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Numerical study of Schramm-Loewner Evolution in the random 3-state Potts model

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Abstract. We have numerically studied the properties of the interface induced in the ferromagnetic random-bond three-state Potts model by symmetry-breaking boundary conditions. The fractal dimension d_f of the interface was determined. The corresponding SLE parameter κ was estimated to be $\kappa \simeq 3.18(6)$, compatible with previous estimate. On the other hand, we estimated κ independently from the probability of passage of the interface at the left of a given point. The numerical data are well reproduced by the Schramm theoretical prediction and the fit leads to $\kappa \simeq 3.245(10)$, in agreement with the first estimate. This provides evidences that the geometric properties of spin interfaces in the random 3-state Potts model may be described by chordal SLE_κ .

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Introduction

The assumption of conformal invariance in two-dimensional critical systems has been very fruitful [1, 2, 3]. Exact expressions for correlation functions, as well as values of critical exponents for minimal models, are among the most stringent predictions of conformal theory. Recently, the interest has been focused on fractal curves that may be realized as interfaces induced in a critical conformal-invariant system by imposing symmetry-breaking boundary conditions [4, 5, 6]. It is believed that the geometric properties of this kind of curves can be described by a family of self-avoiding stochastic processes introduced by Schramm. The self-avoidance property is obtained by iteratively removing, during the growth of the curve, the hull formed from the curve itself and all non-reachable domains enclosed by the curve. This removal is performed by a conformal transformation. As a consequence, the construction relies on both the assumption of conformal invariance and on Markovian property. The family of curves obtained in this way is parameterized by a single parameter κ corresponding to the intensity of the noise in the underlying Brownian process.

In a conformal field theory characterized by a central charge c , it has been predicted that the interfaces induced by symmetry-breaking boundary conditions are described by the chordal Schramm-Löwner process of parameter κ with

$$c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}. \quad (1)$$

The length ℓ of the interface is expected to scale anomalously with the lattice size as $\ell \sim L^{d_f}$ where the fractal dimension is related to κ by

$$d_f = 1 + \frac{\kappa}{8}. \quad (2)$$

The probability that a given point be on the left of the interface is given by the Schramm formula:

$$\wp_{\text{left}}(x, y) = \frac{1}{2} - \frac{\Gamma(4/\kappa)}{\sqrt{\pi}\Gamma((8-\kappa)/2\kappa)} \left(\frac{x}{y}\right) {}_2F_1\left(\frac{1}{2}, \frac{4}{\kappa}; \frac{3}{2}; -\frac{x^2}{y^2}\right). \quad (3)$$

This formula is valid in the upper half-plane $y > 0$ with fixed boundary conditions on the real axis, changing for $x = 0$. It can easily be extended to other geometries by taking advantage of the conformal symmetry of the critical system that allows to apply a conformal transformation from the upper-half plane to another geometry. In the following, we will consider a critical system confined in a square. The mapping of the point $z = x + iy$ in upper-half plane to the point ω in the square is realized by the Schwartz-Christoffel conformal transformation

$$\omega = F(z, k)/K(k) \Leftrightarrow z = \text{sn } K(k)\omega \quad (4)$$

where $F(z, k)$ is the incomplete elliptic integral, $K(k)$ is the complete elliptic integral and k is related to the aspect ratio s of the square by

$$sK(k) = K'(k) \quad (5)$$

where $K'(k)$ is the associated elliptic integral.

The critical properties of the q -state Potts model ($q \leq 4$) defined by the Hamiltonian [7]

$$-\beta\mathcal{H} = J \sum_{(i,j)} \delta_{\sigma_i, \sigma_j}, \quad \sigma_i \in \{0, \dots, q-1\} \quad (6)$$

are described by the conformal theory with central charge

$$c = 1 - \frac{6}{m(m+1)} \quad (7)$$

where the integer m is related to the number of states q by the Nienhuis formula

$$q = 4 \cos^2\left(\frac{\pi}{m+1}\right). \quad (8)$$

One thus expects the geometric properties of the interfaces induced in the Potts model to be described in the scaling limit by the chordal SLE_κ with the two possible values

$$\kappa_1 = \frac{4}{1 - \frac{1}{\pi} \text{Arccos}(\sqrt{q}/2)}, \quad \kappa_2 = 16/\kappa_1. \quad (9)$$

This prediction has been proved exactly by Smirnov in the case of percolation [8] ($q = 1$) and latter for the Ising model [9] ($q = 2$) in the Fortuin-Kasteleyn representation. In the case of the $q = 3$ Potts model, the fractal dimension of the interfaces between spin clusters has been shown to be compatible within numerical accuracy with equation (2) and $\kappa_2 = 10/3$ [10]. Predictions of SLE_κ have been tested more specifically for this model by Gamsa and Cardy [11]. These authors considered the interfaces between the spin clusters and between the Fortuin-Kastelyn clusters and they extracted the parameter κ both from the scaling of the length of the interface and from the Schramm formula for the left-passage probability. Several geometries and

boundary conditions were considered. As expected since $c = 4/5$, Gamsa and Cardy measured the parameter $\kappa_2 = 10/3$ for the spin interfaces and $\kappa_1 = 4.8$ for the interfaces between the Fortuin-Kastelyn clusters, in agreement with the values expected for $c = 4/5$ model. Unlike in the Ising model, certain ambiguities arise for the interfaces in the Potts model. When fixing two different Potts states on the boundaries at the left and at the right of the system, two interfaces are usually induced since a cluster of the third state may appear between the left and right clusters. Gamsa and Cardy generalized the Schramm formula for the left-passage probability to the case of two SLE curves [12]. Unfortunately, in the 3-state Potts model, the two curves sometimes collapse in a finite system and the latter formula for the left-passage probability does not hold. To circumvent this problem, Gamsa and Cardy have also studied fluctuating boundary conditions: the left boundary is fixed to one of the Potts states while at the right boundary, spins are free to take any of the two other Potts states. With these boundary conditions, only one interface is induced and the original Schramm formula for the left-passage probability can be applied. Recently, the fractal dimension of the interface has been numerically shown to be compatible with SLE_κ even for non-integer values of the number of states q [13].

The ferromagnetic random-bond 3-state Potts model is defined by the Hamiltonian

$$-\beta\mathcal{H} = \sum_{(i,j)} J_{ij} \delta_{\sigma_i, \sigma_j}, \quad \sigma_i \in \{0, \dots, q-1\}, \quad J_{ij} > 0 \quad (10)$$

When the couplings are distributed according to the binary distribution

$$\wp(J_{ij}) = \frac{1}{2} [\delta(J_{ij} - J_1) + \delta(J_{ij} - J_2)] \quad (11)$$

the model is critical on the self-dual line

$$(e^{J_1} - 1)(e^{J_2} - 1) = q \quad (12)$$

Randomness is a relevant perturbation and thus alters the critical behavior. In numerical simulations, the ratio $r = J_1/J_2$ is usually fixed to the value that minimizes the scaling corrections due to the cross-over with the pure and percolation fixed points. The optimum r^* can be determined as the value leading to the largest central charge extracted from the scaling of the free energy density on the strip. Even though conformal symmetry is broken for a single disorder realization, it is restored after averaging over randomness. Magnetization profiles or correlation functions in confined geometries have indeed been shown to transform covariantly under conformal transformations [14]. Recently, the interface induced by boundaries has been studied in the perspective of SLE_κ [15]. The scaling of the interface length has been determined both by transfer matrix techniques and Monte Carlo simulations. From the fractal dimension d_f , Jacobsen *et al.* estimated $\kappa_2 \simeq 3.208(24)$ for the spin interface. Duality, i.e. $\kappa_1 \kappa_2 = 16$, is satisfied within error bars. The results are compatible with renormalization group arguments. Note that this estimate of κ_2 is not compatible with the value $\kappa \simeq 3.3370(4)$ given by relation (1) with the estimate of the central charge $c \simeq 0.8024(3)$.

The Schramm formula has never been tested in the random Potts model. It has already been considered in other random systems: Ising spin-glasses [16] and the Solid-On-Solid model on a random substrate [17]. In the second case, a large

discrepancy was observed between the two estimates of κ obtained from the scaling of the interface length and from the left-passage probability. Our aim is to check whether such a discrepancy exists for the random-bond Potts model too. We will restrict ourselves to a square geometry with fluctuating boundary conditions and to interfaces between spin clusters. In the first section, we present the numerical details of the Monte Carlo simulations that we have performed. Then, the fractal dimension of the interfaces is studied with an emphasis on the influence of disorder on the value of κ . In the last section, we present our results for the left-passage probability and the estimation of κ that can be made from it.

1. Details of the Monte Carlo simulations

We have studied the random-bond 3-state Potts model using large-scale Monte Carlo simulations. Square lattices were considered with fluctuating boundary conditions and sizes $L = 64, 90, 128, 180, 256, 360$ and 512 . Each Monte Carlo step consists into one iteration of the Metropolis algorithm and one of the Swendsen-Wang algorithm [18]. This choice is motivated by the fact that the dynamics of an interface with the Swendsen-Wang algorithm is very slow and may even be slower than with local Monte Carlo algorithms (see for example Table 1 of [19]). The combination of the two algorithms may help the system to escape dynamical traps. For equilibration of the system, between 25000 ($L = 64$) and 57810 ($L = 512$) Monte Carlo steps were discarded (Table 1). Measurements were taken every five Monte Carlo steps. Between 15000 ($L = 64$) and 34692 ($L = 512$) measurements were made for each random realization. These values were chosen in order to achieve a good thermalization and then a sufficient sampling of the equilibrium probability distribution. The number of measurements may appear at first sight unnecessarily large since it may be argued that the average over randomness will smooth thermal fluctuations. Smaller numbers of measurements have indeed led to accurate values of the critical exponents in the past. However, in our problem, preliminary tests have shown that a smaller number of Monte Carlo steps leads to systematic deviations on the estimate of κ due to the large autocorrelation time of the interface.

The measurements were then averaged over $N = 5000$ different random configurations of the exchange couplings J_{ij} with the optimal ratio $r^* \simeq 4.0(3)$ determined by Jacobsen *et al.* [15]. Since the random realizations are uncorrelated, the measurements obtained for each of them are statistically independent random variables so that the error can be estimated, according to the central-limit theorem, as $\sqrt{\sigma^2/N}$ where σ^2 is the mean square deviation of the observable among the random configurations. Additional simulations have been performed for other values r of the amplitude of randomness. The same number of Monte Carlo steps were used but only 3000 random realizations were produced and the lattices sizes were limited to $L = 64, 90$ and 128 .

For a given spin configuration, the interface is determined by applying a simple algorithm: starting from the upper point of the boundary where the conditions change and thus the interface is fixed, the interface is built on the dual lattice by going through the broken bonds between neighboring spins (see figure 1 for an example). The interface goes straight as long as the spin at the right is in the same state as the left boundary and the one at the left in one of the two other states. Ambiguous

L	Thermalization	Measurement
64	25000	75000
90	28750	86250
128	33060	99185
180	38015	114060
256	43715	131165
360	50270	150835
512	57810	173460

Table 1. Number of Monte Carlo iterations used for thermalization and measurements for the lattice sizes L considered. For all lattice sizes, the data were then averaged over 5000 random realizations. The total number of Monte Carlo iterations is thus of order 10^9 for $L = 512$ and represents 3 years of single-CPU time (6 years for all lattice sizes).

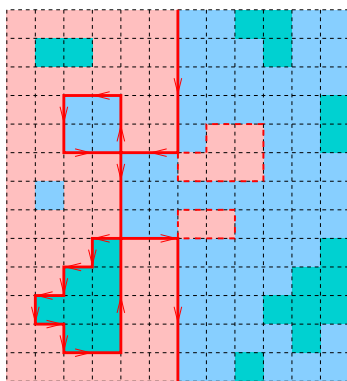


Figure 1. Example of a spin configuration and the interface as determined by the algorithm. The three different spin states are depicted as pink, blue and green pixels. The interface is the bold red line. The dashed line corresponds to paths that are not followed by the algorithm.

situations may arise: when the interface cannot go straight, the spin at the right of the interface can be in some cases in the same state as the left boundary whether the interface turns on the right or on the left. The algorithm is thus not able to choose one of the two possible directions †. On figure 1, four of these situations are depicted. We arbitrarily imposed the rule that the interface should always turn on the right in these cases (*turn-right tie-breaking rule*). As a consequence, the algorithm may fail to detect the correct interface. Because of the rule of always turning on the right in case of ambiguity, the interface may touch itself and thus form loops enclosing a cluster in a different state. As shown on figure 1, this can only happen on the left of the interface. Direct visualization of spin configurations shows that the number of these loops is of order $\mathcal{O}(1)$ and that the cluster sizes are most of the time limited to a few spins and are always much smaller than the lattice size.

In order to gauge the influence of these unwanted loops on the scaling dimension

† To avoid these situations, an hexagonal lattice, and thus a triangular dual lattice, is often considered.

of the interface and on the left-passage probability, we estimated κ for the pure 3-state model and compared with the expected value $10/3$. As will be seen in the following, the scaling dimension is not affected by the presence of these loops. We note that the absence of noticeable effect on the scaling dimension has also been reported recently in the case of the pure Ising model [20]. It is not the case however for the left-passage probability. A small but systematic deviation from the Schramm formula is observed on the left part of the lattice. This deviation affects the estimation of κ . Following the procedure presented in section 3, we obtain $\kappa \simeq 3.284$ for $L = 512$. Since the loops appears only on the left of the interface, it creates a left-right asymmetry. Fortunately, the restriction of the interpolation of the data with the Schramm formula to the right third of the lattice, i.e. for $(x, y) \in [2L/3; L - 1] \times [0; L - 1]$ only turns out to give $\kappa \simeq 3.312$ for $L = 512$ in much better agreement with the expected value $10/3$ for the pure model.

2. Scaling of the interface length

The scaling of the interface length was already determined by Jacobsen *et al.*. Besides reproducing their result, we present here a study of the influence of disorder. As already mentioned, we will restrict ourselves to the interfaces between spin clusters. For the pure model, we obtain $d_f \simeq 1.4164(4)$ corresponding to $\kappa \simeq 3.331(3)$, in very good agreement with the expected value $10/3$. Our data for the disorder amplitude $r^* = 4$, i.e. at the random fixed point, is plotted on figure 2. A simple power-law fit $\ell \sim L^{d_f}$ leads to the fractal dimension $d_f \simeq 1.407(5)$ and thus to the value $\kappa = 3.26(4)$. However, a deviation from a pure power-law can be observed, for example, by removing the smallest lattices sizes. In the inset of figure 2, the fractal dimension obtained when taking into account in the fit only the data with lattice sizes larger than L is plotted with respect to $1/L$. While the effective fractal dimension for the pure system is relatively stable, a tendency to lower values of d_f is clearly observed in the random case. The last points are very noisy because the interpolation is made only with the two or three largest lattice sizes. The extrapolation to the thermodynamic limit is then delicate. Should the two last points be interpreted as a real trend or a statistical fluctuation? Our final estimate will be $d_f \simeq 1.397(7)$ which correspond to $\kappa \simeq 3.18(6)$. This value is less accurate but compatible within error bars with the estimate given by Jacobsen *et al.* (3.208(24)).

The fractal dimension d_f and thus κ are sensitive to the estimate r^* of the disorder amplitude at the random fixed point. As shown on figure 3, d_f and thus κ diminishes when the disorder gets stronger. A linear interpolation of d_f versus r allows for a rough estimation of the additional error Δd_f due to the uncertainty Δr^* on r^* . With the value $\Delta r^* \simeq 0.3$ given by Jacobsen *et al.*, we obtained $\Delta d_f \simeq 0.0006$ and thus $\Delta \kappa \simeq 0.005$, an order of magnitude smaller than the statistical error.

Besides the necessity to average over random realizations, numerical simulations of random systems present the difficulty that an observable ϕ may not be self-averaging. In such a situation, the average of ϕ over N disorder realizations goes to $\langle \phi \rangle$ in the limit $N \rightarrow +\infty$ but the relative fluctuations $R_\phi = [\langle \phi^2 \rangle - \langle \phi \rangle^2] / \langle \phi^2 \rangle$ do not vanish. The probability distribution of the length ℓ of the interface is shown on Figure 4 for different lattice sizes L . When plotted with respect to $\ell / \langle \ell \rangle$, the probability distribution appears the same for all lattice sizes. It means that the mean

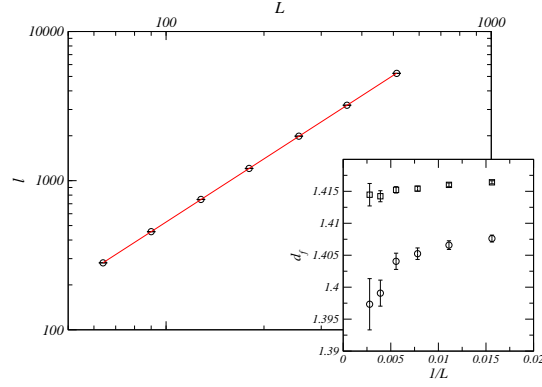


Figure 2. Scaling of the interface length with respect to the lattice size L . The red line is the power-law fit taking into account all the points. In the inset, an effective fractal dimension d_f obtained from a power-law fit of the points in the interval of lattice sizes $[L; 512]$ is plotted with respect to $1/L$. The squares above correspond to the pure system and the circles to the random system with $r = 4$.

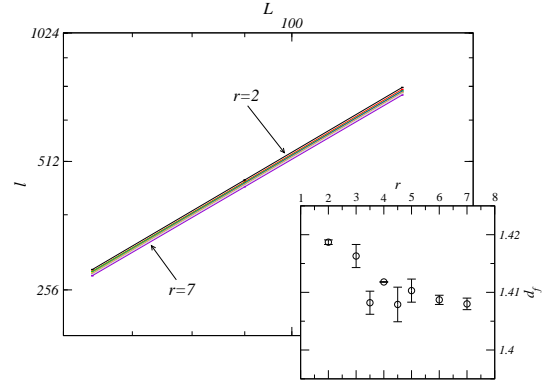


Figure 3. Scaling of the length of the interface with respect to the lattice size L for different disorder amplitudes r . In the inset, the fractal dimension d_f is plotted with respect to r .

square deviation $\langle \ell^2 \rangle - \langle \ell \rangle^2$ scales as $\langle \ell \rangle^2$. The cumulant R_ℓ is thus constant: the interface length is non self-averaging. Another important feature of random systems is that the typical event may differ from the average. As seen on figure 4, the probability distribution of the interface length is not symmetric and thus these events indeed differ slightly. However, the distribution does not present any long tail with rare events difficult to observe in numerical simulations. We rather observe a distribution close to a Gaussian. Finally, random systems have the peculiar property to possibly display multifractality: different moments of an observable ϕ may present a different scaling behavior, i.e. $\langle \phi^n \rangle^{1/n} \sim L^{x_\phi(n)}$ where x_ϕ depends of the order n of the moment. The first moments $\langle \ell^n \rangle^{1/n}$ have been computed. All fractal dimensions $d_f(n)$ corresponding to the scaling behavior $\langle \ell^n \rangle^{1/n} \sim L^{d_f(n)}$ are compatible within error bars. We conclude that the interface length is not multifractal.

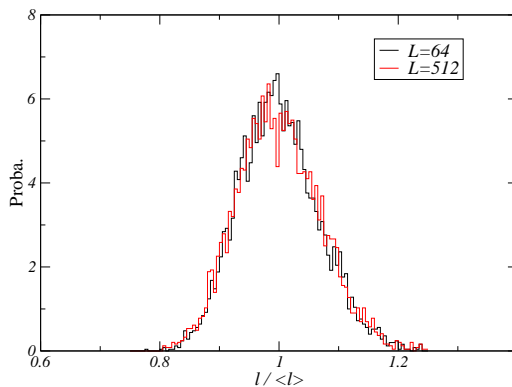


Figure 4. Density probability of the length l of the interface normalized by its average value $\langle l \rangle$. The two different curves correspond to different lattice sizes: $L = 64$ (black) and $L = 512$ (red).

3. Left-passage probability

We now determine κ from the Schramm formula of the left-passage probability. We follow the same procedure as [17]. The left-passage probability $\varphi(x, y)$ was estimated numerically for all points (x, y) of the lattice. To extract κ , the square deviation from the Schramm formula $S_\kappa(x, y)$ at any point of the lattice

$$\chi^2(x, y) = [\varphi(x, y) - S_\kappa(x, y)]^2$$

was computed and then averaged only in the right third of the lattice:

$$\overline{\chi^2} = \frac{1}{\frac{2}{3}L^2} \sum_{x=2L/3}^L \sum_{y=0}^L \chi^2(x, y).$$

The value of κ leading to the smallest mean square deviation is then searched as follows: the mean square deviation $\overline{\chi^2}$ is first computed for a set of values κ and then interpolated with a 4th-order polynomial using a weight $1/\overline{\chi^2}$ in order to give a higher weight to the points in the neighborhood of the minimum. The latter is then determined numerically by searching for the root of the derivative of the 4th-order polynomial. An example of this procedure is shown on figure 6. For the largest lattice size $L = 512$, the mean square deviation is minimal for $\kappa \simeq 3.2452$. The monitoring of $\chi^2(x, y)$ shows that for this value of κ , the square deviation is of order 10^{-6} in the right third of the lattice (figure 5). The square of the statistical error $\Delta\varphi^2(x, y)$ is of the same order. The left-right asymmetry due to the loops is clearly seen: χ^2 is larger by two orders of magnitude in the left part of the lattice. The much better agreement of the data with the Schramm formula in the right third of the lattice justifies *a-posteriori* the restriction of the interpolation to this region.

On figure 7 (left), κ is plotted versus the number of disorder realizations taken into account in the average for different lattice sizes. No abrupt jump that would be the signature of rare events with large contribution is observed. The estimates of κ after 5000 disorder realizations are compatible within statistical fluctuations for lattice

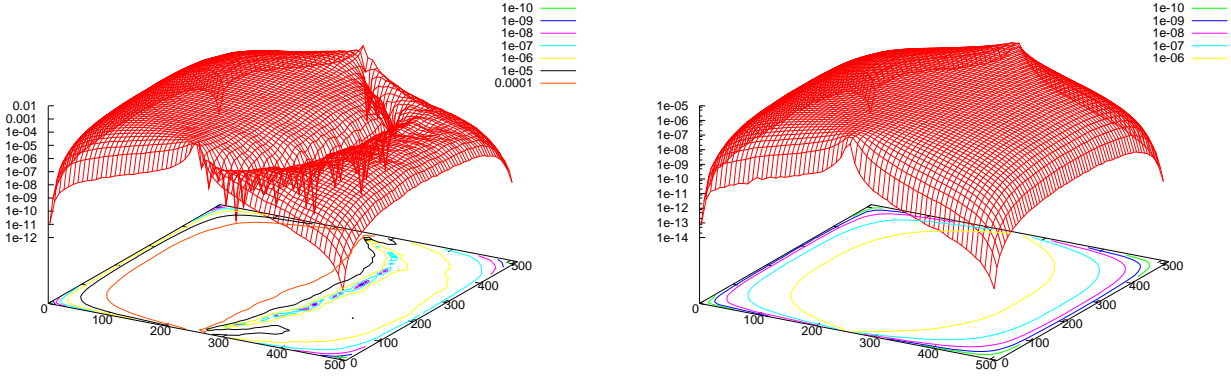


Figure 5. On the right, square deviation $\chi^2(x, y)$ of the data with the Schramm formula with $\kappa = 3.2452$ at each point of a 512×512 lattice. The contour plot at the bottom shows that χ^2 is of order 10^{-6} in the right part of the lattice. The largest discrepancy (of the order of 0.002) is encountered close to the points where the interface meets the boundary. On the right, square of the statistical error $\Delta\phi^2(x, y)$ of the numerical estimate of the left-passage probability for the same system.

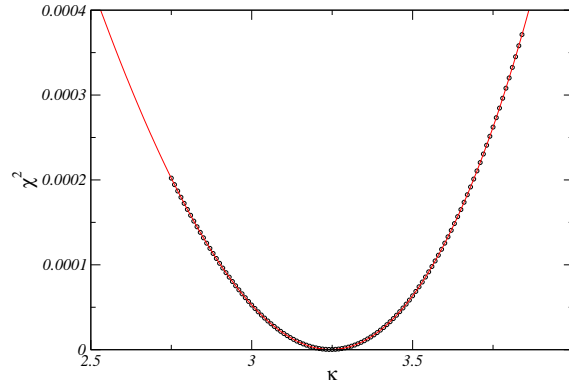


Figure 6. Mean square deviation $\overline{\chi^2}$ of the data at the largest lattice size $L = 512$ with the Schramm formula with respect to κ . The line is the interpolation with a polynomial of fourth order ($\overline{\chi^2} = 4.549281 \cdot 10^{-3} - 1.791033 \cdot 10^{-3} \kappa - 1.542747 \cdot 10^{-5} \kappa^2 - 1.270670 \cdot 10^{-5} \kappa^3 + 1.677021 \cdot 10^{-5} \kappa^4$) from which the minimum (3.245225) is then found numerically.

sizes larger than 128. Our final estimate is $\kappa \simeq 3.245(10)$. For small lattice sizes, κ takes a value close to $10/3$, i.e. the one of the pure 3-state Potts model, suggesting that a cross-over between the pure and the random fixed points occurs for $L^* \sim 100$. The data have been analyzed using other fractions of the lattice. The estimate of κ tends to increase when going from a large fraction of the lattice to a smaller one : 3.222 for an interpolation over the right half of the lattice, 3.255 for an interpolation over the right quarter of the lattice. A compromise has to be found between a too large fraction for which systematic deviations due to loops may be important and a

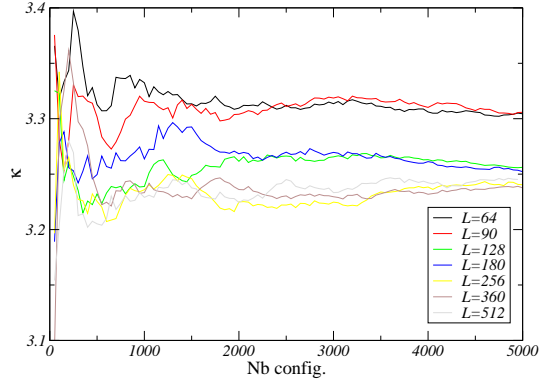


Figure 7. Estimates of κ when averaging over different numbers of random realizations.

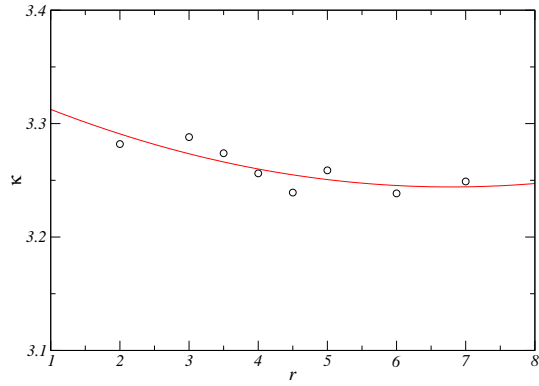


Figure 8. Estimates of κ for the lattice $L = 128$ with respect to the disorder amplitude r . The red curve is a quadratic interpolation which gives $\kappa(r = 1) = 3.312$ for the pure model (we independently obtained 3.325 for the pure model for this lattice size) and is intended to be only a guide to the eye. For small amplitude $r < r^* \simeq 4$, a cross-over with the pure value $10/3$ is observed. In the strong-disorder regime, κ seems stable.

too small fraction where lattice effects due to the boundaries are important. We note that, as expected, applying blindly the interpolation procedure to the whole lattice gives a significantly different estimate $\kappa \simeq 3.4$.

Like for the case of the interface length, we still need to show that a small variation of the disorder amplitude will not induce a large variation of κ so that κ would still be compatible with the estimate 3.18(2) obtained from the scaling of the interface length. Simulations have been made for other disorder amplitudes r but only for lattice sizes $L \leq 128$ and with 3000 disorder realizations. We have shown in the case of $r^* = 4$ that this lattice size still leads to a small overestimate of κ . Nevertheless, it should be possible to get a correct order of magnitude of the variation of κ with r . As shown on figure 8, this variation remains small. With a linear interpolation, one can estimate this variation to be $\Delta\kappa \sim 0.003$.

Conclusions

Although our interface presents loops due to the use of a simple right-turn tie-breaking algorithm, we have shown that they do not affect the scaling dimension of the interface in the pure case. However, systematic deviations between the left-passage probability and the Schramm formula were observed. The parameter κ can be nevertheless estimated by restricting the interpolation to the right part of the lattice. In the case of the 3-state random bond Potts model, we have measured numerically the SLE parameter κ in two independent ways. First, we estimated $\kappa \simeq 3.18(6)$ from the fractal dimension d_f of the interface. This value is compatible, although less accurate, with the estimate obtained in [15]. Then, the left-passage probability has been fitted with to the Schramm formula in the right third of the lattice. We obtained $\kappa \simeq 3.245(10)$. These two compatible values provide further evidences that chordal SLE_κ appropriately describes the geometric properties of interfaces between spin clusters in the 3-state random Potts model. We have not observed the large discrepancy observed in the case of the SOS model on a random substrate [17].

Acknowledgments

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