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Localized and Extended States in a Disordered Trap

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Abstract

We study Anderson localization in a disordered potential combined with a harmonic trap. We show that the spectrum displays both localized and extended states, which coexist at intermediate energies. In the region of coexistence, we find that the extended states result from confinement by the trap and are weakly affected by the disorder. Conversely, the localized states correspond to eigenstates of the disordered potential, which are only affected by the trap via an inhomogeneous energy shift. These results are directly relevant to disordered quantum gases and we propose a realistic scheme to observe the coexistence of localized and extended states in these systems.

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Disorder underlies many fields in physics, such as electronics, superfluid helium, and optics [1–3]. Understanding the effects of disorder poses challenging questions, regarding quantum transport [4] and the interplay of disorder and interactions [5], for instance. In this respect, ultracold quantum gases offer exceptionally-well controlled simulators for condensed-matter physics [6] and are particularly promising for disordered systems [7]. They recently paved the way to the direct observation of the Anderson localization of matter-waves [8–11].

Ultracold atoms do not only mimic standard models of condensed-matter physics, but also raise new issues which require special analysis in its own right. Most importantly, ultracold atoms are almost always confined in spatial traps, which has significant consequences. On one hand, retrieving information about bulk properties requires specific algorithms [12]. On the other hand, trapping induces novel effects, for instance the existence of Bose-Einstein condensates in low dimensions [13], and the appearance of localized single-particle states in optical lattices due to suppression of quantum tunneling [14–16].

Consider the problem of Anderson localization (AL) [17]. In homogeneous disorder, linear waves can localize owing to coherent multiple scattering, with properties depending on the system dimension and the disorder strength [1]. A paradigm of AL is that localized and extended states generally do not coexist in energy. This relies on Mott’s *reductio ad absurdum* [1]: Should there exist a localized state and an extended state with infinitely close energies for a given configuration of disorder, an infinitesimal change of the configuration would hybridize them, forming two extended states. Hence, for a given energy, *almost all* states should be either localized or extended. This statement holds for most models of homogeneous disorder, except for peculiar ones, such as binary alloys [18] or binary off-diagonal disorder [19], which both display strong local symmetries, and multi-channel systems [20]. Then, a question arises: Can inhomogeneous trapping modify this picture so that localized and extended states coexist in energy ?

In this Letter, we study localization in a disordered potential combined with a harmonic trap. The latter is not only generic for inhomogeneous trapping, but also directly relevant to disordered systems of ultracold atoms [11, 21–24]. As can be expected, we find that the lowest-energy states are strongly localized near the trap center, owing to the confinement. Conversely, the highest-energy states extend over the full (energy-dependent) classically-allowed region, similarly as in a homogeneous system of finite size. The central result of this work is that localized and extended states do coexist at intermediate energies. We give

numerical evidence of this effect and show that while the extended states are confined by the trap and weakly affected by the disorder, the localized states correspond to eigenstates of the disordered potential, which are only affected by the trap via an inhomogeneous energy shift. Finally, we propose a realistic scheme based on energy-selective time-of-flight techniques to observe the coexistence of localized and extended states with ultracold Fermi gases.

Let us consider a d -dimensional gas of non-interacting particles of mass m , confined into a spatial trap $V_{\text{T}}(\mathbf{r})$ and subjected to a homogeneous disordered potential $V(\mathbf{r})$ of zero average ($\langle V \rangle = 0$), amplitude V_{R} , and correlation length σ_{R} . Hereafter, we use so-called ‘red-detuned’ speckle potentials ($V_{\text{R}} < 0$), similar to those used for disordered quantum gases [7, 25]. The trap is either a homogeneous box of linear length L ($V_{\text{T}}(\mathbf{r})=0$ for $|r_j| < L$ with $j \in \{x, y, z\}$, and $V_{\text{T}}(\mathbf{r}) = +\infty$ otherwise) or a harmonic trap of angular frequency ω ($V_{\text{T}}(\mathbf{r}) = m\omega^2|\mathbf{r}|^2/2$). We numerically compute the eigenstates $|\psi_n\rangle$ and eigenenergies E_n of the Hamiltonian

$$\hat{H} = -\hbar^2\nabla^2/2m + V(\mathbf{r}) + V_{\text{T}}(\mathbf{r}) \quad (1)$$

and characterize the eigenstates by their center of mass, $\mathbf{r}_n \equiv \langle \psi_n | \hat{\mathbf{r}} | \psi_n \rangle$, and their spatial extension (rms size), $\Delta r_n \equiv (\langle \psi_n | \hat{\mathbf{r}}^2 | \psi_n \rangle - \mathbf{r}_n^2)^{1/2}$. The quantity Δr_n quantifies localization: the smaller, the more localized [26].

Numerical results for the one-dimensional (1D, $d = 1$) case are reported in Fig. 1. In infinite, homogeneous disorder, all states $|\psi_n\rangle$ are localized, they are uniformly distributed in space, and their extension Δz_n monotonically increases with the energy [28]. As Figs. 1(a)-(c) show, finite-size effects have only a trivial effect: For a finite box of length L , the states of energy low enough that $\Delta z_n \ll L$ are not affected by the finite size of the box. For larger energies however, boundary effects come into the picture: The states are centered closer to the box center and their extension saturates to $\Delta z \simeq L/2\sqrt{3}$, that is the value found for extended plane waves. An important outcome of these results is that the curve giving Δz_n versus E displays a single branch. In particular, there is no energy region where localized and extend states coexist. This finding is in agreement with Mott’s statement [1] and holds independently of the size of the box.

For a harmonic potential, we find a completely different behavior. The curves giving Δz_n and z_n versus E now display two clearly separated branches [Figs. 1(b)-(d)]. For low energies, the states are strongly localized and, for $E > 0$, they are roughly uniformly distributed in a region bounded by the (energy-dependent) classical turning points, $z_{\text{cl}}(E) = \pm\sqrt{2E/m\omega^2}$.

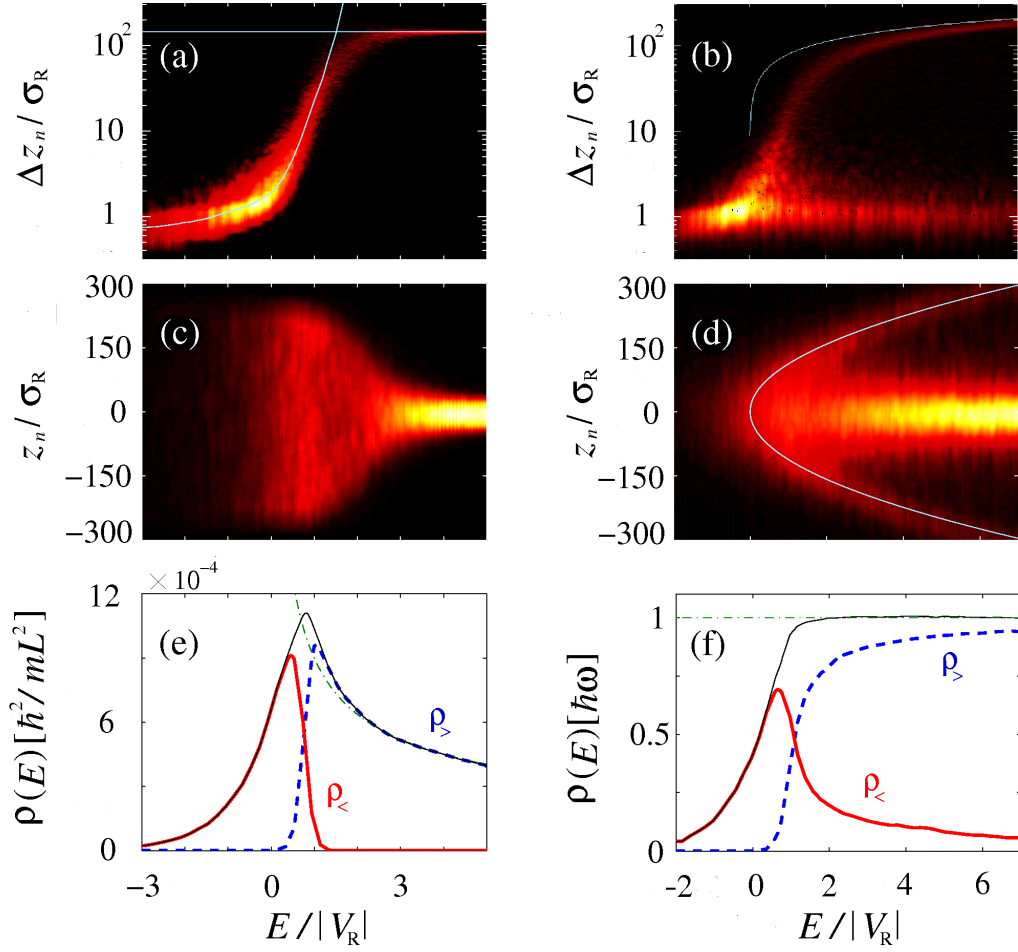


Figure 1: Extension (a)-(b), center of mass (c)-(d) and DOS (e)-(f) versus energy in a 1D disordered trap. The plots result from accumulation of numerical data over 500 realizations of a red-detuned speckle potential with $m\sigma_R^2|V_R|/\hbar^2=0.256$. Panels (a)-(c)-(e) refer to a flat box of length $L=500\sigma_R$; the curved line in (a) corresponds to an infinite system [27] and the horizontal line is $\Delta z=L/2\sqrt{3}$. Panels (b)-(d)-(f) refer to a harmonic trap with $\omega=0.025|V_R|/\hbar$; the solid lines correspond to the non-disordered case. Panels (e)-(f) show the full DOS $\rho(E)$ (solid black line), as well as the DOS restricted to localized ($\rho_<$) and extended ($\rho_>$) states. The dot-dashed green lines are the high-energy limits: $\rho(E)=\sqrt{mL^2/2\pi^2\hbar^2E}$ for the box, and $\rho(E)=1/\hbar\omega$ for the harmonic trap.

For higher energies, the states corresponding to the upper branch in Fig. 1(b) are less localized, their extension grows and eventually saturates to that of the eigenstates of the non-disordered trap, $\Delta z \simeq \sqrt{E/m\omega^2}$. The centers of mass of these states approach the trap center and form the horizontal branch in Fig. 1(d) [29]. This branch, corresponding to

extended states, is hence easily interpreted in terms of finite-size effects, similarly as for a finite, flat box. The lower branch in Fig. 1(b) is more surprising and has no equivalent in the homogeneous box (even of finite size). It corresponds to strongly-localized states with relatively large energies. We find that these states are located close to the classical turning points $z_{\text{cl}}(E)$ and thus correspond to the outer branches in Fig. 1(d) [29].

These results show that localized and extended states coexist in energy in the disordered trap, while they do not in the homogeneous disorder. This is confirmed more quantitatively in Figs. 1(e)-(f) which show the full densities of states (DOS, solid black line), as well as the densities of localized ($\rho_{<}$, solid red line) and extended ($\rho_{>}$, dashed blue line) states [30]. For the flat box [Fig. 1(e)], the crossover from localized to extended states is very sharp and solely due to the width of the statistical distribution of Δz . For the harmonic trap [Fig. 1(f)], the crossover is much smoother and the density of localized states displays a long tail at high energies, due to the branch of localized states. For instance, we find that $\rho_{<}/\rho \simeq 12\%$ of the states are localized for $E \simeq 4|V_{\text{R}}|$.

Figure 2(a) shows the spatial density $|\psi_n(z)|^2$ of all states found for a *single realization* of the disorder, in a narrow slice of the spectrum around $E \sim 4|V_{\text{R}}|$. We can clearly distinguish localized (thick red lined) and extended (thin blue lines) states, which shows that they coexist in energy for each realization of the disorder. Here, finite energy separation and weak spatial overlap between localized and extended states suppresses hybridation. The localized states are very narrow and present no node (e.g. states A and E) or a few nodes (e.g. states C and H). They may be identified as bound states of the local deep wells of the disordered potential, similarly as the lowest-energy states creating the Lifshits tail in bare disordered potentials [28]. Let us decompose the eigenstates $|\psi_n\rangle$ of the disordered trap onto the basis of the eigenstates $|\chi_p\rangle$ of the bare disordered potential [i.e. Hamiltonian (1) with $V_{\text{T}} \equiv 0$], associated to the eigenenergies ϵ_p . Figure 2(b) shows $|\langle \chi_p | \psi_n \rangle|^2$ as a function of $\epsilon'_p = \epsilon_p + \langle \chi_p | V_{\text{T}}(z) | \chi_p \rangle$, i.e. the eigenenergy of $|\chi_p\rangle$ shifted by the potential energy associated to the harmonic trap, for states C and D of Fig. 2(a). It is characteristic of a general behavior: For the localized state (C, thick red line), the plot displays a single peak with amplitude of the order of unity at energy $\epsilon'_p \simeq E_n$. A localized state $|\psi_n\rangle$ of the disordered trap thus corresponds to a strongly-localized state $|\chi_p\rangle$ of the bare disordered potential, which is affected by the trap by just the energy shift $\langle \chi_p | V_{\text{T}}(z) | \chi_p \rangle$. We generally find that $|\epsilon_p| \ll \langle \chi_p | V_{\text{T}}(z) | \chi_p \rangle \simeq \langle \psi_n | V_{\text{T}}(z) | \psi_n \rangle$, and, due to the reduced spatial extension of $|\psi_n\rangle$, we

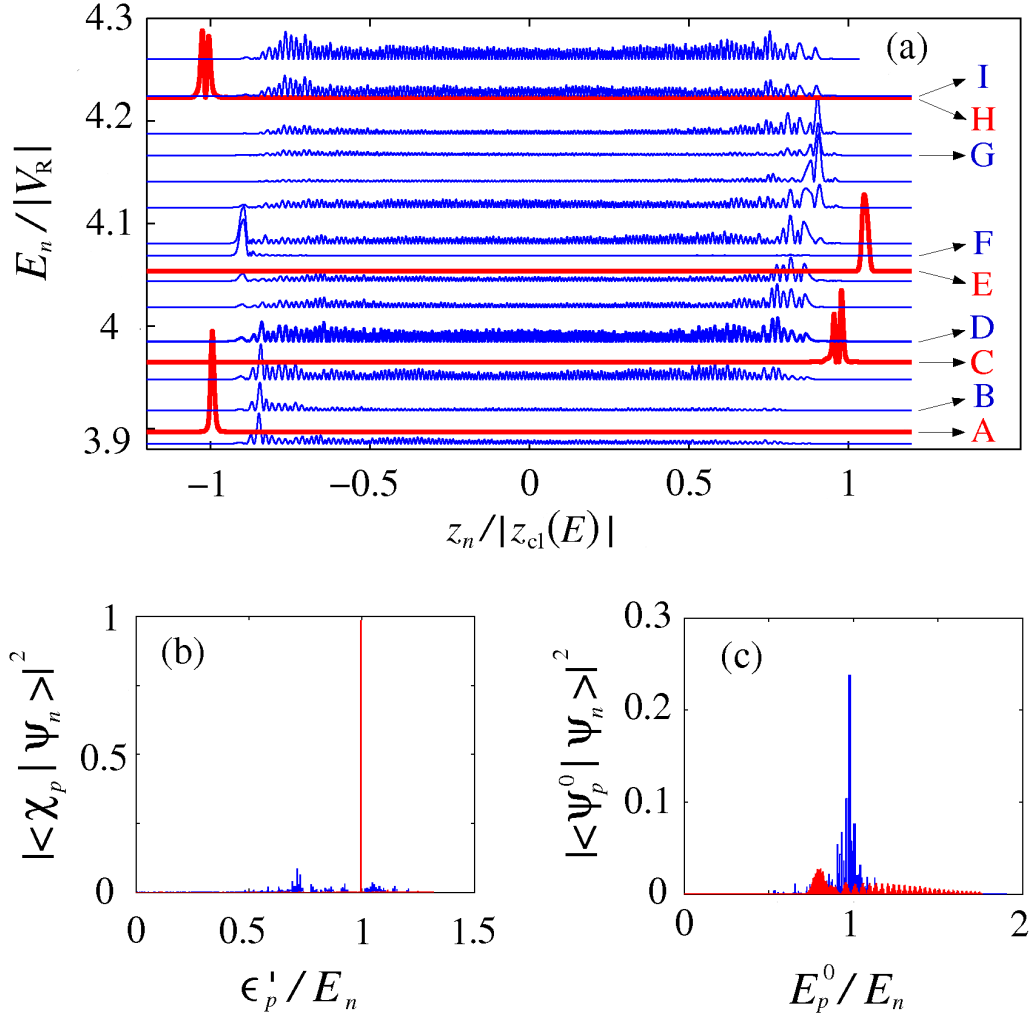


Figure 2: Eigenstates for a single realization of a 1D disordered trap. a) Non-normalized spatial densities, $|\psi_n(z)|^2$, vertically displaced at their eigenenergies E_n . Thick red lines correspond to localized states, and blue lines to extended states [30]. Note that extended and localized states may occupy almost-degenerate energy levels (e.g. H and I). The states C and D are projected: b) over the eigenstates of the disordered potential, $|\chi_p\rangle$, and c) over those of the harmonic trap, $|\psi_p^0\rangle$. The parameters are as in Fig. 1.

get $E_n \simeq m\omega^2 z_n^2/2$. This explains that the localized states are located close to the classical turning points, as observed in Figs 1(d) and 2(a). Conversely, the same decomposition for the extended state (D, thin blue line) shows a broad distribution of amplitude much smaller than unity. An extended state is thus not associated to an eigenstate of the bare disordered potential.

Hence, strongly bound states of the disordered potential can ‘survive’ in the trap. For more weakly bound states, the trap may promote tunneling through the barriers of the disordered potential and lead to extended states with a pronounced peak, a long tail and many nodes (e.g. states B, F and G). Fully extended states occupy the whole classically-allowed region [$|z| \lesssim z_{\text{cl}}(E)$], similarly as in the bare harmonic trap. Let us now decompose the states $|\psi_n\rangle$ of the disordered trap onto the basis of the eigenstates $|\psi_p^0\rangle$ of the bare harmonic trap [i.e. Hamiltonian (1) with $V = 0$], associated to the eigenenergy E_p^0 . Figure 2(c) shows $|\langle\psi_p^0|\psi_n\rangle|^2$ as a function of E_p^0 for states C and D. It exemplifies a characteristic behavior: For the extended state (D, thin blue line), the distribution is sharp and peaks at $E_p^0 \simeq E_n$ to a value equal to a fraction of unity. This state may be seen as reminiscent of a harmonic oscillator eigenstate, which is weakly affected by the disorder. Still, the main peak in Fig. 2(c) is markedly smaller than one. Inspection of the equations of standard perturbation theory shows that only for significantly higher energy, the extended states $|\psi_n\rangle$ result from weak perturbations of those of the non-disordered trap and display a main peak of the order of unity. Conversely, the localized state (C, thick red line) shows a much broader distribution, and is not related to a particular eigenstate of the non-disordered trap.

We now discuss a possible scheme to observe the coexistence of localized and extended states in a disordered trap. Consider a gas of non-interacting ultracold fermions prepared in a given internal state, at temperature T and chemical potential μ . A class of energies $|E_n - E| \lesssim \Delta$ [see Fig. 3(a)] deep in the Fermi sea (i.e. with $\mu - E \ll k_B T$) can be selected by applying a spin-changing radio-frequency (rf) field of frequency $\nu = E/h$ and duration $\tau \sim h/\Delta$ (with h the Planck constant) [16, 31, 32]. The rf field transfers the atoms of corresponding energies to another internal state, which can be chosen insensitive to the disordered trap. Then, the transferred atoms expand freely, which provides their momentum distribution [time-of-flight (TOF) technique]:

$$\mathcal{D}_{E,\Delta}(k) = \sum_{|E_n - E| \lesssim \Delta} |\hat{\psi}_n(k)|^2, \quad (2)$$

where $\hat{\psi}_n(k)$ is the Fourier transform of $\psi_n(z)$. In the coexistence region, $\mathcal{D}_{E,\Delta}(k)$ has two significantly different kinds of contributions: For localized states, $|\hat{\psi}_n(k)|^2$ is centered around $k \simeq 0$ with tails of width $\Delta k_n \sim \Delta z_n^{-1}$. Conversely, for extended states, $|\hat{\psi}_n(k)|^2$ is peaked at $k \simeq \sqrt{2mE}/\hbar$ with long tails towards small momenta. Exhibiting these two peaks would demonstrate the coexistence of localized and extended states at energy E . We however

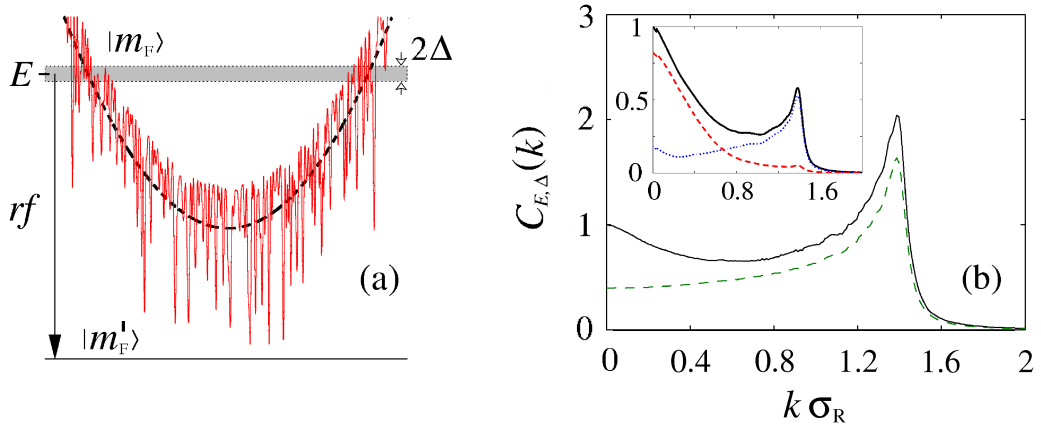


Figure 3: Scheme to observe the coexistence of localized and extended states in disordered traps (solid red line). a) The atoms occupying the eigenstates of energy $E \pm \Delta$ (shaded region) are transferred to a different internal state via rf coupling. The corresponding momentum distribution is then measured by TOF. b) Correlation function $\mathcal{C}_{E,\Delta}(k)$ (black solid line) and momentum distribution $\langle \mathcal{D}_{E,\Delta}(k) \rangle$ (dashed green line, arbitrary units), for $\Delta = 2\hbar\omega$. Inset: $\mathcal{C}_{E,\Delta}(k)$ of all states (solid black line), and separating localized (dashed red line) and extended (dotted blue line) states [30], for $\Delta \ll \hbar\omega$. Here $E = 4|V_R|$ and the other parameters are as in Fig. 1.

found that averaging over realizations of the disorder blurs the central peak associated to the localized states, and $\langle \mathcal{D}_{E,\Delta}(k) \rangle$ only displays the peak at $k \simeq \sqrt{2mE}/\hbar$ corresponding to the extended states [see dashed green line in Fig. 3(b)]. In order to overcome this effect, we propose to measure the correlation function $\mathcal{C}_{E,\Delta}(k) \equiv \langle \mathcal{D}_{E,\Delta}(k) \times \mathcal{D}_{E,\Delta}(0) \rangle / \langle \mathcal{D}_{E,\Delta}(0)^2 \rangle$. The function $\mathcal{C}_{E,\Delta}(k)$ enhances the contribution of the localized states, which have a strong component at $k = 0$. As shown in Fig. 3(b), $\mathcal{C}_{E,\Delta}(k)$ indeed displays two distinct peaks (solid black line) for a rf pulse of realistic duration ($\Delta = 2\hbar\omega$). The peak at $k = 0$ is more pronounced for narrower pulses. The clearest case is for $\Delta \ll \hbar\omega$. As shown in the inset of Fig. 3(b), selecting only the localized states on one hand, and only the extended states on the other hand [30], confirms that the central peak corresponds to the localized states and the side peak to the extended states.

Finally, can localized and extended states coexist in energy in disordered traps of higher dimensions? In order to address this question, we have performed similar calculations as above in two dimensions (2D). Figure 4(a) shows the centers of mass \mathbf{r}_n of the eigenstates with $E_n \simeq 4|V_R|$, the color scale giving Δr_n . Figure 4(b) shows a density plot of Δr_n versus

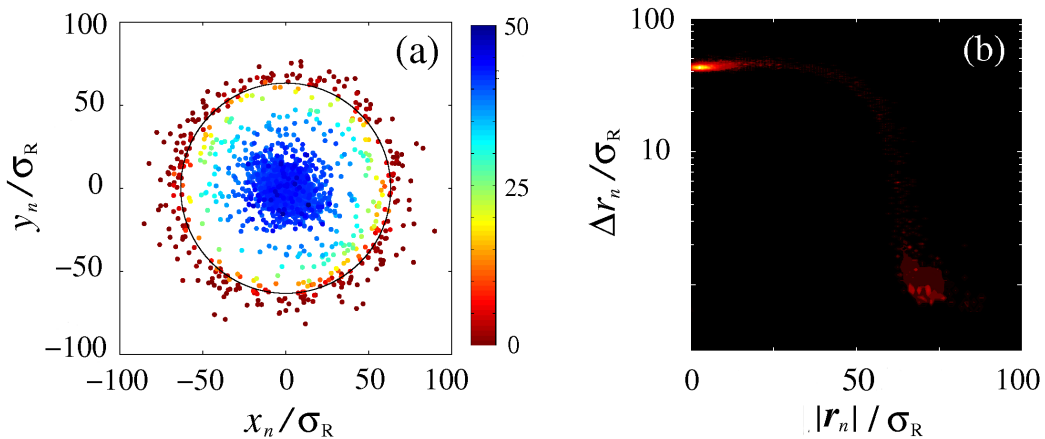


Figure 4: Coexistence of localized and extended states in a 2D disordered trap at energy $E \simeq (4 \pm 0.0003)|V_R|$. The figure results from accumulation of data from 2×10^4 realizations of the disorder, with $m\sigma_R^2|V_R|/\hbar^2 = 0.8$ and $\omega = 0.05|V_R|/\hbar$. a) Centers of mass \mathbf{r}_n of the eigenstates and corresponding values of $\Delta r_n/\sigma_R$ in color scale. The solid black line corresponds to the classical turning points, $r_{\text{cl}}(E) = \sqrt{2E/m\omega^2} \simeq 63.2\sigma_R$. b) Extension Δr_n versus distance from the trap center, $|\mathbf{r}_n|$.

$|\mathbf{r}_n|$ for the same data. Again, the eigenstates clearly separate into two classes: Some states are extended (large Δr_n) and centered nearby the trap center (small $|\mathbf{r}_n|$). The other states are strongly localized (small Δr_n) and located nearby the line of classical turning points ($|\mathbf{r}_n| \simeq r_{\text{cl}}(E) = \sqrt{2E/m\omega^2}$). These results show that localized and extended states coexist at intermediate energies in 2D disordered traps. We expect analogous results in higher dimensions.

In conclusion, we have shown that localized and extended states can coexist in energy in disordered potentials combined with harmonic traps. The localized states correspond to eigenstates of the disordered potential, which are only affected by the trap via an inhomogeneous energy shift. Conversely, the extended states extend over the classically-allowed region of the trap and are weakly affected by the disorder. This effect is directly relevant to present-day experiments with disordered quantum gases, which are most often created in harmonic traps, and we have proposed a realistic scheme to observe it in these systems. We expect these results hold in any dimension and for disordered potentials combined with other kinds of traps or with smooth potentials. The latter case may model certain systems with inhomogeneous disorder.

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