

A -infinity $GL(N)$ -equivariant matrix integrals-III.
Tree-level calculations and variations of nc-Hodge
structure on complex projective manifolds.

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23/06/2010

Tree level BV on complex projective manifold ($g=0$ calculations)

- nc-BV on $Symm(C_\lambda[1+d])$, $C_\lambda = \bigoplus_{j=0}^{\infty} (U[1]^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$

$$\hbar\Delta S + \frac{1}{2}\{S, S\} = 0, \quad S = \sum_{g \geq 0, i} \hbar^{2g-1+i} S_{g,i}, \quad S_{g,i} \in Symm^i(C_\lambda[1+d]),$$

$$\Leftrightarrow \Delta \exp(S/\hbar) = 0$$

$$\{S_{0,1}, S_{0,1}\} = 0, \quad - A_\infty - \text{CY algebra}$$

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- M - smooth projective/ \mathbb{C} , $c_1(T_M) = 0$, $\omega_0 \in \Gamma(M, K_M)$, \rightarrow BV operator on $\Omega^{0,*}(M, \Lambda^* T)$,

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- (tree level) BV equation on $\Omega^{0,*}(M, \Lambda^* T)$

$$u \Delta \gamma + \bar{\partial} \gamma + \frac{1}{2} [\gamma, \gamma] = 0,$$

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- The component

$$\gamma_0^{c'l} \in \Omega^{0,1}(M, T)$$

$\bar{\partial}\gamma_0^{c'l} + \frac{1}{2}[\gamma_0^{c'l}, \gamma_0^{c'l}] = 0$ - deformations of complex structure on M

Noncommutative periods map (B[5])



$$u\Delta\gamma + \bar{\partial}\gamma + \frac{1}{2}[\gamma, \gamma] = 0, \Leftrightarrow (u\Delta + \bar{\partial}) \exp\left(\frac{1}{u}\gamma\right) = 0$$

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$$r_u\gamma = \sum_{p,q} u^{(q-p)/2} \gamma^{p,q}, \quad \gamma^{p,q} \in \Omega^{0,q}(M, \Lambda^p T)$$

$$(u\Delta + \bar{\partial}) \rightarrow \Delta + \bar{\partial}$$

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- $$\begin{aligned} \Omega(t, u) &= \int_M r_u \exp\left(\frac{1}{u}\gamma\right) \lrcorner \omega_0 \\ \Omega(t, u) &\in H_{DR}^*(M)((u)) \hat{\otimes} \mathcal{O}_{\mathcal{M}_{\Delta T}} \end{aligned}$$

$$t \in \mathcal{M}_{\Delta T}, \quad T_0 \mathcal{M}_{\Delta T} = H^*(M, \Lambda^* T)$$

Noncommutative periods map and $g=0$ Gromov-Witten of the mirror



$$\Omega(t, u) = \int_M r_u \exp\left(\frac{1}{u} \gamma\right) \lrcorner \omega_0$$

For γ^W , normalized using a filtration W opposite to F^{Hodge} ([B5], 1999):

$$\frac{\partial^2}{\partial t^i \partial t^j} \Omega^W = u^{-1} C_{ij}^k(t) \frac{\partial}{\partial t^k} \Omega^W$$

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$$C_{kij}(t) = \partial_{kij}^3 (\text{genus} = 0 \text{ GW-potential of } M^{\text{mirror}})$$

Semi-infinite (Noncommutative) Hodge structures ([B5]).

- The class $[\exp(\gamma^W)\omega] = \Omega(t, u)$ is obtained as intersection of moving subspace

$$\Omega(t, u) = \mathcal{L}(t) \cap (\text{Affine space}(W))$$

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where $\gamma_0 \in \Omega^{0,*}(M, \Lambda^* T)$, $\bar{\partial}\gamma_0 + \frac{1}{2}[\gamma_0, \gamma_0] = 0$

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- Over mod space of complex structures it reduces to VHS

Noncommutative Hodge structures ([B5]), cont'd

- $\mathcal{L}(t)$ corresponds via HKR and formality isomorphisms for $C^*(A, A) + k[\zeta, \frac{\partial}{\partial \zeta}]$ -module $C_*(A)$ to

$$\mathcal{L}(t) = HC^-(A_t) \subset HP(A_t)$$

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where

$$rk_{C[[u]]} HC^-(A) = rk_{C((u))} HP$$

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- Real structure on HP in the case of arbitrary A_∞ -algebra?

Summation over trees, BCOV lagrangian and BV cyclic operad.

- For $W = \overline{F^{Hodge}}$, $\gamma(t) = \Sigma \gamma_a t^a + \Delta \alpha(t)$, $\Delta(\gamma) = 0$

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- Meaning of this : the tree-level Feynman transform of the operad of BV algebras / $\Delta = 0 =$ operad of $H_*(\mathcal{M}_{0,n})$

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