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# ON CONVEX OPTIMIZATION WITHOUT CONVEX REPRESENTATION

JB. LASSERRE

ABSTRACT. We consider the convex optimization problem  $\mathbf{P} : \min_{\mathbf{x}} \{f(\mathbf{x}) : \mathbf{x} \in \mathbf{K}\}$  where  $f$  is convex continuously differentiable, and  $\mathbf{K} \subset \mathbb{R}^n$  is a compact convex set with representation  $\{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0, j = 1, \dots, m\}$  for some continuously differentiable functions  $(g_j)$ . We discuss the case where the  $g_j$ 's are not all concave (in contrast with convex programming where they all are). In particular, even if the  $g_j$  are not concave, we consider the log-barrier function  $\phi_\mu$  with parameter  $\mu$ , associated with  $\mathbf{P}$ , usually defined for concave functions  $(g_j)$ . We then show that any limit point of any sequence  $(\mathbf{x}_\mu) \subset \mathbf{K}$  of stationary points of  $\phi_\mu$ ,  $\mu \rightarrow 0$ , is a Karush-Kuhn-Tucker point of problem  $\mathbf{P}$  and a global minimizer of  $f$  on  $\mathbf{K}$ .

## 1. INTRODUCTION

Consider the optimization problem

$$(1.1) \quad \mathbf{P} : f^* := \min_{\mathbf{x}} \{f(\mathbf{x}) : \mathbf{x} \in \mathbf{K}\}.$$

for some convex and continuously differentiable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , and where the feasible set  $\mathbf{K} \subset \mathbb{R}^n$  is defined by:

$$(1.2) \quad \mathbf{K} := \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0, \quad j = 1, \dots, m\},$$

for some continuously differentiable functions  $g_j : \mathbb{R}^n \rightarrow \mathbb{R}$ . We say that  $(g_j)$ ,  $j = 1, \dots, m$ , is a *representation* of  $\mathbf{K}$ . When  $\mathbf{K}$  is convex and the  $(g_j)$  are concave we say that  $\mathbf{K}$  has a convex representation.

In the literature, when  $\mathbf{K}$  is convex  $\mathbf{P}$  is referred to as a convex optimization problem and in particular, every local minimum of  $f$  is a global minimum. However, if on the one hand *convex optimization* usually refers to minimizing a convex function on a convex set  $\mathbf{K}$  without precising its representation  $(g_j)$  (see e.g. Ben-Tal and Nemirovsky [1, Definition 5.1.1] or Bertsekas et al. [3, Chapter 2]), on the other hand *convex programming* usually refers to the situation where the representation of  $\mathbf{K}$  is also convex, i.e. when all the  $g_j$ 's are concave. See for instance Ben-Tal and Nemirovski [1, p. 335], Berkovitz [2, p. 179], Boyd and Vandenberghe [4, p. 7], Bertsekas et al. [3, §3.5.5], Nesterov and Nemirovski [13, p. 217-218], and Hiriart-Urruty [11]. Convex programming is particularly interesting because under Slater's condition<sup>1</sup>, the standard Karush-Kuhn-Tucker (KKT) optimality conditions are not only necessary but also sufficient and in addition, the concavity property of the  $g_j$ 's is used to prove convergence (and rates of convergence) of specialized algorithms.

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<sup>1</sup>Slater's condition holds if  $g_j(\mathbf{x}_0) > 0$  for some  $\mathbf{x}_0 \in \mathbf{K}$  and all  $j = 1, \dots, m$ .

To the best of our knowledge, little is said in the literature for the specific case where  $\mathbf{K}$  is convex but not necessarily its representation, that is, when the functions  $(g_j)$  are *not* necessarily concave. It looks like outside the convex programming framework, all problems are treated the same. This paper is a companion paper to [12] where we proved that if the nondegeneracy condition

$$(1.3) \quad \forall j = 1, \dots, m : \quad \nabla g_j(\mathbf{x}) \neq 0 \quad \forall \mathbf{x} \in \mathbf{K} \text{ with } g_j(\mathbf{x}) = 0$$

holds, then  $\mathbf{x} \in \mathbf{K}$  is a global minimizer of  $f$  on  $\mathbf{K}$  if and only if  $(\mathbf{x}, \lambda)$  is a KKT point for some  $\lambda \in \mathbb{R}_+^m$ . This indicates that for convex optimization problems (1.1), and from the point of view of "first-order optimality conditions", what really matters is the geometry of  $\mathbf{K}$  rather than its representation. Indeed, for *any* representation  $(g_j)$  of  $\mathbf{K}$  that satisfies the nondegeneracy condition (1.3), there is a one-to-one correspondence between global minimizers and KKT points.

But what about from a computational viewpoint? Of course, not all representations of  $\mathbf{K}$  are equivalent since the ability (as well as the efficiency) of algorithms to obtain a KKT point of  $\mathbf{P}$  will strongly depend on the representation  $(g_j)$  of  $\mathbf{K}$  which is used. For example, algorithms that implement Lagrangian duality would require the  $(g_j)$  to be concave, those based on second-order methods would require all functions  $f$  and  $(g_j)$  to be twice continuous differentiable, self-concordance of a barrier function associated with a representation of  $\mathbf{K}$  may or may not hold, etc.

When  $\mathbf{K}$  is convex but not its representation  $(g_j)$ , several situations may occur. In particular, the level set  $\{\mathbf{x} : g_j(\mathbf{x}) \geq a_j\}$  may be convex for  $a_j = 0$  but not for some other values of  $a_j > 0$ , in which case the  $g_j$ 's are not even quasiconcave on  $\mathbf{K}$ , i.e., one may say that  $\mathbf{K}$  is convex *by accident* for the value  $\mathbf{a} = 0$  of the parameter  $\mathbf{a} \geq 0$ . One might think that in this situation, algorithms that generate a sequence of feasible points in the interior of  $\mathbf{K}$  could run into problems to find a local minimum of  $f$ . If the  $-g_j$ 's are all quasiconvex on  $\mathbf{K}$ , we say that we are in the generic convex case because not only  $\mathbf{K}$  but also all sets  $\mathbf{K}_{\mathbf{a}} := \{\mathbf{x} : g_j(\mathbf{x}) \geq \mathbf{a}_j, j = 1, \dots, m\}$  are convex. However, quasiconvex functions do not share some nice properties of the convex functions. In particular, (a)  $\nabla g_j(\mathbf{x}) = 0$  does not imply that  $g_j$  reaches a local minimum at  $\mathbf{x}$ , (b) a local minimum is not necessarily global and (c), the sum of quasiconvex functions is not quasiconvex in general; see e.g. Crouzeix et al. [5, p. 65]. And so even in this case, for some minimization algorithms, convergence to a minimum of  $f$  on  $\mathbf{K}$  might be problematic.

So an interesting issue is to determine whether there is an algorithm which converges to a global minimizer of a convex function  $f$  on  $\mathbf{K}$ , no matter if the representation of  $\mathbf{K}$  is convex or not. Of course, in view of [12, Theorem 2.3], a sufficient condition is that this algorithm provides a sequence (or subsequence) of points  $(\mathbf{x}_k, \lambda_k) \in \mathbb{R}^n \times \mathbb{R}_+^m$  converging to a KKT point of  $\mathbf{P}$ .

With  $\mathbf{P}$  and a parameter  $\mu > 0$ , we associate the *log-barrier* function  $\phi_\mu : \mathbf{K} \rightarrow \mathbb{R} \cup \{+\infty\}$  defined by

$$(1.4) \quad \mathbf{x} \mapsto \phi_\mu(\mathbf{x}) := \begin{cases} f(\mathbf{x}) - \mu \sum_{j=1}^m \ln g_j(\mathbf{x}), & \text{if } g_j(\mathbf{x}) > 0, \forall j = 1, \dots, m \\ +\infty, & \text{otherwise.} \end{cases}$$

By a *stationary point*  $\mathbf{x} \in \mathbf{K}$  of  $\phi_\mu$ , we mean a point  $\mathbf{x} \in \mathbf{K}$  with  $g_j(\mathbf{x}) \neq 0$  for all  $j = 1, \dots, m$ , and such that  $\nabla \phi_\mu(\mathbf{x}) = 0$ . Notice that in general and in contrast with the present paper,  $\phi_\mu$  (or more precisely  $\psi_\mu := \mu \phi_\mu$ ) is usually defined for convex problems  $\mathbf{P}$  where all the  $g_j$ 's are concave; see e.g. Den Hertog [6] and for more details on the barrier functions and their properties, the interested reader is referred to Güler [9] and Güler and Tuncel [10].

**Contribution.** The purpose of this paper is to show that no matter which representation ( $g_j$ ) of a convex set  $\mathbf{K}$  (assumed to be compact) is used (provided it satisfies the nondegeneracy condition (1.3)), any sequence of stationary points ( $\mathbf{x}_\mu$ ) of  $\phi_\mu$ ,  $\mu \rightarrow 0$ , has the nice property that each of its accumulation points is a KKT point of  $\mathbf{P}$  and hence, a global minimizer of  $f$  on  $\mathbf{K}$ . Hence, to obtain the global minimum of a convex function on  $\mathbf{K}$  it is enough to minimize the log-barrier function for nonincreasing values of the parameter, for any representation of  $\mathbf{K}$  that satisfies the nondegeneracy condition (1.3). Again and of course, the efficiency of the method will crucially depend on the representation of  $\mathbf{K}$  which is used. For instance, in general  $\phi_\mu$  will not have the self-concordance property, crucial for efficiency.

Observe that at first glance this result is a little surprising because as we already mentioned, there are examples of sets  $\mathbf{K}_\mathbf{a} := \{\mathbf{x} : g_j(\mathbf{x}) \geq a_j, j = 1, \dots, m\}$  which are non convex for every  $0 \neq \mathbf{a} \geq 0$  but  $\mathbf{K} := \mathbf{K}_0$  is convex (by accident!) and (1.3) holds. So inside  $\mathbf{K}$  the level sets of the  $g_j$ 's are not convex any more. Still, and even though the stationary points  $\mathbf{x}_\mu$  of the associated log-barrier  $\phi_\mu$  are inside  $\mathbf{K}$ , all converging subsequences of a sequence ( $\mathbf{x}_\mu$ ),  $\mu \rightarrow 0$ , will converge to some global minimizer  $\mathbf{x}^*$  of  $f$  on  $\mathbf{K}$ . In particular, if the global minimizer  $\mathbf{x}^* \in \mathbf{K}$  is unique then the whole sequence ( $\mathbf{x}_\mu$ ) will converge. Notice that this happens even if the  $g_j$ 's are not log-concave, in which case  $\phi_\mu$  may not be convex for all  $\mu$  (e.g. if  $f$  is linear). So what seems to really matter is the fact that as  $\mu$  decreases, the convex function  $f$  becomes more and more important in  $\phi_\mu$ , and also that the functions  $g_j$  which matter in a KKT point  $(\mathbf{x}^*, \lambda)$  are those for which  $g_j(\mathbf{x}^*) = 0$  (and so with convex associated level set  $\{\mathbf{x} : g_j(\mathbf{x}) \geq 0\}$ ).

## 2. MAIN RESULT

Consider the optimization problem (1.1) in the following context.

**Assumption 1.** The set  $\mathbf{K}$  in (1.2) is convex and Slater's assumption holds. Moreover, the nondegeneracy condition

$$(2.1) \quad \nabla g_j(\mathbf{x}) \neq 0 \quad \forall \mathbf{x} \in \mathbf{K} \text{ such that } g_j(\mathbf{x}) = 0,$$

holds for every  $j = 1, \dots, m$ .

Observe that when the  $g_j$ 's are concave then the nondegeneracy condition (2.1) holds automatically. Recall that  $(\mathbf{x}^*, \lambda) \in \mathbf{K} \times \mathbb{R}^m$  is a Karush-Kuhn-Tucker (KKT) point of  $\mathbf{P}$  if

- $\mathbf{x} \in \mathbf{K}$  and  $\lambda \geq 0$
- $\lambda_j g_j(\mathbf{x}^*) = 0$  for every  $j = 1, \dots, m$
- $\nabla f(\mathbf{x}^*) - \sum_{j=1}^m \lambda_j \nabla g_j(\mathbf{x}^*) = 0$ .

We recall the following result from [12]:

**Theorem 1** ([12]). *Let  $\mathbf{K}$  be as in (1.2) and let Assumption 1 hold. Then  $\mathbf{x}$  is a global minimizer of  $f$  on  $\mathbf{K}$  if and only if there is some  $\lambda \in \mathbb{R}_+^m$  such that  $(\mathbf{x}, \lambda)$  is a KKT point of  $\mathbf{P}$ .*

The next result is concerned with the log-barrier  $\phi_\mu$  in (1.4).

**Lemma 2.** *Let  $\mathbf{K}$  in (1.2) be convex and compact and assume that Slater's condition holds. Then for every  $\mu > 0$  the log-barrier function  $\phi_\mu$  in (1.4) has at least one stationary point on  $\mathbf{K}$  (which is a global minimizer of  $\phi_\mu$  on  $\mathbf{K}$ ).*

*Proof.* Let  $f^*$  be the minimum of  $f$  on  $\mathbf{K}$  and let  $\mu > 0$  be fixed, arbitrary. We first show that  $\phi_\mu(\mathbf{x}_k) \rightarrow \infty$  as  $\mathbf{x}_k \rightarrow \partial\mathbf{K}$  (where  $(\mathbf{x}_k) \subset \mathbf{K}$ ). Indeed, pick up an index  $i$  such that  $g_i(\mathbf{x}_k) \rightarrow 0$  as  $k \rightarrow \infty$ . Then  $\phi_\mu(\mathbf{x}_k) \geq f^* - \mu \ln g_i(\mathbf{x}_k) - (m-1) \ln C$  (where all the  $g_j$ 's are bounded above by  $C$ ). So  $\phi_\mu$  is coercive and therefore must have a (global) minimizer  $\mathbf{x}_\mu \in \mathbf{K}$  with  $g_j(\mathbf{x}_\mu) > 0$  for every  $j = 1, \dots, m$ ; and so  $\nabla\phi_\mu(\mathbf{x}_\mu) = 0$ .  $\square$

Notice that  $\phi_\mu$  may have several stationary points in  $\mathbf{K}$ . We now state our main result.

**Theorem 3.** *Let  $\mathbf{K}$  in (1.2) be compact and let Assumption 1 hold true. For every fixed  $\mu > 0$ , choose  $\mathbf{x}_\mu \in \mathbf{K}$  to be an arbitrary stationary point of  $\phi_\mu$  in  $\mathbf{K}$ .*

*Then every accumulation point  $\mathbf{x}^* \in \mathbf{K}$  of such a sequence  $(\mathbf{x}_\mu) \subset \mathbf{K}$  with  $\mu \rightarrow 0$ , is a global minimizer of  $f$  on  $\mathbf{K}$ , and if  $\nabla f(\mathbf{x}^*) \neq 0$ ,  $\mathbf{x}^*$  is a KKT point of  $\mathbf{P}$ .*

*Proof.* Let  $\mathbf{x}_\mu \in \mathbf{K}$  be a stationary point of  $\phi_\mu$ , which by Lemma 2 is guaranteed to exist. So

$$(2.2) \quad \nabla\phi_\mu(\mathbf{x}_\mu) = \nabla f(\mathbf{x}_\mu) - \sum_{j=1}^m \frac{\mu}{g_j(\mathbf{x}_\mu)} \nabla g_j(\mathbf{x}_\mu) = 0.$$

As  $\mu \rightarrow 0$  and  $\mathbf{K}$  is compact, there exists  $\mathbf{x}^* \in \mathbf{K}$  and a subsequence  $(\mu_\ell) \subset \mathbb{R}_+$  such that  $\mathbf{x}_{\mu_\ell} \rightarrow \mathbf{x}^*$  as  $\ell \rightarrow \infty$ . We need consider two cases:

*Case when  $g_j(\mathbf{x}^*) > 0, \forall j = 1, \dots, m$ .* Then as  $f$  and  $g_j$  are continuously differentiable,  $j = 1, \dots, m$ , taking limit in (2.2) for the subsequence  $(\mu_\ell)$ , yields  $\nabla f(\mathbf{x}^*) = 0$  which, as  $f$  is convex, implies that  $\mathbf{x}^*$  is a global minimizer of  $f$  on  $\mathbb{R}^n$ , hence on  $\mathbf{K}$ .

*Case when  $g_j(\mathbf{x}^*) = 0$  for some  $j \in \{1, \dots, m\}$ .* Let  $J := \{j : g_j(\mathbf{x}^*) = 0\} \neq \emptyset$ . We next show that for every  $j \in J$ , the sequence of ratios  $(\mu/g_j(\mathbf{x}_{\mu_\ell}), \ell = 1, \dots)$ , is bounded. Indeed let  $j \in J$  be fixed arbitrary. As Slater's condition holds, let  $\mathbf{x}_0 \in \mathbf{K}$  be such that  $g_j(\mathbf{x}_0) > 0$  for all  $j = 1, \dots, m$ ; then  $\langle \nabla g_j(\mathbf{x}^*), \mathbf{x}_0 - \mathbf{x}^* \rangle > 0$ . Indeed, as  $\mathbf{K}$  is convex,  $\langle \nabla g_j(\mathbf{x}^*), \mathbf{x}_0 + \mathbf{v} - \mathbf{x}^* \rangle \geq 0$  for all  $\mathbf{v}$  in some small enough ball  $\mathbf{B}(0, \rho)$  around the origin. So if  $\langle \nabla g_j(\mathbf{x}^*), \mathbf{x}_0 - \mathbf{x}^* \rangle = 0$  then  $\langle \nabla g_j(\mathbf{x}^*), \mathbf{v} \rangle \geq 0$  for all  $\mathbf{v} \in \mathbf{B}(0, \rho)$ , in contradiction with  $\nabla g_j(\mathbf{x}^*) \neq 0$ . Next,

$$(2.3) \quad \langle \nabla f(\mathbf{x}_{\mu_\ell}), \mathbf{x}_0 - \mathbf{x}^* \rangle = \underbrace{\sum_{k \notin J} \frac{\mu_\ell}{g_k(\mathbf{x}_{\mu_\ell})} \langle \nabla g_k(\mathbf{x}_{\mu_\ell}), \mathbf{x}_0 - \mathbf{x}^* \rangle}_{A_\ell} + \underbrace{\sum_{k \in J} \frac{\mu_\ell}{g_k(\mathbf{x}_{\mu_\ell})} \langle \nabla g_k(\mathbf{x}_{\mu_\ell}), \mathbf{x}_0 - \mathbf{x}^* \rangle}_{B_\ell}$$

Observe that in (2.3):

- Every term of the sum  $B_\ell$  is nonnegative for sufficiently large  $\ell$ , say  $\ell \geq \ell_0$ , because  $\mathbf{x}_{\mu_\ell} \rightarrow \mathbf{x}^*$  and  $\langle \nabla g_k(\mathbf{x}^*), \mathbf{x}_0 - \mathbf{x}^* \rangle > 0$  for all  $k \in J$ .
- $A_\ell \rightarrow 0$  as  $\ell \rightarrow \infty$  because  $\mu_\ell \rightarrow 0$  and  $g_k(\mathbf{x}_{\mu_\ell}) \rightarrow g_k(\mathbf{x}^*) > 0$  for all  $k \notin J$ .

Therefore  $|A_\ell| \leq A$  for all sufficiently large  $\ell$ , say  $\ell \geq \ell_1$ , and so for every  $j \in J$ :

$$\langle \nabla f(\mathbf{x}_{\mu_\ell}), \mathbf{x}_0 - \mathbf{x}^* \rangle + A \geq \frac{\mu_\ell}{g_j(\mathbf{x}_{\mu_\ell})} \langle \nabla g_j(\mathbf{x}_{\mu_\ell}), \mathbf{x}_0 - \mathbf{x}^* \rangle, \quad \ell \geq \ell_2 := \max[\ell_0, \ell_1],$$

which shows that for every  $j \in J$ , the nonnegative sequence  $(\mu_\ell/g_j(\mathbf{x}_{\mu_\ell}))$ ,  $\ell \geq \ell_2$ , is bounded from above.

So take a subsequence (still denoted  $(\mu_\ell)$ ,  $\ell \in \mathbb{N}$ , for convenience) such that the ratios  $\mu_\ell/g_j(\mathbf{x}_{\mu_\ell})$  converge for all  $j \in J$ , that is,

$$\lim_{\ell \rightarrow \infty} \frac{\mu_\ell}{g_j(\mathbf{x}_{\mu_\ell})} = \lambda_j \geq 0, \quad \forall j \in J,$$

and let  $\lambda_j := 0$  for every  $j \notin J$ , so that  $\lambda_j g_j(\mathbf{x}^*) = 0$  for every  $j = 1, \dots, m$ . Taking limit in (2.2) as  $\ell \rightarrow \infty$ , yields:

$$(2.4) \quad \nabla f(\mathbf{x}^*) = \sum_{j=1}^m \lambda_j \nabla g_j(\mathbf{x}^*),$$

which shows that  $(\mathbf{x}^*, \lambda) \in \mathbf{K} \times \mathbb{R}_+^m$  is a KKT point for  $\mathbf{P}$ . Finally, invoking Theorem 1,  $\mathbf{x}^*$  is also a global minimizer of  $\mathbf{P}$ .  $\square$

**2.1. Discussion.** The log-barrier function  $\phi_\mu$  or its exponential variant  $f + \mu \sum g_j^{-1}$  has become popular since the pioneer work of Fiacco and McCormick [7, 8], where it is assumed that  $f$  and the  $g_j$ 's are twice continuously differentiable, the  $g_j$ 's are concave<sup>2</sup>, Slater's condition holds, the set  $\mathbf{K} \cap \{\mathbf{x} : f(\mathbf{x}) \leq k\}$  is bounded for every finite  $k$ , and finally, the barrier function is strictly convex for every value of the parameter  $\mu > 0$ . Under such conditions, the barrier function  $f + \mu \sum g_j^{-1}$  has a unique minimizer  $\mathbf{x}_\mu$  for every  $\mu > 0$  and the sequence  $(\mathbf{x}_\mu, (\mu/g_j(\mathbf{x}_\mu))^2) \subset \mathbb{R}^{n+m}$  converges to a Wolfe-dual feasible point.

In contrast, Theorem 3 states that without assuming concavity of the  $g_j$ 's, one may obtain a global minimizer of  $f$  on  $\mathbf{K}$ , by looking at *any* limit point of *any* sequence of stationary points  $(\mathbf{x}_\mu)$ ,  $\mu \rightarrow 0$ , of the log-barrier function  $\phi_\mu$  associated with a representation  $(g_j)$  of  $\mathbf{K}$ , provided that the representation satisfies the non-degeneracy condition (1.3). To us, this comes as a little surprise as the stationary points  $(\mathbf{x}_\mu)$  are all inside  $\mathbf{K}$ , and there are examples of convex sets  $\mathbf{K}$  with a representation  $(g_j)$  satisfying (1.3) and such that the level sets  $\mathbf{K}_a = \{\mathbf{x} : g_j(\mathbf{x}) \geq a_j\}$  with  $a_j > 0$ , are not convex! (See Example 1.) Even if  $f$  is convex, the log-barrier function  $\phi_\mu$  need not be convex; for instance if  $f$  is linear,  $\nabla^2 \phi_\mu = -\mu \sum_j \nabla^2 \ln g_j$ , and so if the  $g_j$ 's are not log-concave then  $\phi_\mu$  may not be convex on  $\mathbf{K}$  for every value of the parameter  $\mu > 0$ .

**Example 1.** Let  $n = 2$  and  $\mathbf{K}_a := \{\mathbf{x} \in \mathbb{R}^2 : g(\mathbf{x}) \geq a\}$  with  $\mathbf{x} \mapsto g(\mathbf{x}) := 4 - ((x_1 + 1)^2 + x_2^2)((x_1 - 1)^2 + x_2^2)$ , with  $a \in \mathbb{R}$ . The set  $\mathbf{K}_a$  is convex only for those values of  $a$  with  $a \leq 0$ ; see in Figure 1. It is even disconnected for  $a = 4$ .

<sup>2</sup>In fact as noted in [7], concavity of the  $g_j$ 's is merely a sufficient condition for the barrier function to be convex.

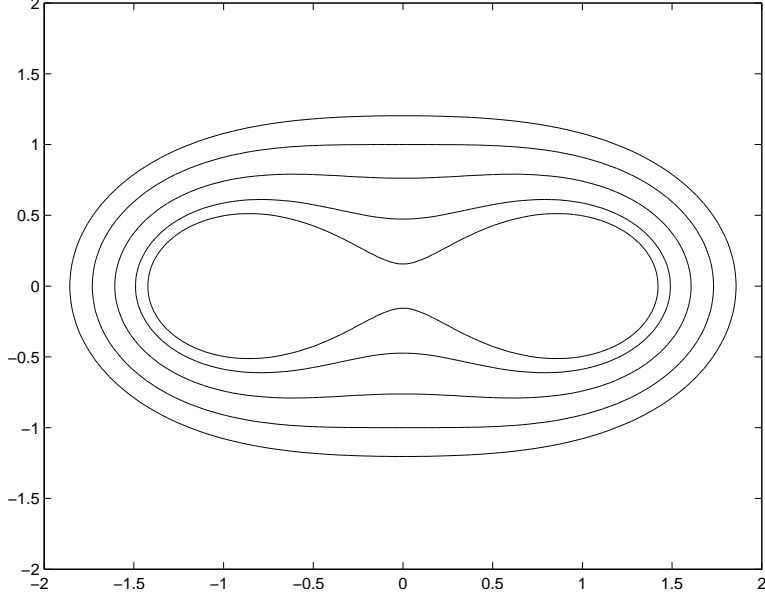


FIGURE 1. Example 1: Level sets  $\{\mathbf{x} : g(\mathbf{x}) = a\}$  for  $a = 2.95, 2.5, 1.5, 0$  and  $-2$

We might want to consider a generic situation, that is, when the set

$$\mathbf{K}_{\mathbf{a}} := \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq a_j, \quad j = 1, \dots, m\},$$

is also convex for every positive vector  $0 \leq \mathbf{a} = (a_j) \in \mathbb{R}^m$ . This in turn would imply that the  $g_j$  are *quasiconcave*<sup>3</sup> on  $\mathbf{K}$ . In particular, if the nondegeneracy condition (1.3) holds on  $\mathbf{K}$  and the  $g_j$ 's are twice differentiable, then at most one eigenvalue of the Hessian  $\nabla^2 g_j$  (and hence  $\nabla^2 \ln g_j$ ) is possibly positive (i.e.,  $\ln g_j$  is *almost* concave). This is because for every  $\mathbf{x} \in \mathbf{K}$  with  $g_j(\mathbf{x}) = 0$ , one has  $\langle \mathbf{v}, \nabla^2 g_j(\mathbf{x}) \mathbf{v} \rangle \leq 0$  for all  $\mathbf{v} \in \nabla g_j(\mathbf{x})^\perp$  (where  $\nabla g_j(\mathbf{x})^\perp := \{\mathbf{v} : \langle \nabla g_j(\mathbf{x}), \mathbf{v} \rangle = 0\}$ ). However, even in this situation, the log-barrier function  $\phi_\mu$  may not be convex. On the other hand,  $\ln g_j$  is "more" concave than  $g_j$  on  $\text{Int } \mathbf{K}$  because its Hessian  $\nabla^2 g_j$  satisfies  $g_j^2 \nabla^2 \ln g_j = g_j \nabla^2 g_j - \nabla g_j (\nabla g_j)^T$ . But still,  $g_j$  might not be log-concave on  $\text{Int } \mathbf{K}$ , and so  $\phi_\mu$  may not be convex at least for values of  $\mu$  not too small (and for all values of  $\mu$  if  $f$  is linear).

**Example 2.** Let  $n = 2$  and  $\mathbf{K} := \{\mathbf{x} : g(\mathbf{x}) \geq 0, \mathbf{x} \geq 0\}$  with  $\mathbf{x} \mapsto g(\mathbf{x}) = x_1 x_2 - 1$ . The representation of  $\mathbf{K}$  is not convex but the  $g_j$ 's are log-concave, and so the log-barrier  $\mathbf{x} \mapsto \phi_\mu(\mathbf{x}) := f(\mathbf{x}) - \mu(\ln g(\mathbf{x}) - \ln x_1 - \ln x_2)$  is convex.

**Example 3.** Let  $n = 2$  and  $\mathbf{K} := \{\mathbf{x} : g_1(\mathbf{x}) \geq 0; a - x_1 \geq 0; 0 \leq x_2 \leq b\}$  with  $\mathbf{x} \mapsto g_1(\mathbf{x}) = x_1 / (\epsilon + x_2^2)$  with  $\epsilon > 0$ . The representation of  $\mathbf{K}$  is not convex and  $g_1$  is not log-concave. If  $f$  is linear and  $\epsilon$  is small enough, the log-barrier

$$\mathbf{x} \mapsto \phi_\mu(\mathbf{x}) := f(\mathbf{x}) - \mu(\ln x_1 + \ln(a - x_1) - \ln(\epsilon + x_2^2) + \ln x_2 + \ln(b - x_2))$$

<sup>3</sup>Recall that on a convex set  $O \subset \mathbb{R}^n$ , a function  $f : O \rightarrow \mathbb{R}$  is quasiconvex if the level sets  $\{\mathbf{x} : f(\mathbf{x}) \leq r\}$  are convex for every  $r \in \mathbb{R}$ . A function  $f : O \rightarrow \mathbb{R}$  is said to be quasiconcave if  $-f$  is quasiconvex; see e.g. [5].

is not convex for every value of  $\mu > 0$ .

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