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# Slow and fast scales for superprocess limits of age-structured populations

Sylvie Méléard\* and Viet Chi Tran<sup>†</sup>

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## Abstract

A superprocess limit for an interacting birth-death particle system modelling a population with trait and physical age-structures is established. Traits of newborn offspring are inherited from the parents except when mutations occur, while ages are set to zero. Because of interactions between individuals, standard approaches based on the Laplace transform do not hold. We use a martingale problem approach and a separation of the slow (trait) and fast (age) scales. While the trait marginals converge in a pathwise sense to a superprocess, the age distributions, on another time scale, average to equilibria that depend on traits. The convergence of the whole process depending on trait and age, only holds for finite-dimensional time-marginals. We apply our results to the study of examples illustrating different cases of trade-off between competition and senescence.

**Keywords:** Interacting particle system ; age-structure ; superprocess ; slow and fast scales ; trait-structured density-dependent population. **MSC:** 60J80 ; 60K35 ; 60G57.

## 1 Introduction

We consider an asexual population in which the survival probability and reproduction rate of each individual are characterized by a quantitative trait, such as for example the body size, or the rate of food intake. As emphasized by Charlesworth [7], most of these abilities also depend on age. In this paper, we are interested in studying the joint effects of age and trait structures in the interplay between ecology and evolution. Evolution, acting on the distribution of traits in the population, is the consequence of three basic mechanisms: heredity, which transmits traits to new offspring, mutation, which drives the variation in the trait values, and selection between these different trait values, which is due to ecological interactions. Some questions on evolution are strongly related to the age structure. For example, we would like to understand how the age influences the persistence of the population or the trait's evolution, or which age structure will appear in long time scales for a given trait.

Our model relies on an individual-based birth and death process with age and trait introduced in Méléard and Tran [26] (see also Ferrière and Tran [14]). It generalizes the trait-structured case developed in Champagnat *et al.* [5, 6] and the age-structured case in Jagers and Klebaner [18, 19], Tran [34]. Here, each individual is characterized by a quantitative trait and by its physical age, that is the time since its birth. We describe the dynamics of large populations composed of small individuals with short lives. Life-lengths and durations between reproductions are assumed to be proportional to the individuals' weights, and these weights are inversely proportional to the population size. At each birth, the age is reset to 0, inducing a large asymmetry between

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the mother and her daughters. When a mutation occurs, the new mutant trait is close to its ancestor's one, yielding a slow variation of the trait. Hence, these two mechanisms lead to a difference of time scales between age and trait. Moreover, we take resource constraints into account by including competition between individuals.

Our aim is to study an approximation of this model when the population size tends to infinity. Our main result shows that two qualitatively different asymptotic behaviors arise from the separation of the fast age and slow trait time scales. While the trait marginals converge in a pathwise sense to a superprocess, the age distributions stabilize into deterministic equilibria that depend on the traits. To our knowledge, nothing has been done before in the setting we are interested in, with slow-fast variables and nonlinearity. The techniques we use are based on martingale properties and generalize to this infinite dimensional setting the treatment of the slow-fast scales for diffusion processes, developed by Kurtz [23], and by Ball *et al.* [2].

Our results generalize Athreya *et al.* [1], Bose and Kaj [3], where averaging phenomena are proved in the case where birth and death rates do not depend on age. In our case with dependence, the lifespan of an individual cannot be governed by an age distribution function independent of trait, or by a positive continuous additive functional, as in Bose and Kaj [4], Dawson *et al.* [9], Dynkin [10], Kaj and Sagitov [22], Fleischmann *et al.* [16], Wang [35]. Moreover the Laplace characterization that these authors use extensively does not hold anymore when interactions between particles are allowed. In Evans and Steinsaltz [13], the damage segregation at cell fissioning is considered as an age. Nevertheless in their model there is no interaction between cells, and at each birth, the daughters' ages (as damages) are not reset to zero, but distributed asymmetrically following a distribution centered on the mother's age.

In Section 2, the population dynamics is described by an individual-based point measure-valued birth and death process with an age and trait structures. In Section 3, we establish our main limit theorem (Theorem 3.1) where the averaging phenomenon is obtained. In Section 4, we consider as an illustration two models where the population is structured by size and physical or biological age. We present simulations and comment the different behaviors.

**Notation:** For a given metric space  $E$ , we denote by  $\mathbb{D}([0, T], E)$  the space of right continuous and left limited (càd-làg) functions from  $[0, T]$  to  $E$ . This space is embedded with the Skorohod topology (*e.g.* [29, 20]).

If  $\mathcal{X}$  is a subset of  $\mathbb{R}^d$ , we denote by  $\mathcal{M}_F(\mathcal{X})$  the set of finite measures on  $\mathcal{X}$ , which will be usually embedded with the topology of weak convergence. Nevertheless, if  $\mathcal{X}$  is unbounded, we will also consider the topology of vague convergence. If we need to differentiate both topological spaces, we will denote by  $(\mathcal{M}_F(\mathcal{X}), w)$ , respectively  $(\mathcal{M}_F(\mathcal{X}), v)$ , the space of measures endowed by the weak (resp. vague) topology. For a measurable real bounded function  $f$ , and a measure  $\mu \in \mathcal{M}_F(\mathcal{X})$ , we will denote

$$\langle \mu, f \rangle = \int_{\mathcal{X}} f(x) \mu(dx).$$

For  $\ell \in \mathbb{N}$ , we denote by  $\mathcal{C}_b^\ell(\mathcal{X}, \mathbb{R})$  the space of real bounded functions  $f$  of class  $\mathcal{C}^\ell$  with bounded derivatives. In the sequel, the space  $\mathcal{C}_b^{0,1}(\mathcal{X} \times \mathbb{R}_+, \mathbb{R})$  (resp.  $\mathcal{C}_c(\mathcal{X}, \mathbb{R})$ ,  $\mathcal{C}_c^1(\mathbb{R}_+, \mathbb{R})$ ) denotes the space of continuous bounded real functions  $\varphi(x, a)$  on  $\mathcal{X} \times \mathbb{R}_+$  of class  $\mathcal{C}^1$  with respect to  $a$  with bounded derivatives (resp. of continuous real functions on  $\mathcal{X}$  with compact support, of  $\mathcal{C}^1$  real functions on  $\mathbb{R}_+$  with compact support in  $[0, +\infty)$ ).

## 2 Microscopic age and trait structured particle system

We consider a discrete population in continuous time where the individuals reproduce, age and die with rates depending on a hereditary trait and on their age. An individual is characterized by a quantitative trait  $x \in \mathcal{X}$  where  $\mathcal{X}$  is a closed subset of  $\mathbb{R}^d$  and by its physical age  $a \in \mathbb{R}_+$ , *i.e.* the time since its birth. The individuals reproduce asexually during their lives, and the trait from the parent is transmitted to its offspring except when a mutation occurs. Resources are shared by the individuals, implying an interaction described by a kernel comparing the competitors' traits and ages. Senescence, which quantifies the decrease of fertility or survival probability with age, is also taken into account. These two phenomena create selection pressure.

We are interested in approximating the dynamics of a large population whose size is parametrized by some integer  $n$ . This parameter can be seen as the order of the carrying capacity, when the total amount of resources is assumed to be fixed. If the parameter  $n$  is large, there will be many individuals with little per capita resource and we renormalize the individual biomass by the weight  $1/n$ .

We consider here allometric demographies where the lifetime and gestation length of each individual are proportional to its biomass. Thus the birth and death rates are of order  $n$ , while preserving the demographic balance. As a consequence the right scale to observe a nontrivial limit in the age structure, as  $n$  increases, is of order  $1/n$ .

The population at time  $t$  is represented by a point measure as follows:

$$X_t^n = \frac{1}{n} \sum_{i=1}^{N_t^n} \delta_{(X_i(t), A_i(t))}, \quad (2.1)$$

where  $N_t^n = \langle nX_t^n, 1 \rangle$  is the number of individuals alive at time  $t$ , and  $X_i(t)$  and  $A_i(t)$  denote respectively the trait and age of individual  $i$  at time  $t$  (individuals are ranked in lexicographical order for instance).

The dynamics of  $X^n$  is given as follows:

- The birth of an individual with trait  $x \in \mathcal{X}$  and age  $a \in \mathbb{R}_+$  is given by  $n r(x, a) + b(x, a)$ . The new offspring is of age 0 at birth. Moreover, it inherits of the trait  $x$  of its ancestor with probability  $1 - p(x, a) \in [0, 1]$  and is a mutant with probability  $p(x, a) \in [0, 1]$ . The mutant trait is then  $x + h$ , where the variation  $h$  is randomly chosen following the distribution  $\pi^n(x, dh)$ .
- Individuals age with velocity  $n$ , so that the physical age at time  $t$  of an individual born at time  $c$  is  $a = n(t - c)$ .
- The intrinsic death rate of an individual with trait  $x \in \mathcal{X}$  and age  $a \in \mathbb{R}_+$  is given by  $n r(x, a) + d(x, a)$ . The competition between individuals  $(x, a)$  and  $(y, \alpha)$  is described by the value  $U((x, a), (y, \alpha))$  of a kernel  $U$ . In a population described by the measure  $X \in \mathcal{M}_F(\mathcal{X} \times \mathbb{R}_+)$ , the total interaction on an individual  $(x, a)$  is thus:

$$XU(x, a) = \int_{\mathcal{X} \times \mathbb{R}_+} U((x, a), (y, \alpha)) X(dy, d\alpha), \quad (2.2)$$

and its total death rate is  $n r(x, a) + d(x, a) + XU(x, a)$ .

**Assumption 2.1.** 1. The birth and death rates  $b$  and  $d$  are continuous on  $\mathcal{X} \times \mathbb{R}_+$  and bounded respectively by  $\bar{b}$  and  $\bar{d}$ .

2. The function  $r$  is continuous on  $\mathcal{X} \times \mathbb{R}_+$ . There exist a positive constant  $\bar{r}$  and a non-negative real function  $\underline{r}$  such that  $\forall(x, a) \in \mathcal{X} \times \mathbb{R}_+$ ,  $\underline{r}(a) \leq |r(x, a)| \leq \bar{r}$  with

$$\int_0^{+\infty} \underline{r}(a) da = +\infty. \quad (2.3)$$

3. The competition kernel  $U$  is continuous on  $(\mathcal{X} \times \mathbb{R}_+)^2$  and is bounded by  $\bar{U}$ .  $\square$

Assumption 2.1-(2) implies that any individual  $i$  from the population  $X_t^n$  has a finite lifetime that is stochastically upper-bounded by a random variable  $D_i^n(t)$  with survival function

$$S^n(\ell) = \mathbb{P}(D_i^n(t) > \ell) = \exp\left(-\int_0^\ell n \underline{r}(nu) du\right), \quad (2.4)$$

where we recall that the aging velocity is equal to  $n$ .

If the competition kernel  $U$  is positive on  $(\mathcal{X} \times \mathbb{R}_+)^2$ , it can model a competition of the logistic type: the more important the size of the population is and the higher the death rate by competition is. For examples of such kernels we refer to [26].

**Example 2.2.** Let us illustrate the condition (2.3).

1. If the function  $r$  is lower bounded by a positive constant  $\underline{r}$ , then (2.3) is satisfied and so is (2.4), with exponential random variable  $D_i^n(t)$  of parameter  $n\underline{r}$ .
2. Another example is when the trait  $x$  is linked to the rate of metabolism, which measures the energy expended by individuals. Ageing may result from toxic by-products of the metabolism and we can define a biological age,  $xa$ . If  $x \in [x_1, x_2]$  with  $x_1, x_2 > 0$  and if we define  $r(x, a) = xa$ , then Condition (2.3) is satisfied with  $\underline{r}(a) = x_1 a$ . The example of Section 4.2 deals with this case.
3. If we consider  $r(x, a) = \gamma/(1+a)$  with  $\gamma \in (0, 1)$ , then (2.3) is also satisfied and the probability of observing an age higher than  $a$  is equivalent to  $a^{-\gamma}$  when  $a$  tends to infinity. Such cases with distributions in the domain of attraction of a stable law, but without interaction, have been considered for instance in [16].

**Assumption 2.3.** For any  $x \in \mathcal{X}$ , the mutation kernel  $\pi^n(x, dh)$  has its support in  $\mathcal{X} - \{x\} = \{h \in \mathbb{R}^d \mid x + h \in \mathcal{X}\}$ . We consider two cases:

1. The trait space  $\mathcal{X}$  is a compact subset of  $\mathbb{R}^d$  and there exists a generator  $A$  of a Feller semi-group on  $\mathcal{C}_b(\mathcal{X}, \mathbb{R})$  with domain  $\mathcal{D}(A)$  dense in  $\mathcal{C}_b(\mathcal{X}, \mathbb{R})$  such that:

$$\forall f \in \mathcal{D}(A), \quad \lim_{n \rightarrow +\infty} \sup_{x \in \mathcal{X}} \left| n \int_{\mathcal{X} - \{x\}} (f(x+h) - f(x)) \pi^n(x, dh) - Af(x) \right| = 0. \quad (2.5)$$

2. The trait space  $\mathcal{X}$  is a closed subset of  $\mathbb{R}^d$  and we assume in addition that there exists  $\ell_1 \geq \ell_0 \geq 2$  with  $\mathcal{C}_b^{\ell_1}(\mathcal{X}, \mathbb{R}) \subset \mathcal{D}(A)$  and  $\forall f \in \mathcal{C}_b^{\ell_1}(\mathcal{X}, \mathbb{R})$ ,  $\forall x \in \mathcal{X}$ .

$$|Af(x)| \leq C \sum_{\substack{|k| \leq \ell_0 \\ k=(k_1, \dots, k_d)}} |D^k f(x)| \quad (2.6)$$

and

$$\sup_{x \in \mathcal{X}} \left| n \int_{\mathcal{X} - \{x\}} (f(x+h) - f(x)) \pi^n(x, dh) - Af(x) \right| \leq \varepsilon_n \sum_{\substack{|k| \leq \ell_1 \\ k=(k_1, \dots, k_d)}} \|D^k f\|_\infty, \quad (2.7)$$

where  $D^k f(x) = \partial_{x_1}^{k_1} \dots \partial_{x_d}^{k_d} f(x)$ ,  $\varepsilon_n$  is a sequence tending to 0 as  $n$  tends to infinity and  $C$  is a constant.

**Remark 2.4.** Both Assumptions (2.5) and (2.7) describe small mutation steps. The stronger hypothesis (2.7) is required when  $\mathcal{X}$  is not compact, to obtain the tightness in the proof of Theorem 3.1.

**Example 2.5.** Let us give some examples of mutation kernels satisfying (2.5) or (2.6) and (2.7).

1. In the case where  $\mathcal{X} = [x_1, x_2]$ , the mutation kernel  $\pi^n(x, dh)$  is a Gaussian distribution with mean 0 and variance  $\sigma^2/n$ , conditioned to  $[x_1 - x, x_2 - x]$ . In this case, elementary computation shows that for  $f \in \mathcal{C}_b^2([x_1, x_2], \mathbb{R})$  such that  $f'(x_1) = f'(x_2) = 0$ ,  $Af(x) = \frac{\sigma^2}{2}f''(x)$ , which satisfies (2.5).

2. In the case where  $\mathcal{X} = \mathbb{R}^d$ , a possible choice of mutation kernel  $\pi^n(x, dh)$  is a Gaussian distribution with mean 0 and covariance matrix  $\Sigma(x)/n$ , with  $\Sigma(x) = (\Sigma_{ij}(x), 1 \leq i, j \leq d)$ . The generator  $A$  is given for  $f \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$  by  $Af(x) = \frac{1}{2} \sum_{i,j=1}^d \Sigma_{ij}(x) \partial_{ij}^2 f(x)$ . If the function  $\Sigma$  is bounded, then Assumption (2.6) is fulfilled. If moreover, the third moments of  $\pi^n(x, dh)$  are bounded (in  $x$ ), then (2.7) is satisfied.  $\square$

Let us now describe the generator  $L^n$  of the  $\mathcal{M}_F(\mathcal{X} \times \mathbb{R}_+)$ -valued Markov process  $X^n$ , which sums the aging phenomenon and the ecological dynamics of the population. As developed in Dawson [8] Theorem 3.2.6, the set of cylindrical functions defined for each  $\mu \in \mathcal{M}_F(\mathcal{X} \times \mathbb{R}_+)$  by  $F_\varphi(\mu) = F(\langle \mu, \varphi \rangle)$ , with  $F \in \mathcal{C}_b^1(\mathbb{R}, \mathbb{R})$  and  $\varphi \in \mathcal{C}_b^{0,1}(\mathcal{X} \times \mathbb{R}_+, \mathbb{R})$ , generates the set of bounded measurable functions on  $\mathcal{M}_F(\mathcal{X} \times \mathbb{R}_+)$ . For such function, for  $\mu \in \mathcal{M}_F(\mathcal{X} \times \mathbb{R}_+)$ ,

$$\begin{aligned} L^n F_\varphi(\mu) &= n \langle \mu, \partial_a \varphi(\cdot) \rangle F'_\varphi(\mu) \\ &\quad + n \int_{\mathcal{X} \times \mathbb{R}_+} \left[ (nr(x, a) + d(x, a) + \mu U(x, a)) \left( F_\varphi\left(\mu - \frac{1}{n} \delta_{(x,a)}\right) - F_\varphi(\mu) \right) \right. \\ &\quad \left. + (nr(x, a) + b(x, a)) \int_{\mathbb{R}^d} \left( F_\varphi\left(\mu + \frac{1}{n} \delta_{(x+h,0)}\right) - F_\varphi(\mu) \right) K^n(x, a, dh) \right] \mu(dx, da), \end{aligned} \quad (2.8)$$

where

$$K^n(x, a, dh) = p(x, a) \pi^n(x, dh) + (1 - p(x, a)) \delta_0(dh). \quad (2.9)$$

Under the condition  $\sup_{n \in \mathbb{N}^*} \mathbb{E}(\langle X_0^n, 1 \rangle) < +\infty$ , it has been proved in Méléard-Tran [26] (see also the case without age in Champagnat-Ferrière-Méléard [6] and the case without trait in Tran [34]), that there exists for any  $n$ , a càd-làg Markov process with generator  $L^n$ , which can be obtained as solution of a stochastic differential equation driven by a Point Poisson measure. Trajectorial uniqueness also holds for this equation. The construction provides an exact individual-based simulation algorithm (see [14]).

A slight adaptation of the proofs in [6] allows us to get the

**Proposition 2.6.** (i) Under Assumptions 2.1, and if

$$\sup_{n \in \mathbb{N}^*} \mathbb{E}(\langle X_0^n, 1 \rangle^3) < +\infty, \quad (2.10)$$

then for all  $T > 0$ ,

$$\sup_{n \in \mathbb{N}^*} \sup_{t \in [0, T]} \mathbb{E}(\langle X_t^n, 1 \rangle^3) < +\infty \text{ and } \sup_{n \in \mathbb{N}^*} \mathbb{E} \left( \sup_{t \in [0, T]} \langle X_t^n, 1 \rangle^2 \right) < +\infty. \quad (2.11)$$

(ii) Moreover, for  $n \in \mathbb{N}^*$  and a test function  $\varphi \in \mathcal{C}_b^{0,1}(\mathcal{X} \times \mathbb{R}_+, \mathbb{R})$ , the process  $M^{n,\varphi}$  defined by

$$\begin{aligned} M_t^{n,\varphi} &= \langle X_t^n, \varphi \rangle - \langle X_0^n, \varphi \rangle - n \int_0^t \langle X_s^n, \partial_a \varphi(x, a) \rangle ds \\ &\quad - \int_0^t \int_{\mathcal{X} \times \mathbb{R}_+} \left( (nr(x, a) + b(x, a)) \int_{\mathbb{R}^d} \varphi(x + h, 0) K^n(x, a, dh) \right. \\ &\quad \left. - (nr(x, a) + d(x, a) + X_s^n U(x, a)) \varphi(x, a) \right) X_s^n(dx, da) ds \end{aligned} \quad (2.12)$$

is a square integrable martingale started at 0 with quadratic variation:

$$\begin{aligned} \langle M^{n,\varphi} \rangle_t &= \frac{1}{n} \int_0^t \int_{\mathcal{X} \times \mathbb{R}_+} \left( (nr(x, a) + b(x, a)) \int_{\mathbb{R}^d} \varphi^2(x + h, 0) K^n(x, a, dh) \right. \\ &\quad \left. + (nr(x, a) + d(x, a) + X_s^n U(x, a)) \varphi^2(x, a) \right) X_s^n(dx, da) ds. \end{aligned} \quad (2.13)$$

Notice that in (2.12), the mutation rate is hidden in the kernel  $K^n$ :

$$\begin{aligned} (nr(x, a) + b(x, a)) \int_{\mathbb{R}^d} \varphi(x + h, 0) K^n(x, a, dh) &= (nr(x, a) + b(x, a))(1 - p(x, a))\varphi(x, 0) \\ &\quad + (nr(x, a) + b(x, a))p(x, a) \int_{\mathbb{R}^d} \varphi(x + h, 0) \pi^n(x, dh). \end{aligned} \quad (2.14)$$

### 3 Superprocess limit

We now investigate the limit when  $n$  increases to  $+\infty$ . In the limit, we obtain a continuum of individuals in which the individualities are lost. It is in particular difficult to keep track of the age-distribution when  $n$  tends to infinity. Because of the non-local branching (a mother of age  $a > 0$  gives birth to a daughter of age 0) and because the aging velocity tends to infinity, it is impossible to obtain directly the uniform tightness on  $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_F(\mathcal{X} \times \mathbb{R}_+))$  of the sequence of laws of the measure-valued processes  $(X_t^n(dx, da))_{n \in \mathbb{N}^*}$ , as it can be observed considering (2.12). Indeed, assuming that the function  $\varphi$  only depends on  $a$ , the term of the form

$$\int_0^t \int_{\mathcal{X} \times \mathbb{R}_+} nr(x, a) (\varphi(0) - \varphi(a)) X_s^n(dx, da) ds$$

cannot be tight if  $r(x, a)$  is bounded and  $X^n$  is tight. Therefore, we will be led to firstly show the uniform tightness of the trait marginal of the process  $X^n$  and then to prove that in the limit, an averaging phenomenon appears for the age dynamics. Indeed, this "fast" evolving component stabilizes in an equilibrium that depends on the dynamics of the "slow" trait component.

We generalize to measure-valued processes, averaging techniques of Ball *et al.* [2], Kurtz [23] for diffusion processes. A specificity in our case is that the fast-scaling is related to time, since age is involved. In addition, notice that the competition between individuals creates a large dependence between the age and trait distributions. At our knowledge this dependence has never been investigated before in the literature.

Let us introduce the marginal  $\bar{X}_t^n(dx)$  of  $X_t^n(dx, da)$  defined for any bounded and measurable function  $f$  on  $\mathcal{X}$  and for any  $t \in \mathbb{R}_+$  by

$$\int_{\mathcal{X}} f(x) \bar{X}_t^n(dx) = \int_{\mathcal{X} \times \mathbb{R}_+} f(x) X_t^n(dx, da). \quad (3.1)$$

Our main result states the convergence of the sequence  $(\bar{X}^n)_{n \in \mathbb{N}^*}$  to a nonlinear super-process. The nonlinearity remains at the slow time scale in the growth rate, which is preserved in this asymptotics. Moreover, fast mutations are compensated by small mutation steps. Fast births and deaths provide stochastic fluctuations in the limit.

**Theorem 3.1.** *Assume Hypotheses 2.1 and 2.3, (2.10) and assume that there exists  $X_0 \in \mathcal{M}_F(\mathcal{X} \times \mathbb{R}_+)$  such that  $\lim_{n \rightarrow +\infty} X_0^n = X_0$  in  $(\mathcal{M}_F(\mathcal{X} \times \mathbb{R}_+), w)$ , the limit being in probability for the sake of simplicity.*

*For any  $x \in \mathcal{X}$ , let us introduce the age probability density*

$$\hat{m}(x, a) = \frac{\exp\left(-\int_0^a r(x, \alpha) d\alpha\right)}{\int_0^{+\infty} \exp\left(-\int_0^a r(x, \alpha) d\alpha\right) da}. \quad (3.2)$$

*Then, for each  $T > 0$ , the sequence  $(\bar{X}^n)_{n \in \mathbb{N}^*}$  converges in law in  $\mathbb{D}([0, T], (\mathcal{M}_F(\mathcal{X}), w))$  to the unique superprocess  $\bar{X} \in \mathcal{C}([0, T], \mathcal{M}_F(\mathcal{X}))$  such that for any function  $f \in \mathcal{D}(A)$ ,*

$$\begin{aligned} M_t^f = \langle \bar{X}_t, f \rangle - \langle \bar{X}_0, f \rangle - \int_0^t \int_{\mathcal{X}} \left( (\widehat{p r})(x) A f(x) \right. \\ \left. + [\widehat{b}(x) - (\widehat{d}(x) + \bar{X}_s \widehat{U}(x))] f(x) \right) \bar{X}_s(dx) ds \end{aligned} \quad (3.3)$$

*is a square integrable martingale with quadratic variation:*

$$\langle M^f \rangle_t = \int_0^t \int_{\mathcal{X}} 2\widehat{r}(x) f^2(x) \bar{X}_s(dx) ds. \quad (3.4)$$

*Here, any  $\widehat{\psi}(x)$  is defined for a bounded function  $\psi(x, a)$  by*

$$\widehat{\psi}(x) = \int_{\mathbb{R}_+} \psi(x, a) \hat{m}(x, a) da,$$

*and  $\bar{X}_t \widehat{U}(x)$  is given by*

$$\bar{X}_t \widehat{U}(x) = \int_{\mathcal{X}} \left( \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} U((x, a), (y, \alpha)) \hat{m}(y, \alpha) d\alpha \hat{m}(x, a) da \right) \bar{X}_t(dy).$$

□

Theorem 3.1 states that in the limit, an averaging phenomenon happens and the "fast" age component finally submits to the dynamics of the "slow" trait component. Since the fast-scaling (involving age) is related to the time, the stable age distribution  $\hat{m}(x, a) da$  given in (3.2) is obtained for each trait  $x$  as the long time limit in the age-structured population where all coefficients except  $r(x, a)$  are zero.

Before proving this slow-fast limit theorem, let us insist on the main difficulty created by the competition mechanism. Indeed, the branching property fails and it impedes the use of Laplace-transform techniques, as it had almost systematically been done in the past papers studying particle pictures with age-structures. Our model generalizes the age-structure population process studied in Athreya *et al.* [1], Bose and Kaj [3], in which birth and death rates are equal to a constant  $\lambda$ . In that case, the limiting behaviour of the renormalized critical birth and death process appears as a particular case of Theorem 3.1 with  $\hat{m}(x, a) da = \lambda e^{-\lambda a} da$ . In Dawson *et al.* [9], Dynkin [10], and Kaj and Sagitov [22], the age dependence is modelled through an additive functional of the motion process. In that way, the age "accumulates" along the lineage.

In our case, the age is set to zero at each birth, inducing a renewal phenomenon. The life-length does not have a fixed probability distribution anymore, unless there is no interaction. In [4], the authors consider a particle system with a different scaling, which favors large reproduction events. The limit in this case is not a superprocess anymore but behaves as the solution of a McKendrick-Forster equation perturbed by random immigration events created by the large rare birth events.

The proof of Theorem 3.1 is the aim of Section 3. We firstly establish the tightness of the sequence  $(\bar{X}^n)_{n \in \mathbb{N}^*}$  (Section 3.1). To identify its limiting values  $\bar{X}$ , we need an intermediary step. We consider the measures  $(X_t^n(dx, da)dt)_{n \in \mathbb{N}^*}$  and show that their limiting values are equal to  $\bar{X}_t(dx)\hat{m}(x, a)dadt$ . This implies that  $\bar{X}$  is solution of the martingale problem given in (3.3), (3.4). Uniqueness in this martingale problem allows us to deduce the convergence of  $(\bar{X}^n)_{n \in \mathbb{N}^*}$ .

### 3.1 Tightness of $(\bar{X}^n)_{n \in \mathbb{N}^*}$

In this subsection, we shall prove that:

**Proposition 3.2.** *Assume Hypotheses 2.1 and 2.3, and (2.10). If the sequence of laws of  $(\bar{X}_0^n)_{n \in \mathbb{N}^*}$  is uniformly tight in  $(\mathcal{M}_F(\mathcal{X}), w)$ , then the sequence  $(\mathcal{L}(\bar{X}^n))_{n \in \mathbb{N}^*}$  is uniformly tight in the space of probability measures on  $\mathbb{D}([0, T], (\mathcal{M}_F(\mathcal{X}), w))$ .*

*Proof.* Recall firstly that for a measurable and bounded function  $f$  on  $\mathcal{X}$ , the process

$$\begin{aligned} M_t^{n,f} &= \langle X_t^n, f \rangle - \langle X_0^n, f \rangle - \int_0^t \int_{\mathcal{X} \times \mathbb{R}_+} \left( (nr(x, a) + b(x, a)) \int_{\mathbb{R}^d} f(x+h) K^n(x, a, dh) \right. \\ &\quad \left. - (nr(x, a) + d(x, a) + X_s^n U(x, a)) f(x) \right) X_s^n(dx, da) ds \\ &= \langle X_t^n, f \rangle - \langle X_0^n, f \rangle - \int_0^t \int_{\mathcal{X} \times \mathbb{R}_+} \left( (b(x, a) - d(x, a) - X_s^n U(x, a)) f(x) \right. \\ &\quad \left. + (nr(x, a) + b(x, a)) p(x, a) \int_{\mathbb{R}^d} (f(x+h) - f(x)) \pi^n(x, dh) \right) X_s^n(dx, da) ds \end{aligned} \quad (3.5)$$

is a square integrable martingale started at 0 with quadratic variation

$$\begin{aligned} \langle M^{n,f} \rangle_t &= \frac{1}{n} \int_0^t \int_{\mathcal{X} \times \mathbb{R}_+} \left( (nr(x, a) + b(x, a)) \int_{\mathbb{R}^d} f^2(x+h) K^n(x, dh) \right. \\ &\quad \left. + (nr(x, a) + d(x, a) + X_s^n U(x, a)) f^2(x) \right) X_s^n(dx, da) ds \\ &= \frac{1}{n} \int_0^t \int_{\mathcal{X} \times \mathbb{R}_+} \left( (2nr(x, a) + b(x, a) + d(x, a) + X_s^n U(x, a)) f^2(x) \right. \\ &\quad \left. + (nr(x, a) + b(x, a)) p(x, a) \int_{\mathbb{R}^d} (f^2(x+h) - f(x))^2 \pi^n(x, dh) \right) X_s^n(dx, da) ds. \end{aligned} \quad (3.6)$$

We divide the proof into several steps.

**Step 1** Firstly, we prove the uniform tightness of  $(\mathcal{L}(\bar{X}^n))_{n \in \mathbb{N}^*}$  in the space of probability measures on  $\mathbb{D}([0, T], (\mathcal{M}_F(\mathcal{X}), v))$ .

Let us consider a continuous bounded function  $f \in \mathcal{D}(A)$ , and show the uniform tightness of the sequence  $(\langle \bar{X}_t^n, f \rangle)_{n \in \mathbb{N}^*}$  in  $\mathbb{D}([0, T], \mathbb{R})$ . We remark that for every fixed  $t \in [0, T]$  and  $n > 0$ ,

$$\mathbb{P}(|\langle \bar{X}_t^n, f \rangle| > k) \leq \frac{\|f\|_\infty \mathbb{E}(\sup_{t \in [0, T]} \langle \bar{X}_t^n, 1 \rangle)}{k}, \quad (3.7)$$

which tends to 0 as  $k$  tends to infinity (cf. (2.11)). This proves the tightness of the family of time-marginals  $(\langle \bar{X}_t^n, f \rangle)_{n \in \mathbb{N}^*}$ . Denoting by  $A^{n,f}$  the finite variation process in the r.h.s. of (3.5) and thanks to Assumption 2.3, we get for all stopping times  $S_n < T_n < (S_n + \delta) \wedge T$ , that

$$\begin{aligned} \mathbb{E}(|A_{T_n}^{n,f} - A_{S_n}^{n,f}|) &\leq \delta \left[ \left( (\|Af(s, \cdot)\|_\infty + 1) \frac{\bar{r}}{2} + \|f\|_\infty (\bar{b} + \bar{d}) \right) \sup_{n \in \mathbb{N}^*} \mathbb{E} \left( \sup_{t \in [0, T]} \langle X_t^n, 1 \rangle \right) \right. \\ &\quad \left. + \|f\|_\infty \bar{U} \sup_{n \in \mathbb{N}^*} \mathbb{E} \left( \sup_{t \in [0, T]} \langle X_t^n, 1 \rangle^2 \right) \right]. \end{aligned} \quad (3.8)$$

The quadratic variation process (3.6) satisfies a similar inequality:

$$\begin{aligned} &\mathbb{E}(|\langle M^{n,f} \rangle_{T_n} - \langle M^{n,f} \rangle_{S_n}|) \\ &\leq \|f\|_\infty^2 \mathbb{E} \left( \int_{S_n}^{T_n} \left[ 2\bar{r} \langle \bar{X}_s^n, 1 \rangle + \frac{\bar{b} \langle X_s^n, 1 \rangle}{n} + \frac{\bar{d} \langle \bar{X}_s^n, 1 \rangle + \bar{U} \langle \bar{X}_s^n, 1 \rangle^2}{n} \right] ds \right) \\ &\leq \|f\|_\infty^2 \delta \left[ \left( 2\bar{r} + \frac{\bar{b} + \bar{d}}{n} \right) \sup_{n \in \mathbb{N}^*} \mathbb{E} \left( \sup_{t \in [0, T]} \langle \bar{X}_t^n, 1 \rangle \right) + \frac{\bar{U}}{n} \sup_{n \in \mathbb{N}^*} \mathbb{E} \left( \sup_{t \in [0, T]} \langle \bar{X}_t^n, 1 \rangle^2 \right) \right]. \end{aligned} \quad (3.9)$$

Then, for  $\varepsilon > 0$ ,  $\eta > 0$ , a sufficiently large  $n$  and small  $\delta$ , we have using (2.11) and Assumption 2.3, that

$$\mathbb{P}(|A_{T_n}^{n,f} - A_{S_n}^{n,f}| > \eta) \leq \varepsilon \quad \text{and} \quad \mathbb{P}(|\langle M^{n,f} \rangle_{T_n} - \langle M^{n,f} \rangle_{S_n}| > \eta) \leq \varepsilon. \quad (3.10)$$

From (3.7), (3.8), (3.9) and the Aldous-Rebolledo criterion (see *e.g.* [20] or [11, Th. 1.17]), we obtain the uniform tightness of the sequence  $(\langle \bar{X}_t^n, f \rangle)_{n \in \mathbb{N}^*}$  in  $\mathbb{D}([0, T], \mathbb{R})$ . Thanks to Roelly's criterion [30], we conclude that  $(\bar{X}^n)_{n \in \mathbb{N}^*}$  is uniformly tight in  $\mathbb{D}([0, T], (\mathcal{M}_F(\mathcal{X}), v))$ .

Let us denote by  $\bar{X}$  a limiting process of  $(\bar{X}^n)_{n \in \mathbb{N}^*}$ . It is almost surely (a.s.) continuous in  $(\mathcal{M}_F(\mathcal{X}), v)$  since

$$\sup_{t \in \mathbb{R}_+} \sup_{f, \|f\|_\infty \leq 1} |\langle \bar{X}_t^n, f \rangle - \langle \bar{X}_{t-}, f \rangle| \leq \frac{1}{n}. \quad (3.11)$$

In the case where  $\mathcal{X}$  is a compact subset of  $\mathbb{R}^d$ , the vague and weak topologies coincide, which fails in the non-compact case. Nevertheless the tightness in  $\mathbb{D}([0, T], (\mathcal{M}_F(\mathcal{X}), w))$  is needed to identify the limiting values of  $(\bar{X}^n)_{n \in \mathbb{N}^*}$ .

**Step 2** Let us now concentrate on the case where  $\mathcal{X}$  is unbounded and let us show the tightness of  $(\bar{X}^n)_{n \in \mathbb{N}^*}$  in  $\mathbb{D}([0, T], (\mathcal{M}_F(\mathcal{X}), w))$ . The same computation as in Step 1 for  $f(x) = 1$  implies that the sequence  $(\langle \bar{X}_t^n, 1 \rangle)_{n \in \mathbb{N}^*}$  is uniformly tight in  $\mathbb{D}([0, T], \mathbb{R}_+)$ . As a consequence, it is possible to extract from  $(\bar{X}^n)_{n \in \mathbb{N}^*}$  a subsequence  $(\bar{X}^{u_n})_{n \in \mathbb{N}^*}$  such that:

- $(\bar{X}^{u_n})_{n \in \mathbb{N}^*}$  converges in distribution to  $\bar{X}$  in  $\mathbb{D}([0, T], (\mathcal{M}_F(\mathcal{X}), v))$ ,
- $(\langle \bar{X}^{u_n}, 1 \rangle)_{n \in \mathbb{N}^*}$  converges in distribution in  $\mathbb{D}([0, T], \mathbb{R}_+)$ .

Let us now show that the limit of  $(\langle \bar{X}^{u_n}, 1 \rangle)_{n \in \mathbb{N}^*}$  is  $\langle \bar{X}, 1 \rangle$ , empedding a loss of mass in the limit. Indeed, as a consequence, a criterion in Méléard and Roelly [25] will prove that  $(\bar{X}^{u_n})_{n \in \mathbb{N}^*}$  converges in distribution to  $\bar{X}$  in  $\mathbb{D}([0, T], (\mathcal{M}_F(\mathcal{X}), w))$ .

By simplicity, we will again denote  $u_n$  by  $n$ .

As in Jourdain-Méléard [21], we introduce a sequence of smooth functions  $\psi_k$  defined on  $\mathbb{R}_+$  and approximating  $\mathbf{1}_{\{u \geq k\}}$ . For  $k \in \mathbb{N}$ , let  $\psi_k(u) = \psi(0 \vee (u - (k - 1)) \wedge 1)$  where  $\psi(y) = 6y^5 - 15y^4 + 10y^3$  is a nondecreasing function such that  $\psi(0) = \psi'(0) = \psi''(0) = 1 - \psi(1) = \psi'(1) = \psi''(1) = 0$ . The function  $u \mapsto \psi_k(u)$  is nondecreasing on  $\mathbb{R}_+$ , equals 0 on  $[0, k - 1]$  and 1 on the complement

of  $[0, k]$ . In particular  $\psi_0 \equiv 1$ . Moreover the sequence  $(\psi_k)_{k \in \mathbb{N}^*}$  is nonincreasing, and satisfies for  $u \geq 0$  and  $p \geq 1$  that

$$\begin{aligned} \mathbf{1}_{\{u \geq k\}} &\leq \psi_k(u) \leq \mathbf{1}_{\{u \geq k-1\}}; \\ \psi_k^{(p)}(u) &\leq \sup_{u \in [k-1, k]} |\psi_k^{(p)}(u)| \mathbf{1}_{\{u \geq k-1\}} \leq \sup_{u \in [k-1, k]} |\psi_k^{(p)}(u)| \psi_{k-1}(u). \end{aligned} \quad (3.12)$$

The proof of the following lemma is postponed at the end of Proposition 3.2's proof. We define  $f_k(x) = \psi_k(\|x\|)$ , for all  $x \in \mathcal{X}$ .

**Lemma 3.3.** *Under the assumptions of Proposition 3.2,*

$$\lim_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mathbb{E} \left( \sup_{t \leq T} \langle \bar{X}_t^n, f_k \rangle \right) = 0.$$

From Lemma 3.3, we can deduce that

$$\lim_{k \rightarrow +\infty} \mathbb{E} \left( \sup_{t \leq T} \langle \bar{X}_t, f_k \rangle \right) = 0. \quad (3.13)$$

Indeed, for  $k \in \mathbb{N}$ , the continuous and compactly supported functions  $(f_{k,\ell} \stackrel{\text{def}}{=} f_k(1 - f_\ell))_{\ell \in \mathbb{N}}$  increase to  $f_k$  as  $\ell \rightarrow +\infty$ . By continuity of  $\nu \mapsto \sup_{t \leq T} \langle \nu_t, f_{k,\ell} \rangle$  on  $\mathbb{D}([0, T], (\mathcal{M}_F, v))$  and uniform integrability deduced from the uniform square moment estimates (2.11), one has

$$\mathbb{E} \left( \sup_{t \leq T} \langle \bar{X}_t, f_{k,\ell} \rangle \right) = \lim_{n \rightarrow +\infty} \mathbb{E} \left( \sup_{t \leq T} \langle \bar{X}_t^n, f_{k,\ell} \rangle \right) \leq \liminf_{n \rightarrow +\infty} \mathbb{E} \left( \sup_{t \leq T} \langle \bar{X}_t^n, f_k \rangle \right).$$

Taking the limit  $\ell \rightarrow +\infty$  in the left-hand-side by the monotone convergence theorem, one concludes that for  $k = 0$

$$\mathbb{E} \left( \sup_{t \leq T} \langle \bar{X}_t, 1 \rangle \right) = \mathbb{E} \left( \sup_{t \leq T} \langle \bar{X}_t, f_0 \rangle \right) < +\infty, \quad (3.14)$$

and from Lemma 3.3, that (3.13) holds for any  $k$ .

As a consequence one may extract a subsequence of  $(\sup_{t \leq T} \langle \bar{X}_t, f_k \rangle)_k$  that converges to 0 a.s., and since the process  $(\bar{X}_t)_{t \leq T}$  is continuous from  $[0, T]$  into  $(\mathcal{M}_F(\mathcal{X}), v)$ , one deduces that it is also continuous from  $[0, T]$  into  $(\mathcal{M}_F(\mathcal{X}), w)$ .

We can now prove the convergence of  $\langle X^{u_n}, 1 \rangle$  to  $\langle \bar{X}, 1 \rangle$ . For  $F$  a Lipschitz continuous and bounded function on  $\mathbb{D}([0, T], \mathbb{R})$ , we have

$$\begin{aligned} \limsup_{n \rightarrow +\infty} |\mathbb{E}(F(\langle \bar{X}^n, 1 \rangle) - F(\langle \bar{X}, 1 \rangle))| &\leq \limsup_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} |\mathbb{E}(F(\langle \bar{X}^n, 1 \rangle) - F(\langle \bar{X}^n, 1 - f_k \rangle))| \\ &\quad + \limsup_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} |\mathbb{E}(F(\langle \bar{X}^n, 1 - f_k \rangle) - F(\langle \bar{X}, 1 - f_k \rangle))| \\ &\quad + \limsup_{k \rightarrow +\infty} |\mathbb{E}(F(\langle \bar{X}, 1 - f_k \rangle) - F(\langle \bar{X}, 1 \rangle))|. \end{aligned}$$

Since  $|F(\langle \nu, 1 - f_k \rangle) - F(\langle \nu, 1 \rangle)| \leq C \sup_{t \leq T} \langle \nu_t, f_k \rangle$  by Lipschitz property, the first and the third terms in the r.h.s. are equal to 0 respectively according to Lemma 3.3 and to (3.13). The second term is 0 by continuity of  $\nu \mapsto \langle \nu, 1 - f_k \rangle$  in  $\mathbb{D}([0, T], (\mathcal{M}_F(\mathcal{X}), v))$ .

This ends the proof of Proposition 3.2. ■

*Proof of Lemma 3.3.* Firstly, let us show that for each  $t \in [0, T]$ ,

$$\lim_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mathbb{E}(\langle \bar{X}_t^n, f_k \rangle) = 0. \quad (3.15)$$

The boundedness of  $r$  and Assumption 2.3-2 ensure the existence of a sequence  $(\varepsilon_n)_{n \in \mathbb{N}^*}$  converging to 0 such that

$$\begin{aligned} \mathbb{E}(\langle \bar{X}_t^n, f_k \rangle) &\leq \mathbb{E}(\langle \bar{X}_0^n, f_k \rangle) + \bar{b} \int_0^t \mathbb{E}(\langle \bar{X}_s^n, f_k \rangle) ds + \varepsilon_n \int_0^t \mathbb{E}(\langle \bar{X}_s^n, 1 \rangle) ds \\ &\quad + \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}_+} r(x, a) p(x, a) A f_k(x) X_s^n(dx, da) ds \right), \end{aligned} \quad (3.16)$$

and we have by (2.6) and (3.12)

$$\left| \int_{\mathbb{R}^d \times \mathbb{R}_+} r(x, a) p(x, a) A f_k(x) X_s^n(dx, da) \right| \leq \bar{r} \sum_{|\ell| \leq \ell_0} \|D^\ell f_k\|_\infty \langle \bar{X}_s^n, f_{k-1} \rangle.$$

Since moreover, the sequence  $(f_k)_{k \in \mathbb{N}^*}$  is non-increasing,  $\langle \bar{X}_s^n, f_k \rangle \leq \langle \bar{X}_s^n, f_{k-1} \rangle$  and there is a constant  $C > 0$  independent of  $k \geq 2$  such that

$$\mathbb{E}(\langle \bar{X}_t^n, f_k \rangle) \leq \mathbb{E}(\langle \bar{X}_0^n, f_k \rangle) + C \int_0^t \mathbb{E}(\langle \bar{X}_s^n, f_{k-1} \rangle) ds + \varepsilon_n \int_0^t \mathbb{E}(\langle \bar{X}_s^n, 1 \rangle) ds. \quad (3.17)$$

Let  $\mu_s^{n,k} = \mathbb{E}(\langle \bar{X}_s^n, f_k \rangle) \leq \mu_s^n = \mathbb{E}(\langle \bar{X}_s^n, 1 \rangle)$  which is bounded uniformly in  $n \in \mathbb{N}^*$  and  $s \in [0, T]$  according to (2.11). There exist two positive constants  $C_1$  and  $C_2$  such that

$$\mu_t^{n,k} \leq \mu_0^{n,k} + C_1 \int_0^t \mu_s^{n,k-1} ds + C_2 \varepsilon_n.$$

Iteration of this inequality yields

$$\begin{aligned} \mu_t^{n,k} &\leq \sum_{\ell=0}^{k-1} \mu_0^{n,(k-\ell)} \frac{(C_1 t)^\ell}{\ell!} + \frac{(C_1 \int_0^t \mu_s^n ds)^k}{k!} + \varepsilon_n C_2 \sum_{\ell=0}^{k-1} \frac{(C_1 t)^\ell}{\ell!} \\ &\leq \mu_0^{n, \lfloor k/2 \rfloor} e^{C_1 t} + \mu_0^n \sum_{\ell=\lfloor k/2 \rfloor + 1}^{+\infty} \frac{(C_1 t)^\ell}{\ell!} + \frac{(C_1' t)^k}{(k)!} + \varepsilon_n C_2 e^{C_1 t}. \end{aligned}$$

where we used the monotonicity of  $\mu_0^{n,k}$  w.r.t.  $k$  for the second inequality. Given the moment condition (2.10), the assumption of tightness in  $(\mathcal{M}_F(\mathcal{X}), w)$  of the initial conditions  $(\bar{X}_0^n)_{n \in \mathbb{N}^*}$  is equivalent to

$$\lim_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mu_0^{n,k} = 0. \quad (3.18)$$

Hence

$$\lim_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mu_t^{n,k} \leq \sup_{n \in \mathbb{N}^*} \mu_0^n \lim_{k \rightarrow +\infty} \sum_{\ell=\lfloor k/2 \rfloor + 1}^{+\infty} \frac{(C_1 t)^\ell}{\ell!} + \lim_{k \rightarrow +\infty} \frac{(C_1' t)^k}{(k)!}.$$

We deduce immediately that

$$\lim_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mathbb{E}(\langle \bar{X}_t^n, f_k \rangle) = \lim_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mu_t^{n,k} = 0. \quad (3.19)$$

Let us now consider the martingale  $M_t^{n,k}$  defined by (3.5) with  $f_k$  instead of  $f$ , and with quadratic variation given in (3.6). Similar arguments as above allow us to prove that

$$\mathbb{E}(\langle M^{n,k} \rangle_t) \leq C_1 \int_0^t \mathbb{E}(\langle X_s^n, f_{k-1} \rangle) ds + \varepsilon_n C_2 \int_0^t \mathbb{E}(\langle X_s^n, 1 \rangle) ds.$$

Thus, using that  $f_k \leq 1$ , Doob's inequality, (3.15), (2.11) and the dominated convergence theorem, we get

$$\lim_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mathbb{E} \left( \sup_{t \leq T} |M_t^{n,k}| \right) = 0.$$

Let us now come back to the process  $\langle \bar{X}^n, f_k \rangle$ . As before, we can get

$$\begin{aligned} \langle \bar{X}_t^n, f_k \rangle &\leq \langle \bar{X}_0^n, f_k \rangle + M_t^{n,k} + \bar{b} \int_0^t \langle \bar{X}_s^n, f_k \rangle ds + \varepsilon_n \int_0^t \langle \bar{X}_s^n, 1 \rangle ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}_+} r(x, a) p(x, a) A f_k(x) X_s^n(dx, da) ds \\ &\leq \langle \bar{X}_0^n, f_k \rangle + M_t^{n,k} + C_1 \int_0^t \langle \bar{X}_s^n, f_{k-1} \rangle ds + \varepsilon_n C_2 \int_0^t \langle \bar{X}_s^n, 1 \rangle ds, \end{aligned} \quad (3.20)$$

for constants  $C_1$  and  $C_2$ . Let  $\alpha_t^{n,k} = \mathbb{E} \left( \sup_{s \leq t} \langle \bar{X}_s^n, f_k \rangle \right)$  and  $\alpha_t^n = \mathbb{E} \left( \sup_{s \leq t} \langle \bar{X}_s^n, 1 \rangle \right)$  which is bounded uniformly in  $n \in \mathbb{N}^*$  and  $t \in [0, T]$  according to (2.11). One deduces that

$$\alpha_t^{n,k} \leq \alpha_0^{n,k} + C_1 \int_0^t \mu_s^{n,k-1} ds + C_2 \varepsilon_n + \mathbb{E} \left( \sup_{t \leq T} |M_t^{n,k}| \right).$$

An iteration as before allows us to prove that

$$\lim_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \mathbb{E} \left( \sup_{t \leq T} \langle \bar{X}_t^n, f_k \rangle \right) = \lim_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \alpha_t^{n,k} = 0,$$

which concludes the proof of Lemma 3.3 and thus the one of Proposition 3.2.  $\blacksquare$

## 3.2 Identification of the limiting values

To obtain the convergence stated in Theorem 3.1, we show that the limiting value  $\bar{X}$  of the uniformly tight sequence  $(\bar{X}^n)_{n \in \mathbb{N}^*}$  is unique. We establish a martingale problem satisfied by  $\bar{X}$  in which there are integration terms with respect to the equilibrium (3.2) involved in the averaging phenomenon for the ages. The uniqueness of the solution to the martingale problem is then proved.

### 3.2.1 Averaging phenomenon

We begin with establishing the form of the limiting values of the time-marginal distributions  $(X_t^n(dx, da))_{n \in \mathbb{N}^*}$  for  $t \in [0, T]$ . Since the sequence  $(\bar{X}^n)_{n \in \mathbb{N}^*}$  is uniformly tight, there exists a subsequence of  $(X_t^n(dx, da))_{n \in \mathbb{N}^*}$ , with trait-marginals converging in law to a limiting value  $\bar{X}$ , that by simplicity, we denote again by  $(X_t^n(dx, da))_{n \in \mathbb{N}^*}$ .

We have already explained why the uniform tightness of the sequence  $(t \mapsto X_t^n(dx, da))_{n \in \mathbb{N}^*}$  in  $\mathbb{D}([0, T], \mathcal{M}_F(\mathcal{X} \times \mathbb{R}_+))$  cannot hold. However, following Kurtz [23], we will prove the uniform tightness of the sequence of random measures

$$\Gamma^n(dt, dx, da) = X_t^n(dx, da) dt \quad (3.21)$$

on  $\mathcal{M}_F([0, T] \times \mathcal{X} \times \mathbb{R}_+)$ . Proceeding in this way allows us to escape the difficulties created by the degeneracies due to the rapid time scale for age, when one tries to follow individual paths.

**Proposition 3.4.** *Under the assumptions of Theorem 3.1, the sequence  $(\Gamma^n)_{n \in \mathbb{N}^*}$  converges in law to  $\bar{X}_t(dx) \hat{m}(x, da) dt$  in  $\mathcal{M}_F([0, T] \times \mathcal{X} \times \mathbb{R}_+)$ .*

*As a consequence, for dt-almost every (a.e.)  $t \in [0, T]$ , the sequence  $(X_t^n(dx, da))_{n \in \mathbb{N}^*}$  converges weakly to  $\hat{m}(x, da) \bar{X}_t(dx)$ , with  $\hat{m}$  defined in (3.2).*

The proof of Proposition 3.4 is inspired by Kurtz [23]. To establish the result, we need to consider the random measure defined in (3.21).

Firstly, we prove the uniform tightness of the sequence  $(X_t^n(dx, da))_{n \in \mathbb{N}^*}$ , for fixed  $t \in [0, T]$  (Lemma 3.5), as well as the one of the sequence of measures  $(\Gamma^n)_{n \in \mathbb{N}^*}$  (Lemma 3.6), where the pathwise and individual points of view have been forgotten. The techniques to disentangle the traits and individuals' time scales appear strikingly in the proof of Lemma 3.5, where different treatments are used for the trait marginal and for the ages, with the introduction of the individuals' lifelengths. Then, in the proof of Proposition 3.4, a factor  $n$  appears in (3.39), when changing from the macroscopic scale to the microscopic scale. The next part of the proof consists in identifying the limiting martingale problem.

**Lemma 3.5.** *For  $dt$ -a.e.  $t \in [0, T]$ , the sequence  $(X_t^n)_{n \in \mathbb{N}^*}$  is uniformly tight on  $\mathcal{M}_F(\mathcal{X} \times \mathbb{R}_+)$ .*

*Proof.* Let  $\varepsilon > 0$ . Since the family  $(\bar{X}_t^n)_{n \in \mathbb{N}^*}$  is tight, there exists a compact set  $K \subset \mathbb{R}^d$  such that

$$\sup_{n \in \mathbb{N}^*} \mathbb{P}(\bar{X}_t^n(K^c) > \varepsilon) < \varepsilon. \quad (3.22)$$

Moreover, thanks to Point 2 of Assumption 2.1 and using a coupling argument, the life-lengths of the individuals in the population  $X_t^n$  born after time 0 are dominated, uniformly in  $x \in \mathcal{X}$ , by independent random variables  $D_i^n(t)$  with survival function  $S^n$  defined in (2.4). Because the aging velocity is  $n$ , the ages of these individuals satisfy  $A_i(t) \leq n D_i^n(t)$ . For an individual  $i$  alive at time 0, conditionally on the state at time 0, the remaining time it has to live can be dominated by an independent random variable  $\Delta_i^n$  such that

$$\mathbb{P}(\Delta_i^n > \ell) = \exp\left(-\int_0^\ell n \underline{r}(A_i(0) + nu) du\right).$$

Thus, for  $A > 0$  and  $n_0, N \in \mathbb{N}^*$ ,

$$\begin{aligned} \sup_{n \geq n_0} \mathbb{P}(X_t^n((K \times [0, A])^c) > 3\varepsilon) &\leq \sup_{n \geq n_0} \mathbb{P}(\bar{X}_t^n(K^c) > \varepsilon) + \sup_{n \geq n_0} \mathbb{P}\left(\frac{1}{n} \sum_{i=1}^{N_t^n} \mathbb{1}_{\{A_i(t) > A\}} > 2\varepsilon\right) \\ &\leq \varepsilon + \sup_{n \geq n_0} \mathbb{P}\left(\sum_{i=1}^{nN} \mathbb{1}_{\{n D_i^n(t) > A\}} > n\varepsilon\right) + \sup_{n \geq n_0} \mathbb{P}(N_t^n > nN) \\ &\quad + \sup_{n \geq n_0} \mathbb{P}\left(\sum_{i=1}^{N_0^n} \mathbb{1}_{\{A_i(0) + n\Delta_i^n > A\}} > n\varepsilon\right). \end{aligned} \quad (3.23)$$

By (2.10), it is possible to find  $N$  such that:

$$\sup_{n \geq n_0} \mathbb{P}(N_t^n > nN) = \sup_{n \geq n_0} \mathbb{P}(\langle X_t^n, 1 \rangle > N) \leq \frac{\sup_{n \geq n_0} \mathbb{E}(\sup_{t \in [0, T]} \langle X_t^n, 1 \rangle)}{N} \leq \varepsilon. \quad (3.24)$$

Let us consider the second term of (3.23). Notice that

$$\mathbb{E}(\mathbb{1}_{\{n D_i^n(t) > A\}}) = S^n(A/n) = \exp\left(-\int_0^A \underline{r}(a) da\right),$$

which converges to 0 when  $A$  tends to infinity by (2.3). For  $N$  as in (3.24), there hence exists

$A_1$  sufficiently large so that for  $A > A_1$ ,  $S^n(A/n) < \varepsilon/2N$ . Then

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{nN} \mathbb{1}_{\{D_i^n(t) > A/n\}} > n\varepsilon\right) &= \mathbb{P}\left(\sum_{i=1}^{nN} (\mathbb{1}_{\{D_i^n(t) > A/n\}} - S^n(A/n)) > n(\varepsilon - NS^n(A/n))\right) \\ &\leq \mathbb{P}\left(\sum_{i=1}^{nN} (\mathbb{1}_{\{D_i^n(t) > A/n\}} - S^n(A/n)) > n\varepsilon/2\right) \leq \exp\left(-\frac{n\varepsilon^2}{8(Ne^{-\int_0^A \underline{r}(a)da}(1 - e^{-\int_0^A \underline{r}(a)da}) + \varepsilon/3)}\right) \end{aligned} \quad (3.25)$$

by Bernstein's inequality (*e.g.* [32] p.855). For a sufficiently large  $n_1$  and for  $n \geq n_1$ , the r.h.s. of (3.25) is smaller than  $\varepsilon$ .

Let us now upper bound the last term of (3.23). Let  $A \mapsto \phi(A)$  be such that  $\phi(A)$  and  $\int_{\phi(A)}^A \underline{r}(u)du$  tend to infinity when  $A$  tends to infinity. We firstly notice that:

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{N_0^n} \mathbb{E}(\mathbb{1}_{\{A_i(0) + n\Delta_i^n > A\}} | X_0^n) &= \frac{1}{n} \sum_{i=1}^{N_0^n} 1 \wedge e^{-\int_{A_i(0)}^A \underline{r}(u)du} = \int_{\mathcal{X} \times \mathbb{R}_+} 1 \wedge e^{-\int_a^A \underline{r}(u)du} X_0^n(dx, da) \\ &\leq X_0^n(\mathcal{X} \times [\phi(A), +\infty)) + \langle X_0^n, 1 \rangle e^{-\int_{\phi(A)}^A \underline{r}(u)du}. \end{aligned} \quad (3.26)$$

Recall that  $X_0$  is the limit, in probability and for the weak convergence, of  $(X_0^n)_{n \in \mathbb{N}^*}$ . There hence exists  $n_2(A)$  sufficiently large such that for  $n \geq n_2(A)$ , the right hand side of (3.26) is smaller than

$$X_0(\mathcal{X} \times [\phi(A), +\infty)) + \langle X_0, 1 \rangle e^{-\int_{\phi(A)}^A \underline{r}(u)du} + \frac{\varepsilon}{4}$$

with probability  $1 - \varepsilon$ . By choice of  $\phi(A)$ , there exists  $A_2$  sufficiently large such that for  $A > A_2$ , and  $n \geq n_2(A_2)$  this upper bound is smaller than  $\varepsilon/2$ . For such  $A$  and  $n$ , we have

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{N_0^n} \mathbb{1}_{\{A_i(0) + n\Delta_i^n > A\}} > n\varepsilon\right) &\leq \mathbb{E}\left(\mathbb{P}\left(\frac{1}{n} \sum_{i=1}^{N_0^n} (\mathbb{1}_{\{A_i(0) + n\Delta_i^n > A\}} - 1 \wedge e^{-\int_{A_i(0)}^A \underline{r}(u)du}) > \frac{\varepsilon}{2} | X_0^n\right)\right) + \varepsilon \\ &\leq \mathbb{E}\left(\exp\left(-\frac{n\varepsilon^2}{2\langle X_0^n, 1 \rangle + 8\varepsilon/3}\right)\right) + \varepsilon. \end{aligned} \quad (3.27)$$

by applying Bernstein's inequality again. There exists  $n_3$  sufficiently large so that for every  $n \geq n_3$ , the right hand side of (3.27) is smaller than  $2\varepsilon$ .

The tightness of  $(X_t^n)_{n \in \mathbb{N}^*}$  is thus a consequence of (3.23), (3.24), (3.25) and (3.27), with the choices of  $A \geq \max(A_1, A_2)$  and  $n_0 \geq \max(n_1, n_2, n_3)$ .  $\blacksquare$

**Lemma 3.6.** *The family  $(\Gamma^n)_{n \in \mathbb{N}}$  is tight in  $\mathcal{M}_F([0, T] \times \mathcal{X} \times \mathbb{R}_+)$ .*

*Proof.* Following Kurtz [23, Lemma 1.3], a sufficient condition for the tightness of the family  $(\Gamma^n)_{n \in \mathbb{N}}$  is that for all  $\varepsilon > 0$ , there exists a compact set  $\Xi$  of  $\mathcal{X} \times \mathbb{R}_+$  such that

$$\sup_{n \in \mathbb{N}^*} \mathbb{E}\left(\Gamma^n([0, T] \times \Xi^c)\right) \leq C(T)\varepsilon. \quad (3.28)$$

Let us establish (3.28). From the proof of Lemma 3.5, it appears that the upperbounds (3.22), (3.23) and (3.25) are uniform in  $t \in [0, T]$  so that:

$$\sup_{t \in [0, T]} \sup_{n \in \mathbb{N}^*} \mathbb{P}\left(X_t^n((K \times [0, A])^c) > 3\varepsilon\right) < 5\varepsilon. \quad (3.29)$$

We are now ready to upperbound

$$\mathbb{E}\left(\Gamma^n([0, T] \times (K \times [0, A])^c)\right) = \mathbb{E}\left(\int_0^T \langle X_t^n, \mathbb{1}_{(K \times [0, A])^c} \rangle dt\right) = \int_0^T \mathbb{E}\left(X_t^n((K \times [0, A])^c)\right) dt.$$

Indeed:

$$\begin{aligned} \mathbb{E}\left(X_t^n((K \times [0, A])^c)\right) &\leq 3\varepsilon \mathbb{P}\left(X_t^n((K \times [0, A])^c) \leq 3\varepsilon\right) + \mathbb{E}\left(\langle X_t^n, 1 \rangle \mathbb{1}_{X_t^n((K \times [0, A])^c) > 3\varepsilon}\right) \\ &\leq 3\varepsilon + \sqrt{\mathbb{E}\left(\langle X_t^n, 1 \rangle^2\right)} \sqrt{\mathbb{P}\left(X_t^n((K \times [0, A])^c) > 3\varepsilon\right)} \leq C(T)(\varepsilon + \sqrt{\varepsilon}), \end{aligned} \quad (3.30)$$

by Cauchy-Schwarz inequality and (2.11). This proves (3.28) and finishes the proof.  $\blacksquare$

Before proving Proposition 3.4, we provide a lemma characterizing  $\widehat{m}(x, a)$ .

**Lemma 3.7.** *Let  $x \in \mathcal{X}$  be fixed. There exists a unique probability measure  $\widehat{m}(x, da)$  on  $\mathbb{R}_+$ , solution of the following equation: For  $\psi \in \mathcal{C}_c^1(\mathbb{R}_+, \mathbb{R})$  with compact support in  $[0, +\infty)$ ,*

$$\int_{\mathbb{R}_+} \partial_a \psi(a) \widehat{m}(x, da) = \int_{\mathbb{R}_+} \psi(a) r(x, a) \widehat{m}(x, da) - \psi(0) \int_{\mathbb{R}_+} r(x, a) \widehat{m}(x, da). \quad (3.31)$$

The probability measure  $\widehat{m}(x, da)$  is absolutely continuous with respect to the Lebesgue measure and its density is given in (3.2).

*Proof.* Let us consider the test function  $\psi(a) = \int_a^{+\infty} f(\alpha) d\alpha$ , where  $f \in \mathcal{C}_c(\mathbb{R}_+, \mathbb{R}_+)$  is positive. Then  $\partial_a \psi(a) = -f(a)$  and  $\psi(0) = \int_0^{+\infty} f(\alpha) d\alpha$ . Equation (3.31) gives by Fubini's theorem:

$$\begin{aligned} \int_{\mathbb{R}_+} f(a) \widehat{m}(x, da) &= \int_{\mathbb{R}_+} \int_0^a f(\alpha) d\alpha r(x, a) \widehat{m}(x, da) \\ &= \int_{\mathbb{R}_+} f(\alpha) \int_{\alpha}^{+\infty} r(x, a) \widehat{m}(x, da) d\alpha. \end{aligned} \quad (3.32)$$

This entails that  $\widehat{m}(x, da)$  is absolutely continuous with respect to the Lebesgue measure with density  $\widehat{m}(x, a) = \int_a^{+\infty} r(x, \alpha) \widehat{m}(x, \alpha) d\alpha$ . The latter implies that  $a \mapsto \widehat{m}(x, a)$  is a function of class  $\mathcal{C}^1$ . Using further an integration by part in (3.31), and since  $\widehat{m}(x, a)$  tends to 0 when  $a$  grows to infinity, we get for all  $\psi \in \mathcal{C}_c^1(\mathbb{R}_+, \mathbb{R})$

$$-\psi(0) \widehat{m}(x, 0) - \int_{\mathbb{R}_+} \psi(a) \partial_a \widehat{m}(x, a) da = \int_{\mathbb{R}_+} (\psi(a) - \psi(0)) r(x, a) \widehat{m}(x, a) da. \quad (3.33)$$

By identification, we obtain that  $\widehat{m}(x, a)$  is a solution of

$$\begin{aligned} \partial_a \widehat{m}(x, a) &= -r(x, a) \widehat{m}(x, a) \\ \widehat{m}(x, 0) &= \int_{\mathbb{R}_+} r(x, a) \widehat{m}(x, a) da, \end{aligned} \quad (3.34)$$

which is solved by

$$\widehat{m}(x, a) = \widehat{m}(x, 0) \exp\left(-\int_0^a r(x, \alpha) d\alpha\right). \quad (3.35)$$

Since  $\widehat{m}(x, a) da$  is a probability measure, necessarily

$$\widehat{m}(x, 0) = \int_{\mathbb{R}_+} r(x, a) \widehat{m}(x, a) da = \frac{1}{\int_{\mathbb{R}_+} \exp\left(-\int_0^a r(x, \alpha) d\alpha\right) da}. \quad (3.36)$$

This provides existence and uniqueness of the solution of (3.34) and hence of (3.31).  $\blacksquare$

**Remark 3.8.** Notice that the system (3.34) defines the stable age equilibrium of the McKendrick-Von Foerster equation [24, 17] (see also [36]) when the birth and death rates equal to  $r(x, a)$  and the trait  $x$  is fixed.  $\square$

**Remark 3.9.** The space  $\mathcal{C}_c^1(\mathbb{R}_+, \mathbb{R})$  is separable and there exists a denumerable dense family  $(\psi_k)_{k \in \mathbb{N}}$  in this set. To obtain the result of Lemma 3.7, it is sufficient to have (3.31) for these functions  $(\psi_k)_{k \in \mathbb{N}}$ .

We are now able to prove Proposition 3.4.

*Proof of Proposition 3.4.* From (3.21), we can see that the marginal measure of  $\Gamma^n(ds, dx, da)$  on  $[0, T] \times \mathcal{X}$  is  $\bar{X}_s^n(dx)ds$ . For any real bounded test function  $\varphi : (s, x) \mapsto \varphi_s(x)$  on  $[0, T] \times \mathcal{X}$ ,

$$\int_0^t \int_{\mathcal{X} \times \mathbb{R}_+} \varphi_s(x) \Gamma^n(ds, dx, da) = \int_0^t \langle \bar{X}_s^n, \varphi_s \rangle ds. \quad (3.37)$$

The sequence  $(\bar{X}^n)_{n \in \mathbb{N}^*}$  is uniformly tight by Proposition 3.2, as well as  $(\Gamma^n)_{n \in \mathbb{N}^*}$ , by Lemma 3.6 (ii). Using Prohorov's theorem, we thus deduce that  $(\Gamma^n(ds, dx, da), \bar{X}_s^n(dx)ds)_n$  is relatively compact and there exists a subsequence that converges in distribution to a limiting value, say  $(\Gamma(ds, dx, da), \bar{X}_s(dx)ds)$ . Taking (3.37) to the limit, we obtain that  $\bar{X}_s(dx)ds$  is necessarily the marginal measure of  $\Gamma(ds, dx, da)$  on  $[0, T] \times \mathcal{X}$  up to a null-measure set. We deduce from this (*e.g.* Lemma 1.4 of Kurtz [23]) that there exists a (random) probability-valued process  $(\gamma_{s,x}(da), s \in [0, T], x \in \mathcal{X})$  that is predictable in  $(\omega, s)$  and such that for all bounded measurable function  $\varphi(s, x, a)$  on  $[0, T] \times \mathcal{X} \times \mathbb{R}_+$ ,

$$\int_0^t \int_{\mathcal{X} \times \mathbb{R}_+} \varphi(s, x, a) \Gamma(ds, dx, da) = \int_0^t \int_{\mathcal{X}} \int_{\mathbb{R}_+} \varphi(s, x, a) \gamma_{s,x}(da) \bar{X}_s(dx) ds. \quad (3.38)$$

We now want to characterize the limiting value  $\Gamma(ds, dx, da) = \gamma_{s,x}(da) \bar{X}_s(dx)ds$ . Applying (2.12) for a test function  $\varphi(x, a) \in \mathcal{C}_b^{0,1}(\mathcal{X} \times \mathbb{R}_+, \mathbb{R})$  and dividing by  $n$  gives that:

$$\begin{aligned} \frac{M_t^{n,\varphi}}{n} &= \frac{\langle X_t^n, \varphi \rangle - \langle X_0^n, \varphi \rangle}{n} - \int_0^t \int_{\mathcal{X} \times \mathbb{R}_+} \left[ \partial_a \varphi(x, a) + r(x, a) (\varphi(x, 0) - \varphi(x, a)) \right. \\ &\quad \left. + p(x, a) \left( r(x, a) + \frac{b(x, a)}{n} \right) \int_{\mathbb{R}^d} (\varphi(x+h, 0) - \varphi(x, 0)) \pi^n(x, a, dh) \right. \\ &\quad \left. + \frac{b(x, a)}{n} \varphi(x, 0) - \left( \frac{d(x, a) + X_s^n U(x, a)}{n} \right) \varphi(x, a) \right] \Gamma^n(ds, dx, da) \end{aligned} \quad (3.39)$$

is a martingale. Using (2.11), we can easily prove that  $(M_t^{n,\varphi}/n)_{n \in \mathbb{N}^*, t \in [0, T]}$  is uniformly integrable and that

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left( \left| \frac{M_t^{n,\varphi}}{n} - \widetilde{M}_t^\varphi \right| \right) = 0,$$

where

$$\widetilde{M}_t^\varphi = \int_0^t \int_{\mathcal{X} \times \mathbb{R}_+} \left[ \partial_a \varphi(x, a) + r(x, a) (\varphi(x, 0) - \varphi(x, a)) \right] \gamma_{s,x}(da) \bar{X}_s(dx) ds. \quad (3.40)$$

Moreover, the uniform integrability of  $(M_t^{n,\varphi}/n)$  also provides that the process  $(\widetilde{M}_t^\varphi)_t$  is a martingale. As it is also a continuous and finite variation process, it must hence be almost surely zero. Since this holds for every  $t \in \mathbb{R}_+$ , we have proved that a.s.,  $dt$ -a.e.

$$\int_{\mathcal{X} \times \mathbb{R}_+} \left[ \partial_a \varphi(x, a) + r(x, a) (\varphi(x, 0) - \varphi(x, a)) \right] \gamma_{t,x}(da) \bar{X}_t(dx) = 0. \quad (3.41)$$

Choosing  $\varphi(x, a) = \phi_\ell(x)\psi_k(a)$  with  $(\phi_\ell)_{\ell \in \mathbb{N}}$  and  $(\psi_k)_{k \in \mathbb{N}}$  dense families in  $\mathcal{C}_c(\mathcal{X}, \mathbb{R})$  and  $\mathcal{C}_c^1(\mathbb{R}_+, \mathbb{R})$ , respectively, we obtain from (3.41) that, for all  $\ell, k \in \mathbb{N}$ ,

$$\int_{\mathcal{X}} \phi_\ell(x) H_k(t, x) \bar{X}_t(dx) = 0$$

where  $H_k(t, x) = \int_{\mathbb{R}_+} \left[ \partial_a \psi_k(a) + r(x, a)(\psi_k(0) - \psi_k(a)) \right] \gamma_{t,x}(da)$ . Almost surely, the function  $H_k(t, x)$  is bounded and is thus  $dt$ -a.e.  $\bar{X}_t(dx)$ -integrable. We obtain that for all  $k \in \mathbb{N}$ , a.s.,  $dt$ -a.e.,  $\bar{X}_t(dx)$ -a.e.,

$$\int_{\mathbb{R}_+} \left[ \partial_a \psi_k(a) + r(x, a)(\psi_k(0) - \psi_k(a)) \right] \gamma_{t,x}(da) = 0. \quad (3.42)$$

By Lemma 3.7 and Remark 3.9, we deduce that a.s.,  $dt$ -a.e., and  $\bar{X}_t(dx)$ -a.e.,  $\gamma_{t,x}(da) = \hat{m}(x, a)da$  and as a consequence, any limiting value of  $(X_t^n)_{n \in \mathbb{N}^*}$  is of the form  $\bar{X}_t(dx) \otimes \hat{m}(x, a)da$ .  $\blacksquare$

### 3.2.2 Characterization of the limiting values

In the previous sections, we have proved that the sequence  $(\bar{X}^n)_{n \in \mathbb{N}^*}$  is tight and that for a given limiting value  $\bar{X}$ , the associated subsequence  $(\Gamma^n(dt, dx, da) = X_t^n(dx, da) dt)_{n \in \mathbb{N}^*}$  converges in  $(\mathcal{M}_F([0, T] \times \mathcal{X} \times \mathbb{R}_+), w)$  to  $\bar{X}_t(dx) \hat{m}(x, da) dt$ . Now, we are ready to prove that:

**Lemma 3.10.** *The limiting values  $\bar{X}$  of the sequence  $(\bar{X}^n)_{n \in \mathbb{N}^*}$  are solution of the martingale problem (3.3)-(3.4).*

*Proof.* Let  $0 < s_1 \leq \dots \leq s_k < s < t$ , and let us introduce for  $Y \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}_F(\mathcal{X}))$ :

$$\Psi_{s,t}(Y) = \phi_1(Y_{s_1}) \dots \phi_k(Y_{s_k}) \left\{ \langle Y_t, f \rangle - \langle Y_s, f \rangle - \int_s^t du \int_{\mathcal{X}} Y_u(dx) \left[ \widehat{(pr)}(x) Af(x) + (\widehat{b}(x) - \widehat{d}(x) - Y_s \widehat{U}(x)) f(x) \right] \right\}, \quad (3.43)$$

where  $\phi_1, \dots, \phi_k$  are bounded continuous functions on  $\mathcal{M}_F(\mathcal{X})$  and  $f \in \mathcal{D}(A)$ . Our purpose is to prove that  $\mathbb{E}(\Psi_{s,t}(\bar{X})) = 0$  for any limiting value  $\bar{X}$  of  $(\bar{X}^n)_{n \in \mathbb{N}^*}$ .

Let  $\bar{X}$  be a limiting value of  $(\bar{X}^n)_{n \in \mathbb{N}^*}$  and let  $(\bar{X}^{u_n})_{n \in \mathbb{N}^*}$  be a subsequence converging to  $\bar{X}$ . On the one hand, thanks to Proposition 3.4, (2.11) and (3.11):

$$\begin{aligned} \mathbb{E}(\Psi_{s,t}(\bar{X})) &= \lim_{n \rightarrow +\infty} \mathbb{E} \left( \phi_1(\bar{X}_{s_1}^{u_n}) \dots \phi_k(\bar{X}_{s_k}^{u_n}) \left\{ \langle \bar{X}_t^{u_n}, f \rangle - \langle \bar{X}_s^{u_n}, f \rangle - \int_s^t du \int_{\mathcal{X} \times \mathbb{R}_+} X_u^{u_n}(dx, da) \left[ \right. \right. \right. \\ &\quad \left. \left. \left. p(x, a) r(x, a) Af(x) + (b(x, a) - d(x, a) - \int_{\mathcal{X} \times \mathbb{R}_+} U((x, a), (y, \alpha)) X_u^{u_n}(dy, d\alpha)) f(x) \right] \right\} \right). \end{aligned} \quad (3.44)$$

On the other hand, the term under the expectation in the r.h.s. of (3.44) equals:

$$\phi_1(X_{s_1}^{u_n}) \dots \phi_k(X_{s_k}^{u_n}) \left\{ M_t^{u_n, f} - M_s^{u_n, f} + A_{u_n} + B_{u_n} \right\}, \quad (3.45)$$

where  $M^{u_n, f}$  has been defined in (3.5) and where:

$$\begin{aligned} A_{u_n} &= \int_s^t du \int_{\mathcal{X} \times \mathbb{R}_+} X_u^{u_n}(dx, da) r(x, a) \left[ u_n \int_{\mathcal{X}} (f(x+h) - f(x)) K^{u_n}(x, a, dh) - p(x, a) Af(x) \right] \\ &= \int_s^t du \int_{\mathcal{X} \times \mathbb{R}_+} X_u^{u_n}(dx, da) r(x, a) p(x, a) \left[ u_n \int_{\mathcal{X}} (f(x+h) - f(x)) \pi^{u_n}(x, dh) - Af(x) \right] \\ B_{u_n} &= \int_s^t du \int_{\mathcal{X} \times \mathbb{R}_+} X_u^{u_n}(dx, da) b(x, a) \left[ \int_{\mathbb{R}^d} f(x+h) K^{u_n}(x, a, dh) - f(x) \right] \end{aligned}$$

Firstly, using (2.11) and the fact that the process  $M^{n, f}$  is a martingale we obtain that:

$$\mathbb{E}(\phi_1(X_{s_1}^{u_n}) \dots \phi_k(X_{s_k}^{u_n}) [M_t^{u_n, f} - M_s^{u_n, f}]) = 0. \quad (3.46)$$

Secondly, from Assumption 2.3,  $|\int_{\mathbb{R}^d} f(x+h) K^n(x, a, dh) - f(x)| = o(1/n)$  and using (2.11) again provides:

$$\lim_{n \rightarrow +\infty} \mathbb{E}(\phi_1(X_{s_1}^{u_n}) \dots \phi_k(X_{s_k}^{u_n}) [A_{u_n} + B_{u_n}]) = 0. \quad (3.47)$$

From (3.44), (3.45), (3.46) and (3.47), we deduce that  $\mathbb{E}(\Psi_{s,t}(\bar{X})) = 0$  and hence  $(M_t^f)_{t \in \mathbb{R}_+}$  defined in (3.3) is a martingale. From this, using Itô's formula with localization arguments and Proposition 2.6 (i), we obtain that the following process is a martingale:

$$\begin{aligned} \langle \bar{X}_t, f \rangle^2 - \langle \bar{X}_0, f \rangle^2 - \int_0^t 2 \langle \bar{X}_s, f \rangle \int_{\mathcal{X}} \left[ \widehat{(pr)}(x) Af(x) + \widehat{b}(x) \right. \\ \left. - \widehat{d}(x) - \bar{X}_s U(x) \right] f(x) \bar{X}_s(dx) ds - \langle M^f \rangle_t \end{aligned} \quad (3.48)$$

is a martingale. Moreover, using the results of Proposition 2.6, we obtain that the following process is a martingale:

$$\begin{aligned} \langle \bar{X}_t^n, f \rangle^2 - \langle \bar{X}_0^n, f \rangle^2 \\ - \int_0^t \left[ 2 \langle \bar{X}_s^n, f \rangle \int_{\mathcal{X} \times \mathbb{R}_+} \left( nr(x, a) \int_{\mathbb{R}^d} (f(x+h) - f(x)) K^n(x, a, dh) \right. \right. \\ \left. \left. + b(x, a) \int_{\mathbb{R}^d} f(x+h) K^n(x, a, dh) - (d(x, a) + X_s^n U(x, a)) f(x) \right) X_s^n(dx, da) \right] ds \\ - \int_0^t \int_{\mathcal{X} \times \mathbb{R}_+} \left[ r(x, a) \left( \int_{\mathbb{R}^d} f^2(x+h) K^n(x, a, dh) + f^2(x) \right) \right. \\ \left. + \frac{b(x, a)}{n} \int_{\mathbb{R}^d} f^2(x+h) K^n(x, a, dh) + \frac{d(x, a) + X_s^n U(x, a)}{n} f^2(x) \right] X_s^n(dx, da) ds. \end{aligned}$$

By using arguments similar as those in the beginning of the proof, we deduce that

$$\begin{aligned} \langle \bar{X}_t, f \rangle^2 - \langle \bar{X}_0, f \rangle^2 - \int_0^t 2 \langle \bar{X}_s, f \rangle \int_{\mathcal{X}} \left[ \widehat{(pr)} Af(x) \right. \\ \left. + \widehat{b}(x) - d(x) - \bar{X}_s U(x) \right] f(x) \bar{X}_s(dx) ds - \int_0^t \int_{\mathcal{X}} 2 \widehat{r}(x) f^2(x) \bar{X}_s(dx) ds \end{aligned} \quad (3.49)$$

is a martingale. Comparing (3.48) and (3.49) yields the bracket of the martingale  $M^f$ . This ends the proof.  $\blacksquare$

### 3.2.3 Uniqueness of the martingale problem

We have shown that the limiting values of the uniformly tight sequence  $(\bar{X}^n)_{n \in \mathbb{N}^*}$  satisfy the martingale problem (3.3)-(3.4). To conclude the proof of Theorem 3.1, it remains to prove the uniqueness of the solution of this martingale problem.

**Proposition 3.11.** *There is a unique solution to the martingale problem of Theorem 3.1.*

*Proof.* We start with getting rid of the non-linearity by using Girsanov's formula (see Dawson [8] Theorem 7.2.2). There exists a probability measure  $\mathbb{Q}$  on the path space such that for all  $f \in \mathcal{D}(A)$ :

$$\begin{aligned} \widetilde{M}_t^f &= M_t^f + \int_0^t \int_{\mathbb{R}^d} \left( \widehat{b}(x) - \widehat{d}(x) - \bar{X}_s U(x) \right) f(x) \bar{X}_s(dx) ds \\ &= \langle \bar{X}_t, f \rangle - \langle \bar{X}_0, f \rangle - \int_0^t \int_{\mathbb{R}^d} \widehat{(pr)}(x) A f(x) \bar{X}_s(dx) ds \end{aligned} \quad (3.50)$$

is a square integrable martingale with bracket (3.4).

The uniqueness of the solution the martingale problem (3.50)-(3.4) is proved by Roelly and Rouault [31]. It is based on the branching property of  $\bar{X}$  under  $\mathbb{Q}$  which allows us to characterize the Laplace functional  $L_t f$  of  $X_t$  by its cumulant  $U_t f$ :

$$L_t(f) = \mathbb{E} \left( e^{\langle \bar{X}, f \rangle} \right) = \mathbb{E} \left( e^{\langle \bar{X}_0, U_t f \rangle} \right). \quad (3.51)$$

The latter is the unique positive solution of the following PDE:

$$\frac{\partial u}{\partial t}(t, x) = Au(t, x) - \widehat{r}(x)u^2(t, x), \quad u(0, x) = f(x), \quad (3.52)$$

(see *e.g.* Pazy [28, Th. 1.4 and 1.5 p.185 and 187]).

From the uniqueness of the solution of (3.50)-(3.4), we deduce classically the uniqueness of the solution of (3.3)-(3.4). (See for example Evans-Perkins [12], Fitzsimmons [15]).  $\blacksquare$

The proof of Theorem 3.1 is now complete.

## 4 Examples

Let us develop and compare two examples, which only differ by the function  $r(x, a)$ .

### 4.1 Logistic physical-age and size-structured population

In Méléard and Tran [26], the following example for a population structured by age  $a \in \mathbb{R}_+$  and size  $x \in \mathcal{X} = [0, x_0]$  is considered:

$$\begin{aligned} b(x, a) &= x(x_0 - x)e^{-a} \mathbb{1}_{[0, x_0]}(x) \text{ for } x_0 > 0, \\ d(x, a) &= d_0, \quad U((x, a), (y, \alpha)) = \eta(x_0 - x), \end{aligned} \quad (4.1)$$

with  $x_0 = 4$ ,  $d_0 = 1/4$  and  $\eta = 1.7$ . Because reproduction needs energy, and since this energy depends on the size of the created offspring, very small or big individuals are disadvantaged. Individuals of intermediate size  $x = 2$  have the highest birth rate. The competition term in contrast favors bigger individuals. Hence there is a trade-off between competitiveness and reproduction. The decreasing exponential in age describes a senescence phenomenon: older

individuals reproduce less than their young competitors. In [26], partial differential equation limits, Trait substitution sequence and Canonical equations are considered. Here we consider the superprocess approximation described in the above sections, with  $r(x, a) = 1$  and  $\pi^n(x, dh)$  a centered Gaussian kernel with variance  $\frac{\sigma^2}{n}$  conditioned on  $[0, x_0]$ , as in Example 2.5.

Computation gives  $\widehat{m}(x, a) = e^{-a}$  so that  $X_t(dx, da) = \bar{X}_t(dx) \otimes e^{-a} da$  becomes in this particular case a product measure. As soon as the population survives, the age distribution "stabilizes" around an exponential distribution with parameter 1, as seen on the simulations of Figure 1. With the age distribution  $\widehat{m}(x, a) = e^{-a}$ , we get

$$\widehat{b}(x) = x(x_0 - x) \int_{\mathbb{R}_+} e^{-2a} da = \frac{x(x_0 - x)}{2} \quad ; \quad \widehat{d}(x) = d_0 \quad \widehat{U}(x, y) = \eta(x_0 - x). \quad (4.2)$$

The martingale problem (3.3) becomes here:

$$\begin{aligned} M_t^f &= \langle \bar{X}_t, f \rangle - \langle \bar{X}_0, f \rangle - \int_0^t \int_{\mathcal{X}} \left( p \frac{\sigma^2}{2} \Delta f(x) \right. \\ &\quad \left. + \left[ \frac{x(x_0 - x)}{2} - (d_0 + \eta(x_0 - x) \langle \bar{X}_s, 1 \rangle) \right] f(x) \right) \bar{X}_s(dx) ds, \\ \langle M^f \rangle_t &= \int_0^t \int_{\mathcal{X}} 2f^2(x) \bar{X}_s(dx) ds. \end{aligned} \quad (4.3)$$

In Figure 1, two sets of simulations are presented, depending on two different mutation variances  $\sigma^2$ . As expected, when  $\sigma$  increases, the traits vary more rapidly, and the irregularity of the trait support appears more strikingly. On both simulations of Fig. 1, extinction happens in a fast time. Almost-sure extinction is due to the logistic interaction, as proved in the following proposition.

**Proposition 4.1.** *There is almost-sure extinction of the superprocess (4.3).*

*Proof.* The mass of the super-process satisfies the following equation:

$$\begin{aligned} \langle \bar{X}_t, 1 \rangle &= \langle \bar{X}_0, 1 \rangle + \int_0^t \int_{[0, x_0]} \Lambda(x, \langle X_s, 1 \rangle) \bar{X}_s(dx) ds + M_t^1 \\ \langle M^1 \rangle_t &= \int_0^t 2 \langle \bar{X}_s, 1 \rangle ds, \\ \text{where } \Lambda(x, Z) &= \frac{x(x_0 - x)}{2} - (d_0 + \eta(x_0 - x)Z). \end{aligned} \quad (4.4)$$

This equation is not closed for the mass process, since the drift depends on the trait distribution. Our purpose is to upper-bound  $\Lambda(x, Z)$  so that  $\langle \bar{X}, 1 \rangle$  can be stochastically dominated by a Feller diffusion with negative drift, that goes extinct almost surely.

In the case where  $x > x_0 - (\frac{2d_0}{x_0} - \zeta)$  with  $\zeta \in (0, \frac{2d_0}{x_0} \wedge 1)$  and since  $x \in (0, x_0)$ , one gets

$$\begin{aligned} \Lambda(x, Z) &= -d_0 + (x_0 - x) \left( \frac{x}{2} - \eta Z \right) \\ &\leq -d_0 + \left( \frac{2d_0}{x_0} - \zeta \right) \times \frac{x_0}{2} = -\frac{\zeta x_0}{2}. \end{aligned} \quad (4.5)$$

In the case where  $x < x_0 - (\frac{2d_0}{x_0} - \zeta)$ , then  $0 < \frac{2d_0}{x_0} - \zeta \leq x_0 - x \leq x_0$  and depending on the sign of  $x/2 - \eta Z$ :

$$\begin{aligned} \Lambda(x, Z) &\leq -d_0 + \max \left( x_0 \left( \frac{x_0}{2} - \eta Z \right) ; \left( \frac{2d_0}{x_0} - \zeta \right) \left( \frac{x_0}{2} - \eta Z \right) \right) \\ &\leq \frac{x_0^2}{2} - d_0 - \eta \left( \frac{2d_0}{x_0} - \zeta \right) Z. \end{aligned} \quad (4.6)$$

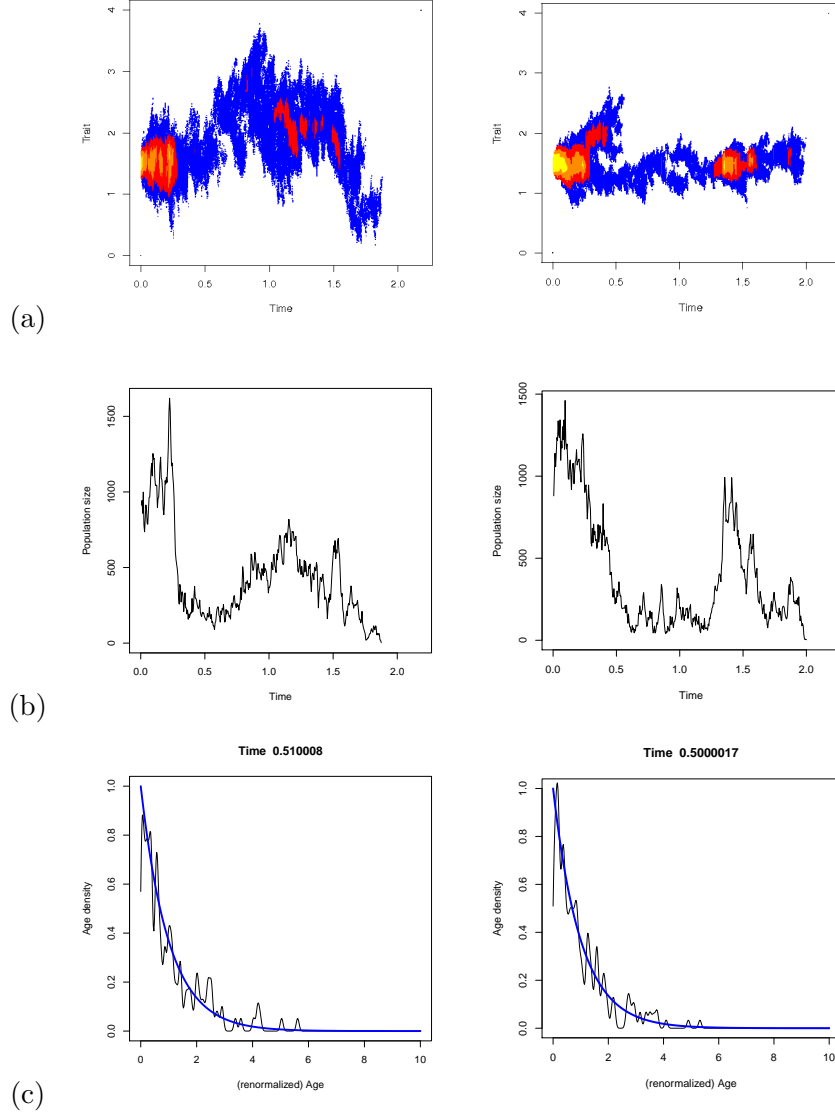


Figure 1: *Simulation of the individual-based process  $X^n$ , with  $n = 1000$  and discretization step  $\Delta t = 0.005$ . The system is started with 1000 particles of trait  $x = 1.5$ . First line:  $\sigma = 1$ . Second line:  $\sigma = 0.8$ . (a): Support of the process  $\bar{X}^n$  (with time in abscissa and trait in ordinate). (b): Evolution of the population size. (c): Age distribution for  $t = 0.5$ . It can be checked that the age distribution converges to an exponential of parameter 1 (plain line).*

Since the upper bounds in (4.5) and (4.6) are equal when the mass  $Z$  equals  $m_0$  defined by

$$m_0 = \frac{\frac{x_0(x_0+\zeta)}{2} - d_0}{\eta\left(\frac{2d_0}{x_0} - \zeta\right)}, \quad (4.7)$$

we thus get in any case that

$$\Lambda(x, Z) \leq -\frac{\zeta x_0}{2} \mathbb{1}_{Z \geq m_0} + \left(\frac{x_0^2}{2} - d_0\right) \mathbb{1}_{Z \leq m_0}. \quad (4.8)$$

Hence, the process  $\langle \bar{X}, 1 \rangle$  can be stochastically dominated by the following positive process:

$$Z_t = \langle \bar{X}_0, 1 \rangle + \int_0^t \left( -\frac{\zeta x_0}{2} Z_s + m_0 \left( \frac{x_0(x_0 + \zeta)}{2} - d_0 \right) \mathbb{1}_{Z_s \leq m_0} \right) ds + \int_0^t \sqrt{2Z_s} dB_s \quad (4.9)$$

where  $B$  is a standard Brownian motion.

We can adapt the results of Meyn and Tweedie [27] to prove almost sure extinction. For  $u \in \mathbb{R}_+$ , let us denote by  $\tau_u = \inf\{t \geq 0, Z_t \leq u\}$  and let  $z = \langle \bar{X}_0, 1 \rangle > 0$ . Either  $z \leq m_0$  and then  $\tau_{m_0} = 0$ , or  $z > m_0$ . In the latter case, let us consider  $M > z > m_0$  and let  $\rho_M = \inf\{t \geq 0, Z_t \geq M\}$ . We have

$$\mathbb{E}_z \left( Z_{\tau_{m_0} \wedge \rho_M} - z + \frac{\zeta x_0}{2} \int_0^{\tau_{m_0} \wedge \rho_M} Z_s ds - \int_0^{\tau_{m_0} \wedge \rho_M} \sqrt{2Z_s} dB_s \right) = 0. \quad (4.10)$$

By the uniform integrability of the fourth term and the optional stopping theorem, we have  $\mathbb{E}_z \left( \int_0^{\tau_{m_0} \wedge \rho_M} \sqrt{2Z_s} dB_s \right) = 0$ . Since moreover  $Z_s \geq m_0$  for  $s \in [0, \tau_{m_0} \wedge \rho_M]$ , we deduce that

$$m_0 \mathbb{E}_z(\tau_{m_0} \wedge \rho_M) \leq \mathbb{E}_z \left( \int_0^{\tau_{m_0} \wedge \rho_M} Z_s ds \right) \leq \frac{2z}{\zeta x_0}. \quad (4.11)$$

It can easily be proved that for all  $T > 0$ ,  $\mathbb{E}(\sup_{t \leq T} (Z_t)^2) < \infty$ , implying that  $\rho_M$  tends to infinity with  $M$ . Thus, (4.11) provides that for all  $z > 0$ ,  $\mathbb{P}_z(\tau_{m_0} < +\infty) = 1$ . By Girsanov's theorem, there exists a probability measure under which the process  $Z$  is a sub-critical Feller diffusion. It turns out that  $\mathbb{P}_{m_0}(\tau_0 \wedge \rho_M < +\infty) = 1$ . Standard computation using the strong Markov property yields  $\mathbb{P}_z(\tau_0 < +\infty) = 1$ .  $\blacksquare$

## 4.2 Logistic biological-age and size-structured population

In this section, the trait  $x \in [x_1, x_2] \subset (0, x_0)$  (with  $x_1, x_2 > 0$ ) is linked to the rate of metabolism, which measures the energy expended by individuals, and is often an increasing function of the body size. Ageing may result from toxic by-products of the metabolism. This leads us to introduce a biological age  $xa$ , where  $a$  is the physical age and where  $x$  can be interpreted as the ageing velocity equals. In this example, we consider  $r(x, a) = xa$  so that biologically older individuals give birth and die with higher rate, the other parameters being chosen as in Subsection 4.1. For a review on body size, energy metabolism and ageing, we refer the reader to [33]. In our example:

$$\hat{m}(x, a) = \frac{2\sqrt{x}e^{-\frac{xa^2}{2}}}{\sqrt{2\pi}} \mathbb{1}_{[0, +\infty)}(a). \quad (4.12)$$

We recognize the Gaussian distribution with variance  $1/x$  conditioned on being positive. Then:

$$\begin{aligned} \hat{b}(x) &= \frac{2x^{3/2}(x_0 - x)}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{x(a^2 + 2a/x)}{2}} da = \frac{2x(x_0 - x)e^{\frac{1}{2x}}}{\sqrt{2\pi}} \int_{1/\sqrt{x}}^{+\infty} e^{-\frac{\alpha^2}{2}} d\alpha \\ &= 2x(x_0 - x)e^{\frac{1}{2x}} \Phi\left(-\frac{1}{\sqrt{x}}\right), \end{aligned} \quad (4.13)$$

$$\hat{r}(x) = \frac{2x^{3/2}}{\sqrt{2\pi}} \int_0^{+\infty} ae^{-\frac{xa^2}{2}} da = \sqrt{\frac{2x}{\pi}} \int_0^{+\infty} \alpha e^{-\frac{\alpha^2}{2}} d\alpha = \sqrt{\frac{2x}{\pi}}, \quad (4.14)$$

where  $\Phi$  is the cumulative distribution function of the standard Gaussian distribution. The functions  $\hat{d}(x)$  and  $\hat{U}(x, y)$  are unchanged and given by (4.2). The martingale problem (3.3)

becomes here:

$$\begin{aligned}
M_t^f &= \langle \bar{X}_t, f \rangle - \langle \bar{X}_0, f \rangle - \int_0^t \int_{\mathcal{X}} \left( p \sqrt{\frac{x}{2\pi}} \sigma^2 \Delta \varphi(x) + [2x(x_0 - x) e^{\frac{1}{2x}} \Phi\left(-\frac{1}{\sqrt{x}}\right) \right. \\
&\quad \left. - (d_0 + \eta(x_0 - x) \langle \bar{X}_s, 1 \rangle)] f(x) \right) \bar{X}_s(dx) ds \\
\langle M^f \rangle_t &= \int_0^t \int_{\mathcal{X}} 2 \sqrt{\frac{2x}{\pi}} f^2(x) \bar{X}_s(dx) ds.
\end{aligned} \tag{4.15}$$

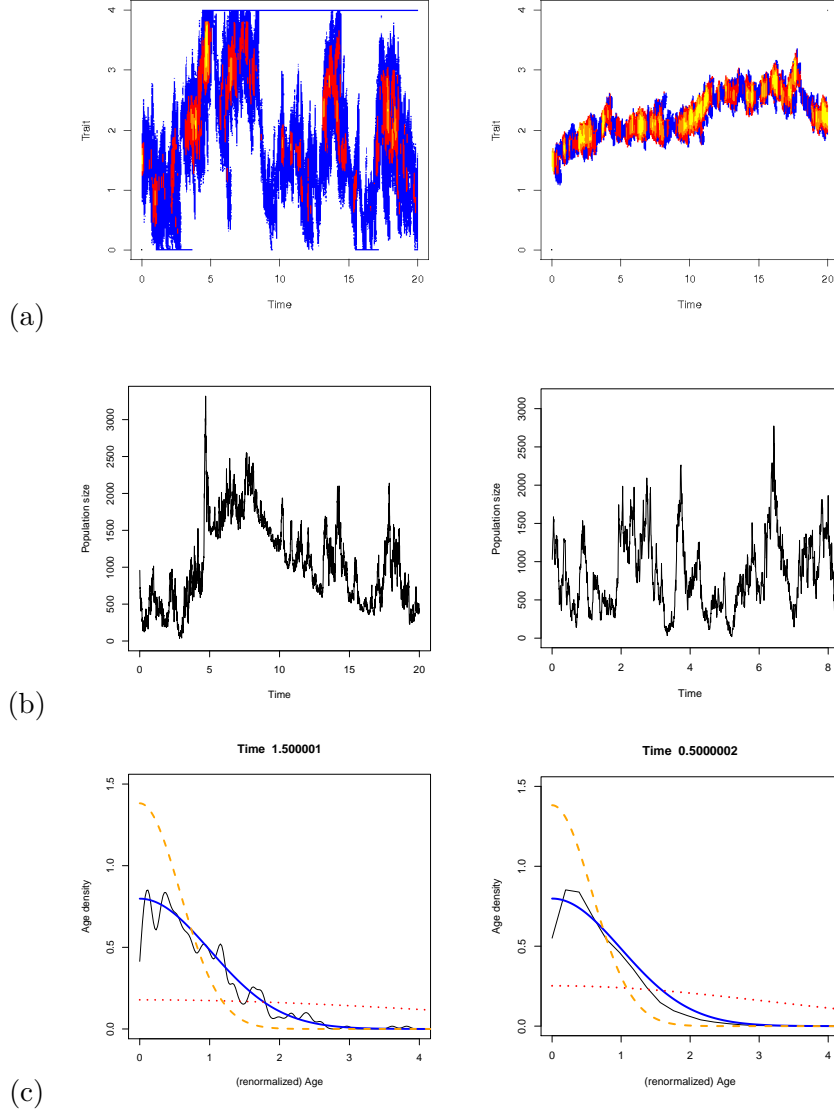


Figure 2: Simulation of the individual-based process  $X^n$ , with  $n = 1000$  and discretization step  $\Delta t = 0.005$ . The system is started with 1000 particles of trait  $x = 1.5$ . First line:  $\sigma = 1$ , Second line:  $\sigma = 0.2$ . (a): Support of the process  $\bar{X}^n$  (with time in abscissa and trait in ordinate). (b): Evolution of the population size. (c): Marginal age distribution for  $t = 0.5$ . For comparison, we draw the density  $\hat{m}(1, a)$  (plain line),  $\hat{m}(0.5, a)$  (dotted line) and  $\hat{m}(3, a)$  (dashed line).

In this example, there is a higher senescence for individuals with trait  $x > 1$ , compared with the example of Section 4.1. The new choice of  $r(x, a)$  influences the age distribution: lifespans are

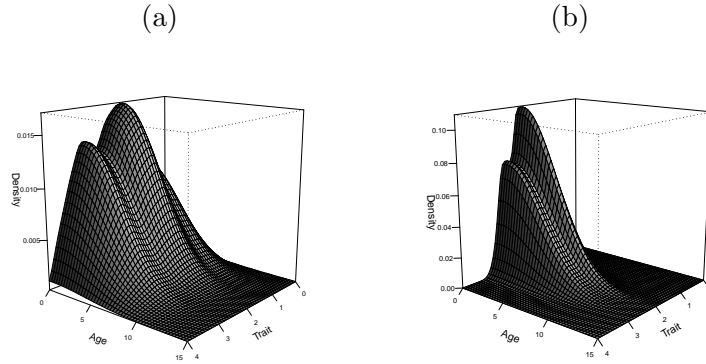


Figure 3: *Joint distributions in trait and age for the simulation of Fig. 2. We see that contrarily to the example of Section 4.1, we do not have here independence of traits and age. (a):  $\sigma = 1$ . (b):  $\sigma = 0.2$*

shortened. This can be seen on the smaller support of the age distribution (compare Fig. 2-(c) with Fig. 1-(c)).

However, the populations are more persistent in the example of this section, although it can be proved similarly to Prop. 4.1 that there is almost sure extinction. Indeed, contrary to the populations in Example 1 which are extinct at  $t = 2$ , the population of Example 2 still survives at  $t = 20$ . One reason is that the growth rate in the finite variation term of (4.15) is bigger than the one in Example 1. Indeed, for many values of  $x$ , the factor  $2 \exp(1/2x) \Phi(-1/\sqrt{x})$  in the birth rate  $\hat{b}(x)$  is bigger than the factor  $1/2$ . For  $x = 1.5$ ,  $2 \exp(1/2x) \Phi(-1/\sqrt{x}) = 0.58 > 0.5$  and for  $x = 3$ ,  $2 \exp(1/2x) \Phi(-1/\sqrt{x}) = 0.67 > 0.5$ .

When comparing Fig. 1-(b) and Fig. 2-(b), we observe more fluctuations of the population size in Example 2. The bracket of the martingale in (4.15) presents a multiplicative  $x$  term, compared to (4.3). As soon as  $x > \frac{\pi}{2}$ , this explains the increased variance. Notice however that this variance tends to zero when the population size tends to zero, which also explains why there is no decrease in the population persistence.

Finally, the multiplicative term  $p\sqrt{x/2\pi\sigma^2}$  in front of the diffusion term  $\Delta\varphi(x)$  explains the large variability of the trait support, which is observed in Fig. 2-(a). When the diffusion coefficient  $\sigma$  is small (second line of Fig. 2), the traits evolve towards a value between  $x = 2$  and  $x = 4$  where the trade-off between reproduction and competition is optimized.

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