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On the nature of Bose-Einstein condensation enhanced by localization

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Abstract

In a previous paper we established that for the perfect Bose gas and the mean-field Bose gas with an external random or weak potential, whenever there is generalized Bose-Einstein condensation in the eigenstates of the single particle Hamiltonian, there is also generalized condensation in the kinetic energy states. In these cases Bose-Einstein condensation is produced or enhanced by the external potential. In the present paper we establish a criterion for the absence of condensation in single kinetic energy states and prove that this criterion is satisfied for a class of random potentials and weak potentials. This means that the condensate is spread over an infinite number of states with low kinetic energy without any of them being macroscopically occupied.

Keywords: Generalized Bose-Einstein Condensation, Random Potentials, Integrated Density of States, Multiscale Analysis, Diagonal Particle Interactions.

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1 Introduction

It can be easily seen from the explicit formula for the occupation numbers in the non-interacting (*perfect*) Bose gas that, the condition for Bose-Einstein Condensation (BEC) to occur, is that the density of states of the one particle Schrödinger operator decreases fast enough near the bottom of the spectrum. In the absence of any external potential, it is known that this happens only in three dimension or higher. This is still true if one introduces a mean-field interaction between particles. It has been known for some time that the behavior of the density of states can be altered by the addition of suitable external potentials, in particular *weak* potentials or *random* potentials. The subject of this paper is the study of models of the Bose gas in the presence of such external potentials. The first case has been extensively studied, see e.g. [1, 2], where sufficient conditions on the external potential were derived for the occurrence of BEC. In the random case, it has been shown in [3] that the so-called *Lifshitz tails*, which are a general feature of disordered systems, see for instance [4], are able to produce BEC. In both cases, it is possible to obtain condensation even in dimensions 1 or 2.
While BEC has historically been associated with the macroscopic occupation of the ground state *only*, it was pointed out in [5] that this phenomena is more thermodynamically stable if it is interpreted as as the macroscopic accumulation of particles into an arbitrarily narrow band of energy above the ground state, or *generalized* BEC. While it is clear that condensation in the ground state implies generalized BEC, there exist many situations in which the converse is not true. For instance, it was shown in [2] that in the case of the weak potential, the condensate can be in one state, in infinitely many states or even not in any state at all, depending on the external potential. These situations correspond respectively to type I, II, III generalized BEC in the classification established by the Van den Berg, Lewis and Pulé, see e.g. [6]. In the random case, far less is known. The only case for which a rigorous proof of the exact type of BEC has been established is the Luttinger-Sy model, see [7], where it was shown that the ground-state only is macroscopically occupied. As far as we know, for more complicated systems this is still an open question. The difficulty lies in the fact that the characterization of the distribution of the condensate in individual states requires much more detailed knowledge about the spectrum than the occurrence of generalized condensate. Indeed, for the latter, it is enough to know the asymptotic behavior of the density of states, while for the former, one needs in addition to know how fast the gap between two eigenvalues vanishes in limit.

In the physics literature the phenomenon of BEC is generally understood to be the macroscopic occupation of the lowest kinetic energy (momentum) state, commonly referred as zero-mode condensation. We refer the reader to [9] for a discussion of the motivation for this type of condensation. This leads naturally to two questions in the case condensation is produced or enhanced by the addition of external potentials.
The first one comes from the fact the condensates referred to here are to be found in the eigenstates of the one-particle Schrödinger operator and not the kinetic energy (momentum) states. Therefore, it is not immediately clear, and in fact counter intuitive in the random case because of the lack of translation invariance, that condensation occurs in the kinetic energy states as well. This problem has been addressed in a previous paper, see [9], where we have shown under fairly general assumptions on the external potential (random or weak) that the amount of generalized BEC in the eigenstates in turn creates a generalized condensate in the kinetic states, and moreover in the perfect gas the densities of condensed particles are iden-

tical. These results were proved for the perfect Bose gas, and can be partially extended to the mean-field Bose gas. Hence, the (generalized) condensation produced in these models by the localization property of the one-particle Schrödinger operator can be correctly described as of "Bose-Einstein" type in the traditional sense. This opens up the possibility of formulating a generalized version of the c-number Bogoliubov approximation ([9], [10]). In the case of the weak external potential, perhaps this result is not so surprising since the model is asymptotically translation invariant, but in the random case, it is less obvious since the system is translation invariant only in the sense that translates of the potential are equally probable and therefore for a given configuration the system is not translation invariant.

Having established generalized BEC in the kinetic states, the next question is about the fine structure of that condensate. In our paper [9], we conjectured that the kinetic generalized BEC is of type III, that is, no single kinetic state is macroscopically occupied, even though the amount of generalized condensation is non-zero. Our motivation came from the fact that the fast decrease of the density of states is usually associated with the corresponding eigenstates becoming *localized* in the infinite volume limit. Hence, since the kinetic states (plane waves) and the (localized) general eigenstates are "asymptotically orthogonal", it should follow that no condensation in any single-mode kinetic energy state could occur, independently of whether the (localized) ground state is macroscopically occupied or not. In [9], we were able to prove this conjecture in a simple example, the Luttinger-Sy model. Our proof in that case used the absence of tunneling effect specific to that model, which we can interpret as "perfect localization".

In this paper we give a proof of the conjecture under a fairly weak localization hypothesis and then we consider a family of continuous random models and a general class of weak external potential for which we are able to establish this localization property. Our results hold for both the perfect and mean-field Bose gas, and for any dimension. Note that, in addition to clarifying the nature of these condensates in low dimensions, we obtain an *unexpected conclusion*. Indeed, we show that the presence of randomness or a weak potential, however small, prevents condensation from occurring in any kinetic state, even if the corresponding *free* Bose gas (without external potential) exhibits zero-mode condensation (isotropic system in dimension 3, for example). This emphasizes the importance of the concept of generalized BEC.

The structure of the paper is as follows: in Section 2 we give the general setting for which our results are applicable and discuss generalized condensation in the kinetic energy states, while in Section 3 we derive a criterion for the absence of condensation into any single kinetic energy state. In Section 4 we establish that this criterion is satisfied for a class of random potentials (Subsection 4.1) and for weak (scaled) potentials (Subsection 4.2).

2 Notation and models

Let $\{\Lambda_l := (-l/2, l/2)^d\}_{l\geqslant 1}$ be a sequence of hypercubes of side l in $\mathbb{R}^d$, centered at the origin of coordinates with volumes $V_l = l^d$. We consider a system of identical bosons, of mass m, contained in Λ_l. For simplicity, we use a system of units such that $\hbar = m = 1$. First we define the self-adjoint one-particle kinetic-energy operator of our system by:

$$h_l^0 := -\tfrac{1}{2}\Delta_D, \tag{2.1}$$

acting in the Hilbert space $\mathscr{H}_l := L^2(\Lambda_l)$, where Δ is the usual Laplacian. The subscript D stands for *Dirichlet* boundary conditions. We denote by $\{\psi_k^l, \varepsilon_k^l\}_{k\geqslant 1}$ the set of normalized eigenfunctions and eigenvalues corresponding to h_l^0. By convention, we order the eigenvalues (counting multiplicity) as $0 < \varepsilon_1^l \leqslant \varepsilon_2^l \leqslant \varepsilon_3^l \ldots$. Note that all kinetic states satisfy the following bound

$$|\psi_k^l(x)| \leqslant l^{-d/2}. \tag{2.2}$$

Next we define the Hamiltonian with an external potential

$$h_l := h_l^0 + v_l, \tag{2.3}$$

also acting in $\mathscr{H}_l$, where $v_l : \Lambda_l \mapsto [0, \infty)$ is positive and bounded. Let $\{\phi_i^l\}_{i\geqslant 1}$ and $\{E_i^l\}_{i\geqslant 1}$ be respectively the sets of normalized eigenfunctions and corresponding eigenvalues of h_l. Again, we order the eigenvalues (counting multiplicity) so that $E_1^l \leqslant E_2^l \leqslant E_3^l \ldots$. Note that the *non-negativity* of the potential implies that $E_1^l > 0$. We shall also assume that the lower end of the spectrums of h_l^0 and h_l coincide in the limit $l \to \infty$, that is, $\lim_{l\to\infty} E_1^l = 0$. This assumption will be proved for the models considered in this paper.

Now, we turn to the many-body problem. Let $\mathscr{F}_l := \mathscr{F}_l(\mathscr{H}_l)$ be the symmetric Fock space constructed over $\mathscr{H}_l$. Then $H_l := \mathrm{d}\Gamma(h_l)$ denotes the second quantization of the *one-particle* Schrödinger operator h_l in $\mathscr{F}_l$. Note that the operator H_l acting in $\mathscr{F}_l$ has the form:

$$H_l = \sum_{i\geqslant 1} E_i^l \; a^*(\phi_i^l) a(\phi_i^l), \tag{2.4}$$

where $a^*(\varphi), a(\varphi)$ are the creation and annihilation operators (satisfying the boson *Canonical Commutation Relations*) for the one-particle state $\varphi \in \mathscr{H}_l$. Then, the grand-canonical Hamiltonian of the perfect Bose gas in an external potential is given by:

$$H_l - \mu N_l \; = \sum_{i\geqslant 1} (E_i^l - \mu) \; N_l(\phi_i^l) \tag{2.5}$$

where $N_l(\phi) := a^*(\phi)a(\phi)$ is the operator for the number of particles in the normalized state ϕ, $N_l := \sum_i N_l(\phi_i^l)$ is the operator for the total number of particles in Λ_l and μ is the chemical potential.

The results in this paper hold for the mean-field Bose gas whose many particle Hamiltonian, $H_l(\mu)$, is obtained by adding a mean-field term to (2.5):

$$H_l(\mu) \; := \; H_l - \mu N_l \; + \; \frac{\lambda}{2V_l} \, N_l^2, \tag{2.6}$$

where λ is a non-negative parameter. Of course the results are valid also for the perfect Bose gas ($\lambda = 0$).

We recall that the thermodynamic equilibrium Gibbs state $\langle - \rangle_l$ associated with the Hamiltonian $H_l(\mu)$ is defined by:

$$\langle A \rangle_l := \frac{\mathrm{Tr}_{\mathscr{F}_l}\{\exp(-\beta H_l(\mu_l))A\}}{\mathrm{Tr}_{\mathscr{F}_l} \exp(-\beta H_l(\mu_l))}$$

where the value of μ, μ_l is determined by fixing the mean density $\overline{\rho} > 0$:

$$\frac{1}{V_l}\langle N_l\rangle_l = \overline{\rho}. \tag{2.7}$$

When referring specifically to the perfect Bose gas state, we shall use the notation $\langle -\rangle_l^0$. For simplicity, in the sequel we shall omit the explicit mention of the dependence on the thermodynamic parameters (β, μ) unless it is necessary to refer to them.

A normalized single particle state φ is macroscopically occupied if

$$\lim_{l\to\infty} \frac{1}{V_l}\langle N_l(\varphi)\rangle_l > 0$$

and in particular there is condensation in the ground state if

$$\lim_{l\to\infty} \frac{1}{V_l}\langle N_l(\phi_1^l)\rangle_l > 0.$$

The concept of generalized condensation consists in considering the possible macroscopic occupation of an arbitrary small band of energies at the bottom of the spectrum. To be more precise, we say that there is generalized condensation in the states ϕ_i^l if

$$\lim_{\delta\downarrow 0}\lim_{l\to\infty} \frac{1}{V_l}\sum_{i:E_i^l\leqslant\delta}\langle N_l(\phi_i^l)\rangle_l > 0.$$

It is clear that the usual one-mode condensation implies generalized condensation, however the converse is not true. Indeed, as was first established by the Dublin School in the eighties [6], it is possible to classify generalized condensation into three types. Type I condensation, when a finite number of states are macroscopically occupied (which includes the most commonly known notion of BEC as condensation in the ground state only), type II condensation, when condensation occurs in an infinite number of states, and finally type III, when, although the amount of generalized condensation is non-zero, *no* individual state is macroscopically occupied. One can easily show that in the perfect Bose gas, under fairly general assumptions, for both random and weak positive potentials, there is indeed generalized condensation in a suitable range of density (or temperature).

In [9] we discussed the possibility of generalized condensation not in the states ϕ_i^l but in the kinetic energy states ψ_k^l. For both random and weak positive potentials, we established that for models which are diagonal in the occupation numbers of the eigenstates of the Hamiltonian (2.3), the density of *generalized BEC* in the kinetic states is never less then that in the eigenstates of the single particle Hamiltonian. To be more precise, we proved that

$$\lim_{\delta\downarrow 0}\lim_{l\to\infty} \frac{1}{V_l}\sum_{k:\varepsilon_k^l<\delta}\langle N_l(\psi_k^l)\rangle_l \geqslant \lim_{\delta\downarrow 0}\lim_{l\to\infty} \frac{1}{V_l}\sum_{i:E_i^l<\delta}\langle N_l(\phi_i^l)\rangle_l.$$

We also showed that in the case of the perfect gas, the two quantities in the above inequality are equal. Here, we shall give a "localization criterion" on the states ϕ_i^l so that the condensation in the kinetic energy states ψ_k^l is of type III, that is no kinetic energy state is macroscopically occupied.

It is easy to see that the mean-field gas satisfies the following commutation relation

$$[H_l(\mu), N_l(\phi_j^l)] = 0, \quad \text{for all } j. \tag{2.8}$$

This property implies that $\langle a^*(\phi_i^l)a(\phi_j^l)\rangle_l = 0$ if $i \neq j$ and allows us to obtain a simple relation between the mean occupation numbers for the ψ_k^l's and ϕ_k^l's:

$$\begin{aligned} \frac{1}{V_l}\langle N_l(\psi_k^l)\rangle_l = \frac{1}{V_l}\langle a^*(\psi_k^l)a(\psi_k^l)\rangle_l &= \frac{1}{V_l}\sum_{i,j}(\phi_i^l,\psi_k^l)\overline{(\phi_j^l,\psi_k^l)}\,\langle a^*(\phi_i^l)a(\phi_j^l)\rangle_l \\ &= \frac{1}{V_l}\sum_i |(\phi_i^l,\psi_k^l)|^2\,\langle N_l(\phi_i^l)\rangle_l. \end{aligned} \tag{2.9}$$

Finally, we want to point out that it may be possible to extend the results of this paper to a more general class of interacting Bose gases. More precisely, consider a class of "diagonal" interactions defined by

$$U_l := \frac{\lambda}{V_l}\sum_{i,j} a_{i,j}\, N_l(\phi_i)N_l(\phi_j)$$

with suitable assumptions on the coefficients $a_{i,j}$ in order to make the associated many-particle Hamiltonian well-defined, that is self-adjoint and bounded below. Note that the mean-field gas (2.6) is a particular case of this class, in which $a_{i,j} = \delta_{i,j}$ (with a shift in the chemical potential). It is easy to see that the condition (2.8) is satisfied. However we shall also need the monotonicity of the mean occupation numbers $\langle N_l(\phi_i^l)\rangle_l$ (see Lemma 3.1), which so far we are unable to prove beyond the mean-field case.

In the next section we use the expansion (2.9) to obtain a localization criterion for the absence of single mode condensation in the kinetic energy states.

3 Localization and kinetic single-state BEC

First we shall prove the following lemma which is trivial for the perfect gas. For the mean-field Bose gas it was proved by Fannes and Verbeure, [8], using correlations inequalities. Here we present an alternative proof, based only on a convexity argument.

Lemma 3.1 *For the mean-field Bose gas, i.e. for a bosonic system with Hamiltonian (2.6), the function $i \to \langle N_l(\phi_i)\rangle_l$ is non-increasing.*

Proof:

Let us define $f : \mathbb{R}_+ \mapsto \mathbb{R}$ by

$$\begin{aligned} f(t) &:= \beta^{-1}\ln \operatorname{Tr} \mathrm{e}^{-\beta H_l(\mu;t)}, \\ \text{where}\quad H_l(\mu;t) &:= H_l(\mu) + t(N_l(\phi_m^l) - N_l(\phi_n^l)), \end{aligned}$$

for some $1 \leqslant m < n$. It follows that

$$f'(0) = \langle N_l(\phi_n^l) - N_l(\phi_m^l)\rangle_l$$

and since the function f is convex, we have the following inequality

$$\langle N_l(\phi_n^l) - N_l(\phi_m^l)\rangle_l \leqslant f'(t), \tag{3.1}$$

for any $t \geqslant 0$. Now we set $t = \frac{1}{2}(E_n^l - E_m^l)$. Note that with this choice $t \geqslant 0$, since we have assumed that $m < n$. From the explicit expression (2.6) for $H_l(\mu)$, we have

$$H_l(\mu;t) = \sum_{i\neq m,n} (E_i^l - \mu)N_l(\phi_i^l) + \frac{\lambda}{2V_l} N_l^2 + (\frac{E_m^l + E_n^l}{2} - \mu)N_l(\phi_m^l) + (\frac{E_m^l + E_n^l}{2} - \mu)N_l(\phi_n^l).$$

Since the mean-field term in (2.6) is symmetric with respect to a permutation of any two eigenstate indices i, j, it follows that $H_l(\mu;t)$ is symmetric with respect to the exchange of m and n. Hence

$$f'(t) = \frac{\mathrm{Tr}\left(N_l(\phi_n^l) - N_l(\phi_m^l)\right)\mathrm{e}^{-\beta H_l(\mu;t)}}{\mathrm{Tr}\,\mathrm{e}^{-\beta H_l(\mu;t)}} = 0,$$

which in view of (3.1) gives

$$\langle N_l(\phi_n^l) - N_l(\phi_m^l)\rangle_l \leqslant 0,$$

and the lemma follows since $m < n$ are arbitrary. □

Let us introduce the notation

$$\rho_i^l := \frac{1}{V_l}\langle N_l(\phi_i^l)\rangle_l.$$

With this notation we can write the standard fixed density condition (2.7) for a given density $\overline{\rho}$, as

$$\sum_i \rho_i^l = \overline{\rho},$$

and so for any $N \in \mathbb{N}$,

$$\sum_{i=1}^N \rho_i^l \leqslant \overline{\rho}.$$

Letting

$$\rho_i := \limsup_{l\to\infty} \rho_i^l,$$

and taking the infinite volume limit, we then get

$$\sum_{i=1}^N \rho_i = \limsup_{l\to\infty} \sum_{i=1}^N \rho_i^l \leqslant \overline{\rho}.$$

Letting N tend to infinity, this gives $\sum_{i=1}^\infty \rho_i \leqslant \overline{\rho}$, and hence, for any $\varepsilon > 0$, there exists $i_0 < \infty$ such that $\rho_{i_0} < \varepsilon$. Splitting up the sum in (2.9) and using the monotonicity

property (see Lemma 3.1), property (2.2) and the fact that the kinetic eigenfunctions ψ_k^l are normalized, we obtain

$$\begin{aligned}
\frac{1}{V_l}\langle N_l(\psi_k^l)\rangle_l &= \sum_{i\leqslant i_0} |(\phi_i^l,\psi_k^l)|^2\,\rho_i^l + \sum_{i>i_0} |(\phi_i^l,\psi_k^l)|^2\,\rho_i^l \\
&\leqslant \sum_{i\leqslant i_0} |(\phi_i^l,\psi_k^l)|^2\,\rho_i^l + \rho_{i_0}^l \sum_{i>i_0} |(\phi_i^l,\psi_k^l)|^2 \\
&\leqslant \overline{\rho}\sum_{i\leqslant i_0} |(\phi_i^l,\psi_k^l)|^2 + \rho_{i_0}^l \\
&\leqslant \overline{\rho}\sum_{i\leqslant i_0} \left(l^{-d/2}\,||\phi_i^l||_1\right)^2 + \rho_{i_0}^l\,.
\end{aligned}$$

Therefore if $l^{-d/2}\,||\phi_i^l||_1 \to 0$ as $l\to\infty$ for each i, then

$$\limsup_{l\to\infty}\frac{1}{V_l}\langle N_l(\psi_k^l)\rangle_l \leqslant \varepsilon,$$

and since ε is arbitrary

$$\lim_{l\to\infty}\frac{1}{V_l}\langle N_l(\psi_k^l)\rangle_l = 0.$$

The above argument leads us to define the following localization criterion for the absence of single mode condensation in the kinetic energy states.

Definition 3.1 *We call an eigenfunction ϕ_i^l localized if it satisfies the following condition*

$$\lim_{l\to\infty}\frac{1}{l^{d/2}}\int_{\Lambda_l} \mathrm{d}x\,|\phi_i^l(x)| \;=\; 0. \tag{3.2}$$

Note that this localization condition is not as strong as the usual localization property, in the following sense. While, localization is frequently understood to be associated with the persistence of a pure point spectrum in the limit $l\to\infty$, at least near the bottom of the spectrum, the presence of a pure point spectrum is not necessary for the condition (3.2) to hold for all eigenfunctions. Indeed it may happen that (3.2) is satisfied and the infinite volume Schrödinger operator has only absolutely continuous spectrum.

In [9] we conjectured that the kinetic generalized BEC observed in the random models is in fact of type III, and gave a proof in a simple case, the Luttinger-Sy model. In the above argument we proved that our conjecture is correct under the fairly weak localization hypothesis (3.2). We formulate this result in the following theorem.

Theorem 3.1 *Assume that the eigenfunctions ϕ_i^l are localized in the sense of (3.2) for all i. Then, for the mean-field Bose gas, no kinetic state ψ_k^l can be macroscopically occupied, that is*

$$\lim_{l\to\infty}\frac{1}{V_l}\langle N_l(\psi_k^l)\rangle_l \;=\; 0, \tag{3.3}$$

which implies in particular that any possible kinetic generalized BEC in these models is of type III.

In this paper, we provide two classes of externals potential for which we can prove localization in the sense of (3.2). The first one is a class of *random* external potentials, the second involves *weak* external potentials.

4 Proof of the localization condition

4.1 Random potentials

Before we specify the random model under consideration, let us emphasize again that the localization property (3.2) is very different from what is usually called "exponential localization" in the literature about random Schrödinger operators (see for example [12]). In the standard literature localization refers to the eigenfunctions of the infinite volume Hamiltonian and requires these functions, with energies in some band, to decay very fast, in many cases exponentially. This implies that the spectrum is pure point in that band. In our case we are dealing with eigenfunctions in finite volume with energies tending to zero as the volume increases and so these bear no relation to the infinite volume eigenfunctions. In particular, our localization condition (3.2) does *not* imply that the spectrum is discrete in the thermodynamic limit. While we only need the L^1 norm not to diverge too fast, because our eigenfunctions depend crucially on the volume and in particular, because we do not work at a fixed energy but with volume dependent eigenvalues, we have to deal with the additional problem of controlling the finite-volume behavior. However, we find that in fact the *multiscale analysis* developed for the infinite volume case can be adapted to establish our localization condition.

The model studied in this section is taken from [12]. It consists of impurities located at points of the lattice $\mathbb{Z}^d$, with appropriate assumptions over the single-impurity potential, mainly designed to obtain independence between regions which are sufficiently far away from each other. Let us make it more explicit by giving some definitions. In the rest of this section, we shall denote by $\Lambda_l(x)$ the cubic box of side l centered at x. The single-site potential f, $\Lambda_1(0) \to \mathbb{R}$ has the following properties:

1. f is bounded;
2. there is $\sigma > 0$ such that $f(x) \geqslant \sigma$ for all $x \in \Lambda_1(0)$.

The randomness in this model is given by varying the strength of each impurity. For this purpose, we define a single-site (probability) measure μ, with $\mathrm{supp}(\mu) = [0, a]$ for a finite a. We will assume that μ is Hölder-continuous, that is for some $\alpha > 0$,

$$\sup_{\{s,t\}} \big\{\mu([s,t]) : 0 \leqslant t - s \leqslant \eta\big\} \;\leqslant\; \eta^\alpha, \quad \forall\, 0 \leqslant \eta \leqslant 1 \tag{4.1}$$

The random potential is then defined by

$$v^\omega(x) \;:=\; \sum_{k \in \mathbb{Z}^d} q^\omega(k)\, f(x-k), \tag{4.2}$$

where the $q^\omega(k)$'s are i.i.d. random variables distributed according to μ. We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the associated probability space, and by $\omega \in \Omega$ a particular realization of the random potential. Note that by property 2 and the fact that $a < \infty$, there exists a non-random $M < \infty$ such that $v^\omega(x) < M$ for any x and all ω.

The one-particle random Schrödinger operator in finite volume is then given as in (2.3) by

$$h_l^\omega \;=\; h_l^0 + v_l^\omega, \tag{4.3}$$

where v_l^ω is the restriction of v^ω to Λ_l. The eigenfunctions and eigenvalues of h_l^ω are denoted by $\phi_i^{\omega,l}$ and $E_i^{\omega,l}$ respectively. We denote by $h_l^\omega(x)$ the restriction of the Schrödinger operator $-\frac{1}{2}\Delta + v^\omega$ to the region $\Lambda_l(x)$, with Dirichlet boundary conditions.

Before we establish the localization criterion (3.2) we prove our assumption that the eigenvalues of h_l^ω tend to zero as l tends to ∞:

Lemma 4.1 *With probability one, for each i*

$$\lim_{l\to\infty} E_i^{\omega,l} = 0. \tag{4.4}$$

Proof: Let ν denote the limiting integrated density of states for the Hamiltonians h_l^ω, that is for any Borel subset $A \subset \mathbb{R}_+$,

$$\nu(A) := \lim_{l\to\infty} \frac{1}{V_l} \sharp\{i \,:\, E_i^{\omega,l} \in A\}. \tag{4.5}$$

Since by ergodicity ν is nonrandom (see for example Theorem 5.18 in [4]), it is clearly sufficient to prove that for every $E > 0$, $\nu([0,E]) > 0$. To do this we start from the following inequality (see equation (4) in [16]),

$$\nu([0,E]) \;\geqslant\; \frac{1}{V_L}\,\mathbb{E}\big(\sharp\{i : E_i^{\omega,L} \leqslant E\}\big) \;\geqslant\; \frac{1}{V_L}\,\mathbb{P}\{\omega : \, E_1^{\omega,L} \;\leqslant\; E\}. \tag{4.6}$$

which is satisfied for any $L > 0$. From the min-max principle, we obtain

$$E_1^{\omega,L} \;\leqslant\; \varepsilon_1^L \;+\; \int_{\Lambda_L} \mathrm{d}x\, |\psi_1^L|^2\, v^\omega(x) \tag{4.7}$$

where ε_1^L is the first kinetic eigenvalue and ψ_1^L the corresponding eigenfunction. Since $|\psi_1^L(x)|^2 \leqslant 1/V_L$, we have

$$E_1^{\omega,L} \;\leqslant\; \varepsilon_1^L \;+\; \frac{1}{V_L}\int_{\Lambda_L} \mathrm{d}x\, v^\omega(x) \;\leqslant\; \varepsilon_1^L \;+\; \frac{A}{V_L}\sum_{k\in\mathbb{Z}^d\cap\Lambda_L} q^\omega(k). \tag{4.8}$$

where $A := \int_{\Lambda_1} \mathrm{d}x\, f(x)$. Letting $L := \pi(E/2)^{-1/2}$ so that $\varepsilon_1^L = E/2$, (4.6) and (4.8) give

$$\nu([0,E]) \;\geqslant\; \frac{1}{V_L}\,\mathbb{P}\{\omega : \sum_{k\in\mathbb{Z}^d\cap\Lambda_L} q^\omega(k) \;\leqslant\; EV_L/2A\}. \tag{4.9}$$

Since the right-hand side of the last inequality is strictly positive, the lemma is proved. □

The rest of this subsection is devoted to proving that this model satisfies our localization assumption (3.2). For this purpose we need a result from multiscale analysis which exists in various forms in the literature (see references in [12]). For convenience here we follow the version in [12].

Adhering to the terminology of [12], we first define so-called "good boxes":

Definition 4.1 *Given $x \in \mathbb{Z}^d$, a scale $l \in 2\mathbb{N}+1$, an energy E, a rate of decay $\gamma > 0$, we call the box $\Lambda_l(x)$ (γ, E)-good for a particular realization ω of the random potential (4.2) if $E \notin \sigma(h_l^\omega(x))$ and*

$$||\chi_l^{out}\,(h_l^\omega(x) - E)^{-1}\,\chi_l^{int}|| \;\leqslant\; e^{-\gamma l}. \tag{4.10}$$

Here $\sigma(h_l^\omega(x))$ denotes the spectrum of $h_l^\omega(x)$, the norm in (4.10) refers to the operator norm in $L^2(\Lambda_l(x))$, and $\chi_l^{int}, \chi_l^{out}$ are the characteristic functions of the regions $\Lambda_l^{int}(x), \Lambda_l^{out}(x)$ respectively, which we define as follows

$$\Lambda_l^{int}(x) \;:=\; \Lambda_{l/3}(x), \quad \Lambda_l^{out}(x) \;:=\; \Lambda_l(x) \setminus \Lambda_{l-2}(x).$$

Our proof depends crucially on the following important multiscale analysis result extracted from [12]. We refer the reader to Theorem 3.2.2 and Corollary 3.2.6 for the general multiscale analysis argument and to Theorems 2.3.2, 2.2.3 and 2.4.1 for proving that this particular model satisfies the necessary conditions required for multiscale analysis.

Proposition 4.1 *Assume that h_l^ω is as above with random potential given by (4.2). Then for any $\zeta > 0$ and any $\alpha \in \big(1, 2 - (4d/(4d+\zeta))\big]$, there exist a sequence $\{l_k\}, k \geqslant 1$, satisfying $l_1 \geqslant 2$ and $l_{k-1}^\alpha \leqslant l_k \leqslant l_{k-1}^\alpha + 6$ for $k \geqslant 2$, and constants $r > 0$ and $\gamma > 0$ such that if $I := [0, r]$,*

$$\mathbb{P}\Big\{\omega :\ \text{for all } E \in I,\ \text{either } \Lambda_{l_k}(x) \text{ or } \Lambda_{l_k}(y) \text{ is } (\gamma, E)\text{-good}\Big\} \;\geqslant\; 1 - (l_k)^{-2\zeta}, \tag{4.11}$$

for all $k \geqslant 1$ and for all $x, y \in \mathbb{Z}^d$, satisfying $|x - y| > l_k$.

For our proof we need also the *Eigenfunction Decay Inequality.* We state it in a convenient form for our purpose, and refer the reader to [12] (Lemma 3.3.2) for a detailed proof. Note that this inequality has to be understood for a given realization ω.

Proposition 4.2 *Let h_l^ω be defined as above, and $\phi_i^{\omega,l}$ to be one eigenfunction with eigenvalue $E_i^{\omega,l}$. Let $x \in \Lambda_l$, such that $\Lambda_{l_k}(x) \subset \Lambda_l$. If $E_i^{\omega,l}$ does not belong to the spectrum of $h_{l_k}^\omega(x)$, then the following inequality holds*

$$\|\chi_{l_k}^{int}(x)\phi_i^{\omega,l}\| \;\leqslant\; \kappa\|\chi_{l_k}^{out}(x)(h_{l_k}^\omega(x) - E_i^{\omega,l})^{-1}\chi_{l_k}^{int}(x)\|, \tag{4.12}$$

where the norms are $L^2(\Lambda_l)$-norm, and κ is a constant depending only on M.

We are now ready to prove that for our model the localization condition (3.2) is satisfied.

Lemma 4.2 *Assume that h_l^ω is as in (4.3) with random potential given by (4.2). Then almost surely, for all i,*

$$\lim_{l\to\infty} \frac{1}{V_l^{1/2}} \int_{\Lambda_l} dx\, |\phi_i^{\omega,l}(x)| \;=\; 0. \tag{4.13}$$

Proof: We first choose $0 < \delta < 1/7$ and $\zeta > (2d+1)/2\delta$ and then we take the constants α, γ and r and the sequence $\{l_k\}$ to be those obtained in Proposition 4.1 for this value of ζ. For a given scale l large enough we pick $k = k(l)$ satisfying

$$\frac{1}{\ln\alpha} \ln\left(\frac{\delta \ln l}{\ln l_1}\right) \;<\; k \;<\; \frac{1}{\ln\alpha} \ln\left(\frac{(1-\delta)\ln l}{\ln(l_1+6)}\right).$$

The fact that $\delta < 1/7$ ensures that there exists such an integer k. Then, by Proposition 4.1, we have

$$l^\delta < l_k < l^{1-\delta}. \tag{4.14}$$

Now let us define $A(\omega, l)$ to be the event in which, for all $E \in I$, for any $x, y \in \Lambda_l \cap \mathbb{Z}^d$ such that $|x-y| > l_k$, either $\Lambda_{l_k}(x)$ or $\Lambda_{l_k}(y)$ are (γ, E)-good.

We shall first use the Borel-Cantelli lemma to show that almost surely $A(\omega, l)$ occurs for all l large enough. Let us define

$$X_l \;:=\; \Big\{\omega : A(\omega, l) \,\text{is not true at scale}\, l\Big\}.$$

Then we can write

$$\begin{aligned} X_l \;&:=\; \Big\{\omega : \exists\, E \in I,\ \exists\, x, y \in \Lambda_l \cap \mathbb{Z} \text{ with } |x-y| > l_k, \\ &\qquad \text{such that both } \Lambda_{l_k}(x) \text{ and } \Lambda_{l_k}(y) \text{ are not } (\gamma, E)\text{-good}\Big\} \\ &=\; \bigcup_{\substack{x,y\in\Lambda_l\cap\mathbb{Z} \\ |x-y|>l_k}} \Big\{\omega : \exists\, E \in I,\ \text{such that both } \Lambda_{l_k}(x) \text{ and } \Lambda_{l_k}(y) \text{ are not } (\gamma, E)\text{-good}\Big\}, \end{aligned}$$

and by Proposition 4.1 we obtain

$$\mathbb{P}(X_l) \;\leqslant\; l^{2d}\,(l_k)^{-2\zeta} \;\leqslant\; l^{-2(\delta\zeta - d)},$$

where the last step follows from (4.14). Since $2(\delta\zeta - d) > 1$, it follows that

$$\sum_l \mathbb{P}(X_l) \;<\; \infty.$$

By the Borel-Cantelli lemma, almost surely there exists $L(\omega) < \infty$ such that the event $A(\omega, l)$ occurs for all $l > L(\omega)$.

Since by Lemma 4.1 with probability one, $E_i^{\omega,l}$ tends to 0 as l tends to ∞, $E_i^{\omega,l} \in I$ for l large enough almost surely. Therefore there exists $\tilde{\Omega} \subset \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that for each $\omega \in \tilde{\Omega}$ there is $L_1(\omega) < \infty$ such that for all $l > L_1(\omega)$ and for any $x, y \in \Lambda_l \cap \mathbb{Z}^d$ satisfying $|x-y| > l_k$, either $\Lambda_{l_k}(x)$ or $\Lambda_{l_k}(y)$ are $(\gamma, E_i^{\omega,l})$-good.

Now we take $\omega \in \tilde{\Omega}$ and $l > L_1(\omega)$ and partition the box $\Lambda_l(0)$ into $\Lambda_l^1 := \Lambda_{l-l_k}(0)$ and $\Lambda_l^2 := \Lambda_l(0) \setminus \Lambda_l^1$. We then split up the integral in (4.13) into the interior cube Λ_l^1 and the corridor Λ_l^2

$$\int_{\Lambda_l} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \;=\; \int_{\Lambda_l^1} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \;+\; \int_{\Lambda_l^2} \mathrm{d}x\, |\phi_i^{\omega,l}(x)|\,. \tag{4.15}$$

In the second term, we can use the Schwarz inequality and the fact that the eigenfunctions are $L^2(\Lambda_l)$-normalized to obtain

$$\int_{\Lambda_l^2} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \;\leqslant\; |\Lambda_l^2|^{1/2} \;\leqslant\; 2^d l^{(d-1)/2}\, l_k^{1/2} \;\leqslant\; 2^d l^{(d-\delta)/2}\,. \tag{4.16}$$

For the first term in (4.15), we shall use the eigenfunction decay inequality (4.12) of Proposition 4.2. We cover the "interior cube" Λ_l^1 by disjoints subcubes Λ_j of side $l_k/3$. Let us call $\{x_j\}$ their respective centers. Then for each j the cube $\Lambda_{l_k}(x_j)$ is included in Λ_l and Λ_j coincides with $\Lambda_{l_k}^{int}(x_j)$.
Using the Schwarz inequality and Proposition 4.2, we obtain for any j the estimate:

$$\begin{aligned}\int_{\Lambda_j} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \;&\leqslant\; l^{d/2} \Big(\int_{\Lambda_l} \mathrm{d}x |\chi_{l_k}^{int}(x_j) \phi_i^{\omega,l}(x)|^2 \Big)^{1/2} \\ &\leqslant\; l^{d/2} \Big(\kappa \|\chi_{l_k}^{int}(x)(h_{l_k}^{\omega}(x_j) - E_i^{\omega,l})^{-1} \chi_{l_k}^{out}(x)\| \Big)^{1/2}.\end{aligned}$$

Hence, for any j such that Λ_j is $(\gamma, E_i^{\omega,l})$-good, one has the following upper bound

$$\int_{\Lambda_j} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \;\leqslant\; l^{d/2} \mathrm{e}^{-\frac{1}{2}\gamma l_k} \;\leqslant\; l^{d/2}\, \mathrm{e}^{-\frac{1}{2}\gamma l^\delta}. \tag{4.17}$$

Now, we distinguish two cases.

The first one corresponds to the situation where all cubes $\Lambda_{l_k}(x_j)$ are $(\gamma, E_i^{\omega,l})$-good. It then follows directly from (4.16) and (4.17) that

$$\begin{aligned} l^{-d/2} \int_{\Lambda_l} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \;&\leqslant\; 2^d \frac{l^{(d-\delta)/2}}{l^{d/2}} + l^{-d/2} \sum_{x_j \in \Lambda_1} l^{d/2}\, \mathrm{e}^{-\frac{1}{2}\gamma l^\delta} \\ &\leqslant\; 2^d l^{-\delta/2} + 3^d\, \frac{(l - l^\delta)^d}{l^{\delta d}}\, \mathrm{e}^{-\frac{1}{2}\gamma l^\delta}. \end{aligned} \tag{4.18}$$

The second case corresponds to the situation when there exists at least one subcube $\Lambda_{l_k}(x_j)$ which is *not* $(\gamma, E_i^{\omega,l})$-good. Let us denote by $\tilde{x}$ the center of one such bad cube. Since $\omega \in \tilde{\Omega}$ and $l > L_1(\omega)$, for $x, y \in \Lambda_l \cap \mathbb{Z}^d$ satisfying $|x - y| > l_k$, either $\Lambda_{l_k}(x)$ or $\Lambda_{l_k}(y)$ are $(\gamma, E_i^{\omega,l})$-good. It therefore follows that, outside of a box of side $2l_k$ centered at $\tilde{x}$, all other $\Lambda_{l_k}(x_j)$ are $(\gamma, E_i^{\omega,l})$-good. We treat the good boxes as above, and deal with $\Lambda_{2l_k}(\tilde{x})$ by using the Schwarz inequality as we did for Λ_l^2, to obtain:

$$\begin{aligned}\int_{\Lambda_l^1} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \;&=\; \int_{\Lambda_l^1 \setminus \Lambda_{2l_k}(\tilde{x})} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \;+\; \int_{\Lambda_{2l_k}(\tilde{x})} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \\ &\leqslant\; \sum_{x_j \in \Lambda_l^1 \setminus \Lambda_{2l_k}(\tilde{x})} l^{d/2}\, \mathrm{e}^{-\frac{1}{2}\gamma l^\delta} \;+\; |\Lambda_{2l_k}(\tilde{x})|^{d/2} \\ &\leqslant\; l^{d/2} 3^d\, \frac{(l - l^\delta)^d}{l^{\delta d}}\, \mathrm{e}^{-\frac{1}{2}\gamma l^\delta} \;+\; (2l)^{d(1-\delta)/2}.\end{aligned}$$

From that last bound and from (4.16), we get

$$l^{-d/2} \int_{\Lambda_l} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \leqslant 2^d l^{-\delta/2} + 3^d \frac{(l-l^\delta)^d}{l^{\delta d}} \mathrm{e}^{-\frac{1}{2}\gamma l^\delta} + 2^{d(1-\delta)/2} l^{-d\delta/2}. \tag{4.19}$$

Therefore for any $\omega \in \tilde{\Omega}$ either (4.18) or (4.19) is satisfied for all l large enough and (4.13) follows.

□

4.2 Weak external potentials

In this section we consider a scaled external potential. Let v be a non-negative, continuous real-valued function defined on the closed unit cube $\overline{\Lambda}_1 \subset \mathbb{R}^d$ which satisfies the following two conditions:

1. There is a finite, nonempty subset of Λ_1, $D := \{y_j\}_{j=1}^n$ such that $v(x) = 0$ if and only if $x \in D$.
2. For each $y_j \in D$ there are strictly positive numbers $\{\alpha_j\}$, $\{c_j\}$ such that

$$\lim_{x \to y_j} \frac{v(x)}{|x-y_j|^{\alpha_j}} = c_j. \tag{4.20}$$

We order the y_j's in such a way that $0 < \alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_n$.

The *one-particle* Schrödinger operator with a *weak* (scaled) external potential in a box Λ_l is defined by:

$$h_l = -\tfrac{1}{2}\Delta_D + v(x_1/l, \ldots, x_d/l)\ . \tag{4.21}$$

We recall that the eigenfunctions and eigenvalues of h_l are denoted by ϕ_i^l and E_i^l respectively. The aim of this section is to prove that our localization condition (3.2) holds for this class of weak potentials.

Lemma 4.3 *Let h_l be as in (4.21). Then, for all i*

$$\lim_{l\to\infty} \frac{1}{l^{d/2}} \int_{\Lambda_l} \mathrm{d}x\, |\phi_i^l(x)| = 0. \tag{4.22}$$

Proof: We start by noting that in view of the condition (4.20), for any $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that for all $j = 1, \ldots, n$

$$(c_j - \varepsilon)|x-y_j|^{\alpha_j} \leqslant v(x) \leqslant (c_j + \varepsilon)|x-y_j|^{\alpha_j}, \tag{4.23}$$

for all $x \in B(y_j, \delta)$, the ball of radius δ centered at y_j. Note also that by continuity there exists a constant $\kappa > 0$ such that $v(x) \geqslant \kappa$, for all $x \in \Lambda_1 \setminus \left(\bigcup_{j=1}^n B(y_j, \delta)\right)$. We let $K := \min(\kappa, c_1 - \varepsilon, \ldots, c_n - \varepsilon)$ and $C := \max(c_1 + \varepsilon, \ldots, c_n + \varepsilon)$.

The first step in our proof is to obtain an estimate for the eigenvalue E_i^l. To this end, let us denote by $h_l^{(n)}$ the restriction of the Schrödinger operator to the region $B(y_n, \delta l)$, with Dirichlet boundary conditions. Then we have

$$h_l \leqslant h_l^{(n)} \tag{4.24}$$

in quadratic form sense (c.f. [14], Chapter VIII, Proposition 4). From the inequality (4.23), we obtain

$$h_l^{(n)} \leqslant \tilde{h}_l^{(n)} := \tfrac{1}{2}\Delta_D + C\left|\frac{x-y_n}{l}\right|^{\alpha_n}, \tag{4.25}$$

where the last operator acts on $L^2\big(B(y_n, \delta l)\big)$. Let $U : L^2\big(B(y_n, \delta l)\big) \mapsto L^2\big(B(0, \delta l^{1-\gamma_n})\big)$ be the unitary transformation defined by

$$(U\varphi)(x) \quad := \quad l^{\gamma_n/2}\,\varphi(l^{\gamma_n}(x-y_n)),$$

where $\gamma_n := \alpha_n/(2+\alpha_n)$. By direct computation, one can check that $\tilde{h}_l^{(n)} = l^{-2\gamma_n}\, U\, \hat{h}_l^{(n)}\, U^{-1}$ where

$$\hat{h}_l^{(n)} := (-\tfrac{1}{2}\Delta + C|x|^{\alpha_n}),$$

acting on $L^2\big(B(0, \delta l^{1-\gamma_n})\big)$. Let $0 < D_1^l \leqslant D_2^l \leqslant \ldots$ be the eigenvalues of $\hat{h}_l^{(n)}$ and $0 < D_1 \leqslant D_2 \leqslant \ldots$ the eigenvalues of $\hat{h}^{(n)}$ where

$$\hat{h}^{(n)} := (-\tfrac{1}{2}\Delta + C|x|^{\alpha_n}),$$

acting on $L^2(\mathbb{R}^d)$. Since for each i, $D_i^l \to D_i$ as $l \to \infty$, there are constants $\tilde{D}_i$ such that $D_i^l \leqslant \tilde{D}_i$ for all l. Using this and the operator inequalities (4.24) and (4.25) we finally get

$$E_i^l \leqslant D_i^l l^{-2\gamma_n} \leqslant \tilde{D}_i l^{-2\gamma_n}. \tag{4.26}$$

The rest of our proof relies on the methods developed in [15]. We start with some definitions. Let Ω^t, for some $t > 0$, to be the set of all continuous trajectories (paths) $\{\xi(s)\}_{s=0}^t$ in $\mathbb{R}^d$ with $\xi(0) = 0$, and let w^t denote the normalized Wiener measure on this set. For a given $x \in \mathbb{R}^d$, we define the following characteristic function

$$\chi_{x,l}(\xi) := \mathbf{1}\big\{\xi : \xi(s) \in \Lambda_l - x, \text{ for all } 0 \leqslant s \leqslant t\big\}.$$

We now use the following identity (c.f. [13]),

$$(\mathrm{e}^{-th_l}\,\phi_i^l)(x) = \int_{\Omega^t} w^t(\mathrm{d}\xi)\,\mathrm{e}^{-\int_0^t \mathrm{d}s v((x+\xi(s))/l)}\,\phi_i^l(x+\xi(t))\,\chi_{x,l}(\xi),$$

from which, since E_i^l is the eigenvalue of h_l corresponding to ϕ_i^l, we get

$$|\phi_i^l(x)| \leqslant \mathrm{e}^{tE_i^l}\int_{\Omega^t} w^t(\mathrm{d}\xi)\,\mathrm{e}^{-\int_0^t \mathrm{d}s v((x+\xi(s))/l)}\,|\phi_i^l(x+\xi(t))|\,\chi_{x,l}(\xi). \tag{4.27}$$

Now, we insert into the right-hand side of (4.27) the following bound proved in [11],

$$|\phi_i^l(x)| \leqslant c_d\,(E_i^l)^{d/4},$$

where $c_d := (e/\pi)^{d/4}$ and we obtain from (4.27) the following estimate

$$\begin{aligned}
|\phi_i^l(x)| &\leqslant c_d\,\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Omega^t} w^t(\mathrm{d}\xi)\,\mathrm{e}^{-\int_0^t \mathrm{d}s v((x+\xi(s))/l)}\,\chi_{x,l}(\xi)\\
&= c_d\,\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Omega^t} w^t(\mathrm{d}\xi)\,\mathrm{e}^{-\frac{1}{t}\int_0^t \mathrm{d}s\, t\, v((x+\xi(s))/l)}\,\chi_{x,l}(\xi)\\
&\leqslant c_d\,\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Omega^t} w^t(\mathrm{d}\xi)\,\frac{1}{t}\int_0^t \mathrm{d}s\,\mathrm{e}^{-tv((x+\xi(s))/l)}\,\chi_{x,l}(\xi),
\end{aligned}$$

where the last step follows from Jensen's inequality. Therefore, integrating over Λ_l with respect to x, and then changing the order of integration, yields

$$\begin{aligned}
l^{-d/2}\int_{\Lambda_l}\mathrm{d}x\,|\phi_i^l(x)| &\leqslant c_d\,l^{-d/2}\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Lambda_l}\mathrm{d}x\int_{\Omega^t} w^t(\mathrm{d}\xi)\,\frac{1}{t}\int_0^t \mathrm{d}s\,\mathrm{e}^{-tv((x+\xi(s))/l)}\,\chi_{x,l}(\xi)\\
&\leqslant c_d\,l^{-d/2}\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Omega^t} w^t(\mathrm{d}\xi)\,\frac{1}{t}\int_0^t \mathrm{d}s\int_{\{x\in\bigcap_{s'}(\Lambda_l-\xi(s'))\}}\mathrm{d}x\,\mathrm{e}^{-tv((x+\xi(s))/l)}.
\end{aligned}$$

Letting $y = x+\xi(s)$ in the second integral we get

$$\begin{aligned}
l^{-d/2}\int_{\Lambda_l}\mathrm{d}x|\phi_i^l(x)| &\leqslant c_d\,l^{-d/2}\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Omega^t} w^t(\mathrm{d}\xi)\,\frac{1}{t}\int_0^t \mathrm{d}s\\
&\quad\cdot\int_{\{y-\xi(s)\in\bigcap_{s'}(\Lambda_l-\xi(s'))\}}\mathrm{d}y\,\mathrm{e}^{-tv(y/l)}.
\end{aligned}$$

Since $\bigcap_{s'}(\Lambda_l-\xi(s')+\xi(s)) \subset \Lambda_l$ for all s, we can now extend the domain of integration over y to Λ_l and use the fact that the Wiener measure w^t is normalized to obtain

$$\begin{aligned}
l^{-d/2}\int_{\Lambda_l}\mathrm{d}x|\phi_i^l(x)| &\leqslant c_d\,\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\,l^{-d/2}\,\frac{1}{t}\int_0^t \mathrm{d}s\int_{\Lambda_l}\mathrm{d}y\,\mathrm{e}^{-tv((y)/l)}\\
&= c_d\,\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\,l^{d/2}\int_{\Lambda_1}\mathrm{d}z\,\mathrm{e}^{-tv(z)}.
\end{aligned}\tag{4.28}$$

Next, we obtain an upper bound for the last integral in (4.28). We have

$$\begin{aligned}
\int_{\Lambda_1}\mathrm{d}z\,\mathrm{e}^{-tv(z)} &\leqslant \sum_{j=1}^n\int_{B(y_j,\delta)}\mathrm{d}z\,\mathrm{e}^{-tv(z)} + \int_{\Lambda_1\setminus\left(\bigcup_{i=1}^n B(y_j,\delta)\right)}\mathrm{d}z\,\mathrm{e}^{-tv(z)}\\
&\leqslant \mathrm{e}^{-tK} + \sum_{j=1}^n\int_{B(y_j,\delta)}\mathrm{d}z\,\mathrm{e}^{-tK|x-y_j|^{\alpha_j}}.
\end{aligned}\tag{4.29}$$

For each j,

$$\int_{B(y_j,\delta)}\mathrm{d}z\,\mathrm{e}^{-tK|x-y_j|^{\alpha_j}} \leqslant t^{-d/\alpha_j}\,K^{d/\alpha_j}\int_{\mathbb{R}^d}\mathrm{d}\tilde{z}\,\mathrm{e}^{-|\tilde{z}|^{\alpha_j}} \leqslant \tilde{K}\,t^{-d/\alpha_j},$$

where $\tilde{K} := K^{d/\alpha_1} \max_j \int_{\mathbb{R}^d} \mathrm{d}\tilde{z}\, e^{-|\tilde{z}|^{\alpha_j}}$, which, in view of (4.29), gives the following bound

$$\int_{\Lambda_1} \mathrm{d}z\, \mathrm{e}^{-tv(z)} \leqslant \mathrm{e}^{-tK} + \tilde{K} \sum_{j=1}^{n} t^{-d/\alpha_j}.$$

Now, fixing $t = (E_i^l)^{-1}$, we get from the last inequality and (4.28)

$$l^{-d/2} \int_{\Lambda_l} \mathrm{d}x |\phi_i^l(x)| \leqslant c_d\, \mathrm{e}(E_i^l)^{d/4}\, l^{d/2} \Big(\mathrm{e}^{-K(E_i^l)^{-1}} + \tilde{K} \sum_{j=1}^{n} (E_i^l)^{d/\alpha_j} \Big).$$

Since by (4.26), $E_i^l \to 0$ as $l \to \infty$, and since we have ordered the α_i's such that $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, there exist new constants A_i such that the following bound holds for l large enough

$$l^{-d/2} \int_{\Lambda_l} \mathrm{d}x |\phi_i^l(x)| \leqslant A_i\, l^{d/2} \big(E_i^l\big)^{d(1/4+1/\alpha_n)} = A_i\, l^{d/2} \big(E_i^l\big)^{d(2-\gamma_n)/(4\gamma_n)}. \tag{4.30}$$

Inserting the bound (4.26), we finally obtain for l large enough

$$l^{-d/2} \int_{\Lambda_l} \mathrm{d}x |\phi_i^l(x)| \leqslant A_i \tilde{D}_i^{d(2-\gamma_n)/(4\gamma_n)}\, l^{-d(1-\gamma_n)/2}$$

and the lemma follows since $\gamma_n < 1$. □

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References

[1] J.V. Pulé, *J. Math. Phys.* **24** (1983), 138–142

[2] M. Van den Berg, J.T. Lewis, *Commun. Math. Phys.* **81** (1981), 475-494

[3] O. Lenoble, L.A. Pastur, V.A. Zagrebnov, *Comptes-Rendus de l'Académie des Sciences (Paris), Physique* **5** (2004), 129-142

[4] L.A. Pastur, A. Figotin, *Spectra of Random and Almost-Periodic Operators*, Springer-Verlag, Berlin, 1992

[5] M. Girardeau, *J. Math. Phys.* **1** (1960), 516-523

[6] M. Van den Berg, J.T. Lewis, J.V. Pulé, *Helv. Phys. Acta* **59** (1986), 1271-1288

[7] O. Lenoble, V.A. Zagrebnov, *Markov Processes and related fields* **13** (2007), 441-468

[8] M. Fannes, A. Verbeure, *J. Math. Phys.* **21** (1980), 1809-1818

[9] T. Jaeck, J.V. Pulé, V.A. Zagrebnov, *J. Stat. Phys.* **137** (2009), 19-55

[10] T. Jaeck, V.A. Zagrebnov, in preparation (2010)

[11] E.B. Davies, *J. Lond. Math. Soc.* **7** (1973), 483

[12] P. Stollmann, *Caught by disorder. Bound states in random media*, Progress in Mathematical Physics, 20. Birkhäuser Boston, Inc., Boston, MA, 2001

[13] D. Ray, *Trans. Amer. Math. Soc.* **77** (1954), 299-321

[14] M. Reed, B. Simon, *Methods of Mathematical Physics, IV: Analysis of Operators*, Academic Press, London, 1978

[15] P. Mac Aonghusa, J.V. Pulé, *Lett. Math. Phys.* **14** (1987), 117-121

[16] W. Kirsch, F. Martinelli, *Commun. Math. Phys.* **89** (1983), 27-40