



HAL
open science

A-infinity $GL(N)$ -equivariant matrix integrals-II. The nc-BV differential, (stable) ribbon graphs and compactified moduli spaces.

Serguei Barannikov

► **To cite this version:**

Serguei Barannikov. A-infinity $GL(N)$ -equivariant matrix integrals-II. The nc-BV differential, (stable) ribbon graphs and compactified moduli spaces.. This is the second part of the series of three lectures which I gave at "D-branes, Effective Actions and Homological Mirror Symmetry" conference, Jun 2010, Vienna, Austria. hal-00494330

HAL Id: hal-00494330

<https://hal.science/hal-00494330>

Submitted on 7 Jul 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



A -infinity $GL(N)$ -equivariant matrix integrals-II. The nc-BV differential, (stable) ribbon graphs and compactified moduli spaces.

Serguei Barannikov

IMJ, CNRS

22/06/2010

The nc-Batalin-Vilkovisky formalism, stable ribbon graphs and compactified moduli spaces (B1,2006)

hal-00494330, version 1 - 22 Jun 2016

- Theorem: stable ribbon graph complex = Feynman transform of the modular operad of "associative algebras with scalar product" (odd/even d)

The nc-Batalin-Vilkovisky formalism, stable ribbon graphs and compactified moduli spaces (B1,2006)

hal-00494330, version 1 - 22 Jun 2016

- Theorem: stable ribbon graph complex = Feynman transform of the modular operad of "associative algebras with scalar product" (odd/even d)
- algebra over this Feynman transform \leftrightarrow 1-to-1 correspondence with solutions to the nc-BV equation.

The nc-Batalin-Vilkovisky formalism, stable ribbon graphs and compactified moduli spaces (B1,2006)

hal-00494330, version 1 - 22 Jun 2016

- Theorem: stable ribbon graph complex = Feynman transform of the modular operad of "associative algebras with scalar product" (odd/even d)
- algebra over this Feynman transform \leftrightarrow 1-to-1 correspondence with solutions to the nc-BV equation.
- solution to nc-BV equation \rightarrow homology classes in the stable ribbon graph complex (\Rightarrow in $H_*(\overline{\mathcal{M}}_{g,n}^{comb})$)

The nc-Batalin-Vilkovisky formalism, stable ribbon graphs and compactified moduli spaces (B1,2006)

hal-00494330, version 1 - 22 Jun 2016

- Theorem: stable ribbon graph complex = Feynman transform of the modular operad of "associative algebras with scalar product" (odd/even d)
- algebra over this Feynman transform \leftrightarrow 1-to-1 correspondence with solutions to the nc-BV equation.
- solution to nc-BV equation \rightarrow homology classes in the stable ribbon graph complex (\Rightarrow in $H_*(\overline{\mathcal{M}}_{g,n}^{comb})$)
- starting from dg-associative (A_∞ -) algebra V with scalar product \rightarrow summation over ribbon graphs defines solution to the nc-BV equation on $H(V)$ ([B6])

The nc-Batalin-Vilkovisky formalism, stable ribbon graphs and compactified moduli spaces (B1,2006)

hal-00494330, version 1 - 22 Jun 2016

- Theorem: stable ribbon graph complex = Feynman transform of the modular operad of "associative algebras with scalar product" (odd/even d)
- algebra over this Feynman transform \leftrightarrow 1-to-1 correspondence with solutions to the nc-BV equation.
- solution to nc-BV equation \rightarrow homology classes in the stable ribbon graph complex (\Rightarrow in $H_*(\overline{\mathcal{M}}_{g,n}^{comb})$)
- starting from dg-associative (A_∞ -) algebra V with scalar product \rightarrow summation over ribbon graphs defines solution to the nc-BV equation on $H(V)$ ([B6])
- I extend this constructions to algebras without scalar product by setting $V = A \oplus A^\vee[d] \rightarrow$ summation over directed ribbon graphs ([B1] (odd d), [B6]).

The nc-Batalin-Vilkovisky formalism, stable ribbon graphs and compactified moduli spaces (B1,2006)

hal-00494330, version 1 - 22 Jun 2016

- Theorem: stable ribbon graph complex = Feynman transform of the modular operad of "associative algebras with scalar product" (odd/even d)
- algebra over this Feynman transform \leftrightarrow 1-to-1 correspondence with solutions to the nc-BV equation.
- solution to nc-BV equation \rightarrow homology classes in the stable ribbon graph complex (\Rightarrow in $H_*(\overline{\mathcal{M}}_{g,n}^{comb})$)
- starting from dg-associative (A_∞ -) algebra V with scalar product \rightarrow summation over ribbon graphs defines solution to the nc-BV equation on $H(V)$ ([B6])
- I extend this constructions to algebras without scalar product by setting $V = A \oplus A^\vee[d] \rightarrow$ summation over directed ribbon graphs ([B1] (odd d), [B6]).
- Let I - an *odd derivation* acting on associative superalgebra V , $\dim_k V < \infty$, with scalar product, $tr([a, \cdot])$ on V_0 , and let $\exists \tilde{I}$, $[I, \tilde{I}] = 1$, \rightarrow Cohomology classes in $H^*(\overline{\mathcal{M}}_{g,n}^{comb})$ ([B2])

The nc-Batalin-Vilkovisky operator and compactified moduli spaces (B1,2006)

• U - $\mathbb{Z}/2\mathbb{Z}$ graded vector space/ \mathbb{C} , l -scalar product on U of degree $d \in \mathbb{Z}/2\mathbb{Z}$,

$$C_\lambda = \bigoplus_{j=0}^{\infty} (U[1]^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$$

-the symmetric (resp. exterior) powers for odd (resp even) d , of cyclic tensors

The nc-Batalin-Vilkovisky operator and compactified moduli spaces (B1,2006)

- U - $\mathbb{Z}/2\mathbb{Z}$ graded vector space/ \mathbb{C} , l -scalar product on U of degree $d \in \mathbb{Z}/2\mathbb{Z}$,

$$C_\lambda = \bigoplus_{j=0}^{\infty} (U[1]^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$$

- the symmetric (resp. exterior) powers for odd (resp even) d , of cyclic tensors

- nc-BV equation

$$\hbar \Delta S + \frac{1}{2} \{S, S\} = 0, \quad S = \sum_{g \geq 0, i} \hbar^{2g-1+i} S_{g,i}, \quad S_{g,i} \in \text{Symm}^i(C_\lambda[1+d]),$$

$$\Leftrightarrow \Delta \exp(S/\hbar) = 0$$

The nc-Batalin-Vilkovisky operator and compactified moduli spaces (B1,2006)

hal-00494330, version 1 - 22 Jun 2016

- U $\mathbb{Z}/2\mathbb{Z}$ graded vector space/ \mathbb{C} , l -scalar product on U of degree $d \in \mathbb{Z}/2\mathbb{Z}$,

$$C_\lambda = \bigoplus_{j=0}^{\infty} (U[1]^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$$

-the symmetric (resp. exterior) powers for odd (resp even) d , of cyclic tensors

- nc-BV equation

$$\begin{aligned} \hbar \Delta S + \frac{1}{2} \{S, S\} &= 0, \quad S = \sum_{g \geq 0, i} \hbar^{2g-1+i} S_{g,i}, \quad S_{g,i} \in \text{Sym}^i(C_\lambda[1+d]), \\ &\Leftrightarrow \Delta \exp(S/\hbar) = 0 \end{aligned}$$

- set of tensors $S_{g,i} \rightarrow$ partition function on stable ribbon graphs: contraction of $\bigotimes_{\text{vertices}} S_{g,i}$ with $\bigotimes_{\text{edges}} l$

The nc-Batalin-Vilkovisky operator and compactified moduli spaces (B1,2006)

hal-00494330, version 1 - 22 Jun 2016

- U $-\mathbb{Z}/2\mathbb{Z}$ graded vector space/ \mathbb{C} , l -scalar product on U of degree $d \in \mathbb{Z}/2\mathbb{Z}$,

$$C_\lambda = \bigoplus_{j=0}^{\infty} (U[1]^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$$

-the symmetric (resp. exterior) powers for odd (resp even) d , of cyclic tensors

- nc-BV equation

$$\begin{aligned} \hbar \Delta S + \frac{1}{2} \{S, S\} &= 0, \quad S = \sum_{g \geq 0, i} \hbar^{2g-1+i} S_{g,i}, \quad S_{g,i} \in \text{Symm}^i(C_\lambda[1+d]), \\ &\Leftrightarrow \Delta \exp(S/\hbar) = 0 \end{aligned}$$

- set of tensors $S_{g,i} \rightarrow$ partition function on stable ribbon graphs: contraction of $\otimes_{\text{vertices}} S_{g,i}$ with $\otimes_{\text{edges}} l$

- Theorem: S satisfies nc-BV equation \Leftrightarrow this is a closed chain in the stable ribbon graph complex (\Rightarrow in $H_*(\overline{\mathcal{M}}_{g,n}^{\text{comb}})$)

Solutions to nc BV equation

hal-00494330, version 1 - 22 Jun 2010

- Theorem ([B6]). Summation over ribbon graphs \rightarrow solution to the nc Batalin-Vilkovisky equation from dg-associative algebras (summation over trees \rightarrow A-infinity algebra structure)

Solutions to nc BV equation

hal-00494330, version 1 - 22 Jun 2010

- Theorem ([B6]). Summation over ribbon graphs \rightarrow solution to the nc Batalin-Vilkovisky equation from dg-associative algebras (summation over trees \rightarrow A-infinity algebra structure)
- Conjecture ([B1]). Counting of holomorphic curves $(\Sigma, \partial\Sigma, p_i) \rightarrow (M, \coprod L_i, \oplus H_*(L_i \cap L_j))$, with $\mathbb{Z}/2\mathbb{Z}$ -graded local systems, gives solution to the nc-BV equations.

- Let I - an *odd derivation* acting on associative superalgebra V , $\dim_k V < \infty$, with scalar product I , remark: in general $I^2 \neq 0$

- Let I - an *odd derivation* acting on associative superalgebra V , $\dim_k V < \infty$, with scalar product I , remark: in general $I^2 \neq 0$
- let $\exists \tilde{I}$, $[I, \tilde{I}] = 1$, and assume that $\text{tr}([a, \cdot]) = 0$ for any $a \in V_0$.

- Let I - an *odd derivation* acting on associative superalgebra V , $\dim_k V < \infty$, with scalar product I , remark: in general $I^2 \neq 0$
- let $\exists \tilde{I}$, $[I, \tilde{I}] = 1$, and assume that $tr([a, \cdot]) = 0$ for any $a \in V_0$.
- Theorem ([B2],[B3]) This data \rightarrow Cohomology classes in $H^*(\overline{\mathcal{M}}_{g,n}^{comb})$

- Let I - an *odd derivation* acting on associative superalgebra V , $\dim_k V < \infty$, with scalar product I , remark: in general $I^2 \neq 0$
- let $\exists \tilde{I}$, $[I, \tilde{I}] = 1$, and assume that $tr([a, \cdot]) = 0$ for any $a \in V_0$.
- Theorem ([B2],[B3]) This data \rightarrow Cohomology classes in $H^*(\overline{\mathcal{M}}_{g,n}^{comb})$
- Construction: contraction of $\otimes_{vertices} tr^{\sigma, g}$ with $\otimes_{edges} I^{\vee}(\tilde{I} \cdot, \cdot)$

Derivations and the cohomology classes

hal-00494330, version 1 - 22 Jun 2010

- Let I - an *odd derivation* acting on associative superalgebra V , $\dim_k V < \infty$, with scalar product I , remark: in general $I^2 \neq 0$
- let $\exists \tilde{I}$, $[I, \tilde{I}] = 1$, and assume that $tr([a, \cdot]) = 0$ for any $a \in V_0$.
- Theorem ([B2],[B3]) This data \rightarrow Cohomology classes in $H^*(\overline{\mathcal{M}}_{g,n}^{comb})$
- Construction: contraction of $\otimes_{vertices} tr^{\sigma, g}$ with $\otimes_{edges} I^\vee(\tilde{I} \cdot, \cdot)$
- \tilde{I} is homotopy inverse to the derivation \rightarrow this defines cohomology class

Strange associative superalgebra with odd trace and π -classes.

hal-00494330, version 1 - 22 Jun 2016

• $q(N) = \{[X, \pi] = 0 \mid X \in gl(N|N)\}$, where π -odd involution, $q(N)$ has *odd trace* otr , $l = [\Xi, \cdot]$, Ξ - odd element $\Xi = \begin{pmatrix} 0 & \\ & diag(\lambda_1, \dots, \lambda_n) \end{pmatrix}$, ($l^2 \neq 0$ (!))

Strange associative superalgebra with odd trace and psi-classes.

hal-00494330, version 1 - 22 Jun 2016

- $q(N) = \{[X, \pi] = 0 \mid X \in gl(N|N)\}$, where π -odd involution, $q(N)$ has *odd trace* otr , $l = [\Xi, \cdot]$, Ξ - odd element $\Xi = (0 \mid \text{diag}(\lambda_1, \dots, \lambda_n))$, ($l^2 \neq 0$ (!))
- Theorem ([B2],[B3]) This is the generating function for products of classes $c_1(T_i)$.

Strange associative superalgebra with odd trace and π -classes.

hal-00494330, version 1 - 22 Jun 2016

- $q(N) = \{[X, \pi] = 0 \mid X \in gl(N|N)\}$, where π -odd involution, $q(N)$ has *odd trace* otr , $I = [\Xi, \cdot]$, Ξ - odd element $\Xi = (0 \mid \text{diag}(\lambda_1, \dots, \lambda_n))$, ($I^2 \neq 0$ (!))
- Theorem ([B2],[B3]) This is the generating function for products of classes $c_1(T_i)$.
- Similarly, with even scalar product and an *odd* derivation, in particular for $gl(N|N)$ and $I = [\Xi, \cdot]$, $\Xi \in gl(N|N)_{\text{odd}}$.

References:

[B1] S.Barannikov, *Modular operads and Batalin-Vilkovisky geometry*. IMRN, Vol. 2007, article ID rnm075. Preprint Max Planck Institute for Mathematics 2006-48 (04/2006),

[B2] S.Barannikov, *Noncommutative Batalin-Vilkovisky geometry and matrix integrals*. «Comptes rendus Mathematique», presented for publication by M.Kontsevich in 05/2009, arXiv:0912.5484; Preprint NI06043 Newton Institute (09/2006), Preprint HAL, the electronic CNRS archive, hal-00102085 (09/2006)

[B3] S.Barannikov, *Supersymmetry and cohomology of graph complexes*. Preprint hal-00429963; (11/2009).

[B4] S.Barannikov, *Matrix De Rham complex and quantum A-infinity algebras*. arXiv:1001.5264, Preprint hal-00378776; (04/2009).

[B5] S.Barannikov, *Quantum periods - I. Semi-infinite variations of Hodge structures*. Preprint ENS DMA-00-19. arXiv:math/0006193 (06/2000), Intern. Math. Res. Notices. 2001, No. 23

[B6] S.Barannikov, *Solving the noncommutative Batalin-Vilkovisky equation*. Preprint hal-00464794 (03/2010). arXiv:1004.2253

