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HIGHER DIMENSIONAL TAUTOLOGICAL INEQUALITIES AND APPLICATIONS

CARLO GASBARRI, GIANLUCA PACIENZA AND ERWAN ROUSSEAU

ABSTRACT. We study the degeneracy of holomorphic mappings tangent to holomorphic foliations on projective manifolds. Using Ahlfors currents in higher dimension, we obtain several degeneracy statements.

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1. INTRODUCTION

In the last decades, many efforts have been done to understand the geometry of subvarieties of varieties of general type. One of the main motivation is the fascinating conjectural relation between analytic aspects and arithmetic ones. On the geometric side, the philosophy (Green-Griffiths, Lang, Vojta, Campana) is that positivity properties of the canonical bundle of a projective manifold should impose strong restrictions on its subvarieties.

One of the first striking results is the following theorem of Bogomolov [1] for surfaces

Theorem 1.1 (Bogomolov). *There are only finitely many rational and elliptic curves on a surface of general type with $c_1^2 > c_2$.*

In this theorem, the hypothesis $c_1^2 > c_2$ ensures that the cotangent bundle is big, so that rational and elliptic curves are shown to be leaves of a foliation and then, one can use results on algebraic leaves of foliations [9].

Several decades later, this result was extended to transcendental leaves of foliations by McQuillan [12]

Theorem 1.2 (McQuillan). *Let X be a surface of general type and \mathcal{F} a holomorphic foliation on X . Then \mathcal{F} has no entire leaf which is Zariski dense.*

As a consequence he obtains the following.

Corollary 1.3 (McQuillan). *On a surface X of general type with $c_1^2 > c_2$, there is no entire curve $f : \mathbb{C} \rightarrow X$ which is Zariski dense.*

It is of course of great interest to generalize these results, even partially, to higher dimension. On the algebraic side, this was investigated by Lu and Miyaoka in [10].

Theorem 1.4 (Lu-Miyaoka). *Let X be a nonsingular projective variety. If X is of general type, then X has only a finite number of nonsingular codimension-one subvarieties having pseudoeffective anticanonical divisor. In particular, X has only a finite number of nonsingular codimension-one Fano, Abelian, and Calabi-Yau subvarieties.*

This can be seen as a generalization to higher dimension of the aforementioned theorem of Bogomolov.

In this paper, we would like to study some generalizations of McQuillan's result. The study of several variables holomorphic maps into X in relation with the hyperbolicity properties of X has been considered in [18], where the extension to this larger framework of the Demailly-Semple jet-spaces technology was given, together with several applications. Here, for $1 \leq p \leq n-1$, we consider non-degenerate holomorphic mapping $f : \mathbb{C}^p \rightarrow X$, in a projective manifold of dimension n , such that the image of f is tangent to a holomorphic foliation \mathcal{F} on X .

If the foliation is smooth we obtain the following.

Theorem A. *Let \mathcal{F} be a smooth foliation of dimension p on a projective manifold X , with $p \leq \dim X = n$. If X is of general type then there is no holomorphic mapping $f : \mathbb{C}^p \rightarrow X$ of generic maximal rank tangent to \mathcal{F} which is Zariski dense.*

Most of the examples of foliations are not smooth and the natural generalizations to the singular case may be highly non-trivial. For instance, in the 2-dimensional case treated by McQuillan the technical core of the proof consists in the detailed study of the contribution of singularities of the foliations. On the other hand, in some cases, we have results reducing the study of non-smooth foliations to some special classes of singularities, see e.g. [20, 4, 15]. In particular, thanks to the work of McQuillan, a class of singularities emerged as a very natural one: the class of canonical singularities (see Definition 6.1). Notice that the more classical logarithmic simple singularities (see (4.1)) fall in this class. In this framework we obtain several results. Supposing that the singularities of the foliation are of logarithmic simple type we show the following.

Theorem B. *Let \mathcal{F} be a holomorphic foliation of codimension one on a projective manifold X of dimension n . Suppose $\text{Sing}\mathcal{F}$ consists only of logarithmic simple singularities. Consider a holomorphic mapping $f : \mathbb{C}^{n-1} \rightarrow X$ of generic maximal rank tangent to \mathcal{F} . If the canonical divisor $K_{\mathcal{F}}$ of the foliation is big, then f is not Zariski dense.*

Theorem B may be seen as an illustration of the Green-Griffiths principle in the setting of holomorphic maps tangent to foliations. As an immediate application we obtain the following.

Corollary C. *Let \mathcal{F} be a holomorphic foliation of codimension one on \mathbb{P}^n of degree $d \geq n$. Suppose $\text{Sing}\mathcal{F}$ consists only of logarithmic simple singularities. Consider a holomorphic mapping $f : \mathbb{C}^{n-1} \rightarrow \mathbb{P}^n$ of generic maximal rank tangent to \mathcal{F} . Then f is not Zariski dense.*

If the singularities are canonical, and the foliation has local first integrals (see (5.1)) we get the following.

Theorem D. *Let \mathcal{F} be a holomorphic foliation of codimension one on a projective manifold of general type X of dimension n . Suppose $\text{Sing}\mathcal{F}$ consists only of canonical singularities with local first integrals. Consider a holomorphic mapping $f : \mathbb{C}^{n-1} \rightarrow X$ of generic maximal rank tangent to \mathcal{F} . Then f is not Zariski-dense.*

As a corollary of the classical result of Malgrange [11] we deduce the following.

Corollary E. *Let \mathcal{F} be a holomorphic foliation of codimension one on a projective manifold X , $\dim X = n$. Suppose X is of general type and $\text{codim}\text{Sing}\mathcal{F} \geq 3$ where all singularities are canonical. Then there is no holomorphic mapping $f : \mathbb{C}^{n-1} \rightarrow X$ of maximal rank tangent to \mathcal{F} which is Zariski dense.*

We construct explicit examples of varieties of general type with foliations satisfying the hypotheses of the preceding theorems.

Let us indicate the methods of the proofs, which may be of independent interest. We recall that McQuillan ([12]) showed that one can associate to a transcendental entire curve f a closed positive current of bidimension $(1, 1)$ called Ahlfors current.

In the second section, we generalize the construction of such currents for arbitrary non-degenerate holomorphic mappings $f : \mathbb{C}^p \rightarrow X$ in compact Kähler manifolds and show how classical Nevanlinna theory (see [7] or [21]) translate into intersection theory for such currents. The problem of associating to several variables holomorphic maps currents with suitable properties has been recently considered by de Thélin and Burns-Sibony in [5, 3]. We refer the reader to these interesting papers for more details on their motivations and for applications in other directions.

The most important part of this theory is developed in the third section which is devoted to the proof of several *tautological inequalities*. They are particularly useful when the holomorphic mappings we study are tangent to holomorphic foliations. This is the object of section 4, where we prove that if the singularities of the foliation are mild, we can control the intersection of the Ahlfors current with the canonical bundle of the foliation. This leads to the theorems stated above proved in section 5. We illustrate these results, in section 6, with examples of foliated varieties having the appropriate singularities.

In the final section, we prove a desingularization statement for Ahlfors currents in dimension 3 which will be used in a sequel of this paper.

The present work leads to several questions. Two of them seem particularly interesting.

Question 1.5. *Is it possible to remove the hypothesis of the existence of local first integrals from Theorem D?*

Notice that a positive answer to the previous question when $\dim(X) = 3$ would lead, thanks to the work of Cano [4], to the non-existence of Zariski dense holomorphic maps from \mathbb{C}^2 into a threefold of general type, which are tangent to a holomorphic foliation.

Question 1.6. *Is it possible to find (numerical/geometrical) conditions insuring the existence of (codimension one) holomorphic foliations on varieties of general type?*

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2. HOLOMORPHIC MAPPINGS AND CLOSED POSITIVE CURRENTS

Let a holomorphic mapping $f : \mathbb{C}^m \rightarrow X$ of maximal rank be given, where X is a complex projective manifold. We want to associate to f a closed positive current of bidimension $(1, 1)$ adapting in higher dimension the ideas of [12] (see also [2]) developed in the one-dimensional case.

Fix a Kähler form ω on X . On \mathbb{C}^m we take the homogeneous metric form

$$\omega_0 := dd^c \ln |z|^2,$$

and

$$\sigma = d^c \ln |z|^2 \wedge \omega_0^{m-1}$$

is the Poincaré form.

Let $\eta \in A^{1,1}(X)$ and $r > 0$ then we define

$$T_{f,r}(\eta) = \int_0^r \frac{dt}{t} \int_{B_t} f^* \eta \wedge \omega_0^{m-1},$$

where $B_t \subset \mathbb{C}^m$ is the ball of radius t . Then we consider the positive currents $\Phi_r \in A^{1,1}(X)'$ defined by

$$\Phi_r(\eta) := \frac{T_{f,r}(\eta)}{T_{f,r}(\omega)}.$$

This gives a family of positive currents of bounded mass from which we can extract a subsequence Φ_{r_n} which converges to $\Phi \in A^{1,1}(X)'$.

Let us prove that

Claim 2.1. *We can choose $\{r_n\}$ such that Φ is moreover closed.*

Proof. Take β a smooth $(1, 0)$ form.

$$T_{f,r}(\bar{\partial}\beta) = \int_0^r \frac{dt}{t} \int_{B_t} f^* \bar{\partial}\beta \wedge \omega_0^{m-1}.$$

$$T_{f,r}(\bar{\partial}\beta) - T_{f,1}(\bar{\partial}\beta) = \int_1^r \frac{dt}{t} \int_{B_t} f^* \bar{\partial}\beta \wedge \omega_0^{m-1},$$

which, by Stokes' formula and then by Fubini's theorem, is equal to

$$\int_1^r \frac{dt}{t} \int_{S_t} f^* \beta \wedge \omega_0^{m-1} = \frac{1}{2} \int_{B_r \setminus B_1} d \ln |z|^2 \wedge f^* \beta \wedge \omega_0^{m-1}.$$

Then Cauchy-Schwartz gives

$$\begin{aligned} & \left| \frac{1}{2} \int_{B_r \setminus B_1} d \ln |z|^2 \wedge f^* \beta \wedge \omega_0^{m-1} \right| \\ & \leq \pi \left| \int_{B_r \setminus B_1} d \ln |z|^2 \wedge d^c \ln |z|^2 \wedge \omega_0^{m-1} \right|^{\frac{1}{2}} \cdot \left| \int_{B_r \setminus B_1} f^* \beta \wedge f^* \bar{\beta} \wedge \omega_0^{m-1} \right|^{\frac{1}{2}} \\ & \leq C \ln(r)^{\frac{1}{2}} \cdot \left(\int_{B_r \setminus B_1} f^* \omega \wedge \omega_0^{m-1} \right)^{\frac{1}{2}} \\ & \leq C \ln(r)^{\frac{1}{2}} \cdot \left(r \frac{d}{dr} T_{f,r}(\omega) \right)^{\frac{1}{2}} \end{aligned}$$

Since f is of maximal rank, $T_{f,r}(\omega)$ is strictly increasing and has at least a logarithmic growth, therefore

$$\liminf_{r \rightarrow +\infty} \frac{r \ln r \frac{d}{dr} T_{f,r}(\omega)}{T_{f,r}(\omega)^2} = 0,$$

which concludes the proof. \square

Remark 2.2. *We remark that all this extends to the setting of meromorphic maps $f : \mathbb{C}^m \rightarrow X$. Indeed, in this case, $f^* \eta$ may have singularities but it has coefficients which are locally integrable.*

Remark 2.3. *One remarks that the above definition can easily be generalized to associate currents of any bidimension (k, k) , $1 \leq k \leq m$ to $f : \mathbb{C}^m \rightarrow X$. But doing this, one loses for $k \neq 1$ the closedness property as discussed in [5] and recently in [3]. Moreover, for our problem, Green-Griffiths' philosophy suggests that the geometry of these holomorphic maps should be determined by a line bundle, K_X .*

We denote by $[\Phi] \in H^{n-1, n-1}(X, \mathbb{R})$ the cohomology class of Φ . A consequence of the *First Main Theorem* of Nevanlinna theory (see [21] p.63) is (see also [3] for the same observation)

Lemma 2.4. *Let $Z \subset X$ an algebraic hypersurface. If $f(\mathbb{C}^m) \not\subset Z$ then $[\Phi].[Z] \geq 0$.*

Proof. We follow the proof of the First Main theorem given in [21] p.61-63. Let s be a section of $\mathcal{O}(Z)$ such that $Z = \{s = 0\}$ and fix a metric h on the line bundle $\mathcal{O}(Z)$. Then by the Poincaré-Lelong formula

$$dd^c \ln \|s\|^2 = Z - \Theta_h,$$

in the sense of currents, where Θ_h is the curvature associated to h . Taking pullbacks by f , assuming $f(0) \notin Z$, and integrating, we obtain

$$\int_{B_t} dd^c \ln \|s \circ f\|^2 \wedge \omega_0^{m-1} = - \int_{B_t} f^*(\Theta_h) \wedge \omega_0^{m-1} + \int_{f^{-1}(Z) \cap B_t} \omega_0^{m-1}.$$

By Stokes formula, one has

$$\begin{aligned} \int_{B_t} dd^c \ln \|s \circ f\|^2 \wedge \omega_0^{m-1} &= \int_{S_t} d^c \ln \|s \circ f\|^2 \wedge \omega_0^{m-1}, \\ \int_0^r \frac{dt}{t} \int_{S_t} d^c \ln \|s \circ f\|^2 \wedge \omega_0^{m-1} &= \int_{S_r} \ln \|s \circ f\| \sigma - \ln \|s \circ f(0)\|, \end{aligned}$$

therefore

$$T_{f,r}(\Theta_h) = \int_0^r \frac{dt}{t} \int_{f^{-1}(Z) \cap B_t} \omega_0^{m-1} + \int_{S_r} \ln \frac{1}{\|s \circ f\|} \sigma + \ln \|s \circ f(0)\|.$$

This can be written using the usual notations of Nevanlinna theory

$$T_{f,r}(\Theta_h) = N_f(Z, r) + m_f(Z, r) + O(1).$$

We can suppose $\|s\| \leq 1$ therefore $m_f(Z, r) \geq 0$ and

$$\Phi(\Theta_h) = \lim_{r_n \rightarrow +\infty} \frac{T_{f,r_n}(\Theta_h)}{T_{f,r_n}(\omega)} \geq 0.$$

□

3. TAUTOLOGICAL INEQUALITIES

3.1. The compact case. Let $X_1 := G(m, TX)$ be the Grassmannian bundle, $\pi : X_1 \rightarrow X$ the natural projection, S_1 the tautological bundle on X_1 and $L := \bigwedge^m S_1$.

For $f : \mathbb{C}^m \rightarrow X$ a holomorphic mapping of maximal rank, we have a natural lifting $f_1 : \mathbb{C}^m \rightarrow X_1$ defined by $f_1 = (f, [\frac{\partial f}{\partial t_1} \wedge \dots \wedge \frac{\partial f}{\partial t_m}])$. Then we can associate a closed positive current Φ_1 to f_1 . The tautological inequality becomes

Theorem 3.1.

$$[\Phi_1].L \geq 0.$$

Proof. ω induces a metric on L of curvature Θ . $\frac{\partial f}{\partial t_1} \wedge \dots \wedge \frac{\partial f}{\partial t_m} : \mathbb{C}^m \rightarrow \bigwedge^m T_X$ defines a section $s \in H^0(\mathbb{C}^m, f_1^*L)$. By the Poincaré-Lelong formula, we have:

$$dd^c \ln \|s\|^2 = (s = 0) - f_1^*\Theta.$$

Therefore, denoting $\xi := \left\| \frac{\partial f}{\partial t_1} \wedge \cdots \wedge \frac{\partial f}{\partial t_m} \right\|^2$, we have

$$-T_{f_1,r}(\Theta) \leq \int_0^r \frac{dt}{t} \int_{B_t} dd^c \ln \xi \wedge \omega_0^{m-1},$$

and as above, using moreover the convexity of the logarithm, we obtain

$$-T_{f_1,r}(\Theta) \leq C + \frac{m}{2} \ln \int_{S_r} \xi^{\frac{1}{m}} \sigma.$$

Using the Euclidean metric form φ_0 which satisfies $\frac{\varphi_0^m}{m!} = r^{2m-1} \sigma \wedge dr$, we have

$$\int_{S_r} \xi^{\frac{1}{m}} \sigma = \frac{1}{r^{2m-1}} \frac{d}{dr} \frac{1}{m!} \int_{B_r} \xi^{\frac{1}{m}} \varphi_0^m = \frac{1}{m!} \frac{1}{r^{2m-1}} \frac{d}{dr} \left(r^{2m-1} \frac{d\tilde{T}}{dr} \right),$$

where we denote

$$\tilde{T}(r) := \int_0^r \frac{dt}{t^{2m-1}} \int_{B_t} \xi^{\frac{1}{m}} \varphi_0^m.$$

Now, we use the following classical lemma in Nevanlinna theory

Lemma 3.2. *Let $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a positive, increasing, derivable function. For every $\epsilon > 0$, there exists $E \subset \mathbb{R}$ satisfying $\int_E dr \leq \int_{1+\epsilon}^{+\infty} \frac{1}{x \ln^{1+\epsilon}(x)} dx < +\infty$ such that for every $x \notin E$,*

$$F'(x) \leq F(x) \ln^{1+\epsilon}(F(x)).$$

We apply this lemma to the function $r^{2m-1} \frac{d\tilde{T}}{dr}$, and we obtain that

$$\frac{d}{dr} \left(r^{2m-1} \frac{d\tilde{T}}{dr} \right) \leq r^{2m-1} \frac{d\tilde{T}}{dr} \ln^{1+\epsilon} \left(r^{2m-1} \frac{d\tilde{T}}{dr} \right),$$

outside E . The lemma applied to \tilde{T} gives

$$\frac{d\tilde{T}}{dr} \leq \tilde{T}(r) \ln^{1+\epsilon}(\tilde{T}(r)),$$

outside E . Now, a consequence of the classical Hadamard inequality is

$$\xi^{\frac{1}{m}} \varphi_0^m \leq f^* \omega \wedge \varphi_0^{m-1}.$$

Therefore we obtain

$$\tilde{T}(r) \leq \int_0^r \frac{dt}{t^{2m-1}} \int_{B_t} f^* \omega \wedge \varphi_0^{m-1} = T_{f,r}(\omega).$$

Finally, these inequalities give

$$(3.1) \quad -T_{f_1,r}(\Theta) \leq O(\ln(T_{f,r}(\omega))),$$

outside E , which proves the theorem. \square

3.2. The logarithmic case. Let X be a smooth projective variety, $D \subset X$ be a simple normal crossing divisor, ω a Kähler metric on X , $f : \mathbb{C}^m \rightarrow X$ a holomorphic mapping of maximal rank such that $f(\mathbb{C}^m) \not\subset D$ and $X_1 := \mathbb{P}(\bigwedge^m T_X(-\log(D)))$. We have a natural map $f_1 : \mathbb{C}^m \rightarrow X_1$ and a tautological line bundle $L := \mathcal{O}_{X_1}(1)$.

Theorem 3.3. $T_{f_1,r}(L) \leq N_f^1(D, r) + O(\ln T_{f,r}(\omega))$

Proof. We follow the approach given in [6] for the 1-dimensional case. We write $D = \sum D_i$. Let s_i be sections of the D_i and we choose hermitian metrics on the associated line bundles. We have a smooth metric on $T_X(-\log(D))$ induced by a singular (1, 1)-form

$$\omega^{sm} := A\omega + \sum \frac{d\|s_i\| \wedge d^c\|s_i\|}{\|s_i\|^2}.$$

And we have a singular hermitian metric on $T_X(-\log(D))$ induced by

$$\tilde{\omega} := \omega + \sum \frac{d\|s_i\| \wedge d^c\|s_i\|}{\|s_i\|^2 (\ln \|s_i\|)^2}.$$

The map f_1 together with the map $\bigwedge^m T_{\mathbb{C}^m} \rightarrow \mathcal{O}_X$ given by $\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_m} \rightarrow 1$ gives a map $g : \mathbb{C}^m \rightarrow Y := \mathbb{P}(\bigwedge^m T_X(-\log(D)) \oplus \mathcal{O}_X)$. Denote by M the tautological line bundle over Y .

We have an inclusion $X_1 \subset Y$ and $\mathcal{O}_Y(X_1) = M$. ω^{sm} and $\tilde{\omega}$ induce respectively smooth and singular metrics on M and L .

Now, we apply the First Main Theorem to g and M with respect to the singular metric:

$$T_{g,r}(c_1(M)^s) = N_g(X_1, r) + m_g(X_1, r) + O(1).$$

g meets X_1 only over D with multiplicity at most 1, therefore

$$N_g(X_1, r) \leq N_f^1(D, r).$$

We consider the blow-up $p : Z \rightarrow Y$ along the zero section of $\bigwedge^m T_X(-\log(D))$. Then we have a holomorphic map $q : Z \rightarrow X_1$ and

$$q^*L = p^*M(-E)$$

where E is the exceptional divisor in Z . Therefore we obtain

$$T_{f_1,r}(c_1(L)^s) \leq N_f^1(D, r) + m_g(X_1, r) - m_g(E, r) + O(1).$$

Comparing the two metrics, we see that

$$T_{f_1,r}(c_1(L)^{sm}) \leq T_{f_1,r}(c_1(L)^s) + O(\ln T_{f,r}(\omega)).$$

We can suppose $m_g(E, r) \geq 0$ so to finish we just have to bound

$$m_g(X_1, r) = \int_{S_r} \ln \frac{1}{\|s \circ f\|} \sigma,$$

where s is a holomorphic section of $\mathcal{O}_Y(X_1)$. Let

$$\xi := f^*(\tilde{\omega}^m) \left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_m} \right)$$

where $\tilde{\omega}^m$ is the metric induced on $\bigwedge^m T_X(-\log(D))$ by $\tilde{\omega}$. To conclude we need to find an upper bound for

$$\int_{S_r} \ln(\xi) \sigma.$$

Following the end of the proof of theorem 3.1, we are reduced to find a bound for

$$\int_0^r \frac{dt}{t} \int_{B_t} f^* \tilde{\omega} \wedge \omega_0^{m-1}.$$

Recall that we have the following formula

$$dd^c \ln(\ln \|s_i\|^2) = \frac{dd^c \ln \|s_i\|^2}{\ln \|s_i\|^2} - \frac{d \ln \|s_i\|^2 \wedge d^c \ln \|s_i\|^2}{(\ln \|s_i\|^2)^2}.$$

Therefore

$$\tilde{\omega} \leq C\omega - \sum dd^c \ln(\ln^2 \|s_i\|)$$

for a constant C , and finally, by Stokes formula

$$\int_0^r \frac{dt}{t} \int_{B_t} f^* \tilde{\omega} \wedge \omega_0^{m-1} \leq O(\ln T_{f,r}(\omega)) - \sum \int_{S_r} \ln(\ln^2 \|s_i\|) \sigma.$$

Since we may assume $\|s_i\| < \delta < 1$, the theorem is proved. \square

Remark 3.4. *We remark that, in the particular case $m = \dim X$, we recover the second main theorems of Griffiths-King [7].*

3.3. Refined inequalities. In the case $m = 1$ of entire curves, McQuillan [12] shows that one can include in the tautological inequality the defect with respect to a finite number of reduced points. Following the approach of [22] to this question, we would like to show that in our situation the previous inequalities can be made more precise by including the defect with respect to submanifolds of codimension at least 2.

Lemma 3.5. *Let $\Delta \subset \mathbb{C}^n$ be an n -dimensional disc with holomorphic coordinates z_1, \dots, z_n , $V \subset \Delta$ be the locus $z_{k+1} = \cdots = z_n = 0$ and $\pi : \tilde{\Delta} \rightarrow \Delta$ be the blow-up of $\tilde{\Delta}$ along V . Let $E = \pi^{-1}(V)$ be the exceptional divisor. Then*

$$\pi^*(\Omega_{\Delta}^p) \subset \Omega_{\tilde{\Delta}}^p(\log E) \otimes \mathcal{I}_E^{p-k},$$

for $p \geq k + 1$.

Proof. $\tilde{\Delta} \subset \Delta \times \mathbb{P}^{n-k-1}$ is defined by $\tilde{\Delta} = \{(z, l) : z_i l_j = z_j l_i, k+1 \leq i, j \leq n\}$. In the affine chart defined by $l_j \neq 0$ we take the coordinates $z_i, z_j, \frac{l_i}{l_j}$ for $1 \leq i \leq k, k+1 \leq q \leq n$ and $q \neq j$. E is defined by $z_j = 0$. We have $\pi^*(dz_i) = dz_i$ for $1 \leq i \leq k$, $\pi^*(dz_j) = dz_j = z_j \left(\frac{dz_j}{z_j} \right)$

and $\pi^*(dz_q) = z_j d\left(\frac{l_q}{l_j}\right) + \frac{l_q}{l_j} dz_j = z_j \left(d\left(\frac{l_q}{l_j}\right) + \frac{l_q}{l_j} \frac{dz_j}{z_j}\right)$. Now take $\omega := dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, where $p \geq k + 1$. From the previous relations, it is obvious that $\pi^*\omega$ is a differential p -form with logarithmic poles on E whose coefficients vanish on E to order at least $p - k$. \square

The next tool we need is the Lemma on logarithmic derivatives. For a meromorphic function on \mathbb{C} it was done by Nevanlinna [16], and then generalized on \mathbb{C}^m by Vitter [23] (see also [17] and [24] for other generalizations).

Theorem 3.6. *Let f be a meromorphic function on \mathbb{C}^m . Then*

$$\int_{S_r} \ln^+ \left| \frac{\frac{\partial}{\partial t_i} f}{f} \right| \sigma \leq O(\ln T_{f,r} + \ln r).$$

Now, we can prove the following

Theorem 3.7. *Let H be an ample line bundle on a projective manifold X of dimension n . Let $Z \subset X$ be a submanifold of codimension $n - k \geq 2$. Let $f : \mathbb{C}^m \rightarrow X$ be a holomorphic map of maximal rank. Let α be a positive rational number and l be a positive integer such that αl is an integer. Let $\sigma \in H^0(X, S^l \Omega_X^m \otimes (\alpha l H))$ such that $f^*\sigma$ is not identically zero. Let W be the zero divisor of σ in $X_1 := \mathbb{P}(\bigwedge^m T_X)$. Then*

$$(3.2) \quad \frac{1}{l} N_{f_1}(W, r) + (m - k)m_f(Z, r) \leq \alpha T_{f,r}(H) + O(\ln T_{f,r}(H) + \ln r).$$

Proof. Let $\pi : \tilde{X} \rightarrow X$ be the blow-up of Z and $E = \pi^{-1}(Z)$. Let $\tilde{f} : \mathbb{C}^m \rightarrow \tilde{X}$ be the lifting of f and let $\tau = \pi^*\sigma$. By lemma 3.5, τ is a holomorphic section of $S^l \Omega_{\tilde{X}}^m(\log E) \otimes \pi^*(\alpha l H)$ which vanishes to order at least $l(m - k)$ on E . Let s_E be the canonical section of E . Let $\tilde{\tau} = \frac{\tau}{s_E^{l(m-k)}}$ which is a holomorphic section of $S^l \Omega_{\tilde{X}}^m(\log E) \otimes \pi^*(\alpha l H) \otimes (-l(m - k)E)$ over \tilde{X} . We consider h_E a smooth hermitian metric on E and h_H a smooth hermitian metric on H .

$$\begin{aligned} 2l(m - k)m_{\tilde{f}}(E, r) &= \int_{S_r} \ln^+ \frac{1}{\|s_E^{l(m-k)} \circ \tilde{f}\|_{h_E^{l(m-k)}}^2} \sigma + O(1) \\ &\leq \int_{S_r} \ln^+ \frac{1}{\|\tau \circ f\|_{h_H^{\alpha l}}^2} \sigma + \int_{S_r} \ln^+ \|\tilde{\tau} \circ \tilde{f}\|_{h_H^{\alpha l} h_E^{-l}}^2 \sigma + O(1) \\ &= - \int_{S_r} \ln \|\tau \circ f\|_{h_H^{\alpha l}}^2 \sigma + \int_{S_r} \ln^+ \|\tau \circ f\|_{h_H^{\alpha l}}^2 \sigma + \int_{S_r} \ln^+ \|\tilde{\tau} \circ \tilde{f}\|_{h_H^{\alpha l} h_E^{-l}}^2 \sigma + O(1). \end{aligned}$$

Taking the logarithms of global meromorphic functions as local coordinates, the lemma on the logarithmic derivative 3.6 implies

$$\begin{aligned} \int_{S_r} \ln^+ \|\tau \circ f\|^2 \sigma &= O(\ln T_{f,r}(H) + \ln r), \\ \int_{S_r} \ln^+ \|\tilde{\tau} \circ \tilde{f}\|^2 \sigma &= O(\ln T_{f,r}(H) + \ln r). \end{aligned}$$

Therefore, by the First Main Theorem, we obtain

$$2l(m-k)m_{\tilde{f}}(E, r) \leq -2N_{f_1}(W, r) + 2\alpha l T_{f,r}(H) + 2l T_{f_1,r}(\mathcal{O}_{X_1}(1)) + O(\ln T_{f,r}(H) + \ln r).$$

The inequality 3.1 implies that $T_{f_1,r}(\mathcal{O}_{X_1}(1)) = O(\ln T_{f,r}(H) + \ln r)$ and this concludes the proof. \square

Corollary 3.8. *Let X be a projective manifold of dimension n , H an ample line bundle on X , $Z \subset X$ a submanifold of codimension $n - k \geq 2$. Let $f : \mathbb{C}^m \rightarrow X$ a holomorphic map of maximal rank. Then*

$$T_{f_1,r}(\mathcal{O}_{X_1}(1)) + (m-k)m_f(Z, r) \leq O(\ln T_{f,r}(H) + \ln r).$$

Proof. Choosing α sufficiently large, $\mathcal{O}_{X_1}(1) \otimes (\alpha H)$ is ample over X_1 . Then if we choose l sufficiently large, $\mathcal{O}_{X_1}(l) \otimes (\alpha l H)$ is very ample. Then there exists $\sigma \in H^0(X, S^l \Omega_X^m \otimes (\alpha l H))$ such that the defect under the mapping f_1

$$\liminf_r \frac{m_{f_1}(W, r)}{T_{f_1,r}(\mathcal{O}_{X_1}(l) \otimes (\alpha l H))} = 0,$$

where $W \subset X_1$ is the zero divisor of σ . Therefore, from inequality 3.2, we deduce

$$T_{f_1,r}(\mathcal{O}_{X_1}(1)) + (m-k)m_f(Z, r) \leq O(\ln T_{f,r}(H) + \ln r).$$

\square

4. HOLOMORPHIC MAPPINGS AND FOLIATIONS

4.1. The smooth case. A smooth foliation of dimension p on X is given by an integrable subbundle $\mathcal{F} \subset T_X$ of rank p . So we have an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow T_X \rightarrow N_{\mathcal{F}} \rightarrow 0,$$

where $N_{\mathcal{F}}$ is called the normal bundle of the foliation. $K_{\mathcal{F}} := \det(\mathcal{F}^*)$ is the canonical bundle of the foliation. We have an isomorphism $K_X = K_{\mathcal{F}} \otimes \det(N_{\mathcal{F}}^*)$.

We consider holomorphic maps $f : \mathbb{C}^p \rightarrow X$ of maximal rank which are leaves of the foliation \mathcal{F} , i.e. tangent to \mathcal{F} .

Then we obtain the following result

Theorem 4.1. *Let \mathcal{F} be a smooth foliation of dimension p on a projective manifold X , $p \leq \dim X = n$. If X is of general type then there is no holomorphic mapping $f : \mathbb{C}^p \rightarrow X$ of generic maximal rank tangent to \mathcal{F} which is Zariski dense.*

Proof. Suppose there exists such a map f . The foliation \mathcal{F} defines a section $F \subset X_1 := \mathbb{P}(\bigwedge^p T_X)$ over X and the tautological line bundle verifies $L|_F = \pi^* K_{\mathcal{F}}^{-1}$. From theorem 3.1 we have

$$0 \leq [\Phi_1].L = [\Phi].K_{\mathcal{F}}^{-1} = [\Phi].K_X^{-1} - [\Phi].\det(N_{\mathcal{F}}).$$

Since the normal bundle $N_{\mathcal{F}}|_{\mathcal{F}}$ is flat and K_X is big, we obtain using lemma 2.4

$$[\Phi].K_X^{-1} - [\Phi].\det(N_{\mathcal{F}}) = -[\Phi].K_X < 0,$$

which is a contradiction. \square

4.2. The singular case. In general, there is no reason that the foliations we are working with should be non-singular. Also, it is important to generalize the preceding result to singular foliations. We consider singular holomorphic foliations \mathcal{F} of codimension one on a projective manifold X of dimension n . It is locally given by a differential equation ω where

$$\omega = \sum_{i=1}^n a_i(z) dz_i$$

is an integrable 1-form, i.e. $\omega \wedge d\omega = 0$, and the coefficients a_i have no common factor. The *singular locus* $Sing \mathcal{F}$ is locally given by the common zeros of the coefficients a_i .

The foliation can be defined by a collection of 1-forms $\omega_j \in \Omega_X^1(U_j)$ such that $\omega_i = f_{ij} \omega_j$ on $U_i \cap U_j$, $f_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$.

As in the smooth case, there are two holomorphic line bundles associated to \mathcal{F} , the normal bundle $N_{\mathcal{F}}$ and the canonical bundle $K_{\mathcal{F}}$ of the foliation \mathcal{F} . $N_{\mathcal{F}}$ is defined by the cocycle f_{ij} and $K_{\mathcal{F}} = K_X \otimes N_{\mathcal{F}}$.

In the case of surfaces, one uses the theorem of resolution of singularities of Seidenberg [20] to work only with reduced singularities. In general, one may expect theorems of resolution of singularities to reduce the problem to *canonical singularities* as introduced by McQuillan [14] following the approach of the Mori program.

Definition 4.2. Let (X, \mathcal{F}) be a pair where X is a projective variety and \mathcal{F} a foliation. Let $p : (\tilde{X}, \tilde{\mathcal{F}}) \rightarrow (X, \mathcal{F})$ be a birational morphism of pairs. We can write

$$K_{\tilde{\mathcal{F}}} = p^* K_{\mathcal{F}} + \sum a(E, X, \mathcal{F}) E.$$

$a(E, X, \mathcal{F})$ is independent of the morphism p and depends only on the discrete valuation that corresponds to E . It is called the *discrepancy* of (X, \mathcal{F}) at E . We define

$$\begin{aligned} \text{discrep}(X, \mathcal{F}) &= \inf \{ a(E, X, \mathcal{F}); E \text{ corresponds to a discrete valuation} \\ &\text{such that } \text{Center}_X(E) \neq \emptyset \text{ and } \text{codim}(\text{Center}_X(E)) \geq 2 \}. \end{aligned}$$

We say that (X, \mathcal{F}) has *canonical singularities* if $\text{discrep}(X, \mathcal{F}) \geq 0$.

Remark 4.3. Up to now, there are few theorems of resolution of singularities: it is known for codimension 1 foliations in dimension 3 [4] and recently for foliations by curves in dimension 3 [15].

First, we consider the situation where we have a singular codimension 1 foliation with *logarithmic simple singularities* i.e. we can write ω in local coordinates

$$(4.1) \quad \omega = \left(\prod_{i=1}^r z_i \right) \sum_{i=1}^r \lambda_i \frac{dz_i}{z_i},$$

where $\sum_{i=1}^r m_i \lambda_i \neq 0$, for any non-zero vector $(m_i) \in \mathbb{N}^r$.

In particular, the only integral hypersurfaces are the components of $z_1 \dots z_r = 0$. We can suppose, up to doing some blow ups, that the foliation have simple singularities adapted to

a normal crossing divisor [4]. This means that we have an invariant simple normal crossing divisor E such that every $P \in \text{Sing} \mathcal{F}$ belongs to at least $r - 1$ irreducible components of E .

We consider $X_1 := \mathbb{P}(\bigwedge^m T_X(-\log(E)))$ with the natural projection $\pi : X_1 \rightarrow X$. The foliation \mathcal{F} defines a section $F \subset X_1$ over X and

$$\mathcal{O}_{X_1}(-1)|_F = \pi^* K_{\mathcal{F}}^{-1}.$$

Lemma 4.4.

$$\lim_{r \rightarrow +\infty} \frac{N_f^1(E, r)}{T_{f,r}(H)} = 0.$$

In order to prove 4.4 we will prove the following Lemma:

Lemma 4.5. *Let (X, H) be a smooth polarized projective variety and C a smooth closed subvariety of codimension two. Let X_C be the formal completion of X around C and $\iota : V \hookrightarrow X_C$ be a smooth formal subvariety containing C of codimension one. Then we can find a sequence of global sections $s_m \in H^0(X, H^{\otimes m})$ such that $\iota^*(s_m) \geq q_m C$ with*

$$\lim_{m \rightarrow \infty} \frac{m}{q_m} = 0.$$

Proof. Observe that C is a divisor in V . Denote by N_C the normal line bundle of C inside V and by C_i the i -th formal neighborhood of C inside V . For every positive integers $i \geq 1$ and $m \geq 1$, we have an exact sequence

$$0 \rightarrow H^{\otimes m}|_C \otimes N_C^{\otimes -(i-1)} \rightarrow H^{\otimes m}|_{C_i} \rightarrow H^{\otimes m}|_{C_{i-1}} \rightarrow 0.$$

Denote by E_m^i the kernel of the composite map

$$H^0(X, H^{\otimes m}) \rightarrow H^0(V, \iota^*(H^{\otimes m})) \rightarrow H^0(C_i, H^{\otimes m}).$$

Fix $\epsilon > 0$ sufficiently small. We will prove that, for $m \gg 0$ we have that $E_m^{m^{1+\epsilon}} \neq \{0\}$, and this will be enough to conclude.

The snake lemma applied to the exact sequence above gives rise to an inclusion

$$\gamma_m^i : E_m^{i-1}/E_m^i \hookrightarrow H^0(C, H|_C^{\otimes m} \otimes N_C^{-(i-1)}).$$

Consequently

$$\dim(E_m^i) \geq \dim(E_m^{i-1}) - h^0(C, H|_C^{\otimes m} \otimes N_C^{-(i-1)}).$$

By Riemann–Roch Theorem (or Hilbert–Samuel Theorem) we can find constants A and A_1 independent of m and i such that $h^0(X, H^{\otimes m}) \geq Am^n$ and $h^0(C, H^{\otimes m} \otimes N_C^{-i}) \leq A_1(m+i)^{n-2}$. Thus we obtain

$$\dim(E_m^{m^{1+\epsilon}}) \geq Am^n - \sum_{i=1}^{m^{1+\epsilon}} A_1(m+i)^{n-2}.$$

Since, for a suitable A_2 independent on m we have that, for $m \gg 0$,

$$\sum_{i=1}^{m^{1+\epsilon}} A_1(m + (i - 1))^{n-2} \leq \int_1^{m^{1+\epsilon}} (m + (t - 1))^{n-2} dt \leq A_2 m^{(n-2)(1+\epsilon)},$$

the conclusion follows. □

Let us show how lemma 4.5 implies lemma 4.4:

Proof. Let E_1 be an irreducible component of E . The two subvarieties $f(\mathbb{C}^{n-1})$ and E_1 are both invariant for the foliation. Let V be the leaf containing $f(\mathbb{C}^{n-1})$. We may suppose that E_1 and V intersect properly on an irreducible C , component of $Sing \mathcal{F}$ which is a smooth closed subvariety of codimension two of X . We apply Lemma 4.5 and we obtain

$$N_f^1(E_1, r) = N_f^1(C, r) \leq \frac{1}{q_m} N_f(\{s_m = 0\}, r) \leq \frac{m}{q_m} T_{f,r}(H) + c_m(H)$$

where $c_m(H)$ is a constant depending on f , H and s_m but independent on r . Lemma 4.4 follows once one divide the inequality above by $T_{f,r}(H)$ and let m and r tend to infinity. □

Now, we use theorem 3.3 to obtain as an immediate consequence

Theorem 4.6. *Let \mathcal{F} be a holomorphic foliation of codimension one on a projective manifold X , $\dim X = n$. Suppose $Sing \mathcal{F}$ consists only of logarithmic simple singularities. Consider a holomorphic mapping $f : \mathbb{C}^{n-1} \rightarrow X$ of generic maximal rank tangent to \mathcal{F} which is Zariski dense. Then*

$$[\Phi].K_{\mathcal{F}}^{-1} \geq 0.$$

This result implies Theorem B of the introduction.

5. APPLICATIONS

As a first application we have

Corollary 5.1. *Let \mathcal{F} be a holomorphic foliation of codimension one on \mathbb{P}^n of degree $d \geq n$. Suppose $Sing \mathcal{F}$ consists only of logarithmic simple singularities. Consider a holomorphic mapping $f : \mathbb{C}^{n-1} \rightarrow \mathbb{P}^n$ of generic maximal rank tangent to \mathcal{F} . Then f is not Zariski dense.*

Proof. We have $K_{\mathcal{F}} = \mathcal{O}(d - n + 1)$. Therefore if $d \geq n$, $K_{\mathcal{F}}$ is positive and we conclude by theorem 4.6. □

As a second application, we deal with foliations with canonical singularities and local holomorphic first integrals i.e. the form ω can be written

$$(5.1) \quad \omega = gdf$$

with g and f holomorphic, g nonvanishing.

Theorem 5.2. *Let \mathcal{F} be a holomorphic foliation of codimension one on a projective manifold of general type X of dimension n . Suppose $\text{Sing}\mathcal{F}$ consists only of canonical singularities with local first integrals. Consider a holomorphic mapping $f : \mathbb{C}^{n-1} \rightarrow X$ of generic maximal rank tangent to \mathcal{F} . Then f is not Zariski-dense.*

Proof. Suppose we have such a map f . Since we have local first integrals, by Hironaka's theorem [8], we have a resolution $(\tilde{X}, \tilde{\mathcal{F}}) \rightarrow (X, \mathcal{F})$ where $(\tilde{X}, \tilde{\mathcal{F}})$ has only logarithmic simple singularities. Consider the (meromorphic) lifting $\tilde{f} : \mathbb{C}^{n-1} \rightarrow \tilde{X}$. Then theorem 4.6 gives

$$[\tilde{\Phi}].K_{\tilde{\mathcal{F}}}^{-1} \geq 0.$$

Since we have canonical singularities,

$$K_{\tilde{\mathcal{F}}} \geq K_{\mathcal{F}}$$

and therefore

$$[\Phi].K_{\mathcal{F}}^{-1} \geq 0.$$

The existence of local integrals $\omega = gdf$ implies that $d\omega = \beta \wedge \omega$, where $\beta = \frac{dg}{g}$ is a holomorphic 1-form.

Let us take an open covering $\{U_j\}_{j \in I}$ of X and holomorphic one-forms $\omega_j \in \Omega_X^1(U_j)$ generating \mathcal{F} such that

$$d\omega_j = \beta_j \wedge \omega_j.$$

On each $U_i \cap U_j$ we also have

$$\omega_i = g_{ij}\omega_j$$

where the cocycle $\{g_{ij}\}$ defines $N_{\mathcal{F}}$. So, we have

$$\beta_i \wedge \omega_i = d\omega_i = dg_{ij} \wedge \omega_j + g_{ij}d\omega_j = \left(\frac{dg_{ij}}{g_{ij}} + \beta_j \right) \wedge \omega_i,$$

and therefore

$$\left(\frac{dg_{ij}}{g_{ij}} + \beta_j - \beta_i \right) \wedge \omega_i = 0.$$

We can find smooth $(1,0)$ -forms $\gamma_j \in A^{1,0}(U_j)$ such that

$$\gamma_j \wedge \omega_j = 0,$$

on U_j , and

$$\frac{dg_{ij}}{g_{ij}} = \beta_i - \beta_j + \gamma_i - \gamma_j,$$

on $U_i \cap U_j$. The 2-form defined by

$$\frac{1}{2i\pi}d(\beta_j + \gamma_j)$$

on U_j represents the first Chern class of $N_{\mathcal{F}}$.

The relation $d\omega_j = \beta_j \wedge \omega_j$ implies that $d\beta_j|_{\mathcal{F}} \equiv 0$. Moreover $d\gamma_j|_{\mathcal{F}} \equiv 0$. Therefore we obtain

$$[\Phi].N_{\mathcal{F}} = 0.$$

Finally, we obtain

$$[\Phi].K_X = [\Phi].K_{\mathcal{F}} + [\Phi].N_{\mathcal{F}}^* \leq 0,$$

which contradicts the fact that K_X is big. \square

As a corollary, we consider the case where $\text{Sing}\mathcal{F}$ has codimension ≥ 3 . Then, we can use the following theorem due to Malgrange [11]

Theorem 5.3. *Let \mathcal{F} be a germ of foliation at $(\mathbb{C}^n, 0)$. If $\text{codim}\text{Sing}\mathcal{F} \geq 3$ then \mathcal{F} has a holomorphic first integral.*

Therefore we obtain

Corollary 5.4. *Let \mathcal{F} be a holomorphic foliation of codimension one on a projective manifold X , $\dim X = n$. Suppose X is of general type and $\text{codim}\text{Sing}\mathcal{F} \geq 3$ where all singularities are canonical. Then there is no holomorphic mapping $f : \mathbb{C}^{n-1} \rightarrow X$ of maximal rank tangent to \mathcal{F} which is Zariski dense.*

6. EXAMPLES

In this section, we would like to provide some examples of foliated pairs (X, \mathcal{F}) satisfying the hypotheses required in the previous results. We will use ramified coverings. First, we need to consider singularities of foliated pairs with a boundary divisor $B = \sum_i \left(1 - \frac{1}{m_i}\right) B_i$ where $m_i \in \mathbb{N} \cup \{\infty\}$. We define the following function

$$\epsilon(E) := \begin{cases} 0 & \text{if } E \text{ is invariant by the foliation} \\ 1 & \text{if } E \text{ is not invariant by the foliation} \end{cases}$$

Definition 6.1. (X, B, \mathcal{F}) is said to be

- (1) *terminal* if $a(E, X, \mathcal{F}) - \sum_i \left(1 - \frac{1}{m_i}\right) \epsilon_i \nu_i(E) > 0$,
- (2) *canonical* if $a(E, X, \mathcal{F}) - \sum_i \left(1 - \frac{1}{m_i}\right) \epsilon_i \nu_i(E) \geq 0$,
- (3) *log terminal* if $a(E, X, \mathcal{F}) - \sum_i \left(1 - \frac{1}{m_i}\right) \epsilon_i \nu_i(E) > -\epsilon(E)$,
- (4) *log canonical* if $a(E, X, \mathcal{F}) - \sum_i \left(1 - \frac{1}{m_i}\right) \epsilon_i \nu_i(E) \geq -\epsilon(E)$,

for all exceptional E , where $\epsilon_i := \epsilon(B_i)$, $\nu_i(E)$ is the multiplicity of B_i .

When a foliation has a holomorphic first integral, it is canonical if and only if the level sets are log-canonical:

Proposition 6.2. *Let (X, \mathcal{F}) be a codimension one foliation. Suppose that, locally on X , the conormal bundle of \mathcal{F} is generated by $\omega = d(f)$; suppose that the divisor $D := \{f = 0\}$ is log-canonical. Then the foliation \mathcal{F} is canonical.*

Proof. Let $\pi : (\tilde{X}, \tilde{D}) \rightarrow (X, D)$ be a Hironaka resolution of the pair (X, D) and let E_1, \dots, E_r be the divisors contracted by π . By construction \tilde{D} is the strict transform of D . Denote by K_X , resp. $K_{\tilde{X}}$, resp. K_D , resp. $K_{\tilde{D}}$ the canonical sheaf of X , resp. \tilde{X} , resp. D , resp. \tilde{D} . By construction, there are positive constants b_i and r_i such that $K_{\tilde{X}} = \pi^*(K_X) + \sum_i b_i E_i$ and $\pi^*(D) = \tilde{D} + \sum_i r_i E_i$.

By adjunction formula we get $K_{\tilde{D}} = \pi^*(K_D) + \sum_i (b_i - r_i) E_i|_{\tilde{D}}$. Since D is log-canonical, we have that $b_i - r_i \geq -1$.

Let $k(X)$ be the function field of X and $R \subset k(X)$ be a discrete valuation ring with fraction field $k(X)$. Let p be a uniformizer of R . Suppose that D is locally given by $p^N u$ with $u \in R^*$. Thus the restriction of ω to $\text{Spec}(R)$ is $(N-1)p^{N-1}ud(p) + p^N d(u)$. This implies the following: Denote by $N_{\mathcal{F}}$ the normal line bundle of the foliation induced by \mathcal{F} on \tilde{X} ; then

$$\pi^*(N_{\mathcal{F}}) = N_{\tilde{\mathcal{F}}} + \sum_i (r_i - 1) E_i.$$

Consequently, since $K_X = K_{\mathcal{F}} - N_{\mathcal{F}}$ and $K_{\tilde{X}} = K_{\tilde{\mathcal{F}}} - N_{\tilde{\mathcal{F}}}$, a straightforward calculation gives

$$K_{\mathcal{F}} - \pi^*(K_{\mathcal{F}}) = \sum_i (b_i - r_i) E_i + \sum_i E_i.$$

The conclusion follows. □

Now, let us consider a ramified cover $\pi : Y \rightarrow X$ and the induced foliation on Y , $\mathcal{G} := \pi^* \mathcal{F}$. We can write

$$K_Y := \pi^*(K_X + \Delta),$$

where $\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) Z_i$.

Then we have

Lemma 6.3. *If (X, Δ, \mathcal{F}) has canonical singularities then (Y, \mathcal{G}) has canonical singularities.*

Proof. We have

$$K_{\mathcal{G}} = \pi^* K_{\mathcal{F}} + \sum_i \epsilon(Z_i) \pi^* \left(1 - \frac{1}{m_i}\right) Z_i.$$

Therefore, if E_Y is an exceptional divisor over Y dominating E_X with multiplicity r , we have for the corresponding discrepancies

$$a(E_Y) = r(a(E_X) - \sum_i \left(1 - \frac{1}{m_i}\right) \epsilon_i \nu_i(E_X)) + \epsilon(E_X)(r - 1).$$

□

The following proposition describes locally how one can produce foliations with canonical singularities.

Proposition 6.4. *Let $(\mathcal{U}, \mathcal{F})$ be a germ of a smooth foliation on a complex manifold of dimension $n \geq 3$ and $D \subset U$ a smooth divisor. Let $\pi : (\mathcal{V}, \mathcal{G}) \rightarrow (\mathcal{U}, \mathcal{F})$ be a covering ramified along D where $\mathcal{G} := \pi^* \mathcal{F}$. Then*

- (1) *If \mathcal{F} is transverse to D then \mathcal{G} is smooth.*
- (2) *If D has only isolated non-degenerate quadratic-type tangencies with \mathcal{F} (i.e. if $D = (h = 0)$ then the restriction of h to a leaf has only isolated non-degenerate critical points), then \mathcal{G} has isolated canonical singularities.*
- (3) *If the degree of the covering π is 2 and D has only quadratic-type tangencies with \mathcal{F} (i.e. if $D = (h = 0)$ then the restriction of h to a leaf has multiplicity less or equal to 2 at any point) then \mathcal{G} has canonical singularities.*

Proof. (1) If \mathcal{F} is transverse to D , we can choose local coordinates (z_1, \dots, z_n) on \mathcal{U} such that \mathcal{F} is given by $dz_1 = 0$ and $D = (z_2 = 0)$. The covering π is given by $(z_1, t, \dots, z_n) \rightarrow (z_1, t^m, \dots, z_n)$ and therefore \mathcal{G} is given by $dz_1 = 0$.

(2) If D has only isolated quadratic-type tangencies with \mathcal{F} , locally by the holomorphic Morse lemma, we can find local coordinates such that h the local equation defining D is written,

$$h(z) = z_1 + \sum_{i=2}^n z_i^2.$$

So, in local coordinates (t, z_2, \dots, z_n) , \mathcal{G} is given by $d(t^m - \sum_{i \geq 2} z_i^2) = 0$. Now, from [19], we know that hypersurfaces given by $t^m - \sum_{i \geq 2} z_i^2 = 0$ have at most canonical singularities if $\frac{1}{m} + \frac{n-1}{2} > 1$. Then we conclude by proposition 6.2

- (3) If S is a germ of leaf, then by the hypotheses the pair $(S, (1 - \frac{1}{2}) D|_S)$ is canonical and we conclude by lemma 6.3. □

Now, let us see how we can ensure globally the conditions of the previous proposition for a smooth divisor D in a complex projective manifold X .

Let $X_1 := G(n-1, T_X)$ be the Grassmannian bundle. Then the inclusion $i : D \hookrightarrow X$ induces a lift $i_1 : D \rightarrow X_1$. A smooth codimension-one foliation \mathcal{F} on X gives a section $F \subset X_1$.

The condition for \mathcal{F} to be transverse to D is equivalent to

$$i_1(D) \cap F = \emptyset.$$

The condition for D to have only isolated non-degenerate quadratic-type tangencies with \mathcal{F} translates into the fact that $i_1(D) \cap F$ is finite and i_1 has some non-degeneracy property over that set.

Let us illustrate these situations in the case where X is an abelian variety.

Proposition 6.5. *Let A be an n -dimensional complex abelian variety and L a line bundle on A . Let D be a smooth divisor in the linear system $|mL|$, where $m > 1$. Let $\pi : X \rightarrow A$ be the degree m cyclic cover of A ramified along D . If \mathcal{F} is a generic linear codimension*

one foliation on A , the induced foliation $\mathcal{G} := \pi^*\mathcal{F}$ on X has at most isolated canonical singularities.

Proof. By the triviality of the tangent bundle T_A , the lifting $i_1 : D \rightarrow X_1$ described above yields a map

$$\varphi : D \rightarrow \mathbb{P}^{n-1},$$

which is locally defined by $[\partial h/\partial z_1 : \dots : \partial h/\partial z_n]$, where $\{h = 0\}$ is a local equation for D . If we choose a base dz_1, \dots, dz_n of invariant differentials, the foliation \mathcal{F} corresponds to a point $p \in \mathbb{P}^{n-1}$. We have two cases. If φ is not dominant, then for a generic choice of \mathcal{F} the preimage $\varphi^{-1}(p)$ will be empty, that is, D is transverse to \mathcal{F} . Therefore the pullback $\pi^*\mathcal{F}$ is smooth, by the local result above, item (1).

If φ is dominant, for a generic choice of \mathcal{F} , we have that $\varphi^{-1}(p)$ is smooth. Notice that we may assume that D is given locally by the equation $\{z_1 + Q(z_1, \dots, z_n) = 0\}$, where $\deg(Q) \geq 2$ (if not, D is transverse to \mathcal{F} and we are done as before). Then, the smoothness of $\varphi^{-1}(p)$ is equivalent to the fact that $(\partial^2 Q/\partial z_i \partial z_j)$ is non-degenerate. Therefore we are done by the local result above, item (2). \square

Example 6.6. *Let L be an ample line bundle on an abelian variety A . Let $D \in |mL|$ be a smooth divisor and $\pi : X \rightarrow A$ be the degree m cyclic cover ramified along D . Observe that X is of general type. Take a generic linear codimension one foliation \mathcal{F} on A . If $n := \dim(A) \geq 3$, by Proposition 6.5 the foliated pair $(X, \pi^*\mathcal{F})$ satisfies the hypotheses of Corollary E. In particular X contains no holomorphic mapping $f : \mathbb{C}^{n-1} \rightarrow X$ tangent to $\pi^*\mathcal{F}$ which is Zariski dense.*

Now we turn back to the condition (3) in the local result above i.e. when D has only quadratic-type tangencies with \mathcal{F} . This condition can be verified considering the second jet-bundle X_2 as defined in [18] which we refer to for details. X_2 is a Grassmannian bundle over X_1 . We have a lifting $i_2 : D \rightarrow X_2$ and the foliation defines a section $s_2 : X \rightarrow X_2$. The previous condition translates as

$$i_2(D) \cap s_2(X) = \emptyset.$$

7. DESINGULARIZATION OF CURRENTS

In the case of surfaces, the following property of the current is quite useful and often used in [12] and [2]. Consider a tower of blow-up mappings $\pi_k : X_k \rightarrow X_{k-1}$, with the convention that $X_0 = X$ the original surface and E_k denotes the corresponding exceptional divisor. Then

$$\lim_{k \rightarrow \infty} [\Phi_k] \cdot [E_k] = 0,$$

where $[\Phi_k]$ is the current associated to the lifting of $f : \mathbb{C} \rightarrow X$.

In fact, McQuillan [13] proves the following stronger result

Theorem 7.1. *Let $(\mathcal{Y}, \mathcal{F})$ be a foliated non-singular surface with canonical singularities and b a formal branch of \mathcal{F} through a canonical singularity z . Let \mathcal{Y}_1 be obtained by blowing up $\mathcal{Y}_0 = \mathcal{Y}$ in z , with b_1 the proper transform of the branch $b_0 = b$, \mathcal{Y}_2 from \mathcal{Y}_1 by blowing up the*

crossing point of b_1 with the exceptional divisor etc., and \mathcal{E}_k the exceptional curve on \mathcal{Y}_k blown down by $\rho_{k,k-1}$ (ρ_{km} from \mathcal{Y}_k to \mathcal{Y}_m , $m < k$ being the projection) then for H ample there is a positive rational constant α such that for all k , every sufficiently large and divisible multiple of $\alpha(\sqrt{(k)}^{-1}\rho_{k0}^*H - \mathcal{E}_k)$ is generated by its global sections outside the exceptional curves other than \mathcal{E}_k .

In this section we would like to prove an analogous desingularization statement for the currents constructed above, in the case $\dim X = 3$, that will be used in a sequel of this paper.

We have to study the following situation. Let (X, \mathcal{F}) be a smooth projective variety of dimension three equipped with a foliation of codimension one. Let Z be a smooth leaf of the foliation and C be a compact curve contained in it.

We perform the following construction. We denote $X = X_0$, $V = V_0$ and $C = C_0$. Let $f_1 : X_1 \rightarrow X_0$ be the blow up along C_0 , E_1 be the exceptional divisor and $Z_1 \subset X_1$ the strict transform of Z_0 . Let $C_1 := E_1 \cdot Z_1$. Inductively we define the sequence (X_k, C_k, E_k, Z_k) where: $f_{k,k-1} : X_k \rightarrow X_{k-1}$ is the blow up of X_{k-1} along C_{k-1} , E_k is the exceptional divisor of $f_{k,k-1}$ and Z_k is the strict transform of Z_{k-1} . We will denote by f_k the natural projection $f_k : X_k \rightarrow X_0$ and, for every $i < k$, we denote by $f_{k,i}$ the natural projection $f_{k,i} : X_k \rightarrow X_i$. Let $i < k$, by abuse of notation, we will denote by E_i the divisor $f_{k,i}^*(E_i)$. We fix an ample divisor H on X_0 and we denote by H the nef divisor $f_k^*(H)$ on X_k .

Then we want to prove

Theorem 7.2. *In the situation above, there is a constant α independent of k such that, for every k and m sufficiently big (possibly depending in k), the divisor on X_k*

$$m \left(\frac{\alpha}{k^{1/3}} H - E_k \right)$$

is effective.

For this we need

Proposition 7.3. *With the notation above, for α sufficiently big, the line bundle*

$$F_{\alpha,k} := \alpha k^{2/3} H - \sum_{i=1}^k E_i$$

is effective on X_k .

The proof consists in four steps which will be stated in the lemmas below. But first let us show that this proposition implies the theorem

Proof. Using the notations below, denoting by \overline{E}_i the strict transform of E_i in X_k and remarking that $\sum_{i=1}^k E_i = \sum_{i=1}^k i \overline{E}_i$, we get

$$\frac{\alpha}{k^{1/3}} H - E_k = \frac{1}{k} \left(\left(\alpha k^{2/3} H - \sum_{i=1}^k E_i \right) + \sum_{j=1}^{k-1} j \overline{E}_j \right).$$

The conclusion easily follows because the two terms on the right are effective divisors. \square

The *first step* is

Lemma 7.4. *If α is sufficiently big independently of k , then $F_{\alpha,k}^3 > 0$.*

Proof. A systematic use of the projection formula gives the following relations:

$$(H^2, E_i) = 0,$$

$$(H, E_i^2) = (H, C_0) = r_0 \text{ for a suitable } r_0 > 0,$$

$$(E_i, E_j, E_k) = 0 \text{ if } k > j > i;$$

$$(E_i^2, E_j) = 0 \text{ if } j > i,$$

and

$$(E_i^2, E_j) = -(E_j, C_j) \text{ if } j < i.$$

Since $f_{i,i-1}^*(Z_{i-1}) = Z_i + E_i$, again the projection formula gives

$$(E_i, C_i) = -E_i^3.$$

The relations above gives the existence of a positive constant s_0 independent on k such that

$$\begin{aligned} F_{\alpha,k}^3 &= \alpha^3 k^2 H^3 - \alpha k^{2/3} r_0 k - \sum_{i=1}^k E_i^3 - 3 \sum_{i=1}^k \sum_{j=i+1}^k (E_i, E_j^2) \\ &= \alpha k^2 s_0 - \sum_{i=1}^k E_i^3 (1 + 3(k - (i + 1))). \end{aligned}$$

Consequently $F_{\alpha,k}^3 > 0$ for a sufficiently big α , as soon as we prove that there is a constant R independent of k such that, for every i we have that $|E_i^3| \leq R$.

Let N_i be the restriction of the normal sheaf of Z_i to C_i . Since C_i is the intersection of E_i and Z_i , we obtain that

$$-E_i^3 = \deg(N_{i-1}(E_{i-1})|_{C_{i-1}}).$$

Again, by projection formula we get

$$-E_i^3 = (N_{i-2}, C_{i-2}).$$

Thus, using the fact that $-E_j^3 = (E_j, C_j)$, by induction we get

$$\begin{aligned} -E_i^3 - E_{i-2}^3 - E_{i-3}^3 - \cdots - E_1^3 &= -a, \\ -E_{i-1}^3 - E_{i-3}^3 - \cdots - E_1^3 &= -a, \\ &\dots \\ -E_3^3 - E_1^3 &= -a, \\ -E_2^3 &= -a, \\ -E_1^3 &= -b. \end{aligned}$$

From this we get the recursive formula

$$\begin{aligned} E_i^3 &= E_{i-1}^3 - E_{i-2}^3, \\ -E_2^3 &= -a, \\ -E_1^3 &= -b. \end{aligned}$$

Consequently the E_i^3 are periodic thus $F_{\alpha,k}^3 > 0$ for $\alpha \gg 0$ independent of k . \square

Now, we state the *second step* as the following lemma which is interesting in its own

Lemma 7.5. *Let X be a smooth projective variety of dimension n . Let L be a line bundle and D a smooth nef and big divisor on X . Suppose that the restriction $L|_D$ of L to D is nef and big. Then, there exist a_0 such that for every $m \gg 0$ and $i > 1$ we have $h^i(X, L^{ma_0}) = 0$.*

Proof. Let K_X be the canonical line bundle of X . By Serre duality it suffices to show that $h^j(X, L^{-ma_0} + K_X) = 0$ for $j \leq n - 2$ and $m > 0$ sufficiently big.

By hypotheses, we may suppose that a_0 is so big that both line bundles $(L^{ma_0} - K_X)|_D$ are nef and big on D . Thus, by Kawamata–Viehweg vanishing theorem, $h^j(D, L^{-ma_0} + K_X) = 0$, as soon as $m > 0$. In the sequel we will denote by L the line bundle L^{a_0} .

Consider the exact sequence

$$0 \longrightarrow (L^{-m} + K_X - (i-1)D)|_D \longrightarrow L^{-m} + K_X|_{iD} \longrightarrow L^{-m} + K_X|_{(i-1)D} \longrightarrow 0.$$

Since, for every $i > 0$ and $m > 1$, the line bundle $(L^m - K_X + iD)|_D$ is nef and big, again by Kawamata–Viehweg vanishing theorem, for every $j < n-1$ we have $h^j(D, L^{-m} + K_X - iD) = 0$. Consequently, by induction on i we have $h^j(iD, L^{-m} + K_X) = 0$.

Fix $m > 0$. We can take i so big that $L^m - K_X + iD$ is nef and big on X (i will depend on m in general).

Kawamata–Viehweg vanishing theorem applied to $L^m - K_X + iD$ and the long exact sequence of cohomology of the exact sequence

$$0 \longrightarrow L^{-m} + K_X - iD \longrightarrow L^{-m} + K_X \longrightarrow (L^{-m} + K_X)|_{mD} \longrightarrow 0$$

allow to conclude. \square

The *third step* is

Lemma 7.6. *Suppose that D is a generic global section of (a power of) H . Let D_k be the strict transform of D in X_k . Then, the restriction of $F_{\alpha,k}$ to D_k is nef and big.*

Proof. Let \mathcal{F}_D be the restriction of the foliation \mathcal{F} to D . Let p_1, \dots, p_r be the intersection points of D with C . They are singular points for the foliation \mathcal{F}_D . Let Z_D be the intersection of Z with D . It is a leaf of the foliation \mathcal{F}_D . We may perform a tower of blow ups $(D_k, (E_D)_k, p_k, (Z_D)_k)$ of D on the points p_j similar to the tower we performed on X . Since D is transverse to the curve C , the strict transform of D in X_k is exactly D_k and the intersection of E_k with D_k is $(E_D)_k$. Since Seidenberg’s desingularization process is obtained by blow up,

as soon as k is sufficiently big, we may suppose that the restriction of the foliation to D_k is with reduced singularities on the points p_k .

The restriction of $F_{\alpha,k}$ to D_k is then nef and big as soon as α is sufficiently big by theorem 7.1. \square

Finally, the *fourth step* is the proof of proposition 7.3

Proof. The Euler characteristic $\chi(F_{\alpha,k})$ is positive because $F_{\alpha,k}$ has positive self intersection. Since, the restriction of $F_{\alpha,k}$ to the nef and big smooth divisor D_k is nef and big, then $h^i(X_k, F_{\alpha,k}^m) = 0$ for $m \gg 0$ and $i > 1$. Thus, as soon as k and m are sufficiently big, we have $h^0(X_k, F_{\alpha,k}^m) > 0$. \square

Remark 7.7. (a) We proved a little bit more: for α and k sufficiently big, the line bundle $\frac{\alpha}{k^{1/3}}H - E_k$ is a big divisor.

(b) If one wants to avoid the Seidenberg theorem on resolution of singularities of foliations on surfaces, one can remark that in the case of logarithmic simple singularities, we may suppose that D intersect the curve C properly and only on smooth points, we may suppose that the foliation \mathcal{F}_D has reduced singularities on D .

As a corollary of theorem 7.2 we obtain the following desingularization statement

Corollary 7.8. In the situation above, consider a Zariski-dense holomorphic map $f : \mathbb{C}^2 \rightarrow X$ and its (meromorphic) liftings $f_k : \mathbb{C}^2 \rightarrow X_k$ with the associated currents $[\Phi]^{(k)}$. Then

$$\lim_{k \rightarrow +\infty} [\Phi]^{(k)}.E_k = 0.$$

Proof. From theorem 7.2, we have

$$0 \leq [\Phi]^{(k)}.E_k \leq \frac{\alpha}{k^{1/3}}[\Phi].H,$$

and we let k tend to infinity. \square

REFERENCES

- [1] F.A. Bogomolov, *Families of curves on a surface of general type*, Soviet. Math. Dokl. 18 (1977), 1294–1297.
- [2] M. Brunella, *Courbes entières et feuilletages holomorphes*, L'Enseignement Mathématique 45 (1999), 195–216.
- [3] D. Burns, N. Sibony, *Limit currents and value distribution of holomorphic maps*, preprint 2010, arXiv:1005.4330.
- [4] F. Cano, *Reduction of the singularities of codimension one singular foliations in dimension three*, Ann. of Math. (2) 160 (2004), no. 3, 907–1011.
- [5] H. De Thelin, *Ahlfors' currents in higher dimension*, to appear in Ann. Fac. Sci. Toulouse.
- [6] C. Gasbarri, *The strong abc conjecture over function fields, after McQuillan and Yamanoi*, Séminaire Bourbaki 60, 2007/08, nj 989.
- [7] P. Griffiths, J. King, *Nevanlinna theory and holomorphic mappings between algebraic varieties*, Acta Math. 130 (1973), 145–220.
- [8] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) 79 (1964), 109–203; *ibid.* (2) 79 1964 205–326.

- [9] J.-P. Jouanolou, *Hypersurfaces solutions d'une équation de Pfaff analytique*, Math. Ann., 332 (1978), 239–248.
- [10] S. S.-Y. Lu, Y. Miyaoka, *Bounding codimension-one subvarieties and a general inequality between Chern numbers*, Amer. J. Math. 119 (1997), no. 3, 487–502.
- [11] B. Malgrange *Frobenius avec singularités. I. Codimension un*, Inst. Hautes Études Sci. Publ. Math. No. 46 (1976), 163–173.
- [12] M. McQuillan, *Diophantine approximations and foliations*, Publ. IHES 87 (1998), 121-174.
- [13] M. McQuillan, *Bloch hyperbolicity*, preprint IHES, 2001.
- [14] M. McQuillan, *Canonical models of foliations*, Pure Appl. Math. Q. 4 (2008), no. 3, part 2, 877–1012.
- [15] M. McQuillan; D. Panazzolo, *Almost étale resolution of foliations*, prépublication IHES, 2009.
- [16] R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Gauthier-Villars, Paris, 1929.
- [17] J. Noguchi, *On the value distribution of meromorphic mappings of covering spaces over C^m into algebraic varieties*, J. Math. Soc. Japan 37 (1985), no. 2, 295–313.
- [18] G. Pacienza, E. Rousseau, *Generalized Demailly-Semple jet bundles and holomorphic mappings into complex manifolds*, preprint, 2008.
- [19] M. Reid, *Canonical threefolds*, Journées de Géométrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, pp. 273–310, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.
- [20] A., Seidenberg, *Reduction of singularities of the differential equation $A dy = B dx$* , Amer. J. Math. 90 1968, 248–269.
- [21] B. V. Shabat, *Distribution of values of holomorphic mappings*, Translated from the Russian by J. R. King. Translation edited by Lev J. Leifman. Translations of Mathematical Monographs, 61. American Mathematical Society, Providence, RI, 1985. v+225 pp.
- [22] Y.-T. Siu, *Hyperbolicity in complex geometry*, The legacy of Niels Henrik Abel, 543–566, Springer, Berlin, 2004.
- [23] A. Vitter, *The lemma of the logarithmic derivative in several complex variables*, Duke Math. J. 44 (1977), no. 1, 89–104.
- [24] K. Yamanoi, *Algebro-geometric version of Nevanlinna's lemma on logarithmic derivative and applications*, Nagoya Math. J. 173 (2004), 23–63.

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