



On Convex optimization without convex representation

Jean-Bernard Lasserre

► To cite this version:

Jean-Bernard Lasserre. On Convex optimization without convex representation. Optimization Letters, 2011, 5 (4), p. 549-556. hal-00495396v3

HAL Id: hal-00495396

<https://hal.science/hal-00495396v3>

Submitted on 28 Jan 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON CONVEX OPTIMIZATION WITHOUT CONVEX REPRESENTATION

JB. LASSERRE

ABSTRACT. We consider the convex optimization problem $\mathbf{P} : \min_{\mathbf{x}} \{f(\mathbf{x}) : \mathbf{x} \in \mathbf{K}\}$ where f is convex continuously differentiable, and $\mathbf{K} \subset \mathbb{R}^n$ is a compact convex set with representation $\{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0, j = 1, \dots, m\}$ for some continuously differentiable functions (g_j) . We discuss the case where the g_j 's are not all concave (in contrast with convex programming where they all are). In particular, even if the g_j are not concave, we consider the log-barrier function ϕ_μ with parameter μ , associated with \mathbf{P} , usually defined for concave functions (g_j) . We then show that any limit point of any sequence $(\mathbf{x}_\mu) \subset \mathbf{K}$ of stationary points of ϕ_μ , $\mu \rightarrow 0$, is a Karush-Kuhn-Tucker point of problem \mathbf{P} and a global minimizer of f on \mathbf{K} .

1. INTRODUCTION

Consider the optimization problem

$$(1.1) \quad \mathbf{P} : \quad f^* := \min_{\mathbf{x}} \{ f(\mathbf{x}) : \mathbf{x} \in \mathbf{K} \}.$$

for some convex and continuously differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, and where the feasible set $\mathbf{K} \subset \mathbb{R}^n$ is defined by:

$$(1.2) \quad \mathbf{K} := \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0, \quad j = 1, \dots, m\},$$

for some continuously differentiable functions $g_j : \mathbb{R}^n \rightarrow \mathbb{R}$. We say that (g_j) , $j = 1, \dots, m$, is a *representation* of \mathbf{K} . When \mathbf{K} is convex and the (g_j) are concave we say that \mathbf{K} has a convex representation.

In the literature, when \mathbf{K} is convex \mathbf{P} is referred to as a convex optimization problem and in particular, every local minimum of f is a global minimum. However, if on the one hand *convex optimization* usually refers to minimizing a convex function on a convex set \mathbf{K} without precising its representation (g_j) (see e.g. Ben-Tal and Nemirovsky [1, Definition 5.1.1] or Bertsekas et al. [3, Chapter 2]), on the other hand *convex programming* usually refers to the situation where the representation of \mathbf{K} is also convex, i.e. when all the g_j 's are concave. See for instance Ben-Tal and Nemirovski [1, p. 335], Berkovitz [2, p. 179], Boyd and Vandenberghe [4, p. 7], Bertsekas et al. [3, §3.5.5], Nesterov and Nemirovski [13, p. 217-218], and Hiriart-Urruty [11]. Convex programming is particularly interesting because under Slater's condition¹, the standard Karush-Kuhn-Tucker (KKT) optimality conditions are not only necessary but also sufficient and in addition, the concavity property of the g_j 's is used to prove convergence (and rates of convergence) of specialized algorithms.

1991 *Mathematics Subject Classification.* 90C25 90C46 65K05.

Key words and phrases. Convex optimization; convex programming; log-barrier.

¹Slater's condition holds if $g_j(\mathbf{x}_0) > 0$ for some $\mathbf{x}_0 \in \mathbf{K}$ and all $j = 1, \dots, m$.

To the best of our knowledge, little is said in the literature for the specific case where \mathbf{K} is convex but not necessarily its representation, that is, when the functions (g_j) are *not* necessarily concave. It looks like outside the convex programming framework, all problems are treated the same. This paper is a companion paper to [12] where we proved that if the nondegeneracy condition

$$(1.3) \quad \forall j = 1, \dots, m : \quad \nabla g_j(\mathbf{x}) \neq 0 \quad \forall \mathbf{x} \in \mathbf{K} \text{ with } g_j(\mathbf{x}) = 0$$

holds, then $\mathbf{x} \in \mathbf{K}$ is a global minimizer of f on \mathbf{K} if and only if (\mathbf{x}, λ) is a KKT point for some $\lambda \in \mathbb{R}_+^m$. This indicates that for convex optimization problems (1.1), and from the point of view of "first-order optimality conditions", what really matters is the geometry of \mathbf{K} rather than its representation. Indeed, for *any* representation (g_j) of \mathbf{K} that satisfies the nondegeneracy condition (1.3), there is a one-to-one correspondence between global minimizers and KKT points.

But what about from a computational viewpoint? Of course, not all representations of \mathbf{K} are equivalent since the ability (as well as the efficiency) of algorithms to obtain a KKT point of \mathbf{P} will strongly depend on the representation (g_j) of \mathbf{K} which is used. For example, algorithms that implement Lagrangian duality would require the (g_j) to be concave, those based on second-order methods would require all functions f and (g_j) to be twice continuous differentiable, self-concordance of a barrier function associated with a representation of \mathbf{K} may or may not hold, etc.

When \mathbf{K} is convex but not its representation (g_j) , several situations may occur. In particular, the level set $\{\mathbf{x} : g_j(\mathbf{x}) \geq a_j\}$ may be convex for $a_j = 0$ but not for some other values of $a_j > 0$, in which case the g_j 's are not even quasiconcave on \mathbf{K} , i.e., one may say that \mathbf{K} is convex *by accident* for the value $\mathbf{a} = 0$ of the parameter $\mathbf{a} \geq 0$. One might think that in this situation, algorithms that generate a sequence of feasible points in the interior of \mathbf{K} could run into problems to find a local minimum of f . If the $-g_j$'s are all quasiconvex on \mathbf{K} , we say that we are in the generic convex case because not only \mathbf{K} but also all sets $\mathbf{K}_{\mathbf{a}} := \{\mathbf{x} : g_j(\mathbf{x}) \geq a_j, j = 1, \dots, m\}$ are convex. However, quasiconvex functions do not share some nice properties of the convex functions. In particular, (a) $\nabla g_j(\mathbf{x}) = 0$ does not imply that g_j reaches a local minimum at \mathbf{x} , (b) a local minimum is not necessarily global and (c), the sum of quasiconvex functions is not quasiconvex in general; see e.g. Crouzeix et al. [5, p. 65]. And so even in this case, for some minimization algorithms, convergence to a minimum of f on \mathbf{K} might be problematic.

So an interesting issue is to determine whether there is an algorithm which converges to a global minimizer of a convex function f on \mathbf{K} , no matter if the representation of \mathbf{K} is convex or not. Of course, in view of [12, Theorem 2.3], a sufficient condition is that this algorithm provides a sequence (or subsequence) of points $(\mathbf{x}_k, \lambda_k) \in \mathbb{R}^n \times \mathbb{R}_+^m$ converging to a KKT point of \mathbf{P} .

With \mathbf{P} and a parameter $\mu > 0$, we associate the *log-barrier* function $\phi_\mu : \mathbf{K} \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$(1.4) \quad \mathbf{x} \mapsto \phi_\mu(\mathbf{x}) := \begin{cases} f(\mathbf{x}) - \mu \sum_{j=1}^m \ln g_j(\mathbf{x}), & \text{if } g_j(\mathbf{x}) > 0, \forall j = 1, \dots, m \\ +\infty, & \text{otherwise.} \end{cases}$$

By a *stationary point* $\mathbf{x} \in \mathbf{K}$ of ϕ_μ , we mean a point $\mathbf{x} \in \mathbf{K}$ with $g_j(\mathbf{x}) \neq 0$ for all $j = 1, \dots, m$, and such that $\nabla \phi_\mu(\mathbf{x}) = 0$. Notice that in general and in contrast with the present paper, ϕ_μ (or more precisely $\psi_\mu := \mu \phi_\mu$) is usually defined for convex problems \mathbf{P} where all the g_j 's are concave; see e.g. Den Hertog [6] and for more details on the barrier functions and their properties, the interested reader is referred to Güler [9] and Güler and Tuncel [10].

Contribution. The purpose of this paper is to show that no matter which representation (g_j) of a convex set \mathbf{K} (assumed to be compact) is used (provided it satisfies the nondegeneracy condition (1.3)), any sequence of stationary points (\mathbf{x}_μ) of ϕ_μ , $\mu \rightarrow 0$, has the nice property that each of its accumulation points is a KKT point of \mathbf{P} and hence, a global minimizer of f on \mathbf{K} . Hence, to obtain the global minimum of a convex function on \mathbf{K} it is enough to minimize the log-barrier function for nonincreasing values of the parameter, for any representation of \mathbf{K} that satisfies the nondegeneracy condition (1.3). Again and of course, the efficiency of the method will crucially depend on the representation of \mathbf{K} which is used. For instance, in general ϕ_μ will not have the self-concordance property, crucial for efficiency.

Observe that at first glance this result is a little surprising because as we already mentioned, there are examples of sets $\mathbf{K}_\mathbf{a} := \{\mathbf{x} : g_j(\mathbf{x}) \geq a_j, j = 1, \dots, m\}$ which are non convex for every $0 \neq \mathbf{a} \geq 0$ but $\mathbf{K} := \mathbf{K}_0$ is convex (by accident!) and (1.3) holds. So inside \mathbf{K} the level sets of the g_j 's are not convex any more. Still, and even though the stationary points \mathbf{x}_μ of the associated log-barrier ϕ_μ are inside \mathbf{K} , all converging subsequences of a sequence (\mathbf{x}_μ), $\mu \rightarrow 0$, will converge to some global minimizer \mathbf{x}^* of f on \mathbf{K} . In particular, if the global minimizer $\mathbf{x}^* \in \mathbf{K}$ is unique then the whole sequence (\mathbf{x}_μ) will converge. Notice that this happens even if the g_j 's are not log-concave, in which case ϕ_μ may not be convex for all μ (e.g. if f is linear). So what seems to really matter is the fact that as μ decreases, the convex function f becomes more and more important in ϕ_μ , and also that the functions g_j which matter in a KKT point (\mathbf{x}^*, λ) are those for which $g_j(\mathbf{x}^*) = 0$ (and so with convex associated level set $\{\mathbf{x} : g_j(\mathbf{x}) \geq 0\}$).

2. MAIN RESULT

Consider the optimization problem (1.1) in the following context.

Assumption 1. The set \mathbf{K} in (1.2) is convex and Slater's assumption holds. Moreover, the nondegeneracy condition

$$(2.1) \quad \nabla g_j(\mathbf{x}) \neq 0 \quad \forall \mathbf{x} \in \mathbf{K} \text{ such that } g_j(\mathbf{x}) = 0,$$

holds for every $j = 1, \dots, m$.

Observe that when the g_j 's are concave then the nondegeneracy condition (2.1) holds automatically. Recall that $(\mathbf{x}^*, \lambda) \in \mathbf{K} \times \mathbb{R}^m$ is a Karush-Kuhn-Tucker (KKT) point of \mathbf{P} if

- $\mathbf{x} \in \mathbf{K}$ and $\lambda \geq 0$
- $\lambda_j g_j(\mathbf{x}^*) = 0$ for every $j = 1, \dots, m$
- $\nabla f(\mathbf{x}^*) - \sum_{j=1}^m \lambda_j \nabla g_j(\mathbf{x}^*) = 0$.

We recall the following result from [12]:

Theorem 1 ([12]). *Let \mathbf{K} be as in (1.2) and let Assumption 1 hold. Then \mathbf{x} is a global minimizer of f on \mathbf{K} if and only if there is some $\lambda \in \mathbb{R}_+^m$ such that (\mathbf{x}, λ) is a KKT point of \mathbf{P} .*

The next result is concerned with the log-barrier ϕ_μ in (1.4).

Lemma 2. *Let \mathbf{K} in (1.2) be convex and compact and assume that Slater's condition holds. Then for every $\mu > 0$ the log-barrier function ϕ_μ in (1.4) has at least one stationary point on \mathbf{K} (which is a global minimizer of ϕ_μ on \mathbf{K}).*

Proof. Let f^* be the minimum of f on \mathbf{K} and let $\mu > 0$ be fixed, arbitrary. We first show that $\phi_\mu(\mathbf{x}_k) \rightarrow \infty$ as $\mathbf{x}_k \rightarrow \partial\mathbf{K}$ (where $(\mathbf{x}_k) \subset \mathbf{K}$). Indeed, pick up an index i such that $g_i(\mathbf{x}_k) \rightarrow 0$ as $k \rightarrow \infty$. Then $\phi_\mu(\mathbf{x}_k) \geq f^* - \mu \ln g_i(\mathbf{x}_k) - (m-1) \ln C$ (where all the g_j 's are bounded above by C). So ϕ_μ is coercive and therefore must have a (global) minimizer $\mathbf{x}_\mu \in \mathbf{K}$ with $g_j(\mathbf{x}_\mu) > 0$ for every $j = 1, \dots, m$; and so $\nabla \phi_\mu(\mathbf{x}_\mu) = 0$. \square

Notice that ϕ_μ may have several stationary points in \mathbf{K} . We now state our main result.

Theorem 3. *Let \mathbf{K} in (1.2) be compact and let Assumption 1 hold true. For every fixed $\mu > 0$, choose $\mathbf{x}_\mu \in \mathbf{K}$ to be an arbitrary stationary point of ϕ_μ in \mathbf{K} .*

Then every accumulation point $\mathbf{x}^ \in \mathbf{K}$ of such a sequence $(\mathbf{x}_\mu) \subset \mathbf{K}$ with $\mu \rightarrow 0$, is a global minimizer of f on \mathbf{K} , and if $\nabla f(\mathbf{x}^*) \neq 0$, \mathbf{x}^* is a KKT point of \mathbf{P} .*

Proof. Let $\mathbf{x}_\mu \in \mathbf{K}$ be a stationary point of ϕ_μ , which by Lemma 2 is guaranteed to exist. So

$$(2.2) \quad \nabla \phi_\mu(\mathbf{x}_\mu) = \nabla f(\mathbf{x}_\mu) - \sum_{j=1}^m \frac{\mu}{g_j(\mathbf{x}_\mu)} \nabla g_j(\mathbf{x}_\mu) = 0.$$

As $\mu \rightarrow 0$ and \mathbf{K} is compact, there exists $\mathbf{x}^* \in \mathbf{K}$ and a subsequence $(\mu_\ell) \subset \mathbb{R}_+$ such that $\mathbf{x}_{\mu_\ell} \rightarrow \mathbf{x}^*$ as $\ell \rightarrow \infty$. We need consider two cases:

Case when $g_j(\mathbf{x}^) > 0, \forall j = 1, \dots, m$.* Then as f and g_j are continuously differentiable, $j = 1, \dots, m$, taking limit in (2.2) for the subsequence (μ_ℓ) , yields $\nabla f(\mathbf{x}^*) = 0$ which, as f is convex, implies that \mathbf{x}^* is a global minimizer of f on \mathbb{R}^n , hence on \mathbf{K} .

Case when $g_j(\mathbf{x}^) = 0$ for some $j \in \{1, \dots, m\}$.* Let $J := \{j : g_j(\mathbf{x}^*) = 0\} \neq \emptyset$. We next show that for every $j \in J$, the sequence of ratios $(\mu/g_j(\mathbf{x}_{\mu_\ell}), \ell = 1, \dots)$, is bounded. Indeed let $j \in J$ be fixed arbitrary. As Slater's condition holds, let $\mathbf{x}_0 \in \mathbf{K}$ be such that $g_j(\mathbf{x}_0) > 0$ for all $j = 1, \dots, m$; then $\langle \nabla g_j(\mathbf{x}^*), \mathbf{x}_0 - \mathbf{x}^* \rangle > 0$. Indeed, as \mathbf{K} is convex, $\langle \nabla g_j(\mathbf{x}^*), \mathbf{x}_0 + \mathbf{v} - \mathbf{x}^* \rangle \geq 0$ for all \mathbf{v} in some small enough ball $\mathbf{B}(0, \rho)$ around the origin. So if $\langle \nabla g_j(\mathbf{x}^*), \mathbf{x}_0 - \mathbf{x}^* \rangle = 0$ then $\langle \nabla g_j(\mathbf{x}^*), \mathbf{v} \rangle \geq 0$ for all $\mathbf{v} \in \mathbf{B}(0, \rho)$, in contradiction with $\nabla g_j(\mathbf{x}^*) \neq 0$. Next,

$$(2.3) \quad \begin{aligned} \langle \nabla f(\mathbf{x}_{\mu_\ell}), \mathbf{x}_0 - \mathbf{x}^* \rangle &= \underbrace{\sum_{k \notin J} \frac{\mu_\ell}{g_k(\mathbf{x}_{\mu_\ell})} \langle \nabla g_k(\mathbf{x}_{\mu_\ell}), \mathbf{x}_0 - \mathbf{x}^* \rangle}_{A_\ell} \\ &\quad + \underbrace{\sum_{k \in J} \frac{\mu_\ell}{g_k(\mathbf{x}_{\mu_\ell})} \langle \nabla g_k(\mathbf{x}_{\mu_\ell}), \mathbf{x}_0 - \mathbf{x}^* \rangle}_{B_\ell} \end{aligned}$$

Observe that in (2.3):

- Every term of the sum B_ℓ is nonnegative for sufficiently large ℓ , say $\ell \geq \ell_0$, because $\mathbf{x}_{\mu_\ell} \rightarrow \mathbf{x}^*$ and $\langle \nabla g_k(\mathbf{x}^*), \mathbf{x}_0 - \mathbf{x}^* \rangle > 0$ for all $k \in J$.
- $A_\ell \rightarrow 0$ as $\ell \rightarrow \infty$ because $\mu_\ell \rightarrow 0$ and $g_k(\mathbf{x}_{\mu_\ell}) \rightarrow g_k(\mathbf{x}^*) > 0$ for all $k \notin J$.

Therefore $|A_\ell| \leq A$ for all sufficiently large ℓ , say $\ell \geq \ell_1$, and so for every $j \in J$:

$$\langle \nabla f(\mathbf{x}_{\mu_\ell}), \mathbf{x}_0 - \mathbf{x}^* \rangle + A \geq \frac{\mu_\ell}{g_j(\mathbf{x}_{\mu_\ell})} \langle \nabla g_j(\mathbf{x}_{\mu_\ell}), \mathbf{x}_0 - \mathbf{x}^* \rangle, \quad \ell \geq \ell_2 := \max[\ell_0, \ell_1],$$

which shows that for every $j \in J$, the nonnegative sequence $(\mu_\ell/g_j(\mathbf{x}_{\mu_\ell}))$, $\ell \geq \ell_2$, is bounded from above.

So take a subsequence (still denoted (μ_ℓ) , $\ell \in \mathbb{N}$, for convenience) such that the ratios $\mu_\ell/g_j(\mathbf{x}_{\mu_\ell})$ converge for all $j \in J$, that is,

$$\lim_{\ell \rightarrow \infty} \frac{\mu_\ell}{g_j(\mathbf{x}_{\mu_\ell})} = \lambda_j \geq 0, \quad \forall j \in J,$$

and let $\lambda_j := 0$ for every $j \notin J$, so that $\lambda_j g_j(\mathbf{x}^*) = 0$ for every $j = 1, \dots, m$. Taking limit in (2.2) as $\ell \rightarrow \infty$, yields:

$$(2.4) \quad \nabla f(\mathbf{x}^*) = \sum_{j=1}^m \lambda_j \nabla g_j(\mathbf{x}^*),$$

which shows that $(\mathbf{x}^*, \lambda) \in \mathbf{K} \times \mathbb{R}_+^m$ is a KKT point for \mathbf{P} . Finally, invoking Theorem 1, \mathbf{x}^* is also a global minimizer of \mathbf{P} . \square

2.1. Discussion. The log-barrier function ϕ_μ or its exponential variant $f + \mu \sum g_j^{-1}$ has become popular since the pioneer work of Fiacco and McCormick [7, 8], where it is assumed that f and the g_j 's are twice continuously differentiable, the g_j 's are concave², Slater's condition holds, the set $\mathbf{K} \cap \{\mathbf{x} : f(\mathbf{x}) \leq k\}$ is bounded for every finite k , and finally, the barrier function is strictly convex for every value of the parameter $\mu > 0$. Under such conditions, the barrier function $f + \mu \sum g_j^{-1}$ has a unique minimizer \mathbf{x}_μ for every $\mu > 0$ and the sequence $(\mathbf{x}_\mu, (\mu/g_j(\mathbf{x}_\mu)^2)) \subset \mathbb{R}^{n+m}$ converges to a Wolfe-dual feasible point.

In contrast, Theorem 3 states that without assuming concavity of the g_j 's, one may obtain a global minimizer of f on \mathbf{K} , by looking at *any* limit point of *any* sequence of stationary points (\mathbf{x}_μ) , $\mu \rightarrow 0$, of the log-barrier function ϕ_μ associated with a representation (g_j) of \mathbf{K} , provided that the representation satisfies the non-degeneracy condition (1.3). To us, this comes as a little surprise as the stationary points (\mathbf{x}_μ) are all inside \mathbf{K} , and there are examples of convex sets \mathbf{K} with a representation (g_j) satisfying (1.3) and such that the level sets $\mathbf{K}_a = \{\mathbf{x} : g_j(\mathbf{x}) \geq a_j\}$ with $a_j > 0$, are not convex! (See Example 1.) Even if f is convex, the log-barrier function ϕ_μ need not be convex; for instance if f is linear, $\nabla^2 \phi_\mu = -\mu \sum_j \nabla^2 \ln g_j$, and so if the g_j 's are not log-concave then ϕ_μ may not be convex on \mathbf{K} for every value of the parameter $\mu > 0$.

Example 1. Let $n = 2$ and $\mathbf{K}_a := \{\mathbf{x} \in \mathbb{R}^2 : g(\mathbf{x}) \geq a\}$ with $\mathbf{x} \mapsto g(\mathbf{x}) := 4 - ((x_1 + 1)^2 + x_2^2)((x_1 - 1)^2 + x_2^2)$, with $a \in \mathbb{R}$. The set \mathbf{K}_a is convex only for those values of a with $a \leq 0$; see in Figure 1. It is even disconnected for $a = 4$.

²In fact as noted in [7], concavity of the g_j 's is merely a sufficient condition for the barrier function to be convex.

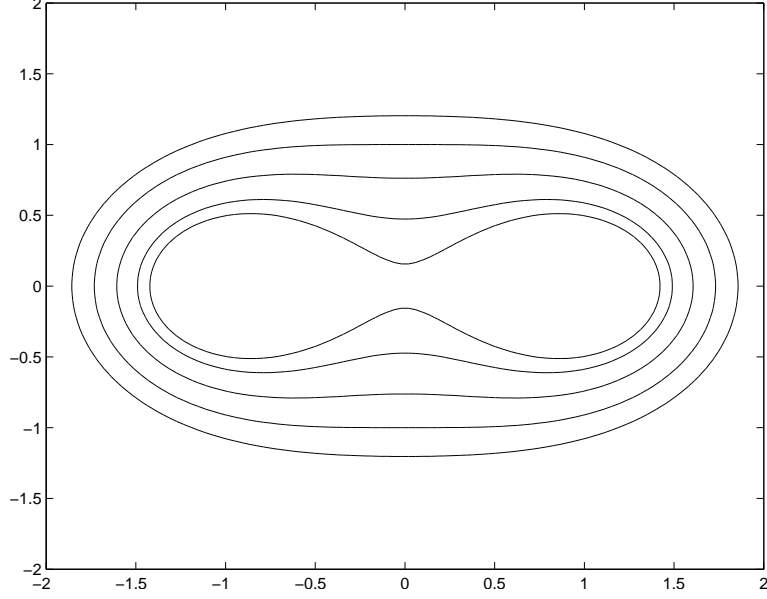


FIGURE 1. Example 1: Level sets $\{\mathbf{x} : g(\mathbf{x}) = a\}$ for $a = 2.95, 2.5, 1.5, 0$ and -2

We might want to consider a generic situation, that is, when the set

$$\mathbf{K}_{\mathbf{a}} := \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq a_j, \quad j = 1, \dots, m\},$$

is also convex for every positive vector $0 \leq \mathbf{a} = (a_j) \in \mathbb{R}^m$. This in turn would imply that the g_j are *quasiconcave*³ on \mathbf{K} . In particular, if the nondegeneracy condition (1.3) holds on \mathbf{K} and the g_j 's are twice differentiable, then at most one eigenvalue of the Hessian $\nabla^2 g_j$ (and hence $\nabla^2 \ln g_j$) is possibly positive (i.e., $\ln g_j$ is *almost* concave). This is because for every $\mathbf{x} \in \mathbf{K}$ with $g_j(\mathbf{x}) = 0$, one has $\langle \mathbf{v}, \nabla^2 g_j(\mathbf{x}) \mathbf{v} \rangle \leq 0$ for all $\mathbf{v} \in \nabla g_j(\mathbf{x})^\perp$ (where $\nabla g_j(\mathbf{x})^\perp := \{\mathbf{v} : \langle \nabla g_j(\mathbf{x}), \mathbf{v} \rangle = 0\}$). However, even in this situation, the log-barrier function ϕ_μ may not be convex. On the other hand, $\ln g_j$ is "more" concave than g_j on $\text{Int } \mathbf{K}$ because its Hessian $\nabla^2 g_j$ satisfies $g_j^2 \nabla^2 \ln g_j = g_j \nabla^2 g_j - \nabla g_j (\nabla g_j)^T$. But still, g_j might not be log-concave on $\text{Int } \mathbf{K}$, and so ϕ_μ may not be convex at least for values of μ not too small (and for all values of μ if f is linear).

Example 2. Let $n = 2$ and $\mathbf{K} := \{\mathbf{x} : g(\mathbf{x}) \geq 0, \mathbf{x} \geq 0\}$ with $\mathbf{x} \mapsto g(\mathbf{x}) = x_1 x_2 - 1$. The representation of \mathbf{K} is not convex but the g_j 's are log-concave, and so the log-barrier $\mathbf{x} \mapsto \phi_\mu(\mathbf{x}) := f(\mathbf{x}) - \mu(\ln g(\mathbf{x}) - \ln x_1 - \ln x_2)$ is convex.

Example 3. Let $n = 2$ and $\mathbf{K} := \{\mathbf{x} : g_1(\mathbf{x}) \geq 0; a - x_1 \geq 0; 0 \leq x_2 \leq b\}$ with $\mathbf{x} \mapsto g_1(\mathbf{x}) = x_1/(\epsilon + x_2^2)$ with $\epsilon > 0$. The representation of \mathbf{K} is not convex and g_1 is not log-concave. If f is linear and ϵ is small enough, the log-barrier

$$\mathbf{x} \mapsto \phi_\mu(\mathbf{x}) := f(\mathbf{x}) - \mu(\ln x_1 + \ln(a - x_1) - \ln(\epsilon + x_2^2) + \ln x_2 + \ln(b - x_2))$$

³Recall that on a convex set $O \subset \mathbb{R}^n$, a function $f : O \rightarrow \mathbb{R}$ is quasiconvex if the level sets $\{\mathbf{x} : f(\mathbf{x}) \leq r\}$ are convex for every $r \in \mathbb{R}$. A function $f : O \rightarrow \mathbb{R}$ is said to be quasiconcave if $-f$ is quasiconvex; see e.g. [5].

is not convex for every value of $\mu > 0$.

Acknowledgement. The author wishes to thank two anonymous referees for pointing out a mistake and providing suggestions to improve the initial version of this paper.

REFERENCES

- [1] A. Ben-Tal, A. Nemirovski. *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, SIAM, Philadelphia, 2001.
- [2] L.D. Berkovitz. *Convexity and Optimization in \mathbb{R}^n* , John Wiley & Sons, Inc., 2002.
- [3] D. Bertsekas, A. Nedić, E. Ozdaglar. *Convex Analysis and Optimization*, Athena Scientific, Belmont, Massachusetts, 2003.
- [4] S. Boyd, L. Vandenberghe. *Convex Optimization*, Cambridge University Press, Cambridge, 2004.
- [5] J-P. Crouzeix, A. Eberhard, D. Ralph. A geometrical insight on pseudoconvexity and pseudomonotonicity, *Math. Program. Ser. B* **123** (2010), 61–83.
- [6] D. den Hertog. *Interior Point Approach to Linear, Quadratic and Convex Programming*, Kluwer, Dordrecht, 1994.
- [7] A.V. Fiacco, G.P. McCormick. The sequential unconstrained minimization technique for nonlinear programming, a primal-dual method, *Manag. Sci.* **10** (1964), 360–366.
- [8] A.V. Fiacco, G.P. McCormick. Computational algorithm for the sequential unconstrained minimization technique for nonlinear programming, *Manag. Sci.* **10** (1964), 601–617.
- [9] O. Güler. Barrier functions in interior point methods, *Math. Oper. Res.* **21** (1996), 860–885.
- [10] O. Güler, L. Tuncel. Characterization of the barrier parameter of homogeneous convex cones, *Math. Progr.* **81** (1998), 55–76.
- [11] J.-B. Hiriart-Urruty. *Optimisation et Analyse Convexe*, Presses Universitaires de France, 1998.
- [12] J.B. Lasserre. On representations of the feasible set in convex optimization, *Optim. Letters* **4** (2010), 1–7.
- [13] Y. Nesterov, A. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming*, SIAM, Philadelphia, 1994.
- [14] B.T. Polyak. *Introduction to Optimization*, Optimization Software, Inc., New York, 1987.
- [15] R. Schneider. *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, Cambridge, UK (1994).

LAAS-CNRS AND INSTITUTE OF MATHEMATICS, UNIVERSITY OF TOULOUSE, LAAS, 7 AVENUE DU COLONEL ROCHE, 31077 TOULOUSE CÉDEX 4, FRANCE
E-mail address: `lasserre@laas.fr`