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A Class of Orthohomological Triangles

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Abstract.

In this article we propose to determine the triangles' class $A_iB_iC_i$ orthohomological with a given triangle ABC , inscribed in the triangle ABC ($A_i \in BC$, $B_i \in AC$, $C_i \in AB$).

We'll remind, here, the fact that if the triangle $A_iB_iC_i$ inscribed in ABC is orthohomologic with it, then the perpendiculars in A_i , B_i , respectively in C_i on BC , CA , respectively AB are concurrent in a point P_i (the orthological center of the given triangles), and the lines AA_i , BB_i , CC_i are concurrent in point (the homological center of the given triangles).

To find the triangles $A_iB_iC_i$, it will be sufficient to solve the following problem.

Problem.

Let's consider a point P_i in the plane of the triangle ABC and $A_iB_iC_i$ its pedal triangle. Determine the locus of point P_i such that the triangles ABC and $A_iB_iC_i$ to be homological.

Solution.

Let's consider the triangle ABC , $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$, and the point $P_i(\alpha, \beta, \gamma)$, $\alpha + \beta + \gamma = 0$.

The perpendicular vectors on the sides are:

$$U_{BC}^{\perp} (2a^2, -a^2 - b^2 + c^2, -a^2 + b^2 - c^2)$$

$$U_{CA}^{\perp} (-a^2 - b^2 + c^2, 2b^2, a^2 - b^2 - c^2)$$

$$U_{AB}^{\perp} (-a^2 + b^2 - c^2, a^2 - b^2 - c^2, 2c^2)$$

The coordinates of the vector \overline{BC} are $(0, -1, 1)$, and the line BC has the equation $x = 0$.

The equation of the perpendicular raised from point P_i on BC is:

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ 2a^2 & -a^2 - b^2 + c^2 & -a^2 + b^2 - c^2 \end{vmatrix} = 0$$

We note $A_i(x, y, z)$, because $A_i \in BC$ we have:

$$x = 0 \text{ and } y + z = 1.$$

The coordinates y and z of A_i can be found by solving the system of equations

$$\begin{cases} \begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ 2a^2 & -a^2 - b^2 + c^2 & -a^2 + b^2 - c^2 \end{vmatrix} = 0 \\ y + z = 0 \end{cases}$$

We have:

$$\begin{aligned} y \cdot \begin{vmatrix} \alpha & \gamma \\ 2a^2 & -a^2 + b^2 - c^2 \end{vmatrix} &= z \cdot \begin{vmatrix} \alpha & \beta \\ 2a^2 & -a^2 - b^2 + c^2 \end{vmatrix}, \\ y \left[\alpha (-a^2 + b^2 - c^2) - 2\gamma a^2 \right] &= z \left[\alpha (-a^2 - b^2 + c^2) - 2\beta a^2 \right], \\ y + y \cdot \frac{\alpha (-a^2 + b^2 - c^2) - 2\gamma a^2}{\alpha (-a^2 - b^2 + c^2) - 2\beta a^2} &= 1, \\ y \cdot \frac{\alpha (-a^2 - b^2 + c^2) - 2\beta a^2 + \alpha (-a^2 + b^2 - c^2) - 2\gamma a^2}{\alpha (-a^2 - b^2 + c^2) - 2\beta a^2} &= 1, \\ y \cdot \frac{-2a^2 (\alpha + \beta + \gamma)}{\alpha (-a^2 - b^2 + c^2) - 2\beta a^2} &= 1, \end{aligned}$$

it results

$$\begin{aligned} y &= \frac{\alpha}{2a^2} (a^2 + b^2 - c^2) + \beta \\ z = 1 - y &= 1 - \beta - \frac{\alpha}{2a^2} (a^2 + b^2 - c^2) = \alpha + \gamma - \frac{\alpha}{2a^2} (a^2 + b^2 - c^2). \end{aligned}$$

Therefore,

$$A_i \left(0, \frac{\alpha}{2a^2} (a^2 + b^2 - c^2) + \beta, \frac{\alpha}{2a^2} (a^2 - b^2 + c^2) + \gamma \right).$$

Similarly we find:

$$\begin{aligned} B_i \left(\frac{\beta}{2b^2} (a^2 + b^2 - c^2) + \alpha, 0, \frac{\beta}{2b^2} (-a^2 + b^2 + c^2) + \gamma \right), \\ C_i \left(\frac{\gamma}{2c^2} (a^2 - b^2 + c^2) + \alpha, \frac{\gamma}{2c^2} (-a^2 + b^2 + c^2) + \beta, 0 \right). \end{aligned}$$

We have:

$$\frac{\overline{A_i B}}{A_i C} = -\frac{\frac{\alpha}{2a^2}(a^2 - b^2 + c^2) + \gamma}{\frac{\alpha}{2a^2}(a^2 + b^2 - c^2) + \beta} = -\frac{\alpha c \cos B + \gamma a}{\alpha b \cos C + \beta a}$$

$$\frac{\overline{B_i C}}{B_i A} = -\frac{\frac{\beta}{2b^2}(a^2 + b^2 - c^2) + \alpha}{\frac{\alpha}{2a^2}(-a^2 + b^2 + c^2) + \gamma} = -\frac{\beta a \cos C + \alpha b}{\beta c \cos A + \gamma b}.$$

$$\frac{\overline{C_i A}}{C_i B} = -\frac{\frac{\gamma}{2c^2}(-a^2 + b^2 + c^2) + \beta}{\frac{\gamma}{2c^2}(a^2 - b^2 + c^2) + \alpha} = -\frac{\gamma b \cos A + \beta c}{\gamma a \cos B + \alpha c}$$

(We took into consideration the cosine's theorem: $a^2 = b^2 + c^2 - 2bc \cos A$).
In conformity with Ceva's theorem, we have:

$$\frac{\overline{A_i B}}{A_i C} \cdot \frac{\overline{B_i C}}{B_i A} \cdot \frac{\overline{C_i A}}{C_i B} = -1.$$

$$\begin{aligned} & (\alpha\gamma + \alpha c \cos B)(b\alpha + \beta a \cos C)(c\beta + \gamma b \cos A) = \\ & = (a\beta + \alpha b \cos C)(b\gamma + \beta c \cos A)(c\alpha + \gamma a \cos B) \\ & a\alpha(b^2\gamma^2 - c^2\beta^2)(\cos A - \cos B \cos C) + b\beta(c^2\alpha^2 - a^2\gamma^2)(\cos B - \cos A \cos C) + \\ & + c\gamma(a^2\beta^2 - b^2\alpha^2)(\cos C - \cos A \cos B) = 0. \end{aligned}$$

Dividing it by $a^2b^2c^2$, we obtain that the equation in barycentric coordinates of the locus \mathcal{L} of the point P_i is:

$$\begin{aligned} & \frac{\alpha}{a} \left(\frac{\gamma^2}{c^2} - \frac{\beta^2}{b^2} \right) (\cos A - \cos B \cos C) + \frac{\beta}{b} \left(\frac{\alpha^2}{a^2} - \frac{\gamma^2}{c^2} \right) (\cos B - \cos A \cos C) + \\ & + \frac{\gamma}{c} \left(\frac{\beta^2}{b^2} - \frac{\alpha^2}{a^2} \right) (\cos C - \cos A \cos B) = 0. \end{aligned}$$

We note $\overline{d_A}$, $\overline{d_B}$, $\overline{d_C}$ the distances oriented from the point P_i to the sides BC , CA respectively AB , and we have:

$$\frac{\alpha}{a} = \frac{\overline{d_A}}{2s}, \quad \frac{\beta}{b} = \frac{\overline{d_B}}{2s}, \quad \frac{\gamma}{c} = \frac{\overline{d_C}}{2s}.$$

The locus' \mathcal{L} equation can be written as follows:

$$\begin{aligned} & \overline{d_A}(\overline{d_C}^2 - \overline{d_B}^2)(\cos A - \cos B \cos C) + \overline{d_B}(\overline{d_A}^2 - \overline{d_C}^2)(\cos B - \cos A \cos C) + \\ & + \overline{d_C}(\overline{d_B}^2 - \overline{d_A}^2)(\cos C - \cos A \cos B) = 0 \end{aligned}$$

Remarks.

1. It is obvious that the triangle's ABC orthocenter belongs to locus \mathcal{L} . The orthic triangle and the triangle ABC are orthohomologic; a orthological center is the orthocenter H , which is the center of homology.
2. The center of the inscribed circle in the triangle ABC belongs to the locus \mathcal{L} , because $\bar{d}_A = \bar{d}_B = \bar{d}_C = r$ and thus locus' equation is quickly verified.

Theorem (Smarandache-Pătrașcu).

If a point P belongs to locus \mathcal{L} , then also its isogonal P' belongs to locus \mathcal{L} .

Proof.

Let $P(\alpha, \beta, \gamma)$ a point that verifies the locus' \mathcal{L} equation, and $P'(\alpha', \beta', \gamma')$ its isogonal

in the triangle ABC . It is known that $\frac{\alpha\alpha'}{a^2} = \frac{\beta\beta'}{b^2} = \frac{\gamma\gamma'}{c^2}$. We'll prove that $P' \in \mathcal{L}$, i.e.

$$\begin{aligned} & \sum \frac{\alpha'}{a} \left(\frac{\gamma'^2}{c^2} - \frac{\beta'^2}{b^2} \right) (\cos A - \cos B \cos C) = 0 \\ & \sum \frac{\alpha'}{a} \left(\frac{\gamma'^2 b^2 - \beta'^2 c^2}{b^2 c^2} \right) (\cos A - \cos B \cos C) = 0 \\ & \sum \frac{\alpha'}{ab^2 c^2} (\gamma'^2 b^2 - \beta'^2 c^2) (\cos A - \cos B \cos C) = 0 \Leftrightarrow \\ & \Leftrightarrow \sum \frac{\alpha'}{ab^2 c^2} \left(\frac{\gamma' \beta' \beta' c^2}{\gamma} - \frac{c^2 \gamma \gamma' \beta'}{\beta} \right) (\cos A - \cos B \cos C) = 0 \Leftrightarrow \\ & \Leftrightarrow \sum \frac{\alpha' \beta' \gamma'}{ab^2 c^2} \left(\frac{\beta c^2}{\gamma} - \frac{\gamma b^2}{\beta} \right) (\cos A - \cos B \cos C) = 0 \Leftrightarrow \\ & \Leftrightarrow \sum \frac{\alpha' \beta' \gamma'}{ab^2 c^2} \left(\frac{\beta^2 c^2 - \gamma^2 b^2}{\beta \gamma} \right) (\cos A - \cos B \cos C) = 0 \Leftrightarrow \\ & \Leftrightarrow \sum \frac{\alpha'}{a} \left(\frac{\alpha' \beta' \gamma'}{\alpha \beta \gamma} \right) \frac{1}{b^2 c^2} \cdot b^2 c^2 \left(\frac{\beta^2}{b^2} - \frac{\gamma^2}{c^2} \right) (\cos A - \cos B \cos C) = 0. \end{aligned}$$

We obtain that:

$$\frac{\alpha' \beta' \gamma'}{\alpha \beta \gamma} \sum \frac{\alpha'}{a} \left(\frac{\gamma^2}{c^2} - \frac{\beta^2}{b^2} \right) (\cos A - \cos B \cos C) = 0,$$

this is true because $P \in \mathcal{L}$.

Remark.

We saw that the triangle 's ABC orthocenter H belongs to the locus, from the precedent theorem it results that also O , the center of the circumscribed circle to the triangle ABC (isogonable to H), belongs to the locus.

Open problem:

What does it represent from the geometry's point of view the equation of locus \mathcal{L} ?

In the particular case of an equilateral triangle we can formulate the following:

Proposition:

The locus of the point P from the plane of the equilateral triangle ABC with the property that the pedal triangle of P and the triangle ABC are homological, is the union of the triangle's heights.

Proof:

Let $P(\alpha, \beta, \gamma)$ a point that belongs to locus \mathcal{L} . The equation of the locus becomes:

$$\alpha(\gamma^2 - \beta^2) + \beta(\alpha^2 - \gamma^2) + \gamma(\beta^2 - \alpha^2) = 0$$

Because:

$$\begin{aligned} \alpha(\gamma^2 - \beta^2) + \beta(\alpha^2 - \gamma^2) + \gamma(\beta^2 - \alpha^2) &= \alpha\gamma^2 - \alpha\beta^2 + \beta\alpha^2 - \beta\gamma^2 + \gamma\beta^2 - \gamma\alpha^2 = \\ &= \alpha\beta\gamma + \alpha\gamma^2 - \alpha\beta^2 + \beta\alpha^2 - \beta\gamma^2 + \gamma\beta^2 - \gamma\alpha^2 - \alpha\beta\gamma = \\ &= \alpha\beta(\gamma - \beta) + \alpha\gamma(\gamma - \beta) - \alpha^2(\gamma - \beta) - \beta\gamma(\gamma - \beta) = \\ &= (\gamma - \beta)[\alpha(\beta - \alpha) - \gamma(\beta - \alpha)] = (\beta - \alpha)(\alpha - \gamma)(\gamma - \beta). \end{aligned}$$

We obtain that $\alpha = \beta$ or $\beta = \gamma$ or $\gamma = \alpha$, that shows that P belongs to the medians (heights) of the triangle ABC .

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