Approximation of Vorob'ev expectation for random closed sets

Philippe Heinrich, Radu Stoica, Viet Chi Tran

Approximation of Vorob'ev expectation for random closed sets

Philippe Heinrich*, Radu S. Stoica* and Viet Chi Tran* †

June 26, 2010

Abstract

Random sets appear in many applications, in particular in image analysis. The issue of a "mean shape" often arises since there is no canonical definition. In this paper, we propose a consistent and ready to use estimator for the Vorob'ev expectation of a random set X. It is a kind of mean closely linked to quantile-like quantities and built from independent copies of X with spatial discretization. The convergence is established through the Strong Law of Large Numbers of Kovyazin. The control of discretization errors is handled with a mild regularity assumption on the boundary of X: a not too large 'box counting' dimension. Some examples, including Boolean models, are studied.

Keywords: Stochastic geometry; random closed sets; Vorob'ev expectation; quantile sets.
AMS Classification: 60D05; 60F15; 28A80.

1 Introduction

This paper is motivated by the practical applications in Stoica *et al.* [7, 8]. These authors used a marked point process in order to detect and characterize cosmic filaments. Thrown points represent positions of galaxies' centers. It is a known fact that the galaxies are not uniformly distributed in the universe. They exhibit intriguing patterns such as filaments, walls and clusters. Among all these patterns it appears that the filamentary network is the most relevant feature. To model shapes and textures of these filaments, marks are defined as interacting randomly oriented cylinders (see Figure 1). The resulting configuration is a random set X. Spatial heterogeneity implies that coverage probabilities by filaments, $p(x) := \mathbb{P}(x \in X)$, strongly depend on the position x. This leads to the natural question of determining an average shape for the random set X.

Several ways to define the expectation of a random set have been developped (see *e.g.* Mochanov [6]). Our investigation through the literature leads us to the Vorob'ev expectation (see *e.g.* Kovyazin [3], Molchanov [5, 6], Stoyan and Stoyan [10]), closely related to quantile-like quantities where regions with high coverage probabilities $p(x)$ are put forward. Despite their very natural definitions, neither these quantile sets nor the approximations in the previous references are tractable for applications. The aim of this paper is to derive a consistent (implementable) estimator for the Vorob'ev expectation, based on independent copies of X and spatial discretizations. Empirically, related statistics had been computed in [7, 8]. The present paper sheds some light on the theoretical justifications of theses works. The almost sure (a.s.) consistency of our set-valued estimator (with respect to the volume of the symmetric difference) is stated in Theorems 2.5 and 3.6. The consistency stems from the Strong Law of Large Numbers (SLLN) for X, provided its boundary ∂X is not too irregular. For the SLLN,

*Laboratoire Paul Painlevé, UMR CNRS 8524, Université des Sciences et Techniques Lille 1, 59 655 Villeneuve d'Ascq Cedex, France.
†CMAP, UMR CNRS 7641, Ecole Polytechnique CNRS, Route de Saclay, 91 128 Palaiseau Cedex France.

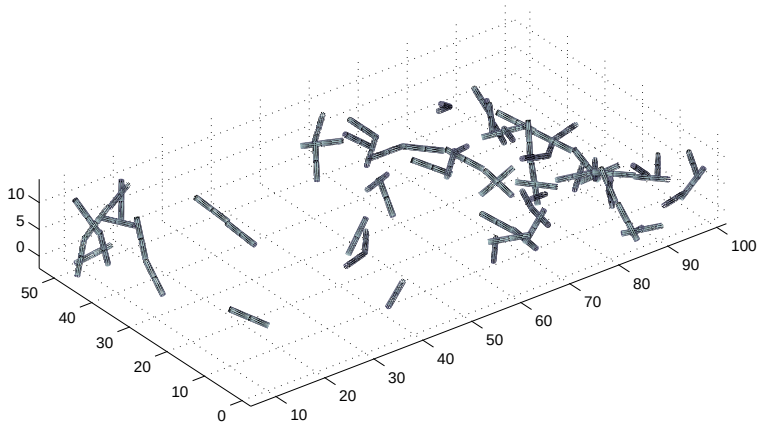

Figure 1: Cylinder configuration for the modelling of cosmic filaments from Stoica *et al.* [7].

we follow the work of Kovyazin [3]. The regularity of the boundary ∂X is ensured if its fractal 'box counting' dimension is less than the space dimension. This is required to control the discretization error.

The paper starts in Section 2 by presenting the quantiles of random closed sets for which consistent estimators are proposed, well suited for applications. Then, the Vorob'ev expectation is introduced in Section 3 and an estimation method is given as well. For both these estimators, the convergence is proved. L^1-convergence rates are provided for the estimators of quantiles in Proposition 2.7. At last, Section 4 of the paper is devoted to the analysis of these estimators for some typical examples of random sets, including Boolean models. For the sake of clarity, some proofs are deferred to the Appendix.

Notation: The following list of useful notation will serve as reference for the reader.

- $\mathbb{E}_v(X)$ will denote the "Vorob'ev expectation" of a random compact set $X \subset [0,1]^d$.
- $X_1, \ldots, X_n, \ldots$ is a sequence of independent copies of X,
- $p(x) = \mathbb{P}(x \in X)$ is the coverage probability of $x \in [0,1]^d$ by the random closed set X, and its empirical counterpart $p_n(x)$ is defined by

$$p_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{x \in X_i\}},$$

- Denote by λ the Lebesgue measure on $[0,1]^d$. $F(\alpha)$ is a decreasing càdlàg function on $[0,1]$ defined by

$$F(\alpha) = \lambda\{x : p(x) > \alpha\} \text{ and } F_-(\alpha) = \lim_{\varepsilon \to 0^+} F(\alpha - \varepsilon),$$

- Λ_n will denote the empirical analogue of $\mathbb{E}\lambda(X)$, namely

$$\Lambda_n = \frac{1}{n} \sum_{i=1}^{n} \lambda(X_i),$$

- For any Borel set B in $[0,1]^d$ and $r \in 2^{-\mathbb{N}}$, B^r denotes the following grid approximation of B by disjoint "cells":

$$B^r = \bigsqcup_{x \in r\mathbb{Z}^d \cap B} [x, x + r)^d,$$

- For two Borel sets A, B in $[0,1]^d$, set

$$\mathrm{d}(A,B) = \lambda(A \triangle B),$$

which defines a pseudo-distance on the Borel σ-field $\mathcal{B}([0,1]^d)$.

2 Quantile or α-level sets

2.1 Background and definitions

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let us consider $[0,1]^d$ endowed with the Borel σ-field $\mathcal{B}([0,1]^d)$ and the Lebesgue measure λ. Let $\mathcal{F}$ be the collection of all closed subsets of $[0,1]^d$ that is endowed with the σ-algebra generated by the system

$$\{\mathcal{F}^K \,:\, K \in \mathcal{F}\} \text{ with } \mathcal{F}^K := \{F \in \mathcal{F} \,:\, F \cap K = \emptyset\}.$$

An application $X : \Omega \mapsto \mathcal{F}$ is said to be a *random closed set* (actually compact) in $[0,1]^d$ if it is measurable. Refer to [6] for more details.

For any $A, B \in \mathcal{B}([0,1]^d)$, define a (classical) pseudo-distance by

$$\mathrm{d}(A,B) = \lambda(A \triangle B). \tag{2.1}$$

It can be easily checked that

- $\mathrm{d}(A,C) \leq \mathrm{d}(A,B) + \mathrm{d}(B,C)$,
- $\mathrm{d}(A,B) = \mathrm{d}(B,A)$,
- $\mathrm{d}(A,B) = 0 \Rightarrow A = B$ up to a set of Lebesgue measure zero.

Definition 2.1. *For $\alpha \in [0,1]$, the* α-level set *of the random closed set X is defined by*

$$Q_\alpha = \{x \in [0,1]^d : p(x) > \alpha\} = \{p > \alpha\},$$

where $p(x) := \mathbb{P}(x \in X)$ is the coverage probability of the point x by the random set X and $\alpha \in [0,1]$.

Note that $Q_0 = \{x \in [0,1]^d : p(x) > 0\}$ and $Q_1 = \emptyset$. The set Q_α contains all the points belonging to X with probability more than α. Therefore, it is equivalently called *quantile set*. The following basic facts are proved in Appendix A.1:

Proposition 2.2. *Let $F : [0,1] \to [0,1]$ be the function attached to the quantile set Q_α that returns its corresponding Lebesgue measure*

$$F(\alpha) = \lambda(Q_\alpha) = \lambda\{p > \alpha\}.$$

With the above notations:

(i) The family $(Q_\alpha)_{\alpha \in [0,1]}$ is decreasing w.r.t. α,

(ii) For every $\alpha \in [0,1)$, $Q_\alpha = \bigcup_{\varepsilon > 0} Q_{\alpha+\varepsilon}$,

(iii) For every $\alpha \in (0,1]$, $\bigcap_{\varepsilon > 0} Q_{\alpha-\varepsilon} = Q_\alpha \cup \{p = \alpha\} = \{p \geq \alpha\}$,

(iv) The function F is decreasing and càdlàg,

(v) The set of discontinuity points of F is $\{\alpha : \lambda\{p = \alpha\} > 0\}$ and is at most countable.

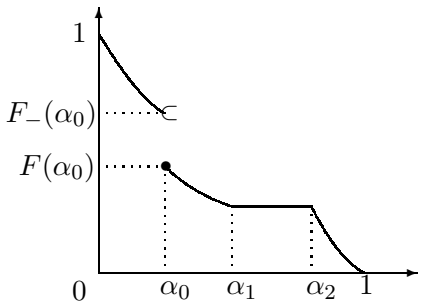

Figure 2: *The dependence of the volume of Q_α sets on the threshold value α is described by the function F.*

As an example, Figure 2 shows how the function F may look like. It can be observed that the plateaus (constant regions) of p make the discontinuities of F while the discontinuities of p provide the plateaus of F. The jump of F at α_0 corresponds to the Lebesgue measure of the points with coverage probability equal to α_0

$$F_-(\alpha_0) - F(\alpha_0) = \lambda\{x \in [0,1]^d : p(x) = \alpha_0\},$$

with the notation $F_-(\alpha_0) = \lim_{\varepsilon \to 0_+} F(\alpha_0 - \varepsilon)$.

The plateau $[\alpha_1, \alpha_2]$ of F means that the set of points with coverage probability between α_1 and α_2 is Lebesgue-negligible. Hence $\lambda(Q_{\alpha_1}) = \lambda(Q_\alpha)$ for all $\alpha \in [\alpha_1, \alpha_2]$.

2.2 Approximation of quantile sets

There are several reasons why quantile sets cannot be computed directly for practical applications. First, the coverage probability $\mathbb{P}(x \in X)$ is not always available in an analytical closed form. Second, quantile sets can not be computed for all the points $x \in [0,1]^d$.

All these aspects require the choice of numerical approximation strategies. In the following, the choices we have opted for are presented. They lead to the construction of a convergent estimator for quantile sets. At the end of this section a result giving the L^1-convergence rate of our estimator is also presented.

2.2.1 Estimation of the coverage probability

Let $X_1, \ldots, X_n$ be n independent realizations of the random set X. A straightforward manner to estimate $p(x)$ is:

$$p_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{x \in X_i\}}. \tag{2.2}$$

The estimator (2.2) is convergent. More precisely,

Proposition 2.3. *if X is a random closed set in $[0,1]^d$, then*

$$\mathbb{P}\left(\forall x \in [0,1]^d, \quad \lim_{n \to \infty} p_n(x) = p(x)\right) = 1.$$

For a proof, see Appendix A.2. A stronger result is available in [5], p.313. Namely, if X is regular closed ($\mathbb{P}(X = \overline{\operatorname{Int} X}) = 1$) and a.s. continuous (for all $x \in [0,1]^d$, $\mathbb{P}(x \in \partial X) = 0$), then

$$\mathbb{P}\left(\lim_{n \to \infty} \sup_{x \in [0,1]^d} |p_n(x) - p(x)| = 0\right) = 1.$$

In fact, the a.s. continuity assumption on X here implies that the coverage function p is continuous. This is a rather restrictive condition as will be shown in some examples in last section of the paper.

2.2.2 r -discretizations of sets

As stated before, it is necessary to approximate Borel sets of $\mathcal{B}([0,1]^d)$ for applications. This can be obtained by using a set discretization procedure.

Let us consider the cube $[0,1]^d$ discretized by a grid of disjoint cells having the same width r. The cell width r takes values in $2^{-\mathbb{N}}$. A natural way to approximate a set $A \in \mathcal{B}(|0,1]^d)$ is to take the following union of disjoint cells

$$A^r := \bigsqcup_{x \in r\mathbb{Z}^d \cap A} [x, x+r)^d,$$

Note the following straightforward property that will be used later:

$$\Big(\bigcap_{n \geq 1} A_n \Big)^r = \bigcap_{n \geq 1} (A_n)^r \,. \tag{2.3}$$

For any $A \in \mathcal{B}([0,1]^d)$ and $r \in 2^{-\mathbb{N}}$, denote by ∂A the boundary of A and define $N_r(\partial A)$ as the number of cells of that intersect ∂A. Following [2], p.38-39, the upper 'box counting' dimension of ∂A is given by

$$\overline{\dim}_{\mathrm{B}}(\partial A) := \limsup_{r \to 0} \frac{\log(N_r(\partial A))}{-\log r}.$$

In this setting, the next result measures the quality of the approximation by cell discretization of Borel sets in $[0,1]^d$.

Proposition 2.4. *Assume that $0 < \overline{\dim}_{\mathrm{B}}(\partial A) < d$. Then, for each $\delta > 0$, there exists $r(\delta) > 0$ such that*

$$0 < r < r(\delta) \Longrightarrow \mathrm{d}(A, A^r) \leq r^{d - \overline{\dim}_{\mathrm{B}}(\partial A) - \delta}.$$

In particular,

$$\lim_{r \to 0} \mathrm{d}(A, A^r) = 0.$$

A proof can be found in Appendix A.3.

Assumption. *From now, for the rest of the paper, it is assumed that X is a random closed set such that there exists $\kappa > 0$ satisfying*

$$\mathbb{P}\left(\overline{\dim}_{\mathrm{B}}(\partial X) \leq d - 2\kappa\right) = 1. \tag{2.4}$$

This hypothesis appears naturally whenever approximating quantiles sets or the Vorob'ev expectation of X and is used to control the discretization error.

2.2.3 Estimation of quantile sets

The proposed estimator for quantile sets Q_α is

$$Q_\alpha^{n,r} = (Q_\alpha^n)^r \quad \text{with} \quad Q_\alpha^n = \{x \in [0,1]^d : p_n(x) > \alpha\}. \tag{2.5}$$

For given n and r, the construction of (2.5) is obtained by computing $p_n(x)$ on the points $x \in r\mathbb{Z}^d$ of the $r-$discretization of $[0,1]^d$ and by taking the union of the cells attached to the points $x \in r\mathbb{Z}^d$ for which the threshold criterion $p_n(x) > \alpha$ is satisfied.

Theorem 2.5. *Let X be a random compact set on $[0,1]^d$ such that (2.4) is satisfied and $\lambda\{p = \alpha\} = 0$. Then, its corresponding estimator $Q_\alpha^{n,r}$ is consistent: with probability one, one has*

$$\lim_{\substack{r \to 0 \\ n \to \infty}} \mathrm{d}\big(Q_\alpha^{n,r}, Q_\alpha\big) = 0.$$

Proof of Theorem 2.5. By the triangular inequality,

$$\mathrm{d}\big(Q_\alpha^{n,r}, Q_\alpha\big) \le \mathrm{d}\big(Q_\alpha^{n,r}, Q_\alpha^n\big) + \mathrm{d}\big(Q_\alpha^n, Q_\alpha\big). \tag{2.6}$$

The first term of the r.h.s. of (2.6) tends to zero with r, as says Proposition 2.4 provided the upper 'box counting'-dimension of ∂Q_α^n is less than d. And this is indeed true: p_n is locally constant on the complementary of $\bigcup_{1\le i\le n} \partial X_i$ and thus one has

$$\partial Q_\alpha^n \subset \bigcup_{1\le i\le n} \partial X_i.$$

Monotonic and stability properties (see [2]) of the upper 'box counting'-dimension yields

$$\overline{\dim}_{\mathrm{B}}\left(\partial Q_\alpha^n\right) \le \overline{\dim}_{\mathrm{B}}\left(\bigcup_{1\le i\le n} \partial X_i\right) = \max_{1\le i\le n} \overline{\dim}_{\mathrm{B}}\left(\partial X_i\right) \le d - 2\kappa \tag{2.7}$$

where the last inequality comes from assumption (2.4). Thus, with probability one, there exists a random $r(\kappa) > 0$ such that

$$0 < r < r(\kappa) \Longrightarrow \mathrm{d}\big(Q_\alpha^{n,r}, Q_\alpha^n\big) \le r^\kappa. \tag{2.8}$$

Note that thanks to the last upper bound (2.7), $r(\kappa)$ does not depend on $\alpha \in [0,1]$.

The second term of the r.h.s. of (2.6) can be bounded by using the following result (proved in Appendix A.4):

Lemma 2.6. *Let $(\alpha_{n,r})_{(n,r)\in\mathbb{N}\times 2^{-\mathbb{N}}}$ be a sequence of random variables in $[0,1]$ having an a.s. non random limit $\alpha \in [0,1]$. Then, with probability one,*

$$\lim_{\substack{n\to\infty \\ r\to 0}} \mathrm{d}(Q_{\alpha_{n,r}}^n, Q_\alpha) = \lambda\{p = \alpha\}.$$

If $\lambda\{p=\alpha\} = 0$ and $\alpha_{n,r} = \alpha$ for all (n,r), one gets

$$\lim_{n\to\infty} \mathrm{d}\big(Q_\alpha^n, Q_\alpha\big) = 0.$$

This completes the proof of Theorem 2.5. ■

When $\alpha \in [0,1]$ is fixed, an L^1-convergence rate can be reached:

Proposition 2.7. *Let X be a random compact set on $[0,1]^d$ such that (2.4) is satisfied together with $\lambda\{p=\alpha\} = 0$. Then, for all $\varepsilon > 0$,*

$$\mathbb{E}\,\mathrm{d}\big(Q_\alpha^{n,r}, Q_\alpha\big) \le r^\kappa + 2\mathrm{e}^{-2n\varepsilon^2} + F(\alpha-\varepsilon) - F(\alpha+\varepsilon).$$

Proof of Proposition 2.7. From (2.6) and (2.8),

$$\mathbb{E}\,\mathrm{d}\big(Q_\alpha^{n,r}, Q_\alpha\big) \le r^\kappa + \mathbb{E}\,\mathrm{d}\big(Q_\alpha^n, Q_\alpha\big). \tag{2.9}$$

Moreover,

$$\begin{aligned} \mathrm{d}(Q_\alpha^n, Q_\alpha) &= \lambda\big\{p_n > \alpha,\, p \le \alpha\big\} + \lambda\big\{p_n \le \alpha,\, p > \alpha\big\} \\ &\le \lambda\big\{x \in [0,1]^d : |p_n(x) - p(x)| \ge |p(x) - \alpha|\big\}. \end{aligned}$$

Taking expectation and using Fubini's theorem and Bernstein's inequality, one gets

$$\begin{aligned} \mathbb{E}\,\mathrm{d}\big(Q_\alpha^n, Q_\alpha\big) &\le \int_{[0,1]^d} \mathbb{P}\{|p_n(x) - p(x)| \ge |p(x) - \alpha|\}\lambda(dx) \\ &\le \int_{\{|\alpha-p|\ge\varepsilon\}} 2\mathrm{e}^{-2n|\alpha-p(x)|^2}\lambda(dx) + \int_{\{|\alpha-p|<\varepsilon\}} \lambda(dx) \\ &\le 2\mathrm{e}^{-2n\varepsilon^2} + F(\alpha-\varepsilon) - F(\alpha+\varepsilon). \end{aligned}$$

■

3 Vorob'ev expectation

The Vorob'ev expectation is based on the quantile sets. It consists in choosing a relevant level α to define a "mean shape". Its mathematical definition following [3, 6, 10, 11] is given below.

Definition 3.1. *Let us define the Vorob'ev threshold as*

$$\alpha_v = \inf\{\alpha \in [0,1] : F(\alpha) \leq \mathbb{E}\lambda(X)\}. \tag{3.1}$$

A Vorob'ev expectation of X is a Borel set $\mathbb{E}_v(X)$ such that

$$Q_{\alpha_v} \subset \mathbb{E}_v(X) \subset \bigcap_{\varepsilon>0} Q_{\alpha_v-\varepsilon} \quad \textit{with} \quad \lambda(\mathbb{E}_v(X)) = \mathbb{E}\lambda(X).$$

Note that, since F is cÃ dlÃ g, one has

$$F(\alpha_v) \leq \mathbb{E}\lambda(X) \leq \lim_{\varepsilon\to 0^+} F(\alpha_v - \varepsilon).$$

The intuition is as follows. From the definition, $Q_1 = \emptyset$ while $Q_0 = \{p > 0\}$ may be equal to $[0,1]^d$. Between these two extremes, it might be possible to choose a set which "would look like" a typical realization of X and would consists in points with the highest coverage probability. The idea of Vorob'ev [11] is to choose the largest α for which the volume $F(\alpha)$ of Q_α is close to $\mathbb{E}\lambda(X)$ *i.e.* the expected volume of X.
The next result states that the Vorob'ev expectation minimizes the expected symmetric difference-distance with X.

Proposition 3.2. *For any $B \in \mathcal{B}([0,1]^d)$, we have:*

$$\lambda(B) = \mathbb{E}\lambda(X) \Longrightarrow \mathbb{E}\mathrm{d}(X, \mathbb{E}_v(X)) \leq \mathbb{E}\mathrm{d}(X, B).$$

This result is for instance proved in [6], Theorem 2.3. p.177. It is the random set counterpart of the classical result for real random variables, stating that for a r.v. ξ, the expectation $\mathbb{E}\xi$ minimizes the function $m \mapsto \mathbb{E}\big((\xi - m)^2\big)$.

Remark 3.3. *The behaviour of the Vorob'ev expectation can be easily explained on Figure 2:*
If $\mathbb{E}\lambda(X) > F_-(\alpha_0)$ then F is strictly decreasing and continuous at α_v, which is such that $F(\alpha_v) = \mathbb{E}\lambda(X)$, hence

$$\mathbb{E}_v(X) = Q_{\alpha_v}.$$

If $F(\alpha_0) < \mathbb{E}\lambda(X) \leq F_-(\alpha_0)$, the expectation of X lies in the discontinuity domain of F, so $\alpha_v = \alpha_0$ and

$$Q_{\alpha_0} \subsetneq \mathbb{E}_v(X) \subset \bigcap_{\varepsilon>0} Q_{\alpha_0-\varepsilon}.$$

And finally, if $\mathbb{E}\lambda(X) = F(\alpha_1)$, the expectation of X lies in a plateau of F, thus $\alpha_v = \alpha_1$ and

$$\mathbb{E}_v(X) = Q_{\alpha_1} = Q_\alpha$$

for any $\alpha \in [\alpha_1, \alpha_2)$.

3.1 Approximation of Vorob'ev expectation

This section proposes an estimator for the Vorob'ev expectation and proves its convergence. Note that our estimators are inspired by the ones proposed by Kovyazin [3], with an additionnal discretization by the grid $r\mathbb{Z}^d \cap [0,1]^d$ so that they can be computed in practice. In the proofs, the Kovyazin's empirical mean will serve as an intermediary between our estimators and $\mathbb{E}_v(X)$.

Vorob'ev expectation, of Definition 3.1, is heuristically close to the α-level set with the $\alpha = \alpha_v$, the Vorob'ev threshold defined in (3.1). Let us first approximate α_v: to do this, we approximate Q_α by $Q_\alpha^{n,r}$ and $\mathbb{E}\lambda(X)$ by

$$\Lambda_n := \frac{1}{n}\sum_{i=1}^{n} \lambda(X_i), \tag{3.2}$$

and choose

$$\alpha^{n,r} = \inf\{\alpha \in [0,1] : \lambda(Q_\alpha^{n,r}) \leq \Lambda_n\}. \tag{3.3}$$

Introduce now the empirical Vorob'ev expectation in view of Definition 3.1:

Definition 3.4. *The empirical Vorob'ev expectation of X is a Borel set s.t.*

$$Q^{n,r}_{\alpha^{n,r}} \subset \widehat{X}^{n,r} \subset \bigcap_{\varepsilon>0} Q^{n,r}_{\alpha^{n,r}-\varepsilon} \quad \textit{with} \quad \lambda(\widehat{X}^{n,r}) = \Lambda_n. \tag{3.4}$$

Remark 3.5.

- *Note that* (3.4) *can be written*

$$\{p_n > \alpha^{n,r}\}^r \subset \widehat{X}^{n,r} \subset \{p_n \geq \alpha^{n,r}\}^r,$$

  where the last inclusion comes from property (2.3)*.*

- *This definition amounts in practice in the following procedure. First, approximate the averaged Lebesgue measure of the random set X by Λ_n, which gives the number of cells to select. The later are chosen according to their estimated coverage, which is given by $p_n(x)$ for the cell $[x, x+r)^d$ with $x \in r\mathbb{Z} \cap [0,1]^d$.*

Theorem 3.6. *Let us recall the definition of α_v and introduce β_v:*

$$\begin{aligned} \alpha_v &= \inf\{\alpha \in [0,1] :\ F(\alpha) \leq \mathbb{E}\lambda(X)\}, \\ \beta_v &= \sup\{\alpha \in [0,1] :\ F(\alpha) \geq \mathbb{E}\lambda(X)\}. \end{aligned} \tag{3.5}$$

Assume that $\lambda\{p = \alpha_v\} = \lambda\{p = \beta_v\} = 0$. Then, with probability one,

$$\lim_{\substack{n\to\infty \\ r\to 0}} \mathrm{d}\big(\widehat{X}^{n,r}, \mathbb{E}_v(X)\big) = 0.$$

Remark 3.7.

- *Heuristically, α_v and β_v are the boundaries of the plateau of F at $\mathbb{E}\lambda(X)$. If $\mathbb{E}\lambda(X)$ has one or zero antecedent by F, then $\alpha_v = \beta_v$. Note that $\alpha_v \leq \beta_v$.*

- *If one considers for instance Figure 2, one sees that $\alpha_v = \beta_v$ except when $\mathbb{E}\lambda(X) = F(\alpha_1)$ in which case $\alpha_v = \alpha_1$ and $\beta_v = \alpha_2$.*

3.2 Proof of Theorem 3.6

The proof of Theorem 3.6 is divided in two parts in order to disentangle the approximation due to the SLLN and the discretization error. One uses the triangular inequality with an intermediary set $\widehat{X}^n$:

$$\mathrm{d}\Big(\widehat{X}^{n,r}, \mathbb{E}_v(X)\Big) \leq \mathrm{d}\Big(\widehat{X}^{n,r}, \widehat{X}^n\Big) + \mathrm{d}\Big(\widehat{X}^n, \mathbb{E}_v(X)\Big), \tag{3.6}$$

where $\widehat{X}^n$ is the 'Kovyazin's empirical mean'. It is defined as follows:

Definition 3.8. *The Kovyazin's empirical mean of X is a Borel set $\widehat{X}^n$ such that*

$$\{p_n > \alpha^n\} \subset \widehat{X}^n \subset \{p_n \geq \alpha^n\} \quad \textit{with} \quad \lambda(\widehat{X}^n) = \Lambda_n,$$

with the threshold

$$\alpha^n := \inf\big\{\alpha \in [0,1] : \lambda\{p_n > \alpha\} \leq \Lambda_n\big\}. \tag{3.7}$$

Remark 3.9. *It is worth to note that the preceding definition is similar to the Vorob'ev expectation's one given in Definition 3.1. Moreover, the Kovyazin's empirical mean is the solution - see Appendix A.5 for a proof - of a minimization problem similar to the one satisfied by the Vorob'ev expectation :*

$$\widehat{X}^n \in \arg \min_{\substack{M \in \mathcal{B}([0,1]^d) \\ \lambda(M) = \Lambda_n}} \frac{1}{n} \sum_{i=1}^{n} \lambda(X_i \triangle M\big). \tag{3.8}$$

In the first part of the proof, the convergence of the Kovyazin's empirical mean towards the Vorob'ev expectation is shown. The second part uses the Kovyazin's mean regularities for obtaining the convergence of the Vorob'ev expectation estimator.

3.2.1 First part: Kovyazin's convergence of $\mathrm{d}(\widehat{X}^n, \mathbb{E}_v(X))$

Here, the approach of Kovyazin [3] is revisited. However, all these do not make the approximation $\widehat{X}^n$ useful in practice since it requires the knowledge of $p_n(x)$ for every $x \in [0,1]^d$. This justifies a further approximation treated in the second part of the proof.

Proposition 3.10. *Let α_v be defined by (3.5) and assume that $\lambda\{p = \alpha_v\} = 0$. One has with probability one*

$$\lim_{n \to \infty} \mathrm{d}\big(\widehat{X}^n, \mathbb{E}_v(X)\big) = 0. \tag{3.9}$$

The proof requires the following lemma (see Appendix A.6 for its proof).

Lemma 3.11. *For α_v, β_v and α^n defined by (3.5) and (3.7), respectively, the following holds:*

(i) If $\alpha_v = \beta_v$ holds, then $(\alpha^n)_{n \in \mathbb{N}}$ converges almost surely to α_v;

(ii) In any case, almost surely,

$$\alpha_v \leq \liminf_{n \to \infty} \alpha^n \leq \limsup_{n \to \infty} \alpha^n \leq \beta_v. \tag{3.10}$$

Proof of Proposition 3.10. We have

$$\begin{aligned}\lambda\big(\widehat{X}^n \triangle \mathbb{E}_v(X)\big) &= \lambda\big(\widehat{X}^n\big) + \lambda\big(\mathbb{E}_v(X)\big) - 2\lambda\big(\widehat{X}^n \cap \mathbb{E}_v(X)\big) \\ &= \lambda\big(\widehat{X}^n\big) - \lambda\big(\mathbb{E}_v(X)\big) + 2\Big[\lambda\big(\mathbb{E}_v(X)\big) - \lambda\big(\widehat{X}^n \cap \mathbb{E}_v(X)\big)\Big] \\ &= \Lambda_n - \mathbb{E}\lambda(X) + 2\lambda\big(\mathbb{E}_v(X) \setminus \widehat{X}^n\big), \end{aligned} \tag{3.11}$$

where the second line is obtained by adding and substracting $\lambda(\mathbb{E}_v(X))$ and in the third line, equalities $\lambda(\mathbb{E}_v(X)) = \mathbb{E}\lambda(X)$ and $\lambda(\widehat{X}^n) = \Lambda_n$ are used.

Similarly, one has

$$\lambda\big(\widehat{X}^n \triangle \mathbb{E}_v(X)\big) = \mathbb{E}\lambda(X) - \Lambda_n + 2\lambda\big(\widehat{X}^n \setminus \mathbb{E}_v(X)\big). \tag{3.12}$$

From inclusions (3.1), (3.8) and Proposition 2.2 (iii), one gets

$$\widehat{X}^n \setminus \mathbb{E}_v(X) \subset \big\{p_n \geq \alpha^n, p \leq \alpha_v\big\} \text{ and } \mathbb{E}_v(X) \setminus \widehat{X}^n \subset \big\{p \geq \alpha_v, p_n \leq \alpha^n\big\}, \tag{3.13}$$

so that (3.13), (3.11) and (3.12) yield

$$\mathrm{d}\big(\widehat{X}^n, \mathbb{E}_v(X)\big) = \lambda\big(\widehat{X}^n \triangle \mathbb{E}_v(X)\big) \leq |\mathbb{E}\lambda(X) - \Lambda_n| + 2m_n, \tag{3.14}$$

with

$$m_n = \lambda\big\{p_n \geq \alpha^n, p \leq \alpha_v\big\} \wedge \lambda\big\{p \geq \alpha_v, p_n \leq \alpha^n\big\}.$$

The first term in the r.h.s. of(3.14) converges to zero by the SLLN. It remains to prove to prove that

$$\lim_{n\to\infty} m_n = 0 \qquad \text{a.s.}$$

Case $\alpha_v = \beta_v$. Note that

$$m_n \leq \mathrm{d}(Q^n_{\alpha^n}, Q_{\alpha_v}) + \lambda\{p_n = \alpha^n\} + \underbrace{\lambda\{p = \alpha_v\}}_{=0}.$$

Thanks to (i) of Lemma 3.11 and Proposition 2.3, the first term converges to zero by Lemma 2.6 and the second term converges to zero by dominated convergence.

Case $\alpha_v < \beta_v$. Thanks to Markov's inequality, we shall bound m_n with expressions using $\int_{[0,1]^d} |p_n - p| d\lambda$. To this end, we distinguish three possible subcases, but we first note the following:

$$\begin{aligned} m_n &\leq \lambda\big\{p_n \leq \alpha^n, p \geq \alpha_v\big\} \\ &\leq \lambda\big\{p_n \leq \alpha^n, p \geq \beta_v\big\} + \lambda\big\{\alpha_v \leq p < \beta_v\big\} \\ &= \lambda\big\{p_n \leq \alpha^n, p \geq \beta_v\big\} + \underbrace{\lambda\{p = \alpha_v\}}_{=0} + \underbrace{F(\alpha_v) - F_-(\beta_v)}_{=0}. \end{aligned} \tag{3.15}$$

Subcase $\alpha^n < \alpha_v$. We can write by using (3.15) together with $\alpha^n < \alpha_v$,

$$\begin{aligned} m_n \leq \lambda\big\{p_n \leq \alpha_v, p \geq \beta_v\big\} &\leq \lambda\big\{|p_n - p| \geq \beta_v - \alpha_v\big\} \\ &\leq \frac{1}{\beta_v - \alpha_v} \int_{[0,1]^d} |p_n - p| d\lambda. \end{aligned}$$

Subcase $\alpha^n > \beta_v$. In a similar way,

$$m_n \leq \lambda\big\{p \leq \alpha_v, p_n \geq \beta_v\big\} \leq \frac{1}{\beta_v - \alpha_v} \int_{[0,1]^d} |p_n - p| d\lambda.$$

Subcase $\alpha_v \le \alpha^n \le \beta_v$**.** Here, we write using (3.15) again

$$\begin{aligned}(\beta_v - \alpha_v)m_n &\le (\beta_v - \alpha^n)\lambda\{p_n \le \alpha^n, p \ge \beta_v\} \\ &\quad +(\alpha^n - \alpha_v)\lambda\{p_n \ge \alpha^n, p \le \alpha_v\} \\ &\le \int_{[0,1]^d} |p_n - p| d\lambda.\end{aligned}$$

Thus (3.15) is in any case bounded by $(\beta_v - \alpha_v)^{-1}\int_{[0,1]^d}|p_n - p|d\lambda$ which converges to 0 a.s. by Proposition 2.3.

The proof of Proposition 3.10 is complete. ■

3.2.2 Second part: control of the r-discretization error

In view of (3.6) and in order to obtain the desired result, it remains to prove that $\lim_{\substack{n\to\infty \\ r\to 0}} \mathrm{d}(\widehat{X}^{n,r}, \widehat{X}^n) = 0$ with probability one.

Before computing this limit, a new lemma is needed. This lemma deals with the behaviour of $\alpha^{n,r}$ for large n and small r and it is analogous to Lemma 3.11.

Lemma 3.12. *We have*

(i) For a given $n \in \mathbb{N}^*$, $\alpha^{n,r}$ converges almost surely to α^n as $r \to 0$.

(ii) If $\alpha_v = \beta_v$ then $\alpha^{n,r}$ converges almost surely to α_v as $n \to \infty$ and $r \to 0$.

(iii) In any case, almost surely,

$$\alpha_v \le \liminf_{\substack{n\to\infty \\ r\to 0}} \alpha^{n,r} \le \limsup_{\substack{n\to\infty \\ r\to 0}} \alpha^{n,r} \le \beta_v. \tag{3.16}$$

For a proof, see Appendix A.7.

Remark 3.13. *Note that (ii) is not a direct consequence of (i) and Lemma 3.11. Indeed, proceeding in this way would only allow us to obtain the convergence of $\alpha^{n,r}$ to α_v by letting first $r \to 0$ then $n \to \infty$.*

Now, let us get back to bound $\mathrm{d}(\widehat{X}^{n,r}, \widehat{X}^n)$. Since $\lambda(\widehat{X}^n) = \lambda(\widehat{X}^{n,r}) = \Lambda_n$, we have

$$\mathrm{d}(\widehat{X}^{n,r}, \widehat{X}^n) = 2\lambda(\widehat{X}^{n,r} \setminus \widehat{X}^n).$$

From definitions we have $\{p_n > \alpha_n\} \subset \widehat{X}^n$ and $\widehat{X}^{n,r} \subset \{p_n \ge \alpha^{n,r}\}^r$. Recall the upper bound (2.7) and let $r(\kappa)$ be as in (2.8). For all $r < r(\kappa)$

$$\begin{aligned}\frac{1}{2}\mathrm{d}(\widehat{X}^{n,r}, \widehat{X}^n) &\le \lambda\left(\{p_n \ge \alpha^{n,r}\}^r \setminus \{p_n > \alpha_n\}\right) \\ &\le \lambda\left(\{p_n \ge \alpha^{n,r}\}^r \setminus \{p_n \ge \alpha^{n,r}\}\right) + \lambda\left(\{p_n \ge \alpha^{n,r}\} \setminus \{p_n > \alpha_n\}\right) \\ &\le r^\kappa + \lambda\{\alpha^{n,r} \le p_n \le \alpha_n\}. \end{aligned} \tag{3.17}$$

The last inequality uses (2.8) applied to $\{p_n \ge \alpha^{n,r}\}$ instead of Q^n_α.

Besides, by (3.10) and (3.16), for all $\varepsilon > 0$, there exists a random $r(\varepsilon) > 0$ and $n(\varepsilon)$ such that for all $r \in (0, r(\varepsilon))$ and all $n > n(\varepsilon)$,

$$\alpha^{n,r} > \alpha_v - \varepsilon \quad \text{and} \quad \alpha^n < \beta_v + \varepsilon. \tag{3.18}$$

One deduces from (3.18) and (3.17) that for all $\varepsilon > 0$, for all $r \in (0, r(\varepsilon) \wedge r(\kappa))$, for all $n > n(\varepsilon)$,

$$\frac{1}{2}\mathrm{d}(\widehat{X}^{n,r}, \widehat{X}^n) \leq r^\kappa + \lambda\{\alpha_v - \varepsilon < p_n < \beta_v + \varepsilon\},$$

and thus, for all $\varepsilon > 0$,

$$\limsup_{\substack{n\to\infty \\ r\to 0}} \mathrm{d}(\widehat{X}^{n,r}, \widehat{X}^n) \leq 2 \limsup_{n\to\infty} \lambda\{\alpha_v - \varepsilon < p_n < \beta_v + \varepsilon\}. \tag{3.19}$$

Since a.s., $p_n(x) \to p(x)$ for each x (see Proposition 2.3), one deduces

$$\begin{aligned} \limsup_{n\to\infty} \mathbb{1}_{\{\alpha_v - \varepsilon < p_n < \beta_v + \varepsilon\}} &\leq \mathbb{1}_{\{\alpha_v - \varepsilon \leq p \leq \beta_v + \varepsilon\}}, \\ \limsup_{n\to\infty} \lambda\{\alpha_v - \varepsilon < p_n < \beta_v + \varepsilon\} &\leq \lambda\{\alpha_v - \varepsilon \leq p \leq \beta_v + \varepsilon\}, \end{aligned}$$

so that, from (3.19),

$$\begin{aligned} \limsup_{\substack{n\to\infty \\ r\to 0}} \mathrm{d}(\widehat{X}^{n,r}, \widehat{X}^n) &\leq 2\lambda\left\{\alpha_v - 2\varepsilon < p < \beta_v + 2\varepsilon\right\} \\ &\leq 2\left[F(\alpha_v - 2\varepsilon) - F_-(\beta_v + 2\varepsilon)\right]. \end{aligned}$$

Now, since F is cÃ dlÃ g, letting $\varepsilon \to 0$ yields

$$\limsup_{\substack{n\to\infty \\ r\to 0}} \mathrm{d}(\widehat{X}^{n,r}, \widehat{X}^n) \leq 2\left[F_-(\alpha_v) - F(\beta_v)\right].$$

It remains to note that

$$F_-(\alpha_v) - F(\beta_v) = 0 \iff \lambda\{p = \alpha_v\} = \lambda\{p = \beta_v\} = 0.$$

The proof of Theorem 3.6 is achieved.

4 Some applications examples

The aim of this last section is to depict situations for which the computation and estimation of the Vorob'ev expectation is appropriate. First, simple one-dimensional random sets are treated. Second, multi-dimensional random sets are studied. For this part, two types of Boolean models are considered, homogeneous and non-homogeneous.

4.1 One-dimensional random sets

Let us consider the random set $X = [0, \xi]$ where ξ is a r.v. on $[0, 1]$ with distribution G. In this case, the coverage function of X is

$$p(x) = \mathbb{P}(x \in X) = \mathbb{P}(\xi \geq x) = 1 - G_-(x). \tag{4.1}$$

In the following, we present two cases where the Vorob'ev expectation can be explicitely computed. A third case shows an example for which the assumptions of Theorem 3.6 are not satisfied.

First example. The distribution G is uniform, so $G(x) = x$ and from (4.1), $Q_\alpha = [0, 1 - \alpha)$. Since in this case $\mathbb{E}\lambda(X) = 1/2$, then $\alpha_v = 1/2$ and $\mathbb{E}_v(X) = [0, 1/2]$. The assumption $\lambda\{p = \alpha\} = 0$ is satisfied for any $\alpha \in [0, 1]$, and Theorems 2.5 and 3.6 apply.

Second example. Assume now that G is the mixing of a uniform and a Dirac distributions given by:

$$G(x) = \gamma x + (1-\gamma)\mathbb{1}_{\frac{1}{2}\leq x\leq 1}, \quad (0<\gamma<1).$$

Here:

$$\begin{aligned} p(x) =& 1-\gamma x-(1-\gamma)\mathbb{1}_{\frac{1}{2}<x\leq 1},\\ F(\alpha) =& \Big(1-\frac{\alpha}{\gamma}\Big)\mathbb{1}_{\alpha<\frac{\gamma}{2}} + \frac{1}{2}\mathbb{1}_{\frac{\gamma}{2}\leq\alpha<1-\frac{\gamma}{2}} + \Big(\frac{1-\alpha}{\gamma}\Big)\mathbb{1}_{1-\frac{\gamma}{2}\leq\alpha}. \end{aligned}$$

one has $\mathbb{E}\lambda(X) = 1/2$, $\alpha_v = \gamma/2$ and $\beta_v = 1-\gamma/2$ with the definition of (3.5). We are in a plateau of the function F. Since $Q_{\gamma/2}(X) = [0,1/2]$ is of Lebesgue measure $1/2$, we can again choose $\mathbb{E}_v(X) = [0,1/2] = Q_{1/2}(X)$. Here $\lambda\{p=\alpha\} = 0$ for any $\alpha\in[0,1]$ since the sets $\{p=\alpha\}$ are all singletons, so Theorems 2.5 and 3.6 apply.

Third example. If we now consider for $\gamma > 1$

$$G(x) = \gamma x\mathbb{1}_{0\leq x<\frac{1}{2\gamma}} + \frac{1}{2}\mathbb{1}_{\frac{1}{2\gamma}\leq x<1-\frac{1}{2\gamma}} + (1-\gamma+\gamma x)\mathbb{1}_{1-\frac{1}{2\gamma}\leq x\leq 1},$$

then:

$$\begin{aligned} p(x) &= (1-\gamma x)\mathbb{1}_{x<\frac{1}{2\gamma}} + \frac{1}{2}\mathbb{1}_{\frac{1}{2\gamma}\leq x<1-\frac{1}{2\gamma}} + \gamma(1-x)\mathbb{1}_{1-\frac{1}{2\gamma}\leq x}\\ F(\alpha) &= \Big(1-\frac{\alpha}{\gamma}\Big)\mathbb{1}_{\alpha<\frac{1}{2}} + \frac{1-\alpha}{\gamma}\mathbb{1}_{\frac{1}{2}\leq\alpha<1}. \end{aligned}$$

Here, $\mathbb{E}\lambda(X) = 1/2$ falls in the discontinuity domain of the function F. Again, $\alpha_v = 1/2$ but $Q_{1/2}(X) = [0,1/(2\gamma))$ which is of Lebesgue measure strictly smaller than $\mathbb{E}\lambda(X) = 1/2$. In this case $\mathbb{E}_v(X)$ is no longer unique: we can choose for $\mathbb{E}_v(X)$ any measurable set of $[0,1]$ that contains $[0,1/(2\gamma))$ and that has Lebesgue measure $1/2$. One possibility is of course $[0,1/2]$ but $[0,1/(2\gamma)]\cup[1/2,1-1/(2\gamma)]$ is another one. Notice that in any case one should have:

$$Q_{1/2}(X) = \Big[0,\frac{1}{2\gamma}\Big) \subset \mathbb{E}_v(X) \subset \Big[0,1-\frac{1}{2\gamma}\Big] = \bigcap_{\varepsilon>0} Q_{1/2-\varepsilon}(X).$$

Moreover, one has $\lambda\{p=\alpha\} = 0$ for all $\alpha\in[0,1]$ except $\alpha = 1/2 = \alpha_v$. Hence Theorem 3.6 does not apply in this case. This is not surprising since there is not uniqueness of $\mathbb{E}_v(X)$ here.

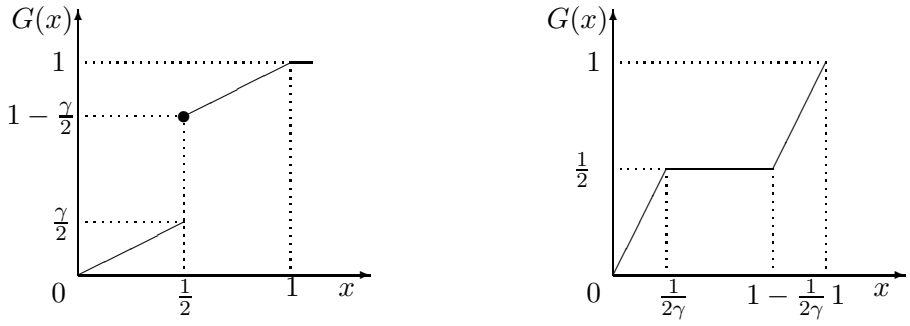

Figure 3: *Repartition function G for the second (left) and third (right) cases.*

Figure 3 shows in the left and right panel, the distribution function G for the second and third examples, respectively. It can be noticed that for the second example the distribution exhibits an atom, while for the third one it exhibits a plateau.

4.2 Multi-dimensional random sets

As an example of multi-dimensional random sets, let us consider the Boolean model [4, 9]. The Boolean model is constructed in two steps. First, we take a Poisson point process Π_μ in $\mathbb{R}^d$ and a sequence of i.i.d compact sets $(\Xi_n)_{n\in\mathbb{N}^*}$, which is also independent of Π_μ. Second, replace each point x_i of Π_μ by the shifted corresponding set $x_i + \Xi_i$. The Boolean model is the resulting union set, that is

$$\Xi = \bigcup_{x_i\in\Pi_\mu} (x_i + \Xi_i).$$

The points x_i are called germs and the random set Ξ_0 is the 'typical' grain of the model. The parameter μ is the intensity of the Poisson point process and it is a locally finite measure on $\mathbb{R}^d$. The Boolean model is also called the Poisson germ-grain model. In order to obtain non-trivial Boolean models, the d-th moment of the radius of the circumscribed circle of Ξ_0 must be finite.

The distribution of the Boolean model is uniquely determined by its capacity functional (see [4, 9])

$$T_\Xi : K \mapsto P(\Xi \cap K \neq \emptyset) = 1 - \exp\left[\mathbb{E}\mu(\check{\Xi}_0 \oplus K)\right], \tag{4.2}$$

where K ranges over compact sets in $\mathbb{R}^d$, $\check{\Xi}_0$ is the symetric of typical grain with respect to the origin and $\oplus$ the Minkowski addition.

4.2.1 Stationary case

This type Boolean model may be easily obtained if the intensity parameter of the Boolean model is proportional to the Lebesgue measure on $\mathbb{R}^d$, that is $\mu(dx) = m\lambda(dx)$ for a given $m > 0$. In the following, the considered grains are balls of random radius whose distribution $G(dR)$ satisfies such that $\int R^d G(dR) < \infty$. Let b_d be the volume of the euclidean ball $B(0, 1)$ in $\mathbb{R}^d$. The covering probability is obtained from (4.2) by taking $K = \{x\}$, hence

$$p(x) = P(x \in \Xi) = T_\Xi(\{x\}),$$

that gives under the present assumptions

$$p(x) = 1 - \exp\left[-mb_d \int R^d G(dR)\right] =: C_{m,d}.$$

The preceding calculation stands if one consider the observation of the considered stationary Boolean model through the finite window $[0, 1]^d$. Hence, we obtain

$$F(\alpha) = \mathbb{1}_{[0,C_{m,d})}(\alpha).$$

Thus, $\alpha_v = C_{m,d}$ and $Q_{\alpha_v}(X) = \emptyset$ while $Q_{\alpha_v-\varepsilon} = [0, 1]^d$ for any $\varepsilon > 0$. In this case, $\mathbb{E}_v(X)$ is not unique and any measurable set with volume $\mathbb{E}\lambda(X)$ is a possible Vorob'ev expectation. Also, the assumption $\lambda\{p = \alpha_v\} = 0$ does not hold here and neither does Theorem 3.6.

4.2.2 Non-stationary case

This type of Boolean model may be obtained for locally finite intensity measures $\mu(dx) = m(x)\lambda(dx)$ with a non-constant positive function $m(x)$.

For building this example some additional hypotheses are required. First, assume that the Boolean model is observed through the window $[0, 1]^d$, hence the measure $\mu(dx)$ vanishes outside this window.

Second, the grains are random balls as described in the previous example with the mention that the radius probability distribution admits a continous density g with respect to the Lebesgue measure such that

$$\forall x \in [0,1]^d,\ \int_{[0,1]^d} g(|x-y|)\frac{x-y}{|x-y|}\mu(dy) \neq 0. \tag{4.3}$$

The coverage probability requires the computation of

$$\begin{aligned}\phi(x) &= \mathbb{E}\mu\left(\check{\Xi}_0 \oplus \{x\}\right)\\ &= \int_{[0,1]^d} \mu(dy) \int_{\mathbb{R}} G(dR)\mathbb{1}_{|x-y|<R}\\ &= \int_{[0,1]^d} \mu(dy) S(|x-y|)\end{aligned}$$

with S the survival function of G.

The level sets $Q_\alpha(X)$ for $\alpha \in (0,1)$ are obtained by solving:

$$p(x) > \alpha \quad \Leftrightarrow \quad \exp\big(-\phi(x)\big) < 1-\alpha \quad \Leftrightarrow \quad \phi(x) > \ln\Big(\frac{1}{1-\alpha}\Big).$$

For a given $y \in [0,1]^d$ and $x \neq y$, $\nabla_x S(|x-y|)$ is the integrand of (4.3). Since g is continuous, ϕ is of class $\mathcal{C}^1$ with derivative (4.3) which does not vanish. Moreover, the boundary $\partial Q_\alpha(X)$ coincide with the set $\{p=\alpha\}$ and is obtained in solving:

$$\phi(x) = \ln\Big(\frac{1}{1-\alpha}\Big). \tag{4.4}$$

Using the implicit function theorem, the boundary of $Q_\alpha(X)$ is a $\mathcal{C}^1$-manifold of dimension $d-1$: it can be locally parameterized by $\mathcal{C}^1$-functions of $d-1$ variables. Assumption (2.4) is satisfied and moreover $\lambda(p=\alpha)=0$. In this case, Theorem 2.5 holds for any $\alpha \in (0,1)$ and Theorem 3.6 applies.

A Auxiliary proofs

A.1 Proof of Proposition 2.2

One only shows that F is càdlàg , the rest being straightforward. One has for $\alpha \in [0,1)$

$$Q_\alpha = \bigcup_{n>0} \uparrow Q_{\alpha+1/n}$$

so that thanks to the continuity from below of measures (*e.g.* [1], Th. 10.2), one gets

$$F(\alpha) = \lim_{n\to\infty} \uparrow F(\alpha+1/n).$$

Since F is decreasing, this is enough to get right-continuity. Since F is a decreasing function lower bounded by 0, it is làg. Besides, for $\alpha \in (0,1]$,

$$Q_\alpha \cup \{p=\alpha\} = \bigcap_{n>0} \downarrow Q_{\alpha-1/n}$$

from which one gets

$$F(\alpha) + \lambda\{p=\alpha\} = \lim_{n\to\infty} \downarrow F(\alpha-1/n).$$

Again this is enough to guarantee

$$F(\alpha) + \lambda\{p=\alpha\} = \lim_{\varepsilon\to 0^+} F(\alpha-\varepsilon).$$

A.2 Proof of Proposition 2.3

Assume first that $\mathbb{P}(X \neq \emptyset) = 1$ and denote by $\mathcal{K}'$ the family of all non-empty compact sets in $[0,1]^d$. It is a known fact that the myopic topology makes $\mathcal{K}'$ separable (since $[0,1]^d$ is, consider all finite sets from $(\mathbb{Q} \cap [0,1])^d$) and metrizable by the Hausdorff distance (see Molchanov, p.402-407). Recall that the myopic topology has a sub-base that consists of

$$\mathcal{K}'^F = \{K \in \mathcal{K}' \,:\, K \cap F = \emptyset\}, \quad F \text{ closed in } [0,1]^d,$$

and

$$\mathcal{K}'_U = \{K \in \mathcal{K}' \,:\, K \cap U \neq \emptyset\}, \quad U \text{ open in } [0,1]^d.$$

Let μ be the distribution of X on $\mathcal{K}'$ and consider now the empirical measure on $\mathcal{K}'$

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i},$$

Since, $\mathcal{K}'$ is a separable metric space, it follows from Varadarajan's theorem (see for example Dudley, p. 309-314) that

$$\mathbb{P}\left(\lim_{N \to \infty} \mu_n = \mu \right) = 1.$$

This means that there exists an event Ω_0 of probability one, such that for all $\omega \in \Omega_0$ and for all bounded continuous functions f on $\mathcal{K}'$, one has

$$\lim_{N \to \infty} \int_{\mathcal{K}'} f d\mu_n(\omega) = \int_{\mathcal{K}'} f d\mu.$$

Note that the functions $f_x(K) = \mathbb{1}_{x \in K}$, $x \in [0,1]^d$, are obviously bounded and continuous on $\mathcal{K}'$; indeed for each closed set C in $\mathbb{R}$, one has that

$$f_x^{-1}(C) = \begin{cases} \emptyset & \text{if } 1 \notin C, \\ \left(\mathcal{K}'^{\{x\}}\right)^c & \text{if } 1 \in C, \end{cases}$$

which implies that $f_x^{-1}(C)$ is closed in $\mathcal{K}'$. It remains to note that

$$\int_{\mathcal{K}'} f_x d\mu_n = \frac{1}{n} \sum_{i=1}^{n} f_x(X_i) = p_n(x),$$
$$\int_{\mathcal{K}'} f_x d\mu = \mu\left(K \in \mathcal{K}' \,:\, K \ni x\right) = \mathbb{P}(X \cap \{x\} \neq \emptyset) = p(x),$$

to achieve the proof under the additionnal assumption $\mathbb{P}(X \neq \emptyset) = 1$. But this can be relaxed by considering $\widetilde{X} = X \cup \{0\}$, $\widetilde{X_i} = X_i \cup \{0\}$ so that the corresponding coverage functions satisfy

$$\widetilde{p}(x) = p(x)\mathbb{1}_{x \neq 0} + \mathbb{1}_{x=0}, \quad \widetilde{p_n}(x) = p_n(x)\mathbb{1}_{x \neq 0} + \mathbb{1}_{x=0},$$

and

$$\widetilde{p_n}(x) \to \widetilde{p}(x) \iff p_n(x) \to p(x).$$

A.3 Proof of Proposition 2.4

Let $y \in A \triangle A^r$. Consider the unique cell $C(y) = [x, x+r)^d$ for some $x \in r\mathbb{Z}^d \cap [0,1]^d$ that contains y. One shows that $C(y)$ necessarily intersects ∂A. Indeed, suppose $y \notin A^r$ and $y \in A$; this implies $C(y) \cap A^r = \emptyset$, $x \notin A$ and $C(y)$ contains the segment $[x, y]$ which necessarily crosses ∂A. Suppose now that $y \in A^r$ and $y \notin A$; this implies $C(y) \subset A^r$, $x \in A$ and $C(y)$ contains the segment $[x, y]$ which crosses again ∂A. As a result $A \triangle A^r$ is included in the union of cells that intersect ∂A so that

$$\lambda(A \triangle A^r) \leq N_r(\partial A) r^d. \tag{A.1}$$

Besides let $\delta > 0$ and $d' = \overline{\dim}_{\mathrm{B}}(\partial A) + \delta$; then for all $r > 0$ small enough,

$$N_r(\partial A) \leq r^{-d'},$$

and combining with (A.1) one gets

$$\lambda(A \triangle A^r) \leq r^{d-d'} \to 0.$$

A.4 Proof of Lemma 2.6

Note first these immediate equalities

$$\begin{aligned} \mathrm{d}(Q^n_{\alpha_{n,r}}, Q_\alpha) &= \lambda\{p_n > \alpha_{n,r},\, p \leq \alpha\} + \lambda\{p_n \leq \alpha_{n,r},\, p > \alpha\} \\ &= \lambda\{p = \alpha\} + \lambda\{p_n > \alpha_{n,r},\, p < \alpha\} \\ &\qquad + \lambda\{p_n \leq \alpha_{n,r},\, p > \alpha\}. \end{aligned} \tag{A.2}$$

Consider the second term in the r.h.s. of (A.2) ; note that

$$\sup_{\substack{n \geq n_0 \\ r \leq r_0}} \lambda\{p_n > \alpha_{n,r},\, p < \alpha\} \leq \int \sup_{\substack{n \geq n_0 \\ r \leq r_0}} \mathbb{1}_{\{p_n > \alpha_{n,r}\}} \mathbb{1}_{\{p < \alpha\}} d\lambda,$$

and by Proposition 2.3,

$$\limsup_{\substack{n \to \infty \\ r \to 0}} \mathbb{1}_{\{p_n > \alpha_{n,r}\}} \leq \mathbb{1}_{\{p \geq \alpha\}}$$

so that by the monotone convergence theorem one gets

$$\limsup_{\substack{n \to \infty \\ r \to 0}} \lambda\{p_n > \alpha_{n,r},\, p < \alpha\} \leq \lambda\{p \geq \alpha,\, p < \alpha\} = 0.$$

Similarly, the third term (A.2) tends to 0 ; one has as before

$$\limsup_{\substack{n \to \infty \\ r \to 0}} \lambda\{p_n \leq \alpha_{n,r},\, p > \alpha\} \leq \lambda\{p \leq \alpha,\, p > \alpha\} = 0.$$

To sum up, one gets from (A.2)

$$\lim_{n,r} \mathrm{d}(Q^n_{\alpha_{n,r}}, Q_\alpha) = \lambda\{p = \alpha\}.$$

A.5 Proof of (3.8)

General considerations on arbitrary bounded Borel sets A, A', B give:

$$(A\triangle B)\setminus(A'\triangle B) = ((A\setminus A')\cap B^c)\cup((A'\setminus A)\cap B),$$

which are disjoint. Using that for any bounded Borel sets C and D,

$$\lambda(C)-\lambda(D) = \lambda(C\setminus D)-\lambda(D\setminus C), \tag{A.3}$$

we obtain with $C = A\triangle B$ and $D = A'\triangle B$ that:

$$\begin{aligned}
\lambda(A\triangle B)-\lambda(A'\triangle B) &= \lambda\big((A\triangle B)\setminus(A'\triangle B)\big)-\lambda\big((A'\triangle B)\setminus(A\triangle B)\big)\\
&= \lambda((A\setminus A')\cap B^c)+\lambda((A'\setminus A)\cap B)\\
&\qquad -\lambda((A'\setminus A)\cap B^c)-\lambda((A\setminus A')\cap B)\\
&= \lambda(A\setminus A')-\lambda(A'\setminus A)\\
&\qquad +2\left[\lambda((A'\setminus A)\cap B)-\lambda((A\setminus A')\cap B)\right]\\
&= \lambda(A)-\lambda(A')\\
&\qquad +2\left[\lambda((A'\setminus A)\cap B)-\lambda((A\setminus A')\cap B)\right]
\end{aligned}$$

by using again (A.3) with $C = A$ and $D = A'$. Now, taking $A = M, A' = \widehat{X}^n$ and $B = X_i$ with M measurable in $[0,1]^d$ such that $\lambda(M) = \lambda(\widehat{X}^n) = \Lambda_n$, we deduce

$$\lambda\big(M\triangle X_i\big)-\lambda\big(\widehat{X}^n\triangle X_i\big) = 2\left[\lambda((\widehat{X}^n\setminus M)\cap X_i)-\lambda((M\setminus\widehat{X}^n)\cap X_i)\right],$$

so that

$$\frac{1}{n}\sum_{i=1}^n\lambda\big(M\triangle X_i\big)-\frac{1}{n}\sum_{i=1}^n\lambda\big(\widehat{X}^n\triangle X_i\big) = \frac{2}{N}\sum_{i=1}^n\left[\lambda((\widehat{X}^n\setminus M)\cap X_i)-\lambda((M\setminus\widehat{X}^n)\cap X_i)\right]. \tag{A.4}$$

Since for any measurable set $A\subset[0,1]^d$,

$$\frac{1}{n}\sum_{i=1}^n\big(\lambda(A\cap X_i)\big) = \frac{1}{n}\sum_{i=1}^n\Big(\int_{[0,1]^d}\mathbb{1}_{X_i}\mathbb{1}_A d\lambda\Big) = \int_A p_n d\lambda,$$

one gets from (A.4):

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^n\lambda\big(M\triangle X_i\big)-\frac{1}{n}\sum_{i=1}^n\lambda\big(\widehat{X}^n\triangle X_i\big) &\geq \int_{\widehat{X}^n\setminus M}p_n d\lambda-\int_{M\setminus\widehat{X}^n}p_n d\lambda\\
&\geq \alpha^n\left[\lambda\big(\widehat{X}^n\setminus M\big)-\lambda\big(M\setminus\widehat{X}^n\big)\right]\\
&= \alpha^n\left[\lambda\big(\widehat{X}^n\big)-\lambda\big(M\big)\right] = 0.
\end{aligned} \tag{A.5}$$

The last inequality holds since from (3.8), on $\widehat{X}^n$, one has $p_n > \alpha^n$ whereas on $(\widehat{X}^n)^c$, we have $p_n\leq\alpha^n$. This ends the proof.

A.6 Proof of Lemma 3.11

For the proof, we follow here the idea of Kovyazin [3], Proposition 4. Let $\varepsilon > 0$ such that F is continuous at $\alpha_v - \varepsilon$ and $\beta_v + \varepsilon$ and let us define:

$$\gamma_1 = F(\alpha_v - \varepsilon) - \mathbb{E}\lambda(X) \tag{A.6}$$

$$\gamma_2 = \mathbb{E}\lambda(X) - F(\beta_v + \varepsilon). \tag{A.7}$$

From the definitions (3.5) of α_v and β_v, we clearly have $\gamma_1 > 0$ and $\gamma_2 > 0$. We shall show that, almost surely, one has

$$\liminf_{n\to\infty} \alpha^n \geq \alpha_v - \varepsilon. \tag{A.8}$$

To this end, suppose that, on an event of positive probability, one has

$$\liminf_{n\to\infty} \alpha^n < \alpha_v - \varepsilon,$$

then, there exists a (random) strictly increasing sequence $(n_k)_{k\in\mathbb{N}^*}$ in $\mathbb{N}$ such that for all k, one has $\alpha^{n_k} < \alpha_v - \varepsilon$. For Λ_{n_k} defined in (3.2), we can then write:

$$\begin{aligned}
\Lambda_{n_k} &= \lambda(X^{n_k}) \\
&\geq \lambda\{p_{n_k} \geq \alpha^{n_k}\} \\
&\geq \lambda\{p_{n_k} \geq \alpha_v - \varepsilon\} \\
&= \lambda\{p_{n_k} \geq \alpha_v - \varepsilon,\, p > \alpha_v - \varepsilon\} + \lambda\{p_{n_k} \geq \alpha_v - \varepsilon,\, p \leq \alpha_v - \varepsilon\} \\
&= F(\alpha_v - \varepsilon) - \lambda\{p_{n_k} < \alpha_v - \varepsilon,\, p > \alpha_v - \varepsilon\} \\
&\qquad + \lambda\{p_{n_k} \geq \alpha_v - \varepsilon,\, p \leq \alpha_v - \varepsilon\} \\
&\geq F(\alpha_v - \varepsilon) + o_k(1),
\end{aligned} \tag{A.9}$$

where we use Lemma 2.6 in the last inequality and where $o_k(1)$ is a quantity that converges to zero as k tends to infinity. Introducing Λ_{n_k} in the definition of γ_1, one gets from (A.9)

$$\gamma_1 \leq |\Lambda_{n_k} - \mathbb{E}\lambda(X)| + o_k(1),$$

which by the SLLN would imply $\gamma_1 = 0$, a contradiction that proves (A.8).

A very similar computation, with γ_2 instead of γ_1 and β_v instead of α_v, gives almost surely

$$\limsup_{n\to\infty} \alpha^n \leq \beta_v + \varepsilon.$$

A.7 Proof of Lemma 3.12

It is very similar to the proof of Lemma 3.11. We split the proof into several steps. In what follows, consider $\varepsilon > 0$.

Step 1. Let us first prove that $\liminf_{r\to 0} \alpha^{n,r} \geq \alpha^n - \varepsilon$. Assume that, on an event of positive probability

$$\liminf_{r\to 0} \alpha^{n,r} < \alpha^n - \varepsilon. \tag{A.10}$$

Then, there exists a random sequence $(r_k)_{k\in\mathbb{N}^*}$ with $r_k \to 0$ satisfying $\alpha^{n,r_k} < \alpha^n - \varepsilon$ and

$$\begin{aligned}
\Lambda_n = \lambda(\widehat{X}^{n,r_k}) \geq \lambda\big(Q^{n,r_k}_{\alpha^{n,r_k}}\big) &\geq \lambda\big(Q^{n,r_k}_{\alpha^n-\varepsilon}\big) \\
&\geq \lambda\big(Q^{n}_{\alpha^n-\varepsilon}\big) - \mathrm{d}\big(Q^{n,r_k}_{\alpha^n-\varepsilon}, Q^{n}_{\alpha^n-\varepsilon}\big) \\
&= \lambda\{p_n > \alpha^n - \varepsilon\} - r_k^{\kappa}
\end{aligned} \tag{A.11}$$

by applying Proposition 2.4 with $A = Q^n_{\alpha^n-\varepsilon} = \{p_n > \alpha^n - \varepsilon\}$ as we did for Q^n_α in subsubsection 2.2.3. But (A.11) implies that $\Lambda_n \geq \lambda\{p_n > \alpha^n - \varepsilon\}$ which contradicts the definition (3.7) of α^n. As a consequence, (A.10) can not be true.

Step 2. Let us assume that $\liminf_{\substack{n\to\infty\\ r\to 0}} \alpha^{n,r} < \alpha_v - \varepsilon$. Following Step 1 and replacing α^n by α_v, one gets on an event of positive probability, a random sequence $(n_k, r_k)_{k\in\mathbb{N}^*}$ satisfying $\alpha^{n_k,r_k} < \alpha_v - \varepsilon$ with $n_k \to \infty$ and $r_k \to 0$ and

$$\begin{aligned}\Lambda_{n_k} &\geq \lambda\{p_{n_k} > \alpha_v - \varepsilon\} - \mathrm{d}\big(Q^{n_k,r_k}_{\alpha^{n_k}-\varepsilon}, Q^{n_k}_{\alpha^{n_k}-\varepsilon}\big)\\ &\geq \lambda\{p_{n_k} > \alpha_v - \varepsilon\} - \sup_n \mathrm{d}\big(Q^{n,r_k}_{\alpha^n-\varepsilon}, Q^n_{\alpha^n-\varepsilon}\big),\end{aligned} \tag{A.12}$$

and since the upper box-dimensions of the $\partial Q^n_{\alpha^n-\varepsilon}$'s are (a.s.) bounded by $d-\kappa$, one gets from Proposition 2.4 that

$$\sup_n \mathrm{d}\big(Q^{n,r_k}_{\alpha^n-\varepsilon}, Q^n_{\alpha^n-\varepsilon}\big) \leq r_k^\kappa. \tag{A.13}$$

From (A.11), we also have

$$\lambda\{p_{n_k} > \alpha_v - \varepsilon\} \geq F(\alpha_v - \varepsilon) + o_k(1). \tag{A.14}$$

Combining (A.12), (A.13) and (A.14), we obtain

$$\Lambda_{n_k} \geq F(\alpha_v - \varepsilon) + o_k(1),$$

which leads to a contradiction on γ_1 as in the proof of Lemma 3.11.

Step 3. Let us now assume that, on an event of positive probability,

$$\limsup_{r\to 0} \alpha^{n,r} > \alpha^n + 2\varepsilon. \tag{A.15}$$

Then there exists a random subsequence $(r_k)_{k\in\mathbb{N}^*}$ such that $r_k \to 0$ and for every k,

$$\alpha^{n,r_k} - \varepsilon > \alpha^n + \varepsilon.$$

From this and the definition (3.3) of $\alpha^{n,r}$, one gets

$$\Lambda_n < \lambda\big(Q^{n,r_k}_{\alpha^{n,r_k}-\varepsilon}\big) \leq \lambda\big(Q^{n,r_k}_{\alpha^n+\varepsilon}\big) \leq \lambda\{p_n \geq \alpha^n + \varepsilon\} + \mathrm{d}\big(Q^{n,r_k}_{\alpha^n+\varepsilon}, Q^n_{\alpha^n+\varepsilon}\big).$$

Arguing as for (A.13) and letting $k \to \infty$ contradicts $\lambda\{p_n \geq \alpha^n + \varepsilon\} \leq \Lambda_n$ so that (A.15) can not be true.

Step 4. Proceeding as in Step 3 with β_v instead of α^n and using the proof of Lemma 3.11 as in Step 2 completes the proof. ■

Acknowledgements. The authors thank the members of the workgroup "Géométrie stochastique" of the University Lille 1 for interesting discussions.

References

[1] Billingsley, P. (1995). *Probability and measure*. John Wiley & Sons.

[2] Falconer, K. (1997). *Fractal Geometry: Mathematical foundations and applications*. Wiley & Sons.

[3] Kovyazin, S. (1986). On the limit behavior of a class of empirical means of a random set. *Theory of Probability and its Applications* **30(4),** 814–820. Translated from Russian by J. Malek.

[4] Molchanov, I. (1997). *Statistics of the Boolean model for practitioners and mathematicians*. John Wiley and Sons.

[5] Molchanov, I. (1999). Random closed sets : results and problems. In *Stochastic geometry, likelihood and computation*. ed. O. Barndorff-Nielsen, W. S. Kendall, and M. N. M. van Lieshout. CRC Press/Chapman and Hall, Boca Raton.

[6] Molchanov, I. (2005). *Theory of random sets*. Springer.

[7] Stoica, R. S., Martinez, V. J. and Saar, E. (2007). A three dimensional object point process for detection of cosmic filaments. *Journal of the Royal Statistical Society : Series C (Applied Statistics)* **55,** 189–205.

[8] Stoica, R. S., Martinez, V. J. and Saar, E. (2010). Filaments in observed and mock catalogs. *Astronomy and Astrophysics* **510,A38,** 1–12.

[9] Stoyan, D., Kendall, W. S. and Mecke, J. (1995). *Stochastic geometry and its applications* 2nd ed. John Wiley and Sons.

[10] Stoyan, D. and Stoyan, H. (1994). *Fractals, random shapes and point fields*. Wiley, Chichester.

[11] Vorob'ev, O. (1977). On set-theoretical characteristics of random processes of spread. *Izv. Sibirsk. Otdel. Akad. Nauk SSSR Ser. Tehn. Nauk* **153,** 3–7. (Russian).