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HOW TO PROVE THAT SOME BERNOULLI CONVOLUTION HAS THE WEAK GIBBS PROPERTY

ÉRIC OLIVIER AND ALAIN THOMAS

ABSTRACT. In this paper we give an example of uniform convergence of the sequence of column vectors $\frac{A_1 \dots A_n V}{\|A_1 \dots A_n V\|}$, $A_i \in \{A, B, C\}$, A, B, C being some $(0, 1)$ -matrices of order 7 with much null entries, and V a fixed positive column vector. These matrices come from the study of the Bernoulli convolution in the base $\beta > 1$ such that $\beta^3 = 2\beta^2 - \beta + 1$, that is, the (continuous singular) probability distribution of the random variable $(\beta - 1) \sum_{n=1}^{\infty} \frac{\omega_n}{\beta^n}$ when the independent random variables ω_n take the values 0 and 1 with probability $\frac{1}{2}$. In the last section we deduce, from the uniform convergence of $\frac{A_1 \dots A_n V}{\|A_1 \dots A_n V\|}$, the Gibbs and the multifractal properties of this measure.

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INTRODUCTION

Given a finite set of nonnegative $d \times d$ matrices, let $\mathcal{A} := \{A(0), \dots, A(s-1)\}$, and a nonnegative d -dimensional column vector V , we associate to any $\omega \in \Omega := \{0, \dots, s-1\}^{\mathbb{N}}$ the sequence of column vectors

$$P_n(\omega, V) := \frac{A(\omega_1) \dots A(\omega_n) V}{\|A(\omega_1) \dots A(\omega_n) V\|} \quad (\|\cdot\| = \text{the norm-sum}).$$

This is not obligatory defined for any $\omega \in \Omega := \{0, \dots, s-1\}^{\mathbb{N}}$, but for

$$\omega \in \Omega_{\mathcal{A}, V} := \{\omega \in \Omega ; \forall n \in \mathbb{N}, A(\omega_1) \dots A(\omega_n) V \neq 0\}.$$

This set is compact because for fixed n , the set $\{\omega \in \Omega ; A(\omega_1) \dots A(\omega_n) V \neq 0\}$ is a finite union of cylinders of order n .

In §1 we prove a straightforward proposition, that may simplify the proof of the uniform convergence of $(P_n(\cdot, V))_{n \in \mathbb{N}}$.

Key words and phrases. Infinite products of nonnegative matrices, Gibbs properties, multifractal analysis of measures, Bernoulli convolutions.

In §2, the set \mathcal{A} we consider has three elements:

$$A = A(0) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = A(1) := \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = A(2) := \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and, using [26, Theorem 1.1], we prove the following

PROPOSITION 0.1. *If V has positive entries, $P_n(\cdot, V)$ converges uniformly on $\{0, 1, 2\}^{\mathbb{N}}$. The set of the indexes of the nonnull entries in the limit vector is $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 5, 6, 7\}$, $\{1, 2, 3, 5, 6\}$, $\{1, 2, 3, 5\}$, $\{1, 3, 4, 5\}$, $\{1, 3, 4\}$ or $\{2, 3, 5, 6, 7\}$.*

The aim of this example is to prove – in §3 – the weak Gibbs property [41] for the (continuous singular) Bernoulli convolution related to the numeration in the base $\beta > 1$ such that $\beta^3 = 2\beta^2 - \beta + 1$. It can be defined as the measure μ , supported by $[0, 1]$, such that $\mu(E) = \lim_{n \rightarrow \infty} \mu_1 * \dots * \mu_n(E)$ for any borelian $E \subset \mathbb{R}$, where

$$\mu_n = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{\frac{\beta-1}{\beta^n}}.$$

Using [6], we conclude that the multifractal formalism holds for this measure (see for instance [32, 5, 30], or the introduction of [35] for an interesting overview about the multifractal formalism).

1. UNIFORM CONVERGENCE OF $P_n(\cdot, V)$

The conditions of uniform convergence are quite different from the ones of simple convergence (see [25] and [39]). Let us show on a trivial example that the uniform convergence of $P_n(\cdot, V)$ is not equivalent to the pointwise convergence of $P_n(\cdot, V)$ to a continuous map.

EXAMPLE 1.1. Let $\mathcal{B} = \{B(0), B(1)\}$ with $B(0) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B(1) = \begin{pmatrix} 1 & 0 & 1 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

and let $V = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Then $\frac{B(\omega_1) \dots B(\omega_n)V}{\|B(\omega_1) \dots B(\omega_n)V\|}$ converges to the continuous map whose

constant value is the vector $\begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix}$; this convergence is not uniform because

$$\frac{B(0)^{n-1}B(1)V}{\|B(0)^{n-1}B(1)V\|} = \begin{pmatrix} 4/5 \\ 1/5 \\ 0 \end{pmatrix}.$$

In the following proposition $\mathcal{A} := \{A(0), \dots, A(s-1)\}$ is a finite set of nonnegative $d \times d$ matrices, V a d -dimensional nonnegative column vector, and we denote the cylinders of $\Omega_{\mathcal{A},V}$ by

$$[\omega_1 \dots \omega_n] := \{\xi \in \Omega_{\mathcal{A},V} ; \xi_1 = \omega_1, \dots, \xi_n = \omega_n\}.$$

PROPOSITION 1.2. $P_n(\cdot, V)$ converges uniformly on $\Omega_{\mathcal{A},V}$ if and only if

$$(1) \quad \forall \omega \in \Omega_{\mathcal{A},V}, \lim_{n \rightarrow \infty} \sup_{\substack{\xi \in [\omega_1 \dots \omega_n] \\ r, s \geq n}} \|P_r(\xi, V) - P_s(\xi, V)\| = 0.$$

Proof: The direct implication is obvious by the Cauchy criterion. Suppose now that (1) holds. Given $\varepsilon > 0$ one can associate to each $\omega \in \Omega_{\mathcal{A},V}$ some cylinder $[\omega_1 \dots \omega_n]$ such that any ξ in this cylinder and any $r, s \geq n$ satisfy

$$(2) \quad \|P_r(\xi, V) - P_s(\xi, V)\| \leq \varepsilon.$$

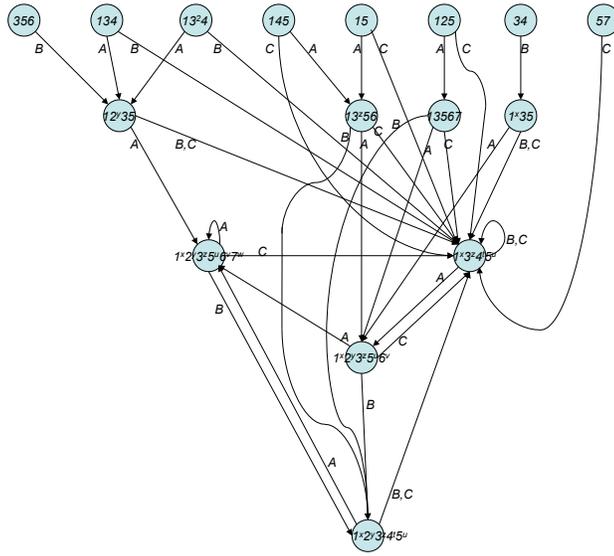
The compact set $\Omega_{\mathcal{A},V}$ is a finite union of such cylinders, let $[\omega_1^{(i)} \dots \omega_{n_i}^{(i)}]$ for $i = 1, \dots, N$. Hence (2) is true for any $\xi \in \Omega_{\mathcal{A},V}$ when $r, s \geq \max_i n_i$, and this proves that $P_n(\cdot, V)$ is uniformly Cauchy. ■

2. PROOF OF PROPOSITION 0.1

In this section \mathcal{A} is the set of the three matrices $A(0), A(1), A(2)$ defined in the introduction, that we call A, B, C respectively. We denote by $\{E_1, \dots, E_7\}$ the canonical basis of the set of the 7-dimensional column vectors and by U the row vector $(1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1)$.

We prove Proposition 0.1 in the following way: for any $\omega \in \{0, 1, 2\}^{\mathbb{N}}$ we search an increasing sequence $(n_k)_{k \in \mathbb{N}}$ such that the sequence of matrices $A'_k = A(\omega_{n_k+1}) \dots A(\omega_{n_{k+1}})$ satisfies the hypotheses of [26, Theorem 1.1]. For this, we use the equivalence classes in some graph associated to $\{A, B, C\}$; although this case is relatively simple, the method we use may be efficient in more complicated ones.

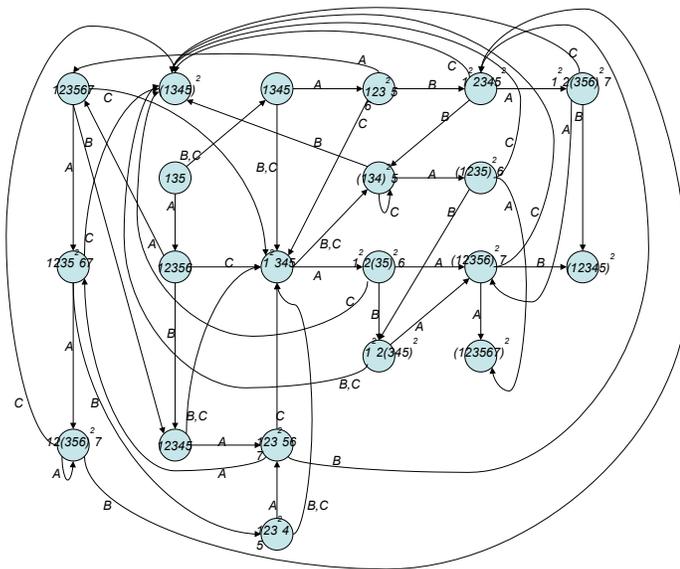
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The notations we use differ for the states of the third graph: for instance the state 123^245

represents all the columns $\begin{pmatrix} x \\ y \\ z \\ t \\ u \\ v \\ w \end{pmatrix}$ for $x, y, t, u \geq 1, z \geq 2$ and $v, w \geq 0$. The initial state

is 135 and the final states are $(12345)^2, (123567)^2, (1345)^2$.



In the sequel we denote the labels of the arrows by 0, 1 and 2 instead of A , B and C , to avoid confusion between words and products of matrices. Let \mathcal{W} be the set of the words $w = \xi_1 \dots \xi_n \in \{0, 1, 2\}^*$ such that the symmetric word $\xi_n \dots \xi_1$ is the label of a path in the third graph, from the initial state 135 to a final state. For instance $20^3 \in \mathcal{W}$ means that the path with label $0^3 2$ from the state 135 has final state $(1345)^2$, and that

$$A(2)A(0)^3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ z \\ t \\ u \\ 0 \\ 0 \end{pmatrix} \text{ with } x, z, t, u \geq 2.$$

We prove by the six following lemmas the existence of an integer $\kappa \in \mathbb{N}$ such that the matrices $A(w) := A(\xi_1) \dots A(\xi_n)$ – for any $n \in \mathbb{N}$ and any word $w = \xi_1 \dots \xi_n \in \{0, 1, 2\}^n$ that can be written $w = w_1 \dots w_\kappa$ with $w_i \in \mathcal{W}$ – satisfy the hypotheses (H_1) , (H_2) and (H^M) of [26, Theorem 1.1]. This is in part due to the existence of synchronizing words in the second graph.

LEMMA 2.1. *In the second graph, the words $w \in \{0, 1, 2\}^3$ are synchronizing from any state whose label contains the digits 1, 3 and 5, that is, the states of the second graph that are not states of the first.*

Proof: We remark that any path starting from a state whose label contains 1, 3 and 5, ends to such a state. Now any word $\xi_1 \xi_2 \xi_3 \in \{0, 1, 2\}^3$ has the factor $0^2, 1^2, 01$ or 2 . Since $0^2, 1^2, 01, 2$ are synchronizing from any state whose label contains 1, 3 and 5, $\xi_1 \xi_2 \xi_3$ also is. ■

LEMMA 2.2. *If some factor of a word $w \in \{0, 1, 2\}^*$ belong to \mathcal{W}^6 , the matrix $A(w)$ has the following property:*

(P): Denoting by $c_j(A(w))$ the set of the indexes of the nonnull entries in the j^{th} column of $A(w)$, either all the nonempty sets among the seven sets $c_1(A(w)), \dots, c_7(A(w))$, are equal, or they take two values c and c' such that $c \supseteq c' \cup \{1, 3, 5\}$.

Proof: Let us check first that it is sufficient to find a factor w' of w such that $A(w')$ has the property (P):

– On the one side the columns of the right product $A(w')X$ for $X \in \{A, B, C\}$ are nonnegative linear combination of columns of $A(w')$; hence if $c_j(A(w')X) \neq \emptyset$, there exists j' such that $c_j(A(w')X) = c_{j'}(A(w'))$ and, if $A(w')$ has the property P, $A(w')X$ also has.

– On the other side the columns of the left product $XA(w')$ are the products of X by the columns of $A(w')$ hence – since any arrow from a state containing 1, 3 and 5 ends to such a state – if $A(w')$ has the property (\mathcal{P}) , $XA(w')$ also has.

We evaluate the length of the words $w \in \{0, 1, 2\}^*$ in two ways: let us denote by $|w|$ the number of letters of w and by $\ell^*(w)$ the number of words $\zeta_i \in \{0^n\}_{n \in \mathbb{N}} \cup \{1\} \cup \{2^n\}_{n \in \mathbb{N}}$ such that $w = \zeta_1 \dots \zeta_{\ell^*(w)}$, without two consecutive 0^n nor two consecutive 2^n . Then any $w \in \mathcal{W}$ satisfies $|w| \geq 4$ and $\ell^*(w) \geq 3$, except $\ell^*(20^n) = \ell^*(12^n) = 2$ for $n \geq 3$.

Let $w \in \{0, 1, 2\}^*$ have the factor $w_1w_2w_3w_4w_5w_6 \in \mathcal{W}^6$. From the above remark, $\ell^*(w_4w_5w_6) \geq 5$. We deduce that, if w_2w_3 contains the digit 0, there exists some factor w' of $w_2w_3w_4w_5w_6$ for which the matrix $A(w')$ is one of the following: we make below the list of the matrices $A(w')$ for $w' \in \mathcal{W}_0$, where the set \mathcal{W}_0 – lexicographically ordered from the left to the right – is chosen in such a way that all the words $w \in \{0, 1, 2\}^*$ of length $\ell^*(w) = 5$ beginning by 01 or 02 has one element of \mathcal{W}_0 as prefix. Nevertheless we omit the word $w' = w''2$ if $w''1 \in \mathcal{W}_0$, because the columns of C are sums of columns of B and consequently $A(w''2)$ has the property (\mathcal{P}) if $A(w''1)$ has.

$$\begin{aligned}
 ABA^2 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & ABAB &= \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
 AB^2A &= \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & AB^3 &= \begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
 ABC &= \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & AC^nA &= \begin{pmatrix} n+1 & 1 & 0 & 0 & 0 & 0 & n \\ n & 1 & 0 & 0 & 0 & 0 & n-1 \\ n+1 & 0 & 0 & 0 & 0 & 0 & n+1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n+1 & 1 & 0 & 0 & 0 & 0 & n \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
 \end{aligned}$$

$$\begin{aligned}
AC^nBA &= \begin{pmatrix} 0 & 0 & 0 & n+1 & n+1 & 0 & 0 \\ 0 & 0 & 0 & n & n & 0 & 0 \\ 1 & 0 & 0 & n+1 & n+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n+1 & n+1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & AC^nB^2A &= \begin{pmatrix} n+2 & 0 & 0 & 0 & 0 & 0 & n+1 \\ n+1 & 0 & 0 & 0 & 0 & 0 & n \\ n+1 & 0 & 0 & 1 & 1 & 0 & n+1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n+2 & 0 & 0 & 0 & 0 & 0 & n+1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
AC^nB^3 &= \begin{pmatrix} n+1 & 0 & n+2 & 1 & 0 & 0 & 0 \\ n & 0 & n+1 & 1 & 0 & 0 & 0 \\ n+2 & 0 & n+1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n+1 & 0 & n+2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & AC^nBC &= \begin{pmatrix} n+1 & 0 & 0 & 1 & 1 & 0 & n+1 \\ n & 0 & 0 & 1 & 1 & 0 & n \\ n+2 & 0 & 0 & 0 & 1 & 0 & n+2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 0 & 1 & 1 & 0 & n+1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

($n \in \mathbb{N}$).

In case w_2w_3 do not contain the digit 0, as seen on the third graph the words w_2 and w_3 belong to $\{1\} \times \{1,2\}^n$ with $n \geq 3$. Hence either $w_2 = 1^4$ or w_2w_3 has a factor in $\{2\} \times \{1\} \times \{1,2\}^3$, so w_2w_3 has a factor $w' \in \mathcal{W}_1 = \{1^4, 21^32, 21^22, 2121, 212^2\}$. This is the list of the matrices $A(w')$ for $w' \in \mathcal{W}_1$, where we omit $A(21^32)$ and $A(2122)$ because $A(1^4)$ and $A(2121)$ are in the list:

$$\begin{aligned}
B^4 &= \begin{pmatrix} 1 & 0 & 1 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & CB^2C &= \begin{pmatrix} 1 & 0 & 0 & 2 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 3 & 0 & 1 \\ 1 & 0 & 0 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
CBCB &= \begin{pmatrix} 1 & 0 & 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

According to Lemma 2.1, for any $X, Y, Z \in \{A, B, C\}$ the product of XYZ by each of these matrices has the property (\mathcal{P}) , hence $A(w)$ also has. ■

LEMMA 2.3. *There exists an integer $\kappa \geq 7$ such that, for any $w \in \{0, 1, 2\}^*$ with a factor in \mathcal{W}^κ , the matrix $A(w)$ has the property (H_2) .*

Proof: Let $w = \xi_1 \dots \xi_n \in \{0, 1, 2\}^n$ be a word with a factor $w_1 \dots w_\kappa \in \mathcal{W}^\kappa$: $w = w'w_1 \dots w_\kappa w''$. We can apply Lemma 2.2 to the word $m = w_{\kappa-5} \dots w_\kappa w''$: the non empty sets, among the $c_j(A(m))$ for $j = 1, \dots, 7$, are equal or they take two values, c for $j \in J$

and c' for $j' \in J'$ with $c \supseteq c' \cup \{1, 3, 5\}$. There is no problem if all the non-empty $c_j(A(w))$ are equal, so we suppose they take two values and consequently the $c_j(A(m))$ also do. For $j \in J$, the obvious property of the final states in the third graph implies that the values of the nonnull entries in the j^{th} column of $A(w)$ are at least $2^{\kappa-6}$. For $j' \in J'$, $A(\xi_4 \dots \xi_n)E_{j'}$ is a state of the first graph, otherwise by Lemma 2.1 the $c_j(A(w))$ should be equal for any $j \in J \cup J'$. Consequently $A(\xi_4 \dots \xi_n)E_{j'}$ has entries at most 2, and $A(w)E_{j'}$ has bounded entries. Choosing κ large enough, $A(w)$ has the property (H_2) . ■

It remains to prove that the matrices $A(w)$, for w in some set specified later, satisfy the condition (H^M) . We first notice that $A(w)$ satisfy (H^3) if w is one of the words $010^{4(n+1)}, 1^20^{4(n+1)}, 210^{4(n+1)}, 20^{4(n+1)}, 0^22^{n+2}, 0^2102^{n+2}, 10102^{n+2}, 20102^{n+2}, 1^202^{n+2}, 2102^{n+2}, 202^{n+2}, 12^{n+2}$, that we denote by $w_1(n), \dots, w_{12}(n)$ for any nonnegative integer n : indeed the $A_{w_i(n)}$ are

$$\begin{aligned}
 ABA^{4(n+1)} &= \begin{pmatrix} n+1 & 0 & 1 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 0 & 1 & 1 & 0 & 0 \\ n+2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 1 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 1 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, & B^2A^{4(n+1)} &= \begin{pmatrix} n+2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n+2 & 0 & 1 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 1 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
 CBA^{4(n+1)} &= \begin{pmatrix} 2n+2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 1 & 0 & 0 & 0 & 0 \\ n+2 & 0 & 1 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & CA^{4(n+1)} &= \begin{pmatrix} 2n+3 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n+2 & 0 & 0 & 0 & 0 & 0 & 1 \\ n+1 & 0 & 0 & 1 & 1 & 0 & 0 \\ n+1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
 A^2C^{n+2} &= \begin{pmatrix} 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 1 & 0 & 0 & 0 & n+1 & 0 & 1 \end{pmatrix}, & A^2BAC^{n+2} &= \begin{pmatrix} 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2n+5 & 0 & 1 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+4 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \end{pmatrix}, \\
 BABAC^{n+2} &= \begin{pmatrix} 1 & 0 & 0 & 1 & 2n+5 & 0 & 1 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2n+5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & CABAC^{n+2} &= \begin{pmatrix} 0 & 0 & 0 & 2 & 2n+7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+4 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
 \end{aligned}$$

$$\begin{aligned}
B^2AC^{n+2} &= \begin{pmatrix} 2 & 0 & 0 & 0 & 2n+4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2n+5 & 0 & 1 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & CBAC^{n+2} &= \begin{pmatrix} 0 & 0 & 0 & 2 & 2n+6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2n+5 & 0 & 1 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
CAC^{n+2} &= \begin{pmatrix} 2 & 0 & 0 & 0 & 2n+4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & BC^{n+2} &= \begin{pmatrix} 1 & 0 & 0 & 1 & 2n+3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 1 & 0 & 0 & 0 & n+1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

The matrix $A((100)^{n+2}) = (BA^2)^{n+2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ also do for any non-

negative integer n .

In the following lemma we extend this property to the matrices that are products of a bounded number of matrices $A, B, C, A(w_i(n))$ and $A((100)^{n+2})$.

LEMMA 2.4. *Let \mathcal{W}_{k_1, k_2} be the set of the words*

$$(3) \quad w = m_0 w_{i_1}(n_1) m_1 w_{i_2}(n_2) \dots w_{i_k}(n_k) m_k$$

with the conditions that $k \leq k_1$, that $m_0 m_1 \dots m_k$ is the concatenation of at most k_2 elements of the alphabet $\{0, 1, 2\} \cup \{(100)^{n+2}\}_{n \in \mathbb{N}}$ and that $n_i \geq 0$. Then for $w \in \mathcal{W}_{k_1, k_2}$, each nonnull entry in the j^{th} column of the matrix $A(w)$ has the form $P(n_1, \dots, n_{k(j)})$ where $k(j) \in \{1, \dots, k\}$ and P is a polynomial with positive coefficients and degree 1 in each variable, or $k(j) = 0$ and $P = \text{constant}$. Moreover $A(w)$ satisfies (H^M) for some constant $M = M(k_1, k_2)$.

Proof: Let $j \in \{1, \dots, 7\}$, one consider the path – in the first graph and then in the second – with initial state j , whose label is the word w read from the right to the left. Let e_i be the final state of the subpath with label $s_i = m_i w_{i_{i+1}}(n_{i+1}) \dots w_{i_k}(n_k) m_k$ (from the right to the left). Notice that only the first column of $A(w_i(n))$, for $i \leq 4$, depends on n , and only the fifth do for $i > 4$. Consequently, if $i_i \leq 4$ and the state e_i do not contain the letter 1, or if $i_i > 4$ and this state do not contain the letter 5, the column $A(w_{i_i}(n_{i_i})) A(s_i) E_j$ do not depend on n_{i_i} and $A(w) E_j$ no more do .

Conversely suppose that ι is the greatest integer such that $i_\iota \leq 4$ and the state e_ι contains the letter 1, or such that $i_\iota > 4$ and this state contains the letter 5. The nonnull entries of $A(w_{i_\iota}(n_\iota)s_\iota)E_j$ have the form $an_\iota + b$ with a and b positive, in particular the entries of indexes 1, 3, 5 have this form. We multiply this column vector by $A(m_{\iota-1}), A(w_{i_{\iota-1}}(n_{\iota-1})), \dots, A(m_0)$ successively; this leads to a column vector whose nonnull entries have the form $a_i n_i + b_i$ for any $i \in \{1, \dots, \iota\}$, with positive a_i and b_i because, in the second graph, each arrow whose initial state contains the digits 1, 3, 5 ends to such a state.

Clearly, a map $f : \mathbb{R}^\iota \rightarrow \mathbb{R}$ which is a polynomial of degree 1 with positive coefficients in each of the variables, is a polynomial with 2^ι positive coefficients, that is,

$$f(X_1, \dots, X_\iota) = a_0 + a_1 X_1 + \dots + a_\iota X_\iota + a_{\iota+1} X_1 X_2 + a_{\iota+2} X_1 X_3 + \dots + a_{2^\iota-1} X_1 \dots X_\iota .$$

In our case, the coefficients of the polynomial belong to some finite set because they only depend on the (at most) $k_1 + k_2$ elements of the decomposition (3) of the word w in letters 0, 1, 2 and words $(001)^n, w_i(n)$. The ratio between two polynomials (with positive coefficients and with the same nonnegative variables) being bounded by the ratio of the greatest coefficient of the first by the lowest of the second, we conclude that the ratio between two nonnull entries of the column $A(w)E_j$ is bounded by some constant that only depends on k_1 and k_2 . ■

LEMMA 2.5. *Let $w \in \mathcal{W}^\kappa$ and let s be the suffix of a word of \mathcal{W}^κ . Then*

- (i) $w \in \mathcal{W}_{2\kappa, 21\kappa}$,
- (ii) $0^2 s$ (if s do not begin by 0) and $2s$ (if s do not begin by 2) belong to $\mathcal{W}_{2\kappa, 21\kappa+4}$,
- (iii) $ws \in \mathcal{W}_{4\kappa, 42\kappa+8}$.

Proof: (i) We consider first a word $w \in \mathcal{W}$, that is, w read from the right to the left is the label of a path of the third graph from the initial state 135 to a final state. We distinguish the four cases: either it has one succession of at least four arrows from $12(356)^2 7$ to itself, or one succession of at least two arrows from $(134)^2 5$ to itself, or it has successively the first and the second subpath, or it has no such subpath. In the first case the arrows from $12(356)^2 7$ to itself are followed by one arrow with label 2, or successively by one arrow with label 1 – without reaching the state $(134)^2 5$ – and one other arrow; so there is a subpath whose label is the symmetrical of $w_i(n)$, $1 \leq i \leq 4$. In the same way, in the second case there exists a subpath whose label is the symmetrical of $w_i(n)$, $5 \leq i \leq 12$. Now if w do not have such subpaths, it cannot be the concatenation of more than 21 words in $\{0, 1, 2\} \cup \{(100)^{n+2}\}_{n \in \mathbb{N}}$. This proves that $w \in \mathcal{W}_{2, 21}$. Of course the concatenation of κ words in $\mathcal{W}_{2, 21}$ gives a word in $\mathcal{W}_{2\kappa, 21\kappa}$.

(ii) Let s be a suffix of a word $w \in \mathcal{W}^\kappa$; we decompose w in the form (3). Suppose that s do not begin by 0. There is no problem if $s = mw_{i_j}(n_j)m_j \dots w_{i_k}(n_k)m_k$ where m is a suffix of m_{j-1} . If not, s begins by some strict and non empty suffix of some $w_{i_j}(n_j)$, more precisely this suffix is $10^{4(n+1)}$, 2^{n+2} or 102^{n+2} for some $n \geq 0$, or it is 2. Hence 0^2s begins with $0w_1(n)$, $w_5(n)$, $w_6(n)$ or 0^22m_j . In this last case we have $|0^22m_j| = |m_j| + 3$, and in all the cases $0^2s \in \mathcal{W}_{2\kappa, 21\kappa+3}$. The proof is similar for $2s$ if s do not begin by 2: if s begins by some strict and non empty suffix of some $w_{i_j}(n_j)$, this suffix is $s' = 10^{4(n+1)}$, $0^{4(n+1)}$, 02^{n+2} , 102^{n+2} , 0102^{n+2} , 0^3 , 0^2 or 0 and, except in the three last cases, $2s'$ is some of the words $w_i(n)$.

(iii) Let now $w, w' \in \mathcal{W}^\kappa$ and let s be a suffix of w' , we decompose w and w' in the form (3) and deduce the following decomposition of ws :

$$ws = m_0w_{i_1}(n_1)m_1w_{i_2}(n_2) \dots w_{i_k}(n_k)m_k s' m'_j w_{i'_{j+1}}(n'_{j+1}) \dots w_{i'_{k'}}(n'_{k'}) m'_{k'}$$

where $j \in \{1, \dots, k'\}$ and s' is a – possibly empty of full – suffix of $m'_{j-1}w_{i'_j}(n'_j)$. This suffix has the form (3) except in case it is a strict suffix of $w_{i'_j}(n'_j)$, and in this case it is sufficient to find a decomposition of $w_{i_k}(n_k)m_k s'$ in the form (3). If the word s' has length at most 4, it ends by 0^4 or 2^2 hence it can be completed in order to obtain a factor of $w_{i_k}(n_k)m_k s'$ of the form $w_i(n)$. We distinguish the cases whenever this factor is disjoint or not from the word $w_{i_k}(n_k)$, and the cases whenever it is or not disjoint from its prefix $w_{i_k}(0)$ (which has length at most 6). The decomposition of $w_{i_k}(n_k)m_k s'$ we obtain is the following, where ξ and ξ' belong to $\{\emptyset, 0, 0^2, 0^3, 2\}$:

$$w_{i_k}(n_k)m_k s' = \begin{cases} w_{i_k}(n_k)m_k s' & (\text{in case } |s'| \leq 3) \\ w_{i_k}(n_k)m w_i(n)\xi & (m \text{ prefix of } m_k) \\ w_{i_k}(n')\xi' w_i(n)\xi & (0 \leq n' < n) \\ m w_i(n)\xi & (m \text{ strict prefix of } w_{i_k}(0)). \end{cases}$$

This proves (iii), since $|s'| \leq 3$ in the first case, $|m| \leq 5$ in the last case, and $|\xi|, |\xi'| \leq 3$. ■

LEMMA 2.6. *The set of the matrices $A(ws)$ for $w \in \mathcal{W}^\kappa$ and s – possibly empty – suffix of a word in \mathcal{W}^κ , has the properties (H_1) , (H_2) and (H^M) .*

Proof: From Lemma 2.2, $A(ws)$ has the property (H_1) because $ws = w'w''$ with $w' \in \mathcal{W}^6$ and $w'' \in \{0, 1, 2\}^*$. It has the property (H_2) from Lemma 2.3 and, from Lemmas 2.4 and 2.5(iii), the set of such matrices $A(ws)$ has the property (H^M) . ■

LEMMA 2.7. *For any word $w \in \{0, 1, 2\}^*$ there exist an integer $\alpha \geq 1$, ζ_1 – possibly empty – strict suffix of a word in \mathcal{W} and $\zeta_2, \dots, \zeta_\alpha \in \mathcal{W}$ such that $w = \zeta_1 \dots \zeta_\alpha$.*

Proof: Reading the word w from the right to the left, we may go many times from the initial state to a final state in the third graph by some paths and we call $\zeta_\alpha, \zeta_{\alpha-1}, \dots, \zeta_2$

the symmetricals of their labels; finally we go from the initial state to a non final state by some path and we call ζ_1 the symmetrical of its label. ■

LEMMA 2.8. *For any sequence $\omega \in \{0, 1, 2\}^{\mathbb{N}}$ that is not eventually 0 nor eventually 2 nor eventually 100 there exist ζ_1 , – possibly empty – strict suffix of a word in \mathcal{W} , and $\zeta_2, \zeta_3, \dots \in \mathcal{W}$ such that*

$$\omega_1\omega_2\omega_3 \cdots = \zeta_1\zeta_2\zeta_3 \dots .$$

Proof: By Lemma 2.7,

$$\omega_1 \dots \omega_n = \zeta_1(n) \dots \zeta_{\alpha(n)}(n)$$

where $\zeta_i(n) \in \mathcal{W}$, except $\zeta_1(n)$ which is a possibly empty strict suffix of a word in \mathcal{W} . Let us prove – for fixed k – that the word $\zeta_k(n)$ can take only a finite number of values when $n \in \mathbb{N}$. By hypothesis there exists one unique sequence of words $\xi_i \in \mathcal{A}^* = \{0^n\}_{n \in \mathbb{N}} \cup \{1\} \cup \{2^n\}_{n \in \mathbb{N}} \cup \{(100)^n\}_{n \in \mathbb{N}}$ such that

$$(4) \quad \omega_1\omega_2\omega_3 \cdots = \xi_1\xi_2\xi_3 \dots$$

and such that for any $i, n, n' \in \mathbb{N}$,

$$(5) \quad \begin{aligned} \xi_i = 0^n &\Rightarrow \xi_{i+1} \neq 0^{n'}, & \xi_i = 2^n &\Rightarrow \xi_{i+1} \neq 2^{n'}, & \xi_i = (100)^n &\Rightarrow \xi_{i+1} \neq (100)^{n'}, \\ \xi_i = 1 &\Rightarrow \xi_{i+1} \neq 0^{n'+1}. \end{aligned}$$

In the same way each word $\zeta_i(n)$ can be written as a concatenation of words of \mathcal{A}^* that satisfy (5); we see on the third graph that the number of such words is at most 11. We deduce the decomposition of $\zeta_1(n) \dots \zeta_k(n)$ – grouping together if necessary some suffix of each $\zeta_i(n)$ with some prefix of $\zeta_{i+1}(n)$ – in at most $11k$ words of \mathcal{A}^* . Since the decomposition (4) is unique, $\zeta_1(n) \dots \zeta_k(n)$ is a prefix of $\xi_1 \dots \xi_{11k}$.

For fixed k , the word $\zeta_k(n)$ – for $n \in \mathbb{N}$ – belongs to the finite set of the factors of $\xi_1 \dots \xi_{11k}$ hence it takes infinitely many times the same value. So we can define by induction the sequence of words ζ_1, ζ_2, \dots : at the k^{th} step we define ζ_k as a word such that $\zeta_1(n) = \zeta_1$, $\zeta_2(n) = \zeta_2$, \dots , $\zeta_k(n) = \zeta_k$ for infinitely many n . ■

Proof of Proposition 0.1: We use Proposition 1.2, that is, given $\omega \in \Omega_{\mathcal{A},V}$ and $\varepsilon > 0$ we prove the existence of $N(\omega, \varepsilon)$ such that

$$(6) \quad n \geq N(\omega, \varepsilon), \quad \xi \in [\omega_1 \dots \omega_n], \quad r, s \geq n \Rightarrow \|P_r(\xi, V) - P_s(\xi, V)\| \leq \varepsilon.$$

Suppose first that the sequence ω is not eventually 0 nor eventually 2 nor eventually 100. Lemma 2.8 implies there exist ζ_1 , possibly empty strict suffix of a word in \mathcal{W}^κ , and $\zeta_2, \zeta_3, \dots \in \mathcal{W}^\kappa$ such that

$$\omega_1\omega_2\omega_3 \cdots = \zeta_1\zeta_2\zeta_3 \dots .$$

According to Lemma 2.6 the matrices $A(\zeta_i)$ satisfy (H_1) , (H_2) and (H^M) for any $i \geq 2$. Since the sequence ω is fixed, one can use the obvious fact that $A(\zeta_1\zeta_2)$ satisfies $(H^{M'})$ when M' is the maximum of M and the ratio of the greatest nonnull entry in $A(\zeta_1\zeta_2)$ by the lowest one. It also satisfies (H_1) and (H_2) by Lemmas 2.2 and 2.3.

We do not exactly apply [26, Corollary 1.2] to the product $A(\zeta_1\zeta_2)A(\zeta_3)\dots A(\zeta_k)$ and the column vectors $A(\xi_{n+1})\dots A(\xi_r)V$ and $A(\xi_{n+1})\dots A(\xi_s)V$, $n = |\zeta_1| + \dots + |\zeta_k|$, because the ratio between two nonnull entries in these column vectors is not necessarily bounded for $r \geq n$, for instance in the case $\xi_{n+1} = \xi_{n+2} = \dots = 0$. There exist – from Lemma 2.7 – $\nu \geq 1$, w_1 – possibly empty – strict suffix of a word in \mathcal{W}^κ and $w_2, \dots, w_\nu \in \mathcal{W}^\kappa$ (if $\nu \neq 1$) such that

$$(7) \quad \xi_{n+1}\dots\xi_r = w_1\dots w_\nu .$$

By Lemma 2.6 $A(\zeta_k w_1)A(w_2)\dots A(w_\nu)$ is a product of matrices which satisfy the conditions (H_1) , (H_2) and (H^M) ; this product is equal to $A(\xi_{n'+1})\dots A(\xi_r)$ for $n' = |\zeta_1| + \dots + |\zeta_{k-1}|$. The square matrix V' whose all columns are equal to V satisfy obviously (H_1) , (H_2) and $(H^{M''})$ when M'' is the maximum of M and the ratio of the greatest entry in V by the lowest one. By [26, Lemma 1.3] the matrix $A(\xi_{n'+1})\dots A(\xi_r)V'$ satisfies $(H^{2M''d})$, hence the ratio between two nonnull entries of the column vector $A(\xi_{n'+1})\dots A(\xi_r)V$ is bounded for any $\xi \in [\omega_1\dots\omega_n]$ and $r \geq n$. We can use [26, Corollary 1.2] for the product $A(\zeta_1\zeta_2)A(\zeta_3)\dots A(\zeta_{k-1})$ and each of the column vectors $A(\xi_{n'+1})\dots A(\xi_r)V$ and $A(\xi_{n'+1})\dots A(\xi_s)V$. Since – by Lemma 2.1 – the emplacement of the nonnull entries in both columns vectors is the same, we conclude that (6) holds for $n = |\zeta_1| + \dots + |\zeta_k|$ when k is large enough.

Suppose now there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $\omega_{n_0+1}\omega_{n_0+2}\dots = \bar{0}$ or $\bar{2}$ or $\overline{100}$; as previously we consider all the sequences $\xi \in [\omega_1\dots\omega_n]$ and we choose n in order that (6) holds. We treat the first case, the second being similar, and the third trivial because in this case $\|P_r(\xi, V) - P_s(\xi, V)\| = 0$ for any $r, s \geq n_0 + 6$. We use the same decomposition $\xi_{n+1}\dots\xi_r = w_1\dots w_\nu$ as in (7). Let 0^i , $i \geq 0$, be the greatest prefix of $\xi_{n+1}\dots\xi_r$ that contains only the letter 0. If $\nu \geq 2$, $w_2 \in \mathcal{W}^\kappa$ contains some other letters than 0 hence 0^i is a strict prefix of $w_1 w_2$. We suppose $n \geq n_0 + 2$ and we make the Euclidean division

$$(8) \quad n + i - n_0 - 2 = 4k + k', \quad k \in \mathbb{N}, \quad k' \in \{0, 1, 2, 3\},$$

so there exists a suffix s of w_1 (if $\nu = 1$) or $w_1 w_2$ (if $\nu \geq 2$) that begins by 1 or 2 (if non empty) and such that

$$\xi_{n_0+1}\dots\xi_{n_0+4k} = 0^{4k} \quad \text{and} \quad \xi_{n_0+4k+1}\dots\xi_r = 0^{k'+2}\xi_{n+i+1}\dots\xi_r = \begin{cases} 0^{k'+2}s & \text{if } \nu \leq 2 \\ 0^{k'+2}s w_3 \dots w_\nu & \text{if } \nu \geq 3. \end{cases}$$

From Lemma 2.5(ii) one has $0^{k'+2}s \in \mathcal{W}_{4\kappa, 42\kappa+7}$, $0^{k'+2}sw_3 \in \mathcal{W}_{6\kappa, 63\kappa+7}$ and, from Lemma 2.5(i), $w_i \in \mathcal{W}_{2\kappa, 21\kappa}$ for any $i \geq 2$. The matrices $A(0^{k'+2}s)$, $A(0^{k'+2}sw_3)$ and $A(w_i)$ for $i \geq 2$ satisfy the condition (H^M) by Lemma 2.4. The second and the third matrix also satisfy (H_1) and (H_2) from Lemmas 2.2 and 2.3; so by [26, Lemma 1.3] the matrix $A(\xi_{n_0+4k+1}) \dots A(\xi_r)V'$ satisfies $(H^{2M''d})$ in all the cases, where V' and M'' are defined as above, and the nonnull entries in the column vector $X = X(\xi, r) := A(\xi_{n_0+4k+1}) \dots A(\xi_r)V$ have bounded ratios independently on $\xi \in [\omega_1 \dots \omega_n]$ and $r \geq n$.

Consequently, denoting by $x_1 = x_1(\xi, r)$ the first entry – obviously nonnull – of X we have

$$A(\xi_{n_0+1} \dots \xi_r)V = A^{4k}X = x_1 \begin{pmatrix} 1 \\ k \\ k \\ 0 \\ k \\ k \\ k \end{pmatrix} + O(x_1).$$

Hence $\|A(\xi_{n_0+1} \dots \xi_r)V\|$, which is equal to $UA(\xi_1 \dots \xi_r)V$, is $x_1(5k+1) + O(x_1)$ and, denoting by σ the shift-map on $\{0, 1, 2\}^{\mathbb{N}}$,

$$P_{r-n_0}(\sigma^{n_0}\xi, V) = \frac{A(\xi_{n_0+1} \dots \xi_r)V}{\|A(\xi_{n_0+1} \dots \xi_r)V\|} = \frac{1}{5} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + O\left(\frac{1}{k}\right).$$

Since $P_r(\xi, V) = P_{n_0}(\omega, P_{r-n_0}(\sigma^{n_0}\xi, V))$ and since the map $P_{n_0}(\omega, \cdot)$ is continuous and

do not vanish at the point $\frac{1}{5} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ – see the first and the second graph – we deduce

$$P_r(\xi, V) = P_{n_0} \left(\omega, \frac{1}{5} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right) + o(1)$$

for $\xi \in [\omega_1 \dots \omega_n]$ and $r \geq n$, where the term $o(1)$ tends to 0 when $n \rightarrow \infty$ because, from (8), $4k \geq n - n_0 - 5$. It follows that (6) holds for n large enough.

It remains to specify the form of the limit vector. In case the sequence ω is not eventually 0 nor eventually 2 nor eventually 100 we use the notations of [26, Theorem 1.1]: by [26, Corollary 1.2] the limit vector is V_1 and, by the second and the fourth assertion in [26, Theorem 1.1], $c(V_1) = c(P'_n V)$ for n large enough. Using Lemmas 2.2, 2.8 and the second graph, $c(P'_n V)$ for n large enough is $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 5, 6, 7\}$, $\{1, 2, 3, 5, 6\}$ or $\{1, 3, 4, 5\}$. Suppose now $\omega = \overline{0, 2}$ or $\overline{100}$; by computation $c\left(\lim_{n \rightarrow \infty} P_n(\omega, V)\right) = \{2, 3, 5, 6, 7\}$, $\{1, 3, 4\}$ or $\{1, 2, 3, 4, 5\}$ respectively. Finally if $\sigma^n \omega \overline{0, 2}$ or $\overline{100}$ for some $n \in \mathbb{N}$ but not for $n = 0$, then – in view of the first and the second graph – $c\left(\lim_{n \rightarrow \infty} P_n(\omega, V)\right) = \{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 5, 6, 7\}$, $\{1, 2, 3, 5, 6\}$, $\{1, 2, 3, 5\}$ or $\{1, 3, 4, 5\}$. ■

3. APPLICATION TO SOME BERNOULLI CONVOLUTION

As we see in §3.4, the matrices we use in the previous section come from the study of some continuous singular measure, which is a Bernoulli convolution. So, before computing the values of this measure on certain intervals, we give the general definitions about the Bernoulli convolutions, the Gibbs measures and the multifractal formalism.

3.1. The Bernoulli convolutions. The probability measures defined by Bernoulli convolutions have been studied from the 30th, see [4, 15, 16, 7], and [29] for an overview on the subject. They also have been considered in view of their application to fractal geometry [21, 20, 19, 33, 18] and ergodic theory [38, 37].

Given a real $\beta > 1$ and a parameter vector $p = (p_0, \dots, p_{s-1})$, $p_i > 0$, $\sum_i p_i = 1$, one calls the *Bernoulli convolution* $\mu_{\beta, p}$ the infinite product (in the sense of the pointwise convergence on the set of the borelian subsets of \mathbb{R}) of the discrete measures

$$p_0 \delta_{\frac{0 \cdot \alpha}{\beta^n}} + \dots + p_{s-1} \delta_{\frac{(s-1) \cdot \alpha}{\beta^n}}$$

for $n = 1, 2, \dots$, where $\alpha = \frac{\beta - 1}{s - 1}$. In other words this measure is defined – for any Borelian $E \subset \mathbb{R}$ – by

$$\mu_{\beta, p}(E) := P \left(\left\{ \omega \in \{0, \dots, s-1\}^{\mathbb{N}} ; \alpha \sum_{n=1}^{\infty} \frac{\omega_n}{\beta^n} \in E \right\} \right)$$

where P is the product-probability defined on $\{0, \dots, s-1\}^{\mathbb{N}}$ by $P[\omega_1 \dots \omega_n] = p_{\omega_1} \dots p_{\omega_n}$ ($[\omega_1 \dots \omega_n]$ is the set of the sequences whose first terms are $\omega_1, \dots, \omega_n$). The measure $\mu_{\beta, p}$ is also the unique probability measure with bounded support that satisfies the self-similarity relation (see[29]):

$$\mu_{\beta, p} = \sum_{i=0}^{s-1} p_i \cdot (\mu_{\beta, p} \circ S_i^{-1}) \quad \text{where } S_i(x) := \frac{x + i\alpha}{\beta}.$$

The Bernoulli convolutions are non-atomic probability measures of support included in $[0, 1]$. From [15], they are either absolutely continuous or purely singular; moreover, according to [4], they are singular when $p = \left(\frac{1}{2}, \frac{1}{2}\right)$ and β is a *Pisot number*, that is, an algebraic integer whose conjugates have modulus less than 1. Some open questions are to know if $\mu_{\beta,p}$ has the weak Gibbs property in this case (see §3.3), and if $\mu_{\beta,p}$ can be singular with support $[0, 1]$ when β is not Pisot.

The Bernoulli convolution $\mu_{\beta,p}$ can also be seen as the probability distribution of the first entry in some infinite product of 2×2 stochastic matrices, let $\lim_{n \rightarrow \infty} S_{\omega_n} \dots S_{\omega_1}$ (easily computable, [40, §1.2]), when the sequence (ω_n) belongs to $\{0, \dots, s-1\}^{\mathbb{N}}$ endowed with the product probability P . More precisely, one obtain $\mu_{\beta,p}$ by setting $S_k = \begin{pmatrix} x_k & 1-x_k \\ y_k & 1-y_k \end{pmatrix}$ with $y_k = \frac{k}{s-1} \left(1 - \frac{1}{\beta}\right)$ and $x_k = y_k + \frac{1}{\beta}$, $0 \leq k \leq s-1$. Mukherjea, Nakassis and Ratti [22] computed the (piecewise polynomial) density of $\mu_{\beta,p}$ in the case $p_0 = \dots = p_{s-1} = \frac{1}{s}$ and $\beta = \sqrt[m]{s}$ for some positive integer m .

3.2. The multifractal analysis. By a simple application of the fixed point theorem, Hutchinson [14] shows that, in any complete metric space, given some contraction maps S_1, \dots, S_N , there exists a unique closed bounded nonempty set K such that

$$K = \bigcup_{i=1}^N S_i(K).$$

This set, called the *attractor of the S_i* , is compact. The most simple example is the Cantor set, with $S_1(x) = \frac{x}{3}$ and $S_2(x) = \frac{x+2}{3}$. But in the example we study in §3.4, K is simply the interval $[0, 1]$; so we are more concerned by the second result of Hutchinson: given some contraction maps S_1, \dots, S_N and some positive reals ρ_1, \dots, ρ_n whose sum is 1, there exists a unique Borel probability measure μ such that

$$\mu = \sum_{i=1}^N \rho_i \cdot (\mu \circ S_i^{-1}).$$

This measure has for support the attractor K of the S_i .

The multifractal analysis is concerned by the *local dimensions*, defined by

$$\dim(x) := \lim_{r \rightarrow 0} \log_r(\mu([x-r, x+r])),$$

where $\log_r(\cdot) := \frac{\log(\cdot)}{\log(r)}$, when this limit exists. One can also define

$$\dim_{\inf}(x) := \liminf_{r \rightarrow 0} \log_r(\mu([x-r, x+r])) \quad \text{and} \quad \dim_{\sup}(x) := \limsup_{r \rightarrow 0} \log_r(\mu([x-r, x+r])).$$

It turns out that, at least in the case where the S_i are affine contractions of \mathbb{R} , the "fat level-sets"

$$F_\alpha := \{x ; d_{\inf}(x) \leq \alpha\} \quad \text{and} \quad G_\alpha := \{x ; d_{\sup}(x) \leq \alpha\}$$

are (non-closed) attractors in the sense that $F_\alpha = \cup_{i=1}^N S_i(F_\alpha)$ and $G_\alpha = \cup_{i=1}^N S_i(G_\alpha)$.

The *level-set* himself is defined by

$$E_\alpha := \{x ; d(x) \text{ exists and } d(x) = \alpha\}.$$

It is likely that E_α has the same Hausdorff dimension as F_α or G_α (see [6, Theorem 3.4 and Appendix B]).

The multifractal analysis studies the relation between the *singularity spectrum* and the *scale spectrum* (or *L^q -spectrum*), let τ_{sing} and τ_{scale} respectively. They are defined by

$$\begin{aligned} \tau_{\text{sing}}(\alpha) &:= \text{H-dim}(E_\alpha) \quad (\text{Hausdorff dimension of } E_\alpha), \\ \tau_{\text{scale}}(q) &:= \liminf_{r \rightarrow 0} \left(\log_r \left(\inf_{\mathcal{I}} \left(\sum_{k=1}^n (\mu(I_k))^q \right) \right) \right), \end{aligned}$$

where \mathcal{I} is the set of the covers of the support of μ by closed intervals of length r [30, 8, 10]. By convention, the empty set has Hausdorff dimension $-\infty$.

The scale spectrum is often computable or approximable, and it is expected (see for instance [9]) that the singularity spectrum is the Legendre transform conjugate of the scale spectrum, which means that, for any $\alpha \in \mathbb{R}$,

$$\tau_{\text{sing}}(\alpha) = \inf \{ \alpha q - \tau_{\text{scale}}(q) ; q \in \mathbb{R} \}.$$

If this holds, one says that μ satisfies the *multifractal formalism*. The multifractal formalism was established for *Gibbs measures* [3, 31, 34] and this was extended in [6, Theorem A'] to the *weak-Gibbs measures* in the sense of Yuri [41, §5]. In the following section we give a definition of the Gibbs and the weak-Gibbs measures related to the Parry expansions.

The multifractal formalism also holds for the *quasi-Bernoulli measures* [2, 11]. An overview can be found in [27]. Notice that the *g-measures* [17] and the *conformal measures* [12] are weak-Gibbs.

The scale spectrum can be considered as a thermodynamic function (see [6]), and any point of discontinuity of its derivative a phase transition [36]. The existence of a phase transition has been proved for the Erdős measure, that is, the Bernoulli convolution $\mu_{\beta,p}$ for $\beta = \frac{1 + \sqrt{5}}{2}$ and $p = \left(\frac{1}{2}, \frac{1}{2}\right)$ [6, Appendix A]. It has also been proved for the 3-fold convolution of the Cantor measure [13], which is – modulo some homotety – the Bernoulli convolution $\mu_{3,p}$ for $p = \left(\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\right)$.

3.3. Gibbs and weak-Gibbs measures in a base $\beta > 1$. The *Parry expansion* [28] of the real number $x \in [0, 1[$ in base $\beta > 1$ is the unique sequence of integers $(\varepsilon_n)_{n \in \mathbb{N}}$ such that

$$(9) \quad \forall n \in \mathbb{N}, x \in \left[s_n, s_n + \frac{1}{\beta^n} \right], \quad \text{where } s_n = \sum_{k=1}^n \frac{\varepsilon_k}{\beta^k}.$$

We denote this sequence by $(\varepsilon_n(x))_{n \in \mathbb{N}}$. The set \mathcal{Adm}_β of the β -admissible sequences is the set of the Parry expansions of the elements of $[0, 1[$. This set is invariant under the shift $\sigma : (\varepsilon_n)_{n \in \mathbb{N}} \mapsto (\varepsilon_{n+1})_{n \in \mathbb{N}}$. We use the partition of $[0, 1[$ by the β -adic intervals of order n , defined by

$$I_{\varepsilon_1 \dots \varepsilon_n} := \{x \in [0, 1[; \varepsilon_1(x) = \varepsilon_1, \dots, \varepsilon_n(x) = \varepsilon_n\}.$$

We say that a measure η supported by $[0, 1]$ is *weak Gibbs*, or has the *weak Gibbs property*, with respect to the β -adic intervals if there exists a map $\Phi : \mathcal{Adm}_\beta \rightarrow \mathbb{R}$, continuous for the product topology, such that

$$(10) \quad \lim_{n \rightarrow \infty} \left(\frac{\eta(I_{\varepsilon_1 \dots \varepsilon_n})}{\exp\left(\sum_{k=0}^{n-1} \Phi(\sigma^k \varepsilon)\right)} \right)^{1/n} = 1 \quad \text{uniformly on } (\varepsilon_n)_{n \in \mathbb{N}} \in \mathcal{Adm}_\beta.$$

It has the *Gibbs property* if the ratio $\frac{\eta(I_{\varepsilon_1 \dots \varepsilon_n})}{\exp\left(\sum_{k=0}^{n-1} \Phi(\sigma^k \varepsilon)\right)}$ is bounded from 0 and ∞ . In

both cases Φ is called a *potential* of η (it may have several potentials).

These definitions recover the classical ones of Bowen [1, Chapter 1.A] and Yuri [41, §5] because, setting $T(x) = \beta x - \lfloor \beta x \rfloor$ and denoting by $X_0 = \left[0, \frac{1}{\beta}\right]$, $X_1 = \left[\frac{1}{\beta}, \frac{2}{\beta}\right]$, \dots , $X_\kappa = \left[\frac{\kappa}{\beta}, 1\right]$ the intervals where T is continuous, we have

$$I_{\varepsilon_1 \dots \varepsilon_n} = X_{\varepsilon_1} \cap T^{-1}(X_{\varepsilon_2}) \cap \dots \cap T^{-(n-1)}(X_{\varepsilon_n}).$$

Let us give a sufficient condition for a measure η to have the weak Gibbs property. For each $\varepsilon \in \mathcal{Adm}_\beta$ we put $\phi_1(\varepsilon) = \log \eta(I_{\varepsilon_1})$ and for $n \geq 2$,

$$(11) \quad \phi_n(\varepsilon) = \log \left(\frac{\eta(I_{\varepsilon_1 \dots \varepsilon_n})}{\eta(I_{\varepsilon_2 \dots \varepsilon_n})} \right).$$

The continuous map $\phi_n : \mathcal{Adm}_\beta \rightarrow \mathbb{R}$ ($n \geq 1$) is called the *n-step potential* of η . Assume the existence of the uniform limit $\Phi = \lim_{n \rightarrow \infty} \phi_n$; then it is straightforward that for $n \geq 1$,

$$(12) \quad \frac{1}{K_n} \leq \frac{\eta(I_{\varepsilon_1 \dots \varepsilon_n})}{\exp\left(\sum_{k=0}^{n-1} \Phi(\sigma^k \varepsilon)\right)} \leq K_n \quad \text{with} \quad K_n = \exp\left(\sum_{k=1}^n \|\Phi - \phi_n\|_\infty\right).$$

By a well known lemma on the Cesàro sums, K_1, K_2, \dots form a subexponential sequence of positive real numbers, that is $\lim_{n \rightarrow \infty} (K_n)^{1/n} = 1$ and thus, (12) means that η has the weak Gibbs property with respect to the β -adic intervals.

In case of the Erdős measure, that is, $\mu_{\beta,p}$ for $\beta = \frac{1 + \sqrt{5}}{2}$ and $p = \left(\frac{1}{2}, \frac{1}{2}\right)$, the weak Gibbs property is proved and the multifractal analysis detailed in [6]. Now in case β is a multinacci number (i.e. $\beta^m = \beta^{m-1} + \dots + \beta + 1$) and $p = (p_0, p_1)$ and except in one special case, $\mu_{\beta,p}$ has the weak Gibbs property [24]. This property seems more complicated to prove for the other Pisot numbers of degree at least 3.

3.4. Proof of the weak Gibbs property in one example. From now $\beta \approx 1.755$ is the unique solution greater than 1 of the equation

$$\beta^3 = 2\beta^2 - \beta + 1$$

and μ is the Bernoulli convolution associated with β and $p = \left(\frac{1}{2}, \frac{1}{2}\right)$ (here $s = 2$ and $\alpha = \beta - 1$).

LEMMA 3.1. *A sequence of integers $(\varepsilon_n)_{n \in \mathbb{N}}$ is β -admissible if and only if its terms are 0 or 1, and each couple of consecutive 1 is followed by a couple of consecutive 0, and $(\varepsilon_k)_{k > n} \neq \overline{1100}$ for any integer $n \geq 0$.*

Proof: The condition (9) is equivalent to $\beta^n(x - s_n) = \beta^n(x - s_{n-1}) - \varepsilon_n \in [0, 1[$, hence $\varepsilon_n = \lfloor \beta^n(x - s_{n-1}) \rfloor -$ with the convention that $s_0 = 0$. This proves that $\varepsilon_n = 0$ or 1 because, using (9) at the rank $n - 1$, one has $\beta^n(x - s_{n-1}) \in [0, \beta[$.

Now (9) also implies

$$\sum_{k=n+1}^{n+4} \frac{\varepsilon_k}{\beta^k} \leq x - s_n < \frac{1}{\beta^n} = \frac{1}{\beta^{n+1}} + \frac{1}{\beta^{n+2}} + \frac{1}{\beta^{n+4}}$$

hence if $\varepsilon_{n+1} = \varepsilon_{n+2} = 1$, ε_{n+3} and ε_{n+4} are different from 1.

Conversely, for any sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ that satisfies the conditions of the lemma, each subsequence $(\varepsilon_k)_{k > n}$ begins by $(1100)^i 0$ or $(1100)^i 10$ for some integer $i \geq 0$. The reals

$$x = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{\beta^k} \text{ and } s_n = \sum_{k=1}^n \frac{\varepsilon_k}{\beta^k} \text{ satisfy}$$

$$0 \leq x - s_n < \frac{1}{\beta^n} \sum_{k=1}^{\infty} \frac{\alpha_k}{\beta^k} \quad \text{where } (\alpha_k)_{k \in \mathbb{N}} = (1100)^i 0\bar{1} \text{ or } (1100)^i 10\bar{1},$$

from what we deduce by a simple computation that $0 \leq x - s_n < \frac{1}{\beta^n}$, that is, $(\varepsilon_n)_{n \in \mathbb{N}}$ is β -admissible. ■

As a consequence of this lemma, each β -admissible sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ can be decomposed from the left to the right in one sequence of words

$$(13) \quad \varepsilon_1 \varepsilon_2 \cdots = w_1 w_2 \cdots \quad \text{where } \forall i, w_i \in \mathcal{A}_1 := \{0, 10, 1100\},$$

and in the sequel we denote $I_{w_1 \dots w_k} := I_{\varepsilon_1 \dots \varepsilon_n}$ when a word $\varepsilon_1 \dots \varepsilon_n \in \{0, 1\}^n$ is the concatenation of words $w_1, \dots, w_k \in \mathcal{A}_1$. We recall that the β -adic interval $I_{\varepsilon_1 \dots \varepsilon_n}$ is the set of the reals x whose expansion in base β begins by $\varepsilon_1 \dots \varepsilon_n$ and, in the two following lemmas, we compute the measure of this set.

LEMMA 3.2. *Setting $i_1 = 0$, $i_2 = 1$, $i_3 = 1 - (\beta - 1)^2$, $i_4 = -(\beta - 1)^2$, $i_5 = \beta - 1$, $i_6 = \beta - (\beta - 1)^2$, $i_7 = \beta(\beta - 1)$ one has, for any $w_1 \dots w_k \in \mathcal{A}_1^*$*

$$(14) \quad \begin{pmatrix} \mu \left(\frac{1}{\beta} (i_1 + I_{w_1 \dots w_k}) \right) \\ \vdots \\ \mu \left(\frac{1}{\beta} (i_7 + I_{w_1 \dots w_k}) \right) \end{pmatrix} = M(w_1) \dots M(w_k) V \quad \text{where } V := \begin{pmatrix} 3/5 \\ 2/5 \\ 13/20 \\ 1/5 \\ 3/5 \\ 3/10 \\ 1/5 \end{pmatrix}$$

$$\text{and } M(0) := \frac{1}{2}A(0), \quad M(10) := \frac{1}{4}A(1), \quad M(1100) := \frac{1}{16}A(2).$$

Proof: For any real γ we evaluate $\mu \left(\frac{1}{\beta} (\gamma + I_{w_1 \dots w_k}) \right)$ in the three cases: $w_1 = 0, 10$ or 1100 . In the first, $(\beta - 1) \sum_{n=1}^{\infty} \frac{\omega_n}{\beta^n}$ belongs to $\frac{1}{\beta} (\gamma + I_{w_1 \dots w_k})$ if and only if $(\beta - 1) \sum_{n=1}^{\infty} \frac{\omega_{n+1}}{\beta^n} \in \frac{1}{\beta} (\gamma' + I_{w_1 \dots w_k})$ with $\gamma' = \gamma\beta - \omega_1\beta(\beta - 1)$ and, since $\omega_1 \in \{0, 1\}$,

$$(15) \quad \begin{aligned} \mu \left(\frac{1}{\beta} (\gamma + I_{w_1 \dots w_k}) \right) &= \frac{1}{2} \sum_{\gamma' \in \Gamma'(\gamma)} \mu \left(\frac{1}{\beta} (\gamma' + I_{w_1 \dots w_k}) \right), \\ \text{where } \Gamma'(\gamma) &= \{\gamma\beta - x\beta(\beta - 1) ; x \in \{0, 1\}\}. \end{aligned}$$

We proceed in the same way if $w_1 = 10$:

$$(16) \quad \begin{aligned} \mu \left(\frac{1}{\beta} (\gamma + I_{w_1 \dots w_k}) \right) &= \frac{1}{4} \sum_{\gamma'' \in \Gamma''(\gamma)} \mu \left(\frac{1}{\beta} (\gamma'' + I_{w_1 \dots w_k}) \right), \\ \text{where } \Gamma''(\gamma)' &= \{\gamma\beta^2 + \beta - (x\beta + y)\beta(\beta - 1) ; x, y \in \{0, 1\}\}. \end{aligned}$$

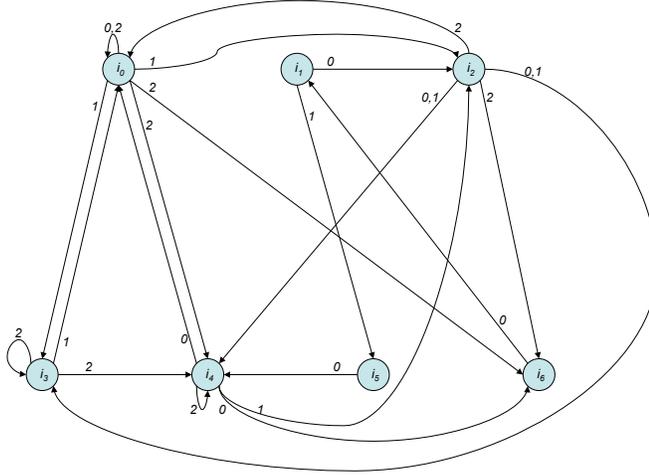
and if $w_1 = 1100$:

(17)

$$\mu \left(\frac{1}{\beta} (\gamma + I_{w_1 \dots w_k}) \right) = \frac{1}{16} \sum_{\gamma''' \in \Gamma'''(\gamma)} \mu \left(\frac{1}{\beta} (\gamma''' + I_{w_1 \dots w_k}) \right),$$

where $\Gamma'''(\gamma) = \{\gamma\beta^4 + \beta^3 + \beta^2 - (x\beta^3 + y\beta^2 + z\beta + t)\beta(\beta - 1) ; x, y, z, t \in \{0, 1\}\}$.

Since the measure μ has support $[0, 1]$, we can restrict the sums in (15), (16) and (17) to the indexes $\gamma^{(i)}$ such that $\mu \left(\frac{1}{\beta} (\gamma^{(i)} + [0, 1]) \right) \neq 0$, that is, $\gamma^{(i)} \in]-1, \beta[$. The relations $\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2$ defined by $\gamma \mathcal{R}_0 \gamma' \Leftrightarrow \gamma' \in \Gamma'(\gamma)$, $\gamma \mathcal{R}_1 \gamma'' \Leftrightarrow \gamma'' \in \Gamma''(\gamma)$ and $\gamma \mathcal{R}_2 \gamma''' \Leftrightarrow \gamma''' \in \Gamma'''(\gamma)$ respectively, are represented below: each relation \mathcal{R}_i is represented by the edges with label i , and the set of states is the set of elements of $] - 1, \beta[$ that can be reached by some path from the initial state 0.



The incidence matrices of the three graphs being $A(0), A(1)$ and $A(2)$, we deduce from (15), (16) and (17) that

$$\begin{pmatrix} \mu \left(\frac{1}{\beta} (i_1 + I_{w_1 \dots w_k}) \right) \\ \vdots \\ \mu \left(\frac{1}{\beta} (i_7 + I_{w_1 \dots w_k}) \right) \end{pmatrix} = M(w_1) \dots M(w_k) V, \quad \text{where } V = \begin{pmatrix} \mu \left(\frac{1}{\beta} (i_1 + [0, 1]) \right) \\ \vdots \\ \left(\frac{1}{\beta} (i_7 + [0, 1]) \right) \end{pmatrix}.$$

We make $k = 1$ in this relation and we sum for $w_1 \in \mathcal{A}_1$. Since – by Lemma 3.1 – the I_{w_1} make a partition of $[0, 1[$, we obtain that V is an eigenvector of $M(0) + M(10) + M(1100)$. Moreover the sum of the two first entries in V is $\mu \left(\frac{1}{\beta} ([0, 2]) \right)$ hence it is $\mu([0, 1]) = 1$. Computing this eigenvector we obtain the expected value for V . ■

As a direct consequence of this lemma we can compute the values of the measure μ on the β -adic intervals:

LEMMA 3.3. (i) For any $w_1 \dots w_k \in \mathcal{A}_1^*$,

$$(18) \quad \mu(I_{w_1 \dots w_k}) = \begin{cases} {}^t E_1 M(w_2) \dots M(w_k) V & \text{if } w_1 = 0 \\ {}^t E_2 M(0) M(w_2) \dots M(w_k) V & \text{if } w_1 = 10 \\ {}^t E_2 M(10) M(0) M(w_2) \dots M(w_k) V & \text{if } w_1 = 1100. \end{cases}$$

(ii) For any $\varepsilon_1 \dots \varepsilon_n \in \{0, 1\}^*$,

$$(19) \quad \mu(I_{\varepsilon_1 \dots \varepsilon_n}) = \begin{cases} \mu(I_{w_1 \dots w_k}) & \text{if } \varepsilon_1 \dots \varepsilon_n = w_1 \dots w_k \\ \mu(I_{w_1 \dots w_k 10}) + \mu(I_{w_1 \dots w_k 1100}) & \text{if } \varepsilon_1 \dots \varepsilon_n = w_1 \dots w_k 1 \\ \mu(I_{w_1 \dots w_k 1100}) & \text{if } \varepsilon_1 \dots \varepsilon_n = w_1 \dots w_k 11 \text{ or } w_1 \dots w_k 110. \end{cases}$$

It remains to prove that μ has the weak-Gibbs property, although we can't use n -step potential of μ because its limit is infinite at the point $\bar{10}$.

THEOREM 3.4. μ has the weak-Gibbs property with respect to the β -adic intervals.

Proof: By the Kolmogorov consistency theorem there exists a unique measure μ' on $[0, 1]$ such that – for any $w_1 \dots w_k \in \mathcal{A}_1^*$

$$(20) \quad \mu'(I_{w_1 \dots w_k}) = \|M(w_1) \dots M(w_k) V\| = UM(w_1) \dots M(w_k) V.$$

We first prove that $\lim_{n \rightarrow \infty} \left(\frac{\mu'(I_{\varepsilon_1 \dots \varepsilon_n})}{\mu(I_{\varepsilon_1 \dots \varepsilon_n})} \right)^{1/n} = 1$ uniformly in $(\varepsilon_n)_{n \in \mathbb{N}} \in \mathcal{Adm}_\beta$, although the ratio $\frac{\mu'(I_{\varepsilon_1 \dots \varepsilon_n})}{\mu(I_{\varepsilon_1 \dots \varepsilon_n})}$ itself tends to ∞ in the cases $\varepsilon = \bar{0}$ and $\varepsilon = \overline{1100}$. Notice that in all cases this ratio is at least 1, in consequence of Lemma 3.3 and the inequalities $UM(0) \geq {}^t E_1$, $UM(10) \geq {}^t E_2 M(0)$, $UM(1100) \geq {}^t E_2 M(10) M(0)$.

It remains to find some upper bound for $\frac{\mu'(I_{\varepsilon_1 \dots \varepsilon_n})}{\mu(I_{\varepsilon_1 \dots \varepsilon_n})}$; it is sufficient to consider only the integers n such that $\varepsilon_1 \dots \varepsilon_n \in \mathcal{A}_1^*$, because μ' as well as μ satisfies (19). From now $\varepsilon_1 \dots \varepsilon_n \in \mathcal{A}_1^*$ and – except in the case $\varepsilon_1 \varepsilon_2 = 10$ – we use the greatest prefix 0^ν or $(1100)^\nu$ of $\varepsilon_1 \dots \varepsilon_n$, with $\nu \in \mathbb{N}$: we have in all cases

$$(21) \quad \varepsilon_1 \dots \varepsilon_n = 0^\nu a w(n) \text{ or } 10 w(n) \text{ or } (1100)^\nu b w(n)$$

where $\nu \in \mathbb{N}$, $w(n) \in \mathcal{A}_1^*$, $a w(n) = \phi$ or $a \in \{10, 1100\}$, $b w(n) = \phi$ or $b \in \{0, 10\}$. Setting

$$(22) \quad V_n = \frac{M(w(n)) V}{\|M(w(n)) V\|}, \text{ or } V_n = \frac{V}{\|V\|} \text{ if } w(n) = \phi, \text{ we deduce from (21), (20) and (18) that}$$

$$\frac{\mu'(I_{\varepsilon_1 \dots \varepsilon_n})}{\mu(I_{\varepsilon_1 \dots \varepsilon_n})} = \frac{UM(0)^\nu M(a) V_n}{{}^t E_1 M(0)^{\nu-1} M(a) V_n} \text{ or } \frac{UM(10) V_n}{{}^t E_2 M(0) V_n} \text{ or } \frac{UM(1100)^\nu M(b) V_n}{{}^t E_2 M(10) M(0) M(1100)^{\nu-1} M(b) V_n}.$$

By direct computation the entries of $UM(0)^\nu$ are at most $\frac{2\nu}{2^\nu}$ and the ones of $UM(1100)^\nu$ at most $\frac{3\nu}{16^\nu}$. On the other side ${}^t E_1 M(0)^{\nu-1} = \frac{1}{2^{\nu-1}} \cdot {}^t E_1$ and ${}^t E_2 M(10) M(0) M(1100)^{\nu-1} =$

$\frac{1}{8 \cdot 16^{\nu-1}} \cdot {}^t E_5$. Finally we obtain

$$(23) \quad \frac{\mu'(I_{\varepsilon_1 \dots \varepsilon_n})}{\mu(I_{\varepsilon_1 \dots \varepsilon_n})} \leq 21\nu \cdot \frac{m_n^+}{m_n^-}$$

where $m_n^+ = \max_i {}^t E_i V_n$ and $m_n^- = \min \left(({}^t E_1 + {}^t E_7) V_n, {}^t E_3 V_n, \min_i \left({}^t E_i \frac{V}{\|V\|} \right) \right)$.

Now we use Proposition 0.1 in the following way: the column vector V_n is equal to $P_{n'}(\omega, V)$ for some $\omega \in \{0, 1, 2\}^{\mathbb{N}}$ and $n' \in \mathbb{N} \cup \{0\}$. The uniform convergence stated in this proposition and the form of the limit vector imply that the sequences

$$m_n^+(\omega) = \max_i {}^t E_i P_n(\omega, V)$$

$$\text{and } m_n^-(\omega) = \min \left(({}^t E_1 + {}^t E_7) P_n(\omega, V), {}^t E_3 P_n(\omega, V), \min_i \left({}^t E_i \frac{V}{\|V\|} \right) \right)$$

converge uniformly and that their limits are two continued maps from the compact set $\{0, 1, 2\}^{\mathbb{N}}$ to $]0, \infty[$, hence $\frac{m_n^+(\omega)}{m_n^-(\omega)}$ is bounded for any $\omega \in \{0, 1, 2\}^{\mathbb{N}}$ and $n \in \mathbb{N} \cup \{0\}$. So

$$(23), \text{ with the inequality } \nu \leq n, \text{ imply that } \lim_{n \rightarrow \infty} \left(\frac{\mu'(I_{\varepsilon_1 \dots \varepsilon_n})}{\mu(I_{\varepsilon_1 \dots \varepsilon_n})} \right)^{1/n} = 1.$$

From the definition of the weak Gibbs property, μ has this property if μ' has. Now, μ' has this property if the exponential of its n -step potential, that is, $\frac{\mu'[\varepsilon_1 \dots \varepsilon_n]}{\mu'[\varepsilon_2 \dots \varepsilon_n]}$, converges uniformly in $(\varepsilon_n)_{n \in \mathbb{N}} \in \mathcal{A}d m_\beta$ to a nonnull limit. For the same reason as above it is sufficient to prove this convergence for the integers n such that $\varepsilon_1 \dots \varepsilon_n$ is a concatenation of words in \mathcal{A}_1^* . We distinguish the cases $\varepsilon_1 \dots \varepsilon_n = 0w'(n)$ or $10w'(n)$ or $1100w'(n)$ with $w'(n) \in \mathcal{A}_1^*$ and we obtain

$$\frac{2\mu'[\varepsilon_1 \dots \varepsilon_n]}{\mu'[\varepsilon_2 \dots \varepsilon_n]} = \frac{\begin{pmatrix} 2 & 1 & 1 & 1 & 2 & 0 & 1 \end{pmatrix} S_n}{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} S_n} \text{ or } \frac{\begin{pmatrix} 1 & 0 & 2 & 2 & 1 & 1 & 0 \end{pmatrix} S_n}{\begin{pmatrix} 2 & 1 & 1 & 1 & 2 & 0 & 1 \end{pmatrix} S_n}$$

$$\text{or } \frac{\begin{pmatrix} 2 & 0 & 0 & 1 & 3 & 0 & 2 \end{pmatrix} S_n}{\begin{pmatrix} 2 & 0 & 0 & 2 & 3 & 0 & 1 \end{pmatrix} S_n}$$

respectively, where $S_n = \frac{M(w'(n))V}{\|M(w'(n))V\|}$. The three ratios converge uniformly to some nonnull limits because, from Proposition 0.1, their numerators and denominators do. ■

COROLLARY 3.5. *μ satisfies the multifractal formalism.*

Proof: In order to apply [6, Theorem A'] we must use some partitions of $[0, 1[$ in intervals $J_{\alpha_1, \dots, \alpha_n}$ defined for any $\alpha_1, \dots, \alpha_n$ in a finite alphabet. Now $I_{\varepsilon_1 \dots \varepsilon_n}$ is not defined if the $(0, 1)$ -sequence $\varepsilon_1 \dots \varepsilon_n \bar{0}$ is not β -admissible. But, using the intervals

$$(24) \quad J_{\alpha_1, \dots, \alpha_n} := I_{w_1, \dots, w_n}, \text{ for } \alpha_1 \dots \alpha_n \in \{0, 1, 2\}^n, \text{ with } w_i = \begin{cases} 0 & \text{if } \alpha_i = 0 \\ 10 & \text{if } \alpha_i = 1 \\ 1100 & \text{if } \alpha_i = 2, \end{cases}$$

it is clear that the weak Gibbs property of μ with respect to the intervals $I_{\varepsilon_1 \dots \varepsilon_n}$, as defined in (10), implies the same property for μ with respect to the intervals $J_{\alpha_1, \dots, \alpha_n}$, with the potential defined – for any $\alpha \in \{0, 1, 2\}^{\mathbb{N}}$ – by

$$\Psi(\alpha) = \begin{cases} \Phi(0\varepsilon') & \text{if } \varepsilon = 0\varepsilon' \\ \Phi(10\varepsilon') + \Phi(0\varepsilon') & \text{if } \varepsilon = 10\varepsilon' \\ \Phi(1100\varepsilon') + \Phi(100\varepsilon') + \Phi(00\varepsilon') + \Phi(0\varepsilon') & \text{if } \varepsilon = 1100\varepsilon', \end{cases}$$

where the sequence ε is the infinite concatenation of the words w_i defined as in (24).

We can apply [6, Theorem A'] to the transformation

$$T(x) = \begin{cases} \beta x & \text{if } x \in J_0 = \left[0, \frac{1}{\beta}\right[\\ \beta^2 x - \beta & \text{if } x \in J_1 = \left[\frac{1}{\beta}, \frac{1}{\beta} + \frac{1}{\beta^2}\right[\\ \beta^4 x - \beta^3 - \beta^2 & \text{if } x \in J_2 = \left[\frac{1}{\beta} + \frac{1}{\beta^2}, 1\right[\end{cases}$$

and conclude that μ satisfies the multifractal formalism. ■

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