



**HAL**  
open science

## A-infinity $GL(N)$ -equivariant matrix integrals-I

Serguei Barannikov

► **To cite this version:**

Serguei Barannikov. A-infinity  $GL(N)$ -equivariant matrix integrals-I. Topological String Theory, Modularity & non-perturbative Physics, Jun 2010, Vienna, Austria. hal-00493919

**HAL Id: hal-00493919**

**<https://hal.science/hal-00493919>**

Submitted on 6 Jul 2010

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



# $A$ -infinity $GL(N)$ -equivariant matrix integrals-I

Serguei Barannikov

IMJ, CNRS

21/06/2010

# The noncommutative Batalin-Vilkovisky equation and $A_\infty$ -infinity algebras

hal-00493919, version 1 - 21 Jun 2010

- $U$  -  $\mathbb{Z}/2\mathbb{Z}$  graded vector space/ $\mathbb{C}$ ,  $l$ -scalar product on  $U$  of degree  $d \in \mathbb{Z}/2\mathbb{Z}$ , (variant:  $\mathbb{Z}$ -graded), consider

$$F = \text{Sym}(C_\lambda[1+d]), \quad C_\lambda = \bigoplus_{j=0}^{\infty} (U[1]^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$$

-the symmetric (resp. exterior) powers for odd (resp even)  $d$ , of cyclic tensors

# The noncommutative Batalin-Vilkovisky equation and $\Lambda$ -infinity algebras

hal-00493919, version 1 - 21 Jun 2010

- $U$   $\mathbb{Z}/2\mathbb{Z}$  graded vector space/ $\mathbb{C}$ ,  $l$ -scalar product on  $U$  of degree  $d \in \mathbb{Z}/2\mathbb{Z}$ , (variant:  $\mathbb{Z}$ -graded), consider

$$F = \text{Sym}(C_\lambda[1+d]), \quad C_\lambda = \bigoplus_{j=0}^{\infty} (U[1]^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$$

- -the symmetric (resp. exterior) powers for odd (resp even)  $d$ , of cyclic tensors ([B1],2006)  $\Delta : F \rightarrow F[1]$ ,  $\Delta^2 = 0$ , defined via dissection-gluing of cyclic tensors, of the *second* order w.r.t. product of cycles.

# The noncommutative Batalin-Vilkovisky equation and $\Delta$ -infinity algebras

- $U$  -  $\mathbb{Z}/2\mathbb{Z}$  graded vector space/ $\mathbb{C}$ ,  $l$ -scalar product on  $U$  of degree  $d \in \mathbb{Z}/2\mathbb{Z}$ , (variant:  $\mathbb{Z}$ -graded), consider

$$F = \text{Sym}(C_\lambda[1+d]), \quad C_\lambda = \bigoplus_{j=0}^{\infty} (U[1]^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$$

-the symmetric (resp. exterior) powers for odd (resp even)  $d$ , of cyclic tensors ([B1],2006)  $\Delta : F \rightarrow F[1]$ ,  $\Delta^2 = 0$ , defined via dissection-gluing of cyclic tensors, of the *second* order w.r.t. product of cycles.

- The noncommutative Batalin-Vilkovisky equation (nc-BV)

$$\hbar \Delta S + \frac{1}{2} \{S, S\} = 0, \quad S = \sum_{g \geq 0, i} \hbar^{2g-1+i} S_{g,i}, \quad S_{g,i} \in \text{Sym}^i(C_\lambda[1+d]),$$

# The noncommutative Batalin-Vilkovisky equation and $\Delta$ -infinity algebras

- $U$ - $\mathbb{Z}/2\mathbb{Z}$  graded vector space/ $\mathbb{C}$ ,  $l$ -scalar product on  $U$  of degree  $d \in \mathbb{Z}/2\mathbb{Z}$ , (variant:  $\mathbb{Z}$ -graded), consider

$$F = \text{Sym}(C_\lambda[1+d]), \quad C_\lambda = \bigoplus_{j=0}^{\infty} (U[1]^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$$

-the symmetric (resp. exterior) powers for odd (resp even)  $d$ , of cyclic tensors ([B1],2006)  $\Delta : F \rightarrow F[1]$ ,  $\Delta^2 = 0$ , defined via dissection-gluing of cyclic tensors, of the *second* order w.r.t. product of cycles.

- The noncommutative Batalin-Vilkovisky equation (nc-BV)

$$\hbar \Delta S + \frac{1}{2} \{S, S\} = 0, \quad S = \sum_{g \geq 0, i} \hbar^{2g-1+i} S_{g,i}, \quad S_{g,i} \in \text{Sym}^i(C_\lambda[1+d]),$$

- nc-BV  $\Leftrightarrow \Delta \exp(S/\hbar) = 0$

# The noncommutative Batalin-Vilkovisky equation and $A_\infty$ -infinity algebras

- $U$ - $\mathbb{Z}/2\mathbb{Z}$  graded vector space/ $\mathbb{C}$ ,  $l$ -scalar product on  $U$  of degree  $d \in \mathbb{Z}/2\mathbb{Z}$ , (variant:  $\mathbb{Z}$ -graded), consider

$$F = \text{Sym}(C_\lambda[1+d]), \quad C_\lambda = \bigoplus_{j=0}^{\infty} (U[1]^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$$

-the symmetric (resp. exterior) powers for odd (resp even)  $d$ , of cyclic tensors ([B1],2006)  $\Delta : F \rightarrow F[1]$ ,  $\Delta^2 = 0$ , defined via dissection-gluing of cyclic tensors, of the *second* order w.r.t. product of cycles.

- The noncommutative Batalin-Vilkovisky equation (nc-BV)

$$\hbar \Delta S + \frac{1}{2} \{S, S\} = 0, \quad S = \sum_{g \geq 0, i} \hbar^{2g-1+i} S_{g,i}, \quad S_{g,i} \in \text{Symm}^i(C_\lambda[1+d]),$$

- nc-BV  $\Leftrightarrow \Delta \exp(S/\hbar) = 0$

$$\{S_{0,1}, S_{0,1}\} = 0,$$

$V = U^\vee$ , and  $S_{0,1} = m_{A_\infty}$  is  $A_\infty$ -algebra structure on  $V$  with invariant scalar product of degree  $d$

hal-00493919, version 1 - 21 Jun 2010

# The noncommutative Batalin-Vilkovisky equation and $A_\infty$ -infinity algebras

hal-00493919, version 1 - 21 Jun 2010

- $U$   $\mathbb{Z}/2\mathbb{Z}$  graded vector space/ $\mathbb{C}$ ,  $l$ -scalar product on  $U$  of degree  $d \in \mathbb{Z}/2\mathbb{Z}$ , (variant:  $\mathbb{Z}$ -graded), consider

$$F = \text{Sym}(C_\lambda[1+d]), \quad C_\lambda = \bigoplus_{j=0}^{\infty} (U[1]^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$$

-the symmetric (resp. exterior) powers for odd (resp even)  $d$ , of cyclic tensors ([B1],2006)  $\Delta : F \rightarrow F[1]$ ,  $\Delta^2 = 0$ , defined via dissection-gluing of cyclic tensors, of the *second* order w.r.t. product of cycles.

- The noncommutative Batalin-Vilkovisky equation (nc-BV)

$$\hbar \Delta S + \frac{1}{2} \{S, S\} = 0, \quad S = \sum_{g \geq 0, i} \hbar^{2g-1+i} S_{g,i}, \quad S_{g,i} \in \text{Symm}^i(C_\lambda[1+d]),$$

- nc-BV  $\Leftrightarrow \Delta \exp(S/\hbar) = 0$

$$\{S_{0,1}, S_{0,1}\} = 0,$$

$V = U^\vee$ , and  $S_{0,1} = m_{A_\infty}$  is  $A_\infty$ -algebra structure on  $V$  with invariant scalar product of degree  $d$

- $A_\infty$ -algebras without scalar product are included in the formalism by setting  $U = A \oplus A^\vee[d]$ , giving an  $A_\infty$ -algebra with scalar product.



# The A-infinity equivariant matrix integrals ([B2],09/2006)

hal-00493919, version 1 - 21 Jun 2010 • ([B2],09/2006) element  $S \in \text{Sym}(C_\lambda[1+d]) \rightarrow$  matrix integral

$$\int \exp \widehat{S}(X, \Lambda) dX$$

$X \in \mathfrak{gl}(N|N) \otimes V[1]$  in the odd  $d$  case,  $X \in \mathfrak{q}(N) \otimes V[1]$  in the even  $d$  case,

# The A-infinity equivariant matrix integrals ([B2],09/2006)

- ([B2],09/2006) element  $S \in \text{Sym}(C_\lambda[1+d]) \rightarrow \text{matrix integral}$

$$\int \exp \widehat{S}(X, \Lambda) dX$$

$X \in \mathfrak{gl}(N|N) \otimes V[1]$  in the odd  $d$  case,  $X \in \mathfrak{q}(N) \otimes V[1]$  in the even  $d$  case,

- If  $S$  satisfies nc-BV equation then

$$(\Delta_{\text{matrix}} + i_{\mathfrak{gl}}) \exp \widehat{S}(X, \Lambda) = 0$$

$\Leftrightarrow \exp \widehat{S}(X, \Lambda) dX$  is  $\mathfrak{gl}$ -equivariantly closed differential form.

# The A-infinity equivariant matrix integrals ([B2],09/2006)

- ([B2],09/2006) element  $S \in \text{Sym}(C_\lambda[1+d]) \rightarrow$  matrix integral

$$\int \exp \widehat{S}(X, \Lambda) dX$$

$X \in \mathfrak{gl}(N|N) \otimes V[1]$  in the odd  $d$  case,  $X \in \mathfrak{q}(N) \otimes V[1]$  in the even  $d$  case,

If  $S$  satisfies nc-BV equation then

$$(\Delta_{\text{matrix}} + i_{\mathfrak{gl}}) \exp \widehat{S}(X, \Lambda) = 0$$

$\Leftrightarrow \exp \widehat{S}(X, \Lambda) dX$  is  $\mathfrak{gl}$ -equivariantly closed differential form.

• In the case of the algebra  $1 \cdot 1 = 1$ , - solution to nc BV for  $V = \{1\}$ , this is the matrix Airy integral  $\int \exp(\frac{1}{6} \text{Tr}(Y^3) - \frac{1}{2} \text{Tr}(\Lambda Y^2)) dY$

# The A-infinity equivariant matrix integrals ([B2],09/2006)

hal-00493919, version 1 - 21 Jun 2010

• ([B2],09/2006) element  $S \in \text{Sym}(C_\lambda[1+d]) \rightarrow$  matrix integral

$$\int \exp \widehat{S}(X, \Lambda) dX$$

•  $X \in \mathfrak{gl}(N|N) \otimes V[1]$  in the odd  $d$  case,  $X \in \mathfrak{q}(N) \otimes V[1]$  in the even  $d$  case,

• If  $S$  satisfies nc-BV equation then

$$(\Delta_{\text{matrix}} + i_{\mathfrak{gl}}) \exp \widehat{S}(X, \Lambda) = 0$$

•  $\Leftrightarrow \exp \widehat{S}(X, \Lambda) dX$  is  $\mathfrak{gl}$ -equivariantly closed differential form.

• In the case of the algebra  $1 \cdot 1 = 1$ , - solution to nc BV for  $V = \{1\}$ , this is the matrix Airy integral  $\int \exp(\frac{1}{6} \text{Tr}(Y^3) - \frac{1}{2} \text{Tr}(\Lambda Y^2)) dY$

• This is the higher genus counterpart of the (nc)Hodge theory integration on CY projective manifolds,  $(\hbar \Delta \gamma + \bar{\partial} \gamma + \frac{1}{2} [\gamma, \gamma] = 0, \gamma \in \Omega^{0,*}(M, \Lambda T))$

# The A-infinity equivariant matrix integrals ([B2],09/2006)

hal-00493919, version 1 - 21 Jun 2010

([B2],09/2006) element  $S \in \text{Symm}(C_\lambda[1+d]) \rightarrow$  matrix integral

$$\int \exp \widehat{S}(X, \Lambda) dX$$

$X \in \mathfrak{gl}(N|N) \otimes V[1]$  in the odd  $d$  case,  $X \in \mathfrak{q}(N) \otimes V[1]$  in the even  $d$  case,

If  $S$  satisfies nc-BV equation then

$$(\Delta_{matrix} + i_{gl}) \exp \widehat{S}(X, \Lambda) = 0$$

$\Leftrightarrow \exp \widehat{S}(X, \Lambda) dX$  is  $\mathfrak{gl}$ -equivariantly closed differential form.

In the case of the algebra  $1 \cdot 1 = 1$ , - solution to nc BV for  $V = \{1\}$ , this is the matrix Airy integral  $\int \exp(\frac{1}{6} \text{Tr}(Y^3) - \frac{1}{2} \text{Tr}(\Lambda Y^2)) dY$

This is the higher genus counterpart of the (nc)Hodge theory integration on CY projective manifolds, ( $\hbar \Delta \gamma + \bar{\partial} \gamma + \frac{1}{2} [\gamma, \gamma] = 0$ ,  $\gamma \in \Omega^{0,*}(M, \Lambda T)$ )

$S$  satisfies nc-BV, asymptotic expansion as  $\Lambda \rightarrow \infty$  -sum over stable ribbon graphs  $\Rightarrow$  cohomology classes in  $H^*(\overline{\mathcal{M}}_{g,n}^K)$  (in  $H^*(\overline{\mathcal{M}}_{g,n}^K, \mathcal{L})$  for odd  $d$ )

# The $A$ -infinity equivariant matrix integrals ([B2],09/2006)

hal-00493919, version 1 - 21 Jun 2010

([B2],09/2006) element  $S \in \text{Sym}(C_\lambda[1+d]) \rightarrow$  matrix integral

$$\int \exp \widehat{S}(X, \Lambda) dX$$

$X \in \mathfrak{gl}(N|N) \otimes V[1]$  in the odd  $d$  case,  $X \in \mathfrak{q}(N) \otimes V[1]$  in the even  $d$  case,

If  $S$  satisfies nc-BV equation then

$$(\Delta_{\text{matrix}} + i_{\mathfrak{gl}}) \exp \widehat{S}(X, \Lambda) = 0$$

$\Leftrightarrow \exp \widehat{S}(X, \Lambda) dX$  is  $\mathfrak{gl}$ -equivariantly closed differential form.

In the case of the algebra  $1 \cdot 1 = 1$ , - solution to nc BV for  $V = \{1\}$ , this is the matrix Airy integral  $\int \exp(\frac{1}{6} \text{Tr}(Y^3) - \frac{1}{2} \text{Tr}(\Lambda Y^2)) dY$

This is the higher genus counterpart of the (nc)Hodge theory integration on CY projective manifolds, ( $\hbar \Delta \gamma + \bar{\partial} \gamma + \frac{1}{2} [\gamma, \gamma] = 0$ ,  $\gamma \in \Omega^{0,*}(M, \Lambda T)$ )

$S$  satisfies nc-BV, asymptotic expansion as  $\Lambda \rightarrow \infty$  -sum over stable ribbon graphs  $\Rightarrow$  cohomology classes in  $H^*(\overline{\mathcal{M}}_{g,n}^K)$  (in  $H^*(\overline{\mathcal{M}}_{g,n}^K, \mathcal{L})$  for odd  $d$ )

My  $A_\infty$  equivariant matrix integrals define an integration framework in the noncommutative (derived algebraic) geometry, particularly adobted to the equation  $\{m_{A_\infty}, m_{A_\infty}\} = 0$

$$\int \exp \widehat{S}(X, \Lambda) \widehat{\varphi} dX, \varphi \in \text{Ker}(\hbar \Delta + \{S, \cdot\}) \subset \text{Sym}(C_\lambda[1+d])$$

# Noncommutative Batalin-Vilkovisky operator ([B1])

hal-00493919, version 1 - 21 Jun 2010

I define my noncommutative BV differential on  $\text{Sym}(C_\lambda[1+d])$  via

$$\begin{aligned} \Delta(x_{\rho_1} \cdots x_{\rho_r})_\lambda (x_{\tau_1} \cdots x_{\tau_t})_\lambda &= \\ &= \sum_{p,q} (-1)^\varepsilon l_{\rho_p \tau_q} (x_{\rho_1} \cdots x_{\rho_{p-1}} x_{\tau_{q+1}} \cdots x_{\tau_{q-1}} x_{\rho_{p+1}} \cdots x_{\rho_r})_\lambda + \end{aligned}$$

$$\sum_{p \pm 1 \neq q} (-1)^{\tilde{\varepsilon}} l_{\rho_p \rho_q} (x_{\rho_1} \cdots x_{\rho_{p-1}} x_{\rho_{q+1}} \cdots x_{\rho_r})_\lambda (x_{\rho_{p+1}} \cdots x_{\rho_{q-1}})_\lambda (x_{\tau_1} \cdots x_{\tau_t})_\lambda$$

$$\sum_{p \pm 1 \neq q} (-1)^{\tilde{\varepsilon}} l_{\tau_p \tau_q} (x_{\rho_1} \cdots x_{\rho_r})_\lambda (x_{\tau_1} \cdots x_{\tau_{p-1}} x_{\tau_{q+1}} \cdots x_{\tau_t})_\lambda (x_{\tau_{p+1}} \cdots x_{\tau_{q-1}})_\lambda$$

$$l_{\rho_p \rho_q} = l(x_{\rho_p}, x_{\rho_q})$$

# Noncommutative Batalin-Vilkovisky operator ([B1])

hal-00493919, version 1 - 21 Jun 2010

I define my noncommutative BV differential on  $\text{Sym}(C_\lambda[1+d])$  via

$$\begin{aligned} \Delta(x_{\rho_1} \cdots x_{\rho_r})_\lambda (x_{\tau_1} \cdots x_{\tau_t})_\lambda &= \\ &= \sum_{p,q} (-1)^\varepsilon l_{\rho_p \tau_q} (x_{\rho_1} \cdots x_{\rho_{p-1}} x_{\tau_{q+1}} \cdots x_{\tau_{q-1}} x_{\rho_{p+1}} \cdots x_{\rho_r})_\lambda + \\ &\sum_{p \pm 1 \neq q} (-1)^{\tilde{\varepsilon}} l_{\rho_p \rho_q} (x_{\rho_1} \cdots x_{\rho_{p-1}} x_{\rho_{q+1}} \cdots x_{\rho_r})_\lambda (x_{\rho_{p+1}} \cdots x_{\rho_{q-1}})_\lambda (x_{\tau_1} \cdots x_{\tau_t})_\lambda \\ &\sum_{p \pm 1 \neq q} (-1)^{\tilde{\varepsilon}} l_{\tau_p \tau_q} (x_{\rho_1} \cdots x_{\rho_r})_\lambda (x_{\tau_1} \cdots x_{\tau_{p-1}} x_{\tau_{q+1}} \cdots x_{\tau_t})_\lambda (x_{\tau_{p+1}} \cdots x_{\tau_{q-1}})_\lambda \end{aligned}$$

$$l_{\rho_p \rho_q} = l(x_{\rho_p}, x_{\tau_q})$$

signs are the standard Koszul signs taking into account that

$$(x_{\rho_1} \cdots x_{\rho_r})_\lambda = (1+d) + \sum \overline{x_{\rho_i}}, x_i \in V[1].$$



# The cyclic tensors, invariant functions and the matrix algebra with odd trace.

- Invariant theory:

$$\begin{aligned} \text{Sym}(C_\lambda) &\rightarrow \text{Sym}((\mathfrak{gl}(N|\tilde{N}) \otimes V[1])^\vee)^{\text{GL}(N|\tilde{N})} \\ \prod (a_1 \dots a_k)_\lambda &\rightarrow \prod \text{sTr}(A_1 \cdot \dots \cdot A_k) \end{aligned}$$

# The cyclic tensors, invariant functions and the matrix algebra with odd trace.

- Invariant theory:

$$\text{Sym}(C_\lambda) \rightarrow \text{Sym}((\mathfrak{gl}(N|\tilde{N}) \otimes V[1])^\vee)^{\text{GL}(N|\tilde{N})}$$

$$\prod (a_1 \dots a_k)_\lambda \rightarrow \prod \text{sTr}(A_1 \cdot \dots \cdot A_k)$$

- This is an isomorphism in degrees  $\leq N$ , it was at the origin of the discovery of cyclic homology, cyclic differential  $\leftrightarrow$  Lie cohomology differential of  $\mathfrak{gl}(V)$ .

# The cyclic tensors, invariant functions and the matrix algebra with odd trace.

- Invariant theory:

$$\text{Sym}(C_\lambda) \rightarrow \text{Sym}((\mathfrak{gl}(N|\tilde{N}) \otimes V[1])^\vee)^{\text{GL}(N|\tilde{N})}$$

$$\prod (a_1 \dots a_k)_\lambda \rightarrow \prod \text{sTr}(A_1 \cdot \dots \cdot A_k)$$

- This is an isomorphism in degrees  $\leq N$ , it was at the origin of the discovery of cyclic homology, cyclic differential  $\leftrightarrow$  Lie cohomology differential of  $\mathfrak{gl}(V)$ .
- To relate this with the nc-BV equation, one needs to solve the problem: for usual algebras (i.e. with scalar product of degree  $d = 0$ ) this is the wrong space: the symmetric instead of the exterior powers of cyclic tensors

# The cyclic tensors, invariant functions and the matrix algebra with odd trace.

- Invariant theory:

$$\text{Sym}(C_\lambda) \rightarrow \text{Sym}((\mathfrak{gl}(N|\tilde{N}) \otimes V[1])^\vee)^{\text{GL}(N|\tilde{N})}$$

$$\prod (a_1 \dots a_k)_\lambda \rightarrow \prod \text{sTr}(A_1 \cdot \dots \cdot A_k)$$

- This is an isomorphism in degrees  $\leq N$ , it was at the origin of the discovery of cyclic homology, cyclic differential  $\leftrightarrow$  Lie cohomology differential of  $\mathfrak{gl}(V)$ .
- To relate this with the nc-BV equation, one needs to solve the problem: for usual algebras (i.e. with scalar product of degree  $d = 0$ ) this is the wrong space: the symmetric instead of the exterior powers of cyclic tensors
- Solution: there must be a matrix algebra with *odd* trace:  
 $\overline{\text{tr}(A_1 \cdot \dots \cdot A_k)} = 1 + \Sigma \overline{A}$

# The cyclic tensors, invariant functions and the matrix algebra with odd trace.

- Invariant theory:

$$\text{Sym}(C_\lambda) \rightarrow \text{Sym}((\mathfrak{gl}(N|\tilde{N}) \otimes V[1])^\vee)^{GL(N|\tilde{N})}$$

$$\prod (a_1 \dots a_k)_\lambda \rightarrow \prod sTr(A_1 \cdot \dots \cdot A_k)$$

- This is an isomorphism in degrees  $\leq N$ , it was at the origin of the discovery of cyclic homology, cyclic differential  $\leftrightarrow$  Lie cohomology differential of  $\mathfrak{gl}(V)$ .

- To relate this with the nc-BV equation, one needs to solve the problem: for usual algebras (i.e. with scalar product of degree  $d = 0$ ) this is the wrong space: the symmetric instead of the exterior powers of cyclic tensors

- Solution: there must be a matrix algebra with *odd* trace:

$$\overline{tr(A_1 \cdot \dots \cdot A_k)} = 1 + \Sigma \overline{A};$$

- Such algebra exists:

$$\mathfrak{q}(N) = \{[X, \pi] = 0 | X \in \mathfrak{gl}(N|N)\}$$

where  $\pi = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}$  is an odd isomorphism,  $\pi^2 = -1$ ,

$$\mathfrak{q}(N) = \left\{ X = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}$$

# Algebra $q(N)$ and nc-BV equation

hal-00493919, version 1 - 21 Jun 2010

- $q(N)$  has *odd trace*

$$\text{otr} \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \text{tr}(B)$$

$$\text{otr}([X_1, X_2]) = 0$$

# Algebra $q(N)$ and nc-BV equation

hal-00493919, version 1 - 21 Jun 2010

- $q(N)$  has *odd trace*

$$\text{otr} \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \text{tr}(B)$$

$$\text{otr}([X_1, X_2]) = 0$$

- $q(N)$  is a simple  $\mathbb{Z}/2\mathbb{Z}$ -graded associative algebra

# Algebra $q(N)$ and nc-BV equation

hal-00493919, version 1 - 21 Jun 2010

- $q(N)$  has *odd trace*

$$\text{otr} \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \text{tr}(B)$$

$$\text{otr}([X_1, X_2]) = 0$$

- $q(N)$  is a simple  $\mathbb{Z}/2\mathbb{Z}$ -graded associative algebra

- The map

$$\text{Sym}(\Pi C_\lambda) \rightarrow \text{Sym}((q(N) \otimes V[1])^\vee)^{Q(N)}$$

$$\prod (a_1 \dots a_k)_\lambda \rightarrow \prod \text{otr}(A_1 \cdot \dots \cdot A_k)$$

is an isomorphism in degrees  $\leq N$



# Algebra $q(N)$ and nc-BV equation

hal-00493919, version 1 - 21 Jun 2010

- $q(N)$  has *odd trace*

$$\text{otr} \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \text{tr}(B)$$

$$\text{otr}([X_1, X_2]) = 0$$

- $q(N)$  is a simple  $\mathbb{Z}/2\mathbb{Z}$ -graded associative algebra

- The map

$$\text{Sym}(\Pi C_\lambda) \rightarrow \text{Sym}((q(N) \otimes V[1])^\vee)^{Q(N)}$$

$$\prod (a_1 \dots a_k)_\lambda \rightarrow \prod \text{otr}(A_1 \dots A_k)$$

is an isomorphism in degrees  $\leq N$

- Theorem([B4]): nc-BV differential  $\Delta$  on  $\text{Sym}(\Pi C_\lambda)$  is identified with  $Q(N)$ -invariant odd BV-operator on  $(q(N) \otimes V[1])^\vee$  corresponding to the odd affine symplectic structure defined by  $\text{otr}(X_1 X_2) \otimes I$

# Algebra $q(N)$ and nc-BV equation

hal-00493919, version 1 - 21 Jun 2010

- $q(N)$  has *odd trace*

$$\text{otr} \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \text{tr}(B)$$

$$\text{otr}([X_1, X_2]) = 0$$

- $q(N)$  is a simple  $\mathbb{Z}/2\mathbb{Z}$ -graded associative algebra

- The map

$$\text{Sym}(\Pi C_\lambda) \rightarrow \text{Sym}((q(N) \otimes V[1])^\vee)^{Q(N)}$$

$$\prod (a_1 \dots a_k)_\lambda \rightarrow \prod \text{otr}(A_1 \dots A_k)$$

is an isomorphism in degrees  $\leq N$

- Theorem([B4]): nc-BV differential  $\Delta$  on  $\text{Sym}(\Pi C_\lambda)$  is identified with  $Q(N)$ -invariant odd BV-operator on  $(q(N) \otimes V[1])^\vee$  corresponding to the odd affine symplectic structure defined by  $\text{otr}(X_1 X_2) \otimes I$
- Corollary: tensor multiplication by  $q(N)$ ,  $gl(N|\tilde{N}) \rightarrow$  super Morita equivalence on solutions to nc-BV.

- $S \in \text{Sym}(\Pi C_\lambda) \rightarrow \widehat{S} \in \text{Sym}((q(N) \otimes V[1])^\vee)^{Q(N)}$ , superfunction on affine BV manifold

- $S \in \text{Sym}(\Pi C_\lambda) \rightarrow \widehat{S} \in \text{Sym}((q(N) \otimes V[1])^\vee)^{Q(N)}$ , superfunction on affine BV manifold
- $\widehat{S}$ - polyvector field on the even part  $(q(N) \otimes V[1])_0$

- $S \in \text{Sym}(\Pi C_\lambda) \rightarrow \widehat{S} \in \text{Sym}((\mathfrak{q}(N) \otimes V[1])^\vee)^{Q(N)}$ , superfunction on affine BV manifold
- $\widehat{S}$ - polyvector field on the even part  $(\mathfrak{q}(N) \otimes V[1])_0$
- canonically defined, up to a sign, affine holomorphic volume element  $dX$  on  $(\mathfrak{q}(N) \otimes V[1])_0$

- $S \in \text{Sym}(\Pi C_\lambda) \rightarrow \widehat{S} \in \text{Sym}((q(N) \otimes V[1])^\vee)^{Q(N)}$ , superfunction on affine BV manifold
- $\widehat{S}$ - polyvector field on the even part  $(q(N) \otimes V[1])_0$
- canonically defined, up to a sign, affine holomorphic volume element  $dX$  on  $(q(N) \otimes V[1])_0$
- polyvector  $\widehat{S} \rightarrow$  differential form

- $S \in \text{Sym}(\Pi C_\lambda) \rightarrow \widehat{S} \in \text{Sym}((q(N) \otimes V[1])^\vee)^{Q(N)}$ , superfunction on affine BV manifold
- $\widehat{S}$ - polyvector field on the even part  $(q(N) \otimes V[1])_0$
- canonically defined, up to a sign, affine holomorphic volume element  $dX$  on  $(q(N) \otimes V[1])_0$
- polyvector  $\widehat{S} \rightarrow$  differential form
- action by the *super*-Lie algebra  $[\Lambda, \cdot] \rightarrow$  extension to  $gl$ -equivariantly closed differential form.

# New integration framework in the noncommutative (derived algebraic) geometry,

hal-00493919, version 1 - 21 Jun 2016

- A. Connes: integration theory based on maps  $C_\lambda \rightarrow$  differential forms- "cycles" on  $HC^*(A)$ ,  $(\Omega, d, \int)$ ,

$$\tau(a_0, \dots, a_n) = \int a_0 da_1 \dots da_n$$

(closely related Karoubi's nc-De Rham complex)



# New integration framework in the noncommutative (derived algebraic) geometry,

hal-00493919, version 1 - 21 Jun 2016

- A. Connes: integration theory based on maps  $C_\lambda \rightarrow$  differential forms- "cycles" on  $HC^*(A)$ ,  $(\Omega, d, \int)$ ,

$$\tau(a_0, \dots, a_n) = \int a_0 da_1 \dots da_n$$

(closely related Karoubi's nc-De Rham complex)

- My nc-BV formalism is an integration framework in the noncommutative geometry, based on maps to polyvectors rather than differential forms, which is particularly adopted to the equation  $\{m_{A_\infty}, m_{A_\infty}\} = 0$  of derived nc-algebraic geometry, variant:  $\{S, S\} = 0$

$$\int \exp \widehat{S}(X, \Lambda) \widehat{\varphi} dX$$

$$\varphi \in \text{Ker}(\hbar\Delta + \{S, \cdot\} + \Delta S), \quad \varphi \in \text{Symm}(C_\lambda[1-d])$$

# New integration framework in the noncommutative (derived algebraic) geometry,

- A. Connes: integration theory based on maps  $C_\lambda \rightarrow$  differential forms- "cycles" on  $HC^*(A)$ ,  $(\Omega, d, \int)$ ,

$$\tau(a_0, \dots, a_n) = \int a_0 da_1 \dots da_n$$

(closely related Karoubi's nc-De Rham complex)

- My nc-BV formalism is an integration framework in the noncommutative geometry, based on maps to polyvectors rather than differential forms, which is particularly adopted to the equation  $\{m_{A_\infty}, m_{A_\infty}\} = 0$  of derived nc-algebraic geometry, variant:  $\{S, S\} = 0$

$$\int \exp \widehat{S}(X, \Lambda) \widehat{\varphi} dX$$

$$\varphi \in \text{Ker}(\hbar\Delta + \{S, \cdot\} + \Delta S), \quad \varphi \in \text{Sym}(C_\lambda[1-d])$$

- Extends to non CY via  $V = A \oplus A^\vee[d]$  etc.

# New integration framework in the noncommutative (derived algebraic) geometry,

- A. Connes: integration theory based on maps  $C_\lambda \rightarrow$  differential forms- "cycles" on  $HC^*(A)$ ,  $(\Omega, d, \int)$ ,

$$\tau(a_0, \dots, a_n) = \int a_0 da_1 \dots da_n$$

(closely related Karoubi's nc-De Rham complex)

- My nc-BV formalism is an integration framework in the noncommutative geometry, based on maps to polyvectors rather than differential forms, which is particularly adopted to the equation  $\{m_{A_\infty}, m_{A_\infty}\} = 0$  of derived nc-algebraic geometry, variant:  $\{S, S\} = 0$

$$\int \exp \widehat{S}(X, \Lambda) \widehat{\varphi} dX$$

$$\varphi \in \text{Ker}(\hbar\Delta + \{S, \cdot\} + \Delta S), \quad \varphi \in \text{Sym}(C_\lambda[1-d])$$

- Extends to non CY via  $V = A \oplus A^\vee[d]$  etc.
- Invariance with respect to  $A_\infty$ -gauge transformation (and more general gauge transformation)

# Example: $V=\{e\}$ and matrix Airy integral

hal-00493919, version 1 - 21 Jun 2010

$V = \{e\}$ ,  $e \cdot e = e$ ,  $I^V(e) = 1$ ,  $\rightarrow$ potential  $(\zeta, \zeta, \zeta)_\lambda$

## Example: $V=\{e\}$ and matrix Airy integral

hal-00493919, version 1 - 21 Jun 2010

- $V = \{e\}$ ,  $e \cdot e = e$ ,  $I^V(e) = 1$ ,  $\rightarrow$ potential  $(\xi, \xi, \xi)_\lambda$
- on  $(q \otimes \Pi V)_0$  this gives the nonhomogenous polyvector  $otr(\Xi^3)$ ,

## Example: $V = \{e\}$ and matrix Airy integral

hal-00493919, version 1 - 21 Jun 2010

- $V = \{e\}$ ,  $e \cdot e = e$ ,  $I^V(e) = 1$ ,  $\rightarrow$  potential  $(\zeta, \zeta, \zeta)_\lambda$
- on  $(\mathfrak{q} \otimes \Pi V)_0$  this gives the nonhomogenous polyvector  $\text{otr}(\Xi^3)$ ,
- action by  $[\Lambda, \cdot]$ ,  $\Lambda \in \mathfrak{q}_{\text{odd}}$ ,  $\rightarrow$  extension of  $\exp \frac{1}{3!} \text{otr}(\Xi^3) dX$  to equivariantly closed differential form  $\exp \frac{1}{3!} \text{otr}(\Xi^3) + \frac{1}{2} \text{otr}([\Lambda, \Xi], \Xi) dX$

## Example: $V=\{e\}$ and matrix Airy integral

hal-00493919, version 1 - 21 Jun 2010

- $V = \{e\}$ ,  $e \cdot e = e$ ,  $I^V(e) = 1$ ,  $\rightarrow$  potential  $(\zeta, \zeta, \zeta)_\lambda$
- on  $(\mathfrak{q} \otimes \Pi V)_0$  this gives the nonhomogenous polyvector  $\text{otr}(\Xi^3)$ ,
- action by  $[\Lambda, \cdot]$ ,  $\Lambda \in \mathfrak{q}_{\text{odd}}$ ,  $\rightarrow$  extension of  $\exp \frac{1}{3!} \text{otr}(\Xi^3) dX$  to equivariantly closed differential form  $\exp \frac{1}{3!} \text{otr}(\Xi^3) + \frac{1}{2} \text{otr}([\Lambda, \Xi], \Xi) dX$
- its highest degree component is the matrix Airy integral

## References:

- [B1] S.Barannikov, *Modular operads and Batalin-Vilkovisky geometry*. IMRN, Vol. 2007, article ID rnm075. Preprint Max Planck Institute for Mathematics 2006-48 (04/2006),
- [B2] S.Barannikov, *Noncommutative Batalin-Vilkovisky geometry and matrix integrals*. «Comptes rendus Mathematique», presented for publication by M.Kontsevich in 05/2009, arXiv:0912.5484; Preprint NI06043 Newton Institute (09/2006), Preprint HAL, the electronic CNRS archive, hal-00102085 (09/2006)
- [B3] S.Barannikov, *Supersymmetry and cohomology of graph complexes*. Preprint hal-00429963; (11/2009).
- [B4] S.Barannikov, *Matrix De Rham complex and quantum A-infinity algebras*. arXiv:1001.5264, Preprint hal-00378776; (04/2009).
- [B5] S.Barannikov, *Quantum periods - I. Semi-infinite variations of Hodge structures*. Preprint ENS DMA-00-19. arXiv:math/0006193 (06/2000), Intern. Math. Res. Notices. 2001, No. 23
- [B6] S.Barannikov, *Solving the noncommutative Batalin-Vilkovisky equation*. Preprint hal-00464794 (03/2010). arXiv:1004.2253

