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A -infinity $GL(N)$ -equivariant matrix integrals-I

Serguei Barannikov

IMJ, CNRS

21/06/2010

The noncommutative Batalin-Vilkovisky equation and A_∞ -infinity algebras

hal-00493919, version 1 - 21 Jun 2010

- U - $\mathbb{Z}/2\mathbb{Z}$ graded vector space/ \mathbb{C} , l -scalar product on U of degree $d \in \mathbb{Z}/2\mathbb{Z}$, (variant: \mathbb{Z} -graded), consider

$$F = \text{Sym}(C_\lambda[1 + d]), \quad C_\lambda = \bigoplus_{j=0}^{\infty} (U[1]^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$$

-the symmetric (resp. exterior) powers for odd (resp even) d , of cyclic tensors

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$$\hbar \Delta S + \frac{1}{2} \{S, S\} = 0, \quad S = \sum_{g \geq 0, i} \hbar^{2g-1+i} S_{g,i}, \quad S_{g,i} \in \text{Sym}^i(C_\lambda[1+d]),$$

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- A_∞ -algebras without scalar product are included in the formalism by setting $U = A \oplus A^\vee[d]$, giving an A_∞ -algebra with scalar product.

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My A_∞ equivariant matrix integrals define an integration framework in the noncommutative (derived algebraic) geometry, particularly adobted to the equation $\{m_{A_\infty}, m_{A_\infty}\} = 0$

$$\int \exp \widehat{S}(X, \Lambda) \widehat{\varphi} dX, \varphi \in \text{Ker}(\hbar \Delta + \{S, \cdot\}) \subset \text{Sym}(C_\lambda[1+d])$$

Noncommutative Batalin-Vilkovisky operator ([B1])

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I define my noncommutative BV differential on $\text{Sym}(C_\lambda[1+d])$ via

$$\begin{aligned} \Delta(x_{\rho_1} \cdots x_{\rho_r})_\lambda (x_{\tau_1} \cdots x_{\tau_t})_\lambda &= \\ &= \sum_{p,q} (-1)^\varepsilon l_{\rho_p \tau_q} (x_{\rho_1} \cdots x_{\rho_{p-1}} x_{\tau_{q+1}} \cdots x_{\tau_{q-1}} x_{\rho_{p+1}} \cdots x_{\rho_r})_\lambda + \end{aligned}$$

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signs are the standard Koszul signs taking into account that

$$(x_{\rho_1} \cdots x_{\rho_r})_\lambda = (1+d) + \sum \overline{x_{\rho_i}}, \quad x_i \in V[1].$$

The cyclic tensors, invariant functions and the matrix algebra with odd trace.

- Invariant theory:

$$\begin{aligned} \text{Sym}(C_\lambda) &\rightarrow \text{Sym}((\mathfrak{gl}(N|\tilde{N}) \otimes V[1])^\vee)^{\text{GL}(N|\tilde{N})} \\ \prod (a_1 \dots a_k)_\lambda &\rightarrow \prod \text{sTr}(A_1 \cdot \dots \cdot A_k) \end{aligned}$$

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- Such algebra exists:

$$\mathfrak{q}(N) = \{[X, \pi] = 0 | X \in \mathfrak{gl}(N|N)\}$$

where $\pi = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}$ is an odd isomorphism, $\pi^2 = -1$,

$$\mathfrak{q}(N) = \left\{ X = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}$$

Algebra $q(N)$ and nc-BV equation

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- Corollary: tensor multiplication by $q(N)$, $gl(N|\tilde{N}) \rightarrow$ super Morita equivalence on solutions to nc-BV.

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- action by the *super*-Lie algebra $[\Lambda, \cdot] \rightarrow$ extension to gl -equivariantly closed differential form.

New integration framework in the noncommutative (derived algebraic) geometry,

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- A. Connes: integration theory based on maps $C_\lambda \rightarrow$ differential forms- "cycles" on $HC^*(A)$, (Ω, d, \int) ,

$$\tau(a_0, \dots, a_n) = \int a_0 da_1 \dots da_n$$

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- My nc-BV formalism is an integration framework in the noncommutative geometry, based on maps to polyvectors rather than differential forms, which is particularly adopted to the equation $\{m_{A_\infty}, m_{A_\infty}\} = 0$ of derived nc-algebraic geometry, variant: $\{S, S\} = 0$

$$\int \exp \widehat{S}(X, \Lambda) \widehat{\varphi} dX$$

$$\varphi \in \text{Ker}(\hbar\Delta + \{S, \cdot\} + \Delta S), \quad \varphi \in \text{Sym}(C_\lambda[1-d])$$

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$$\tau(a_0, \dots, a_n) = \int a_0 da_1 \dots da_n$$

(closely related Karoubi's nc-De Rham complex)

- My nc-BV formalism is an integration framework in the noncommutative geometry, based on maps to polyvectors rather than differential forms, which is particularly adopted to the equation $\{m_{A_\infty}, m_{A_\infty}\} = 0$ of derived nc-algebraic geometry, variant: $\{S, S\} = 0$

$$\int \exp \widehat{S}(X, \Lambda) \widehat{\varphi} dX$$

$$\varphi \in \text{Ker}(\hbar\Delta + \{S, \cdot\} + \Delta S), \quad \varphi \in \text{Sym}(C_\lambda[1-d])$$

- Extends to non CY via $V = A \oplus A^\vee[d]$ etc.

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- Invariance with respect to A_∞ -gauge transformation (and more general gauge transformation)

Example: $V=\{e\}$ and matrix Airy integral

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$V = \{e\}$, $e \cdot e = e$, $I^V(e) = 1$, \rightarrow potential $(\zeta, \zeta, \zeta)_\lambda$

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- its highest degree component is the matrix Airy integral

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