$ and every $0 < h < 1$.

If we compare Condition (3.25) to Condition (3.23), we may infer that if ψ grows fast enough, then (3.23) is always satisfied. Nevertheless, the constant B_ϕ in Proposition 3.33 a priori depends on ϕ , so that it depends on r if we apply it to ϕ_r , which is necessary because of (3.23). In fact, the proof of [11, Proposition 4], which concerns pull-back measures of the weighted Lebesgue measure on the ball, makes it clear that, if $\phi(0) = 0$, the constant B_ϕ does not depend on ϕ . As the proof of Proposition 3.33 does not appear in [11], we prefer to state and prove the following proposition.

Proposition 3.34. *Let $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be holomorphic, such that $\phi(0) = 0$. There exists a constant $B > 0$, independent of ϕ , such that*

$$\mu_\phi(S_f(\xi, h)) \leq B \cdot h$$

for every $\xi \in \mathbb{S}_N$ and every $0 < h < 1$.

Proof. We fix $\xi \in \mathbb{S}_N$ and $0 < h < 1$. We denote by $\chi_{\phi^{-1}(S(\xi, h))}$ the characteristic function of $\phi^{-1}(S(\xi, h))$. The formula of integration by slices (see for example [19, Proposition 1.4.7, (1)]) yields

$$\begin{aligned} \mu_\phi(S(\xi, h)) &= \int_{\mathbb{S}_N} \chi_{(\phi^*)^{-1}(S(\xi, h))}(\zeta) d\sigma(\zeta) \\ &= \int_{\mathbb{S}_N} \int_{\mathbb{T}} \chi_{(\phi^*)^{-1}(S(\xi, h))}(u\zeta) d\lambda(u) d\sigma(\zeta), \end{aligned}$$

where λ is the Lebesgue measure on the torus \mathbb{T} . Let us observe that

$$\chi_{(\phi^*)^{-1}(S(\xi, h))}(u\zeta) = 1$$

is equivalent to

$$(3.26) \quad |1 - \langle \phi^*(u\zeta), \xi \rangle| < h.$$

For every ζ and ξ , we define the function $\varphi_{\zeta, \xi} : \mathbb{D} \rightarrow \mathbb{D}$ such that $\varphi_{\zeta, \xi}(z) = \langle \phi(z\zeta), \xi \rangle$ for any $z \in \mathbb{D}$. $\varphi_{\zeta, \xi}$ is holomorphic and satisfies $\varphi_{\zeta, \xi}(0) = 0$; moreover it is not difficult to verify that $\varphi_{\zeta, \xi}^*(u) = \langle \phi^*(u\zeta), \xi \rangle$ for λ -almost every $u \in \mathbb{T}$, where $\varphi_{\zeta, \xi}^*$ is the λ -almost everywhere radial limit of $\varphi_{\zeta, \xi}$. Inequality (3.26) is then equivalent to

$$\varphi_{\zeta, \xi}^*(u) \in S(1, h)$$

where $S(1, h)$ is here the non-isotropic disk of radius h and centered at 1. Now, according to the Littlewood subordination principle, it is well-known that there exists a constant $B > 0$, independent of ζ and ξ , such that

$$\lambda\left(\left(\varphi_{\zeta, \xi}^*\right)^{-1}(S(1, h))\right) \leq B.h.$$

This concludes the proof. \square

We come back to the existence of some Orlicz function ψ such that there exist some constants $A > 0$ and $C \geq 1$ such that

$$(3.27) \quad Ch \leq \frac{1}{\psi(A\psi^{-1}(1/h^N))}.$$

By a straightforward computation, we can check that Condition (3.27) is equivalent to the existence of $K > 0$ and $C > 1$ such that

$$\psi(x)^N \leq K\psi(Cx)$$

for x large enough. This leads us to introduce what we call the Δ_H^N -Condition:

Definition 3.35. Let ψ be an Orlicz function. We say that ψ satisfies the Δ_H^N -Condition if there exist two constants $K > 0$ and $C > 1$ such that

$$\psi(x)^N \leq K\psi(Cx)$$

for every x large enough.

We have the following theorem:

Theorem 3.36. Let ψ be an Orlicz function. If ψ satisfies both uniform ∇_0 and Δ_H^N -Conditions, then every composition operator is bounded from $H^\psi(\mathbb{B}_N)$ into $H^\psi(\mathbb{B}_N)$.

Proof. Let $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be holomorphic. As every composition operator C_τ induced by an automorphism is bounded from $H^p(\mathbb{B}_N)$ into itself, $1 < p < \infty$, Proposition 3.32 entails that C_τ is also bounded as an operator from $H^\psi(\mathbb{B}_N)$ into itself (let us recall that ψ satisfies uniform ∇_0 -Condition and so ∇_2 -Condition.) Therefore, up to conjugation by automorphism induced composition operators, we may assume that $\phi(0) = 0$. Now, Condition (3.27) is satisfied because ψ satisfies Δ_H^N -Condition, so Theorem 3.29 and Proposition 3.34 ensure that C_ϕ is bounded from $H^\psi(\mathbb{B}_N)$ into itself. \square

Remark 3.37. 1) Let us remark that every $\psi(x) = e^{ax} - 1$ for $a > 0$ belongs to the $\nabla_0^{\text{unif}} \cap \Delta_{H^1}^N$ -Class, so that this latter is not empty.

2) Theorem 3.36 may be proved in a different way, by using Theorem 2.20. Actually, if ψ satisfies both uniform ∇_0 -Condition and Δ_{α}^N -Condition, i. e.

$$(3.28) \quad \psi(x)^{\frac{N+\alpha+1}{\alpha+2}} \leq K\psi(Cx)$$

for some constants $K > 0$, $C \geq 1$ which may depend on α , then

$$(3.29) \quad \|C_{\phi}(f)\|_{A_{\alpha}^{\psi}(\mathbb{B}_N)} \leq C_{\alpha} \|f\|_{A_{\alpha}^{\psi}(\mathbb{B}_N)},$$

for any $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ holomorphic. Moreover, we have

$$(3.30) \quad \|f\|_{H^{\psi}(\mathbb{B}_N)} = \lim_{\alpha \rightarrow -1} \frac{1}{\alpha + 1} \|f\|_{A_{\alpha}^{\psi}(\mathbb{B}_N)}.$$

Indeed, this formula is known for $\psi(x) = x^p$, $1 \leq p < \infty$, and its proof essentially lays on the subharmonicity of $|f|^p$ for f holomorphic. As $\psi(|f|)$ is also subharmonic because ψ is convex, it is plain that this proof works as well in the Orlicz context. Now, it is not difficult, yet fastidious, to check that the constant C_{α} in (3.29) is bounded by a constant which does not depend on α , when $\alpha \in (-1, 0)$. In addition, the decreasing function $\frac{N+\alpha+1}{\alpha+2}$ of the variable $\alpha \in (-1, 0)$ has supremum for $\alpha = -1$, so that (3.28) is satisfied with constant K and C independent of α , as soon as inequality

$$\psi(x)^N \leq K\psi(Cx)$$

holds, which means that ψ satisfies $\Delta_{H^1}^N$ -Condition. Letting α tends to -1 in (3.29), (3.30) then yields Theorem 3.36.

When $N = 1$, thanks to Theorem 4.6, ∇_0^{unif} -Condition is not necessary; moreover, thanks to the factorization of a function $f \in H^{\psi}(\mathbb{D})$ (hence in $H^1(\mathbb{D})$) by a Blaschke product formed from its zeros and a non-zeros function in $H^{\psi}(\mathbb{D})$ (see the remark which precedes the proof of theorem 4.10 in [13], and [1, Section 7, Theorem 1.1],) it is possible to remove the ∇_2 -Condition. Finally, $\Delta_{H^1}^1$ -Condition is clearly satisfied by every Orlicz function so that every composition operator is bounded on every $H^{\psi}(\mathbb{D})$. Remark 3.37 (2) is an other way to prove [13, Proposition 3.12] (the latter one makes use of the Littlewood subordination principle, as it is usual in the context of the unit disk.)

4. SOME COMMENTS

Most of the following remarks hold for both Bergman-Orlicz and Hardy-Orlicz frameworks. For convenience, we just state them for Bergman-Orlicz *or* Hardy-Orlicz spaces. If a remark is specific to one or the other cases, it will be explicitly mentioned.

1) First of all, we start by noticing that inclusion $A_{\alpha}^p(\mathbb{B}_N) \subset L^p(\mu)$, $1 \leq p \leq \infty$, implies that inclusion $A_{\alpha}^{\psi}(\mathbb{B}_N) \subset L^{\psi}(\mu)$ holds. More precisely, if μ is a Carleson measure, then μ satisfies Condition (2.10). This has already been somehow mentioned in Proposition 3.32 for the Hardy-Orlicz case. The following lemma is the reason for such a fact.

Lemma 4.1. *Let ψ be an Orlicz function and let μ be a α -Bergman-Carleson measure on \mathbb{B}_N . Then μ satisfies Condition (3.9).*

Proof. As μ is a Carleson measure, one can find a constant $C \geq 1$ such that $K_{\mu}(h) \leq C$ for every $h \in (0, 1)$. Now, taking $A = 1/C \leq 1$, the convexity of the Orlicz function ψ implies that

$$\psi(A\psi^{-1}(1/h^{N+1+\alpha})) \leq A/h^{N+1+\alpha}.$$

Hence

$$K_{\mu}(h) \leq \frac{1}{A} \leq \frac{1/h^{N+1+\alpha}}{\psi(A\psi^{-1}(1/h^{N+1+\alpha}))}$$

and Condition (2.10) is fulfilled. \square

We immediately deduce from this lemma and Theorem 2.8 the following proposition, already stated in [13] in the case of the disk.

Proposition 4.2. *Let ψ be an Orlicz function and let μ be a α -Bergman-Carleson measure on \mathbb{B}_N . Inclusion $A_\alpha^\psi(\mathbb{B}_N) \subset L^\psi(\mu)$ holds (and is continuous.)*

Without difficulties, the previous also holds for the Hardy cases:

Proposition 4.3. *Let ψ be an Orlicz function which satisfies the ∇_2 -Condition and let μ be a Carleson measure on $\overline{\mathbb{B}_N}$. Inclusion $H^\psi(\mathbb{B}_N) \subset L^\psi(\mu)$ holds (and is continuous.)*

Let us remark that this last results is actually a consequence of the fact that H^ψ is an interpolating space for H^1 and H^∞ (See [2, Theorem V.10.8].)

Concerning composition operators, we immediately deduce from Theorem 2.16 and Proposition 4.2 that every bounded composition operator on some $A_\alpha^\psi(\mathbb{B}_N)$, with ψ satisfying the Δ_2 -Condition, is also bounded on $A_\alpha^\nu(\mathbb{B}_N)$, where ν is any Orlicz function. The converse is false in general since we showed that there are some Orlicz functions ψ such that every composition operator is bounded on $A_\alpha^\psi(\mathbb{B}_N)$, yet it is not the case in $A_\alpha^p(\mathbb{B}_N)$, $1 \leq p < \infty$, $N > 1$. Of course, this argument does not match in the one variable setting.

2) In the classical Bergman spaces, it is well-known that the boundedness or the compactness of a canonical embedding $A_\alpha^p(\mathbb{B}_N) \subset L^p(\mu)$ is proper to the measure μ and does not depend on the exponent p . More precisely, recall for example that $A_\alpha^p(\mathbb{B}_N) \subset L^p(\mu)$ holds (and is bounded) if and only if μ is a Carleson measure, i. e.

$$\mu(W(\xi, h)) \leq h^{N+1+\alpha}$$

for every $\xi \in \mathbb{S}_N$ and every $0 < h < 1$. It clearly appears that this condition does not depend on p . In the context of Bergman-Orlicz, the situation is quite different: the fact that inclusion $A_\alpha^\psi(\mathbb{B}_N) \subset L^\psi(\mu)$ holds for *one* Orlicz function ψ does not systematically entails that inclusion $A_\alpha^\psi(\mathbb{B}_N) \subset L^\psi(\mu)$ holds for *every* Orlicz function ψ . In fact, it seems that this property of independence of inclusion of Bergman-Orlicz spaces into Orlicz spaces is proper to the Bergman spaces $A_\alpha^p(\mathbb{B}_N)$, in the sense that the latter share this property with Bergman-Orlicz spaces which are “comparable” to Bergman spaces. In fact, we need to be careful by using this word “comparable” as it has a very precise sense. Precisely, if ψ and ν are two Orlicz functions both satisfying the Δ_2 -Condition (see Definition 1.4,) then $A_\alpha^\psi(\mathbb{B}_N) \subset L^\psi(\mu)$ holds if and only if $A_\alpha^\nu(\mathbb{B}_N) \subset L^\nu(\mu)$ holds. In fact, this directly proceeds from Theorem 2.8, the fact that if ψ satisfies Δ_2 -Condition, then ψ satisfies

$$\frac{1}{\psi(A\psi^{-1}(1/h^{N+1+\alpha}))} \approx h^{N+1+\alpha}$$

which implies that uniform ∇_0 -Condition is not necessary. In other words, this particular case of Theorem 2.8 is nothing but Carleson’s Theorem, explicitly stated in [10, Theorem 3.5].

Of course, the same kind of remark can be stated in terms of composition operators, or by considering Hardy-Orlicz spaces instead of Bergman-Orlicz spaces, or compactness instead of boundedness.

3) Let ψ be an Orlicz function satisfying the uniform ∇_0 -Condition. Remark 3.37 (2) shows that if C_ϕ is bounded on every $A_\alpha^\psi(\mathbb{B}_N)$, $\alpha \in (-1, 0)$, then C_ϕ is also bounded on $H^\psi(\mathbb{B}_N)$. According to the proof of [6, Theorem 3.3], it is plain that, if ψ satisfies the Δ_2 -Condition, then the boundedness of C_ϕ from $H^\psi(\mathbb{B}_N)$ into itself implies the boundedness of C_ϕ as an operator from $A_\alpha^\psi(\mathbb{B}_N)$ into itself, for every $\alpha > -1$, and therefore is equivalent to the boundedness of C_ϕ on $A_\alpha^\psi(\mathbb{B}_N)$ for every $\alpha > -1$. We can summarize this fact together with the previous one in the following proposition:

Proposition 4.4. *Let $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be holomorphic. If there exists some Orlicz function ψ_1 satisfying the Δ_2 -Condition such that C_ϕ is bounded from $H^{\psi_1}(\mathbb{B}_N)$ into itself, then C_ϕ is bounded from any Hardy-Orlicz space $H^{\psi_2}(\mathbb{B}_N)$ into itself and from any Bergman-Orlicz space $A_\alpha^{\psi_3}(\mathbb{B}_N)$*

into itself, $\alpha > -1$, where ψ_2 is an arbitrary Orlicz function, and where ψ_3 satisfies the Δ_2 -Condition.

It is interesting to state the opposite extreme result, which just comes from both Theorem 2.20 and Theorem 3.36, and the fact that Δ_H^N -Condition implies Δ_α^N -Condition, for every $\alpha > -1$.

Proposition 4.5. *Let ψ be an Orlicz function satisfying the Δ_H^N -Condition. Every holomorphic self-map $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ induces a bounded composition operator from $H^\psi(\mathbb{B}_N)$ into itself, and from $A_\alpha^\psi(\mathbb{B}_N)$ into itself, for every $\alpha > -1$.*

4) The situation concerning compactness is more involved. Few results are known about this question in several variables, yet we think that the study of compact composition operators in the one variable setting may rise questions about both boundedness and compactness when $N > 1$. Anyway, as we mentioned in (2) above, we know that whenever ψ and ν are two Orlicz functions both satisfying the Δ_2 -Condition, the compactness from $H^\psi(\mathbb{D})$ into itself is equivalent to that from $H^\nu(\mathbb{D})$ into itself. However, in the one variable setting, the authors of [14] show that for every $\varepsilon > 0$ and for every $1 \leq p < \infty$, there exists an Orlicz function ψ such that $H^{p+\varepsilon}(\mathbb{D}) \subset H^\psi(\mathbb{D}) \subset H^p(\mathbb{D})$, and a composition operator C_ϕ which is compact on $H^p(\mathbb{D})$ and on $H^{\varepsilon+p}(\mathbb{D})$, yet not compact on $H^\psi(\mathbb{D})$. Concerning boundedness, we know that this phenomenon cannot occur, because of Proposition 3.32. Yet, we can wonder if, given two Orlicz functions ψ_1, ψ_2 and $1 < p < \infty$ such that $H^{\psi_1}(\mathbb{B}_N) \subset H^p(\mathbb{B}_N) \subset H^{\psi_2}(\mathbb{B}_N)$, the boundedness of C_ϕ on both $H^{\psi_1}(\mathbb{B}_N)$ and $H^{\psi_2}(\mathbb{B}_N)$ implies the boundedness of C_ϕ on $H^p(\mathbb{B}_N)$. Similar questions may be investigated about compactness.

In [15], the same authors prove that there are some symbol ϕ which induces a compact composition operator from $H^\psi(\mathbb{D})$ into itself, yet not compact from $A^\psi(\mathbb{D})$ into itself. For any $N \geq 1$, we at least know that this is impossible if $\psi(x) = x^p$ according to [6, Theorem 3.3].

Let us now consider the situation on “small” Hardy-Orlicz spaces. As we said in the introduction, for every Orlicz function ψ , there exists a surjective symbol $\phi : \mathbb{D} \rightarrow \mathbb{D}$ which induces a compact composition operator on $H^\psi(\mathbb{D})$ ([16, Theorem 4.1]. [15, Proposition 4.1] asserts that, under some fast growth condition (satisfied for example by $e^{x^q} - 1$, for $q \geq 1$), every composition operator which is compact from H^ψ into itself is also bounded from $A^\psi(\mathbb{D})$ into itself. This does not say that we cannot find some Orlicz function which defines Bergman-Orlicz spaces on which a criterion for compactness of C_ϕ is the same as for H^∞ , yet it let think that it is unlikely.

5) We make a final remark which directly follows from the most specific difference which appears when we pass from the one-variable case to the several variables case, with the intention to give characterization theorems for boundedness and compactness of composition operators. If we come back to the third points of Theorem 2.8, Theorem 2.12, Theorem 3.14 or Theorem 3.19 and consider a measure μ_ϕ which is the pull-back measure for ϕ analytic from \mathbb{B}_N into itself, we can be a little bit disappointed not to be able to remove the uniform ∇_0 and ∇_0 conditions. Indeed, when $N = 1$, Conditions (2.9) and (2.10) (resp. (2.17) and (2.18)) are equivalent if $\mu = \mu_\phi$ is a pull-back measure for an arbitrary holomorphic function $\phi : \mathbb{D} \rightarrow \mathbb{D}$. In fact, this comes from a general result which does not involve the space that we are dealing with, but the special geometry of the disk together with the geometric properties of holomorphic self-maps of \mathbb{D} ; this is a kind of homogeneity of μ_ϕ according to the non-isotropic disks of the unit disk, which is not trivial at all. This is precisely the content of Theorem 4.19 of [13] and of Theorem 3.1 of [15]; we recall its statement in order to understand why there is rather no hope to give a characterization of the boundedness of composition operators in a general framework. For convenience, we prefer to state [13, Theorem 4.19], that is to say for the pull-back measure μ_ϕ relative to the Lebesgue measure $d\lambda$ on the unit circle, for $\phi : \mathbb{D} \rightarrow \mathbb{D}$.

Theorem 4.6. *There exists a constant $\kappa > 0$ such that, for every holomorphic function $\phi : \mathbb{D} \rightarrow \mathbb{D}$, one has*

$$(4.1) \quad \mu_\phi(S(\xi, \varepsilon h)) \leq \kappa \varepsilon \mu_\phi(S(\xi, h))$$

for every $h \in (0, 1 - |\phi(0)|)$, every $\xi \in \mathbb{T}$, and every $\varepsilon \in (0, 1]$.

Looking at (4.1) from the point of view of composition operators, it appears that the boundedness of composition operators on every $H^p(\mathbb{D})$, $0 < p < \infty$, or every $H^\psi(\mathbb{D})$, ψ an Orlicz function, is a straightforward consequence of this theorem: fixing some $h \in (0, 1 - |\phi(0)|)$, (4.1) entails that $\mu_\phi\left(S\left(\xi, h'\right)\right)$ is a big Oh of h' when h' is small. Thus, such a theorem can not be but specific to the one-variable case, as we know that there are some holomorphic functions which induce not-bounded composition operators on Hardy spaces, or on some Hardy-Orlicz spaces of the ball.

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