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INFINITE PRODUCTS OF NONNEGATIVE 2×2 MATRICES BY NONNEGATIVE VECTORS

ALAIN THOMAS

ABSTRACT. Given a finite set $\{M_0, \dots, M_{d-1}\}$ of nonnegative 2×2 matrices and a non-negative column-vector V , we associate to each $(\omega_n) \in \{0, \dots, d-1\}^{\mathbb{N}}$ the sequence of the column-vectors $\frac{M_{\omega_1} \dots M_{\omega_n} V}{\|M_{\omega_1} \dots M_{\omega_n} V\|}$. We give the necessary and sufficient condition on the matrices M_k and the vector V for this sequence to converge for all $(\omega_n) \in \{0, \dots, d-1\}^{\mathbb{N}}$ such that $\forall n, M_{\omega_1} \dots M_{\omega_n} V \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

2000 Mathematics Subject Classification: 15A48.

INTRODUCTION

Let $\mathcal{M} = \{M_0, \dots, M_{d-1}\}$ be a finite set of nonnegative 2×2 matrices and $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ a non-negative column-vector. We use the notation $Y_n = Y_n^\omega := M_{\omega_1} \dots M_{\omega_n}$ and give the necessary and sufficient condition for the pointwise convergence of $\frac{Y_n V}{\|Y_n V\|}$, $(\omega_n) \in \{0, \dots, d-1\}^{\mathbb{N}}$ such that $Y_n V \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for any n , where $\|\cdot\|$ is the norm-sum. The idea of the proof is that, if the conditions are satisfied, either both columns of Y_n tends to the same limit, or they tend to different limits with different orders of growth, so in case V is positive the limit points of $\frac{Y_n V}{\|Y_n V\|}$ only depend on the limit of the dominant column. This problem is obviously very different from the one of the convergence of $\frac{Y_n}{\|Y_n\|}$, or the convergence of the Y_n itselfs, see the introduction of [5] for some counterexamples and [8, Proposition 1.2] for the infinite products of 2×2 stochastic matrices.

The conditions for the pointwise convergence of $\frac{Y_n V}{\|Y_n V\|}$ also differ from the conditions for its uniform convergence, see [2]. The uniform convergence can be used for the multifractal analysis of some continuous singular measures called Bernoulli convolutions (see [6] for the

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Bernoulli convolutions and [1] for their multifractal analysis). We study such measures in [2], [3] and [4]. The Birkhoff's contraction coefficient [7, Chapter 3] that we use in [3] and [5] but not here, is really not of great help to solve the main difficulties. Moreover the theorem that gives the value of this coefficient is difficult to prove (see [7, §3.4]) even in the case of 2×2 matrices. In [2] we use some other contraction coefficient quite more easy to compute ([2, Proposition 1.3]).

1. CONDITION FOR THE POINTWISE CONVERGENCE OF $\frac{Y_n V}{\|Y_n V\|}$

Proposition 1.1. *The sequence $\frac{Y_n V}{\|Y_n V\|}$ converges for any $\omega \in \{0, \dots, d-1\}^{\mathbb{N}}$ such that*

$\forall n, Y_n V \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, *if and only if at least one of the following conditions holds:*

(i) *V has positive entries and it is an eigenvector of any invertible matrix of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ or $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ that belongs to \mathcal{M} .*

(ii) *Any invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}$ satisfies $a > 0$ and, if $b = c = 0$, $a \geq d$.*

(iii) *Any invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}$ satisfies $d > 0$ and, if $b = c = 0$, $d \geq a$.*

(iv) *V has a null entry and all the invertible matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}$ satisfy $ad > 0$.*

Proof. Let $\omega \in \{0, \dots, d-1\}^{\mathbb{N}}$. If there exists N such that $\det M_{\omega_N} = 0$, the column-vectors $Y_N V, Y_{N+1} V, \dots$ are collinear and $\frac{Y_n V}{\|Y_n V\|}$ is constant for $n \geq N$. So we look only at the $\omega \in \{0, \dots, d-1\}^{\mathbb{N}}$ such that $\forall n, \det M_{\omega_n} \neq 0$. In order to use only matrices with positive determinant we set $\Delta := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$A_n = A_n^\omega := \begin{cases} M_{\omega_n} & \text{if } \det Y_{n-1} > 0 \text{ (or } n = 1) \text{ and } \det M_{\omega_n} > 0 \\ M_{\omega_n} \Delta & \text{if } \det Y_{n-1} > 0 \text{ (or } n = 1) \text{ and } \det M_{\omega_n} < 0 \\ \Delta M_{\omega_n} & \text{if } \det Y_{n-1} < 0 \text{ and } \det M_{\omega_n} < 0 \\ \Delta M_{\omega_n} \Delta & \text{if } \det Y_{n-1} < 0 \text{ and } \det M_{\omega_n} > 0. \end{cases}$$

We set also

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} := A_n \quad \text{and} \quad \begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix} := A_1 \dots A_n = \begin{cases} Y_n & \text{if } \det Y_n > 0 \\ Y_n \Delta & \text{if } \det Y_n < 0. \end{cases}$$

The matrices A_n belong to the set

$$\mathcal{M}^+ := \{M ; \exists i, j, k, M = \Delta^i M_k \Delta^j \text{ and } \det M > 0\}.$$

Since $\det A_n > 0$ we have $a_n d_n p_n s_n \neq 0$. If $\{n; A_n \text{ not diagonal}\}$ is infinite we index this set by an increasing sequence $n_1 < n_2 < \dots$. We have $b_{n_1} \neq 0$ or $c_{n_1} \neq 0$; both cases are equivalent because, using the set of matrices $\mathcal{M}' = \Delta \mathcal{M} \Delta$ and defining similarly Y'_n and $A'_n = \begin{pmatrix} a'_n & b'_n \\ c'_n & d'_n \end{pmatrix}$ from this set, we have $Y'_n = \Delta Y_n \Delta$, $A'_n = \Delta A_n \Delta$ and $b'_{n_1} = c_{n_1}$. So we can suppose $b_{n_1} \neq 0$; we deduce $q_n \neq 0$ by induction on $n \geq n_1$. The sequences defined for any $n \geq n_1$ by

$$u_n = \frac{r_n}{p_n}, \quad v_n = \frac{s_n}{q_n}, \quad w_n = \frac{q_n}{p_n}, \quad x_n = \begin{cases} v_2/v_1 & \text{if } \det Y_n > 0 \\ v_1/v_2 & \text{if not} \end{cases} \quad \text{and} \quad \lambda_n = (1 + w_n x_n)^{-1}$$

satisfy $0 \leq u_n < v_n < \infty$, $0 < w_n < \infty$ and

if the entries of V are positive, $0 < x_n < \infty$ and $0 < \lambda_n < 1$
if not, $x_n \in \{0, \infty\}$ and $\lambda_n \in \{0, 1\}$ according to the sign of $\det Y_n$.

Since we have assumed that $Y_n V \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the ratio $\frac{(Y_n V)_2}{(Y_n V)_1}$ exists in $[0, \infty]$ and we have to prove that it has a finite or infinite limit when $n \rightarrow \infty$. If A_n is not eventually diagonal we have for $n \geq n_1$

$$(1) \quad \frac{(Y_n V)_2}{(Y_n V)_1} = \lambda_n u_n + (1 - \lambda_n) v_n \in I_n := [u_n, v_n] \text{ and } I_n \supseteq I_{n+1}.$$

An immediate consequence is the following lemma:

Lemma 1.1. *Suppose A_n is not eventually diagonal, then*

(i) *the sequences (u_n) and (v_n) converge in \mathbb{R} and the sequence $\left(\frac{(Y_n V)_2}{(Y_n V)_1}\right)$ is bounded;*

(ii) *$\left(\frac{(Y_n V)_2}{(Y_n V)_1}\right)$ converges if $\lim_{n \rightarrow \infty} |I_n| = 0$;*

(iii) *if V has positive entries, $\left(\frac{(Y_n V)_2}{(Y_n V)_1}\right)$ converges if w_n has limit 0 or ∞ ;*

(iv) *if V has a null entry, the necessary and sufficient condition for the convergence of $\left(\frac{(Y_n V)_2}{(Y_n V)_1}\right)$ is that $\lim_{n \rightarrow \infty} |I_n| = 0$ or the sign of $\det Y_n$ is eventually constant.*

We also define for $n > n_1$

$$\alpha_n = \left(1 + \frac{c_n}{a_n} w_{n-1}\right)^{-1}, \quad \beta_n = \left(1 + \frac{b_n}{d_n} (w_{n-1})^{-1}\right)^{-1}, \quad \gamma_n = 1 - \frac{c_n}{a_n} \frac{b_n}{d_n}$$

that belong to $]0, 1]$ and satisfy

$$(2) \quad \begin{aligned} |I_n| &= \alpha_n \beta_n \gamma_n |I_{n-1}| \\ w_n &= \frac{d_n}{a_n} \frac{\alpha_n}{\beta_n} w_{n-1} \end{aligned}$$

so $\prod_{n>n_1} \alpha_n \beta_n \gamma_n = \lim_{n \rightarrow \infty} \frac{|I_n|}{|I_{n_1}|}$ is positive if and only if $\lim_{n \rightarrow \infty} |I_n| > 0$. Using the equivalents of $\log \alpha_n$, $\log \beta_n$ and $\log \gamma_n$,

$$(3) \quad \lim_{n \rightarrow \infty} |I_n| > 0 \Leftrightarrow \sum \frac{c_n}{a_n} w_{n-1} < \infty, \quad \sum \frac{b_n}{d_n} (w_{n-1})^{-1} < \infty \quad \text{and} \quad \sum \frac{c_n b_n}{a_n d_n} < \infty.$$

The set of indexes $\{n; A_n \text{ not diagonal}\}$ is the union of

$$L^\omega = \{n; c_n \neq 0\} \quad \text{and} \quad U^\omega = \{n; b_n \neq 0\}.$$

Moreover, since A_n belongs to the finite set \mathcal{M}^+ there exists $K > 0$ such that

$$L^\omega = \left\{ n; \frac{1}{K} \leq \frac{c_n}{a_n} \leq K \right\} \quad \text{and} \quad U^\omega = \left\{ n; \frac{1}{K} \leq \frac{b_n}{d_n} \leq K \right\}.$$

We deduce a simpler formulation of (3):

$$(4) \quad \lim_{n \rightarrow \infty} |I_n| > 0 \Leftrightarrow \sum_{n \in L^\omega} w_{n-1} < \infty, \quad \sum_{n \in U^\omega} (w_{n-1})^{-1} < \infty \quad \text{and} \quad L^\omega \cap U^\omega \text{ is finite.}$$

In view of Lemma 1.1 we may suppose from now that $\lim_{n \rightarrow \infty} |I_n| > 0$. Since $L^\omega \cap U^\omega = \{n; A_n \text{ positive}\}$ is finite, for n large enough the matrix A_n is lower triangular if $n \in L^\omega$, upper triangular if $n \in U^\omega$, diagonal if $n \notin L^\omega \cup U^\omega$. When A_n is diagonal the second relation in (2) becomes $w_n = \frac{d_n}{a_n} w_{n-1}$; consequently any integer n in an interval $]n_i, n_{i+1}[$ with i large enough satisfies

$$(5) \quad \frac{w_n}{w_{n_i}} = \prod_{n_i < j \leq n} \frac{d_j}{a_j}.$$

Moreover if L^ω is infinite, (4) implies that w_{n-1} has limit to 0 when $L^\omega \ni n \rightarrow \infty$, and w_n also has limit 0 because $w_n = \frac{d_n w_{n-1}}{a_n + c_n w_{n-1}}$ for any $n \in L^\omega \setminus U^\omega$. We have a similar property if U^ω is infinite, so

$$(6) \quad \begin{array}{l} \text{if } L^\omega \text{ is infinite, } w_{n-1} \rightarrow 0 \text{ and } w_n \rightarrow 0 \text{ for } L^\omega \ni n \rightarrow \infty; \\ \text{if } U^\omega \text{ is infinite, } w_{n-1} \rightarrow \infty \text{ and } w_n \rightarrow \infty \text{ for } U^\omega \ni n \rightarrow \infty. \end{array}$$

First case: Suppose that (i) holds. Then the diagonal matrices of \mathcal{M} are collinear to the unit matrix. If at least one matrix of \mathcal{M} has the form $M_k = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ with $bc \neq 0$, its nonnegative eigenvalue – namely $\left(\frac{\sqrt{b}}{\sqrt{c}} \right)$ – is collinear to $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ hence there exists $\lambda \in \mathbb{R}$ such that $M_k = \lambda \begin{pmatrix} 0 & v_1^2 \\ v_2^2 & 0 \end{pmatrix}$.

Notice that if A_n is diagonal from a rank N , the matrix M_{ω_n} has the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ or $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ hence it has V as eigenvector; consequently $\begin{pmatrix} (Y_n V)_2 \\ (Y_n V)_1 \end{pmatrix}$ converges because it is $\frac{(Y_N V)_2}{(Y_N V)_1}$ for any $n \geq N$.

Suppose now A_n is non-diagonal for infinitely many n . We apply (5) on each interval $]n_i, n_{i+1}[$ (if non empty), for i large enough. Among the integers $n \in]n_i, n_{i+1}[$ we consider the ones for which $\det M_{\omega_n} < 0$. For such n the matrix A_n is alternately $M_{\omega_n} \Delta$ and ΔM_{ω_n} , hence alternately proportional to $\begin{pmatrix} v_1^2 & 0 \\ 0 & v_2^2 \end{pmatrix}$ and to $\begin{pmatrix} v_2^2 & 0 \\ 0 & v_1^2 \end{pmatrix}$ and, according to (5),

$$(7) \quad n_i \leq n < n_{i+1} \Rightarrow \frac{w_n}{w_{n_i}} \in \left\{ \frac{v_1^2}{v_2^2}, \frac{v_2^2}{v_1^2}, 1 \right\}.$$

In particular this relation holds for $n = n_{i+1} - 1$. One deduce – according to (6) – that there do not exist infinitely many i such that $n_i \in L^\omega$ and $n_{i+1} \in U^\omega$. Thus $n_i \in L^\omega$ for i large enough (resp. $n_i \in U^\omega$ for i large enough) and, according to (6) and (7), $\lim_{n \rightarrow \infty} w_n = 0$ (resp. $\lim_{n \rightarrow \infty} w_n = \infty$). In view of Lemma 1.1(iii), $\begin{pmatrix} (Y_n V)_2 \\ (Y_n V)_1 \end{pmatrix}$ converges.

Second case: Suppose that (ii) holds (if (iii) holds the proof is similar).

Suppose first the M_{ω_n} are diagonal from a rank N . From the hypothesis (ii) there exists δ_n, δ'_n such that $M_{\omega_N} \dots M_{\omega_n} V = \begin{pmatrix} \delta_n v_1 \\ \delta'_n v_2 \end{pmatrix}$ and $\delta_n \geq \delta'_n$. Since the M_{ω_i} belong to a finite set we have $\lim_{n \rightarrow \infty} \frac{\delta_n}{\delta'_n} = \infty$, or $\frac{\delta_n}{\delta'_n}$ is eventually constant in case M_{ω_n} is eventually the unit matrix, or $\delta'_n = 0 \neq \delta_n$ for n large enough. Denoting by $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ the matrix $M_{\omega_1} \dots M_{\omega_{N-1}}$, we have $\frac{(Y_n V)_2}{(Y_n V)_1} = \frac{r\delta_n v_1 + s\delta'_n v_2}{p\delta_n v_1 + q\delta'_n v_2}$ hence $\begin{pmatrix} (Y_n V)_2 \\ (Y_n V)_1 \end{pmatrix}$ converges in all the cases. Suppose now M_{ω_n} is non-diagonal for infinitely many n . There exists from (6) an integer κ such that

$$(8) \quad i \geq \kappa \Rightarrow \begin{cases} w_{n_{i-1}} < 1 \text{ and } w_{n_i} < 1 & \text{if } n_i \in L^\omega \\ w_{n_{i-1}} > 1 \text{ and } w_{n_i} > 1 & \text{if } n_i \in U^\omega \end{cases}$$

and such that the A_n are diagonal for $n \in]n_i, n_{i+1}[$, $i \geq \kappa$. According to (ii), for such values of n the matrix M_{ω_n} is diagonal and $A_n = M_{\omega_n}$ with $a_n \geq d_n$, or $A_n = \Delta M_{\omega_n} \Delta$ with $a_n \leq d_n$.

If there exists $i \geq \kappa$ such that $n_i \in L^\omega$ and $n_{i+1} \in U^\omega$, $\det Y_{n_i}$ is necessarily negative: otherwise A_n should be equal to M_{ω_n} for $n \in]n_i, n_{i+1}[$, $\frac{d_n}{a_n} \leq 1$ and, by (5), $w_{n_{i+1}-1} \leq w_{n_i} < 1$ in contradiction with (8).

Now $M_{\omega_{n_{i+1}}}$ has positive determinant, otherwise it should have the form $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ and $A_{n_{i+1}} = \Delta M_{\omega_{n_{i+1}}} = \begin{pmatrix} c & 0 \\ a & b \end{pmatrix}$ in contradiction with $n_{i+1} \in U^\omega$.

We have again $\frac{d_n}{a_n} \geq 1$ for $n \in]n_{i+1}, n_{i+2}[$ and consequently $w_{n_{i+2}-1} \geq w_{n_{i+1}} > 1$; so, by induction, $n_j \in U^\omega$ and $\det Y_{n_j} < 0$ for any $j \geq i + 1$. From (5) w_n lies between w_{n_j} and $w_{n_{j+1}-1}$ for any $n \in]n_j, n_{j+1}[$ and j large enough, and from (6) its limit is infinite. Distinguishing the cases where V has positive entries or V has a null entry, $\frac{(Y_n V)_2}{(Y_n V)_1}$ converges by Lemma 1.1.

The conclusion is the same if there do not exist $i \geq \kappa$ such that $n_i \in L^\omega$ and $n_{i+1} \in U^\omega$, because in this case $n_i \in L^\omega$ for i large enough, or $n_i \in U^\omega$ for any $i \geq \kappa$.

Third case: Suppose (iv) holds. As we have seen, from (4) A_n is eventually triangular or diagonal, and M_{ω_n} also is because – by (iv) – \mathcal{M} do not contain invertible matrices of the form $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$ or $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$. We deduce that the sign of $\det Y_n$ is eventually constant. If

A_n is not eventually diagonal the sequence $\left(\frac{(Y_n V)_2}{(Y_n V)_1} \right)$ converges by Lemma 1.1(iv) and, if A_n is, the sequence $\left(\frac{(Y_n V)_2}{(Y_n V)_1} \right)$ is eventually constant.

Fourth case: Suppose that the set \mathcal{M} do not satisfy (i), (ii), (iii) nor (iv), and that at least one matrix of this set, let M_k , has the form $M_k = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ with $bc \neq 0$; let us prove

that $\left(\frac{(Y_n V)_2}{(Y_n V)_1} \right)$ diverges.

– Suppose first there exists a matrix M_k of this form that do not have V as eigenvector; we chose as counterexample the constant sequence defined by $\omega_n = k$ for any n : Y_{2n} is collinear to the unit matrix, hence $Y_{2n} V$ is collinear to V and $Y_{2n+1} V$ to $M_k V$, so $\left(\frac{(Y_n V)_2}{(Y_n V)_1} \right)$ diverges.

– Suppose now that all the matrices of \mathcal{M} of the form $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ with $bc \neq 0$ have V as eigenvector that is, V is collinear to $\begin{pmatrix} \sqrt{b} \\ \sqrt{c} \end{pmatrix}$ for all such matrix. Since (i) do not hold, at least one matrix M_h of \mathcal{M} is diagonal with nonnull and distinct diagonal entries. In

this case $M_h M_k$ has the form $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ but do not have V as eigenvector. We recover the previous case; more precisely the counterexample is defined by $\omega_{2n-1} = h$ and $\omega_{2n} = k$ for any $n \in \mathbb{N}$.

Fifth case: Suppose that \mathcal{M} do not satisfy (i), (ii), (iii) nor (iv), and that no matrix of the form $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ with $bc \neq 0$ belongs to \mathcal{M} . Since (i) do not hold, at least one matrix of this set has the form $M_k = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $ad \neq 0$ and $a \neq d$. We suppose that $a > d$ and we use the negation of (ii) (in case $a < d$ we use similarly the negation of (iii)). According to the negation of (ii) there exists in \mathcal{M} at least one matrix of the form $M_h = \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}$ with $\beta\gamma\delta \neq 0$, or one of the form $M_\ell = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ with $0 < \alpha < \delta$.

_ Consider first the case where \mathcal{M} contains some matrices M_k and M_h as above. Let $(n_i)_{i \in \mathbb{N}}$ be an increasing sequence of positive integers with $n_1 = 1$, and ω the sequence defined by $\omega_n = h$ for $n \in \{n_1, n_2, \dots\}$ and $\omega_n = k$ otherwise.

For i odd, A_{n_i} is lower-triangular and $\forall n \in]n_i, n_{i+1}[$, $A_n = \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}$, $a_n = d$ and $d_n = a$.

For i even, A_{n_i} is upper-triangular and $\forall n \in]n_i, n_{i+1}[$, $A_n = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $a_n = a$ and $d_n = d$.

Using (5) for $n = n_{i+1} - 1$ and choosing $n_{i+1} - n_i$ large enough one has $w_{n_{i+1}-1} \geq 2^i$ if i is odd, $w_{n_{i+1}-1} \leq 2^{-i}$ if i is even, so the three conditions in (4) are satisfied and the interval $\cap I_n$ is not reduced to one point. If the entries of V are positive, the first relation in (1) and the definition of λ_n imply that $\liminf_{n \rightarrow \infty} \left(\frac{(Y_n V)_2}{(Y_n V)_1} \right)$ is the lower bound of this interval

and $\limsup_{n \rightarrow \infty} \left(\frac{(Y_n V)_2}{(Y_n V)_1} \right)$ its upper bound, so the sequence $\left(\frac{(Y_n V)_2}{(Y_n V)_1} \right)$ diverges. If V has a

null entry, the divergence of $\left(\frac{(Y_n V)_2}{(Y_n V)_1} \right)$ results from Lemma 1.1(iv).

_ In case \mathcal{M} contains some matrices M_k and M_ℓ as above, one defines ω from a sequence $i_1 = 1 < i_2 < i_3 < \dots$ by setting, for $j \geq 1$ and $i_j \leq n < i_{j+1}$,

$$\omega_n = \begin{cases} k & \text{if } j \text{ even} \\ \ell & \text{if } j \text{ odd.} \end{cases}$$

The diagonal matrix Y_n can be easily computed, and $\left(\frac{(Y_n V)_2}{(Y_n V)_1} \right)$ obviously diverges if one choose the $i_{j+1} - i_j$ large enough.

If V has a null entry, since (iv) do not hold \mathcal{M} contains at least one matrix of the form $M_h = \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}$, $\beta\gamma\delta \neq 0$ or $M_{h'} = \begin{pmatrix} \alpha & \beta \\ \gamma & 0 \end{pmatrix}$, $\alpha\beta\gamma \neq 0$, or $M_{h''} = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$, $\beta\gamma \neq 0$. We already know that $\left(\frac{(Y_n V)_2}{(Y_n V)_1}\right)$ diverges if \mathcal{M} contains M_k and M_h . Similarly it diverges if \mathcal{M} contains M_ℓ and $M_{h'}$. If \mathcal{M} contains M_k and $M_{h''}$ the counterexample is given – from a sequence $i_1 = 1 < i_2 < i_3 < \dots$ – by $\omega_{i_j} = h''$ and $\omega_n = k$ for $n \in]i_j, i_{j+1}[$, $j \in \mathbb{N}$: $\frac{(Y_n V)_2}{(Y_n V)_1}$ is alternately 0 and ∞ because Y_{i_j} has the form $\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$ for j odd and $\begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix}$ for j even. \square

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