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HOW TO PROVE THAT SOME BERNOULLI CONVOLUTION HAS THE WEAK GIBBS PROPERTY

ÉRIC OLIVIER AND ALAIN THOMAS

Abstract. In this paper we give one example of uniform convergence of the sequence of column vectors $\dfrac{A_1\dots A_nV}{\|A_1\dots A_nV\|}$, $A_i\in\{A,B,C\}$, A,B,C being some $(0,1)$-matrices of order 7 with much null entries, and V a fixed positive column vector. This example comes from the study of a continuous singular measure defined by infinite convolution; in the last section we deduce from the uniform convergence result that this measure has the weak Gibbs property.

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Introduction

Given a finite set of nonnegative $d\times d$ matrices $\mathcal{A}:=\{A(0),\dots,A(s-1)\}$ and a nonnegative d-dimensional column vector V, we associate to any $\omega\in\Omega:=\{0,\dots,s-1\}^{\mathbb{N}}$ the sequence of product matrices

$$P_n(\omega,V)=\frac{A(\omega_1)\dots A(\omega_n)V}{\|A(\omega_1)\dots A(\omega_n)V\|}\quad (\|\cdot\|=\text{ the norm-sum}).$$

This is not obligatory defined for any $\omega\in\Omega:=\{0,\dots,s-1\}^{\mathbb{N}}$ but for

$$\omega\in\Omega_{\mathcal{A},V}:=\{\omega\in\Omega\ ;\ \forall n\ge 0,\ A(\omega_1)\dots A(\omega_n)V\ne 0\}.$$

This set is compact because for fixed n, the set $\{\omega\in\Omega\ ;\ A(\omega_1)\dots A(\omega_n)V\ne 0\}$ is a finite union of cylinders of order n.

In §1 we prove a straightforward proposition, that may simplify the proof of the uniform convergence of $(P_n(\cdot,V))_{n\in\mathbb{N}}$.

Key words and phrases. Infinite products of nonnegative matrices, Gibbs properties, multifractal analysis of mesures, Bernoulli convolutions.

In §2 the set $\mathcal{A}$ we consider has three elements:

$$A = A(0) := \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&1&0&0\\ 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&1\\ 0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0 \end{pmatrix}, \quad B = A(1) := \begin{pmatrix} 0&0&1&1&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&1&1&0&0\\ 1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix},$$

$$C = A(2) := \begin{pmatrix} 1&0&0&0&1&0&1\\ 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&1\\ 0&0&0&1&1&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

and using [8, Theorem 1.1], we prove the following

Proposition 0.1. *If V has positive entries, $P_n(\cdot, V)$ converges uniformly on $\{0,1,2\}^{\mathbb{N}}$. The set of the indexes of the nonnull entries in the limit vector is $\{1,2,3,4,5\}$, $\{1,2,3,5,6,7\}$, $\{1,2,3,5,6\}$, $\{1,2,3,5\}$, $\{1,3,4,5\}$, $\{1,3,4\}$ or $\{2,3,5,6,7\}$.*

In §3 we use this proposition to prove that the Bernoulli convolution related to the numeration in the base $\beta > 1$ defined by $\beta^3 = 2\beta^2 - \beta + 1$ has the weak Gibbs property.

1. Products of matrices that belong to a finite set; uniform convergence of $P_n(\cdot, V)$

The conditions for the uniform convergence are quite different from the ones for the simple convergence (see [7] and [11]). Let us show on a trivial example that the uniform convergence of $P_n(\cdot, V)$ is not equivalent to the pointwise convergence of $P_n(\cdot, V)$ to a continuous map.

Example 1.1. *Let $\mathcal{B} = \{B(0), B(1)\}$ with $B(0) = \begin{pmatrix} 1&0&1\\ 0&1&0\\ 0&0&0 \end{pmatrix}$, $B(1) = \begin{pmatrix} 1&0&1\\ 1/2&0&0\\ 0&0&0 \end{pmatrix}$, and let $V = \begin{pmatrix} 1\\1\\1 \end{pmatrix}$. Then $\dfrac{B(\omega_1)\dots B(\omega_n)V}{\|B(\omega_1)\dots B(\omega_n)V\|}$ converges to the continuous map whose* *constant value is the vector* $\begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix}$*; this convergence is not uniform because*

$$\frac{B(0)^{n-1}B(1)V}{\|B(0)^{n-1}B(1)\|} = \begin{pmatrix} 4/5 \\ 1/5 \\ 0 \end{pmatrix}.$$

In the following proposition $\mathcal{A} := \{A(0), \dots, A(s-1)\}$ is a finite set of nonnegative $d \times d$ matrices, V a d-dimensional nonnegative column vector, and we denote the cylinders of $\Omega_{\mathcal{A},V}$ by

$$[\omega_1 \dots \omega_n] := \{\xi \in \Omega_{\mathcal{A},V} \ ; \ \xi_1 = \omega_1, \dots, \xi_n = \omega_n\}.$$

Proposition 1.2. *$P_n(\cdot, V)$ converges uniformly on $\Omega_{\mathcal{A},V}$ if and only if*

$$\forall \omega \in \Omega_{\mathcal{A},V} \ , \ \lim_{n\to\infty} \sup_{\substack{\xi \in [\omega_1 \dots \omega_n] \\ r,s \geq n}} \|P_r(\xi, V) - P_s(\xi, V)\| = 0. \tag{1}$$

Proof: The direct implication is obvious by the Cauchy criterion. Suppose now that (1) holds. Given $\varepsilon > 0$ one can associate to each $\omega \in \Omega_{\mathcal{A},V}$ some cylinder $[\omega_1 \dots \omega_n]$ such that any ξ in this cylinder and any $r, s \geq n$ satisfy

$$\|P_r(\xi, V) - P_s(\xi, V)\| \leq \varepsilon. \tag{2}$$

The compact set $\Omega_{\mathcal{A},V}$ is a finite union of such cylinders, let $[\omega_1^{(i)} \dots \omega_{n_i}^{(i)}]$ for $i = 1, \dots, N$. Hence (2) is true for any $\xi \in \Omega_{\mathcal{A},V}$ when $r, s \geq \max_i n_i$, and this proves that $P_n(\cdot, V)$ is uniformly Cauchy. ■

2. Proof of the proposition

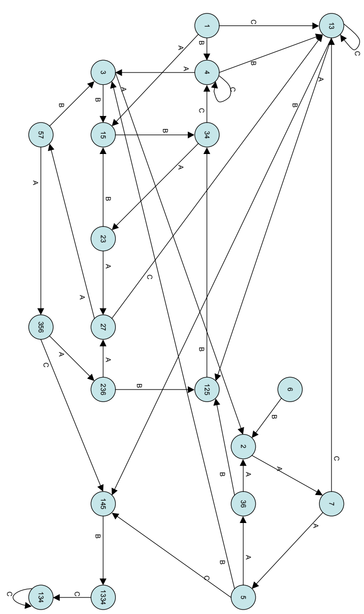
In this section $\mathcal{A}$ is the set of the three matrices $A(0), A(1), A(2)$ defined in the introduction, that we call A, B, C respectively. We denote by $\{E_1, \dots, E_7\}$ the canonical basis of the set of the 7-dimensional column vectors and by U the row vector $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.

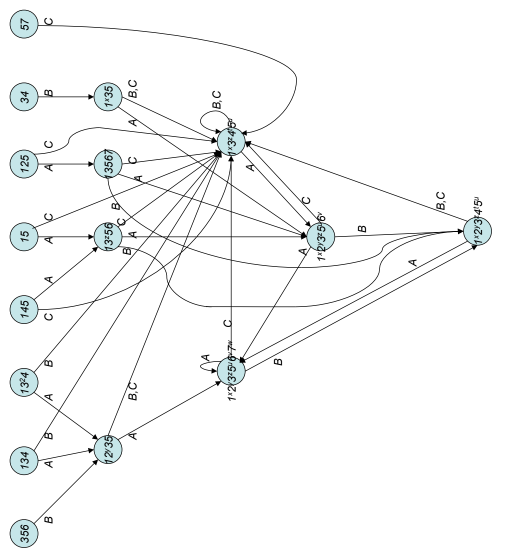
We prove Proposition 0.1 in the following way: for any $\omega \in \{0,1,2\}^{\mathbb{N}}$ we search an increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that the sequence of matrices $A'_k = A(\omega_{n_k+1}) \dots A(\omega_{n_{k+1}})$ satisfies the hypotheses of [8, Theorem 1.1]. For this we use the equivalence classes in some graph associated to $\{A, B, C\}$; although this case is relatively simple, the method we use may be efficient in more complicated ones.

This graph is defined as follows:

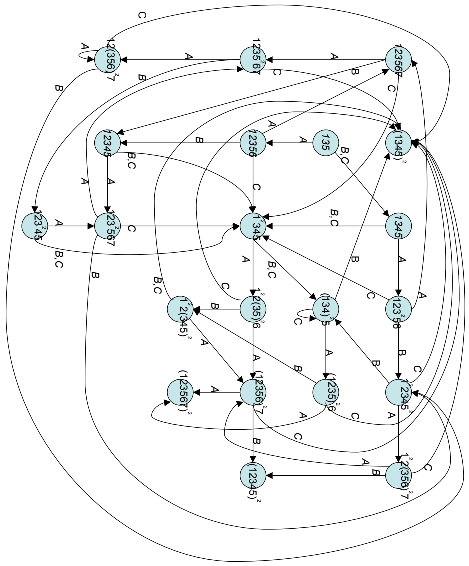
- Each state represents a column: for instance the state 1334 – or 13^24 – represents the column $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and the state 12^x35 represents all the columns $\begin{pmatrix} 1 \\ x \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ for $x \in \mathbb{N}$; now we consider only the columns that appear in the matrices $A(\omega_1)\dots A(\omega_n)$, $\omega \in \{0,1,2\}^{\mathbb{N}}$, $n \in \mathbb{N}$.
- The state X is related to the state Y by one arrow with label A (resp. B or C) if $Y = AX$ (resp. BX or CX).

We present this graph in two parts: in the first are represented all the states except the ones from which any infinite path leads to the four final states that is, to the states $1^x2^y3^z4^t5^u, 1^x2^y3^z5^u6^v, 1^x2^y3^z5^u6^v7^w, 1^x3^z4^t5^u$ where x, y, z, t, u, v, w are some positive integers. The second graph contains the other states and – on the first line – the states of the first graph that are related to them by some arrow.





The notations we use differ for the states of the third graph: for instance the state 123^245 represents all the columns $\begin{pmatrix} x \\ y \\ z \\ t \\ u \\ v \\ w \end{pmatrix}$ for $x, y, t, u \geq 1$, $z \geq 2$ and $v, w \geq 0$. The initial state is 135 and the final states are $(12345)^2, (123567)^2, (1345)^2$.



In the sequel we denote the labels of the arrows by 0, 1 and 2 instead of A, B and C, to avoid confusion between words and products of matrices. Let $\mathcal{W}$ be the set of the words $w = \xi_1 \dots \xi_n \in \{0,1,2\}^*$ such that the symmetric word $\xi_n \dots \xi_1$ is the label of a path in the third graph, from the initial state 135 to a final state. For instance $20^3 \in \mathcal{W}$ means that the path with label $0^3 2$ from the state 135 has final state $(1345)^2$, and that

$$A(2)A(0)^3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ z \\ t \\ u \\ 0 \\ 0 \end{pmatrix} \text{ with } x, z, t, u \geq 2.$$

We prove by the six following lemmas the existence of an integer $\kappa \in \mathbb{N}$ such that the matrices $A(w) := A(\xi_1) \dots A(\xi_n)$ – for any $n \in \mathbb{N}$ and any word $w = \xi_1 \dots \xi_n \in \{0,1,2\}^n$ that can be written $w = w_1 \dots w_\kappa$ with $w_i \in \mathcal{W}$ – satisfy the hypotheses $(H_1), (H_2)$ and $\left(H^M\right)$ of [8, Theorem 1.1]. This is in part due to the existence of synchronizing words in the second graph.

Lemma 2.1. *In the second graph, the words $w \in \{0,1,2\}^3$ are synchronizing from any state whose label contains the digits $1, 3$ and 5 that is, the states of the second graph that are not states of the first.*

Proof: We remark that any path starting from a state whose label contains $1, 3$ and 5, ends to such a state. Now any word $\xi_1\xi_2\xi_3 \in \{0, 1, 2\}^3$ has the factor $0^2, 1^2, 01$ or 2. Since $0^2, 1^2, 01, 2$ are synchronizing from any state whose label contains $1, 3$ and 5, $\xi_1\xi_2\xi_3$ also is. ■

LEMMA 2.2. *If some factor of a word $w \in \{0, 1, 2\}^*$ belong to $\mathcal{W}^6$, the matrix $A(w)$ has the following property:*

$(\mathcal{P})$: Denoting by $c_j(A(w))$ the set of the indexes of the nonnull entries in the j^{th} column of $A(w)$, either all the nonempty sets among the seven sets $c_1(A(w)), \dots, c_7(A(w))$, are equal, or they take two values c and c' such that $c \supseteq c' \cup \{1, 3, 5\}$.

Proof: Let us check first that it is sufficient to find a factor w' of w such that $A(w')$ has the property $(\mathcal{P})$:

_ On the one side the columns of the right product $A(w')X$ for $X \in \{A, B, C\}$ are nonnegative linear combination of columns of $A(w')$; hence if $c_j(A(w')X) \neq \emptyset$, there exists j' such that $c_j(A(w')X) = c_{j'}(A(w'))$ and, if $A(w')$ has the property $\mathcal{P}$, $A(w')X$ also has.

_ On the other side the columns of the left product $XA(w')$ are the products of X by the columns of $A(w')$ hence – since any arrow from a state containing $1, 3$ and 5 ends to such a state – if $A(w')$ has the property $(\mathcal{P})$, $XA(w')$ also has.

We evaluate the length of the words $w \in \{0, 1, 2\}^*$ in two ways: let us denote by $|w|$ the number of letters of w and by $\ell^*(w)$ the number of words $\zeta_i \in \{0^n\}_{n\in\mathbb{N}} \cup \{1\} \cup \{2^n\}_{n\in\mathbb{N}}$ such that $w = \zeta_1 \dots \zeta_{\ell^*(w)}$, without two consecutive 0^n nor two consecutive 2^n. Then any $w \in \mathcal{W}$ satsfies $|w| \geq 4$ and $\ell^*(w) \geq 3$, except $\ell^*(20^n) = \ell^*(12^n) = 2$ for $n \geq 3$.

Let $w \in \{0, 1, 2\}^*$ have the factor $w_1w_2w_3w_4w_5w_6 \in \mathcal{W}^6$. From the above remark, $\ell^*(w_4w_5w_6) \geq 5$. We deduce that, if w_2w_3 contains the digit 0, there exists some factor w' of $w_2w_3w_4w_5w_6$ for which the matrix $A(w')$ is one of the following: we make below the list of the matrices $A(w')$ for $w' \in \mathcal{W}_0$, where the set $\mathcal{W}_0$ – lexicographically ordered from the left to the right – is chosen in such a way that all the words $w \in \{0, 1, 2\}^*$ of length $\ell^*(w) = 5$ beginning by 01 or 02 has one element of $\mathcal{W}_0$ as prefix. Nevertheless we omit the word $w' = w''2$ if $w''1 \in \mathcal{W}_0$, because the columns of C are sums of columns of B and consequently $A(w''2)$ has the property $(\mathcal{P})$ if $A(w''1)$ has.

$$ABA^2=\begin{pmatrix}1&0&0&0&0&0&1\\1&1&0&0&0&0&0\\2&0&0&0&0&0&1\\0&0&0&0&0&0&0\\1&0&0&0&0&0&1\\1&0&0&0&0&0&1\\1&0&0&0&0&0&1\end{pmatrix},\quad ABAB=\begin{pmatrix}1&0&1&0&0&0&0\\0&0&1&1&0&0&0\\1&0&2&1&0&0&0\\0&0&0&0&0&0&0\\1&0&1&0&0&0&0\\1&0&1&0&0&0&0\\0&0&1&0&0&0&0\end{pmatrix},$$

$$AB^2A=\begin{pmatrix}2&0&0&0&0&0&1\\1&0&0&1&1&0&0\\1&0&0&1&1&0&1\\0&0&0&0&0&0&0\\2&0&0&0&0&0&1\\1&0&0&0&0&0&1\\0&0&0&0&0&0&0\end{pmatrix},\quad AB^3=\begin{pmatrix}1&0&2&1&0&0&0\\0&0&1&2&1&0&0\\2&0&1&1&1&0&0\\0&0&0&0&0&0&0\\1&0&2&1&0&0&0\\1&0&1&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$ABC=\begin{pmatrix}1&0&0&1&1&0&1\\0&0&0&1&2&0&0\\2&0&0&0&1&0&2\\0&0&0&0&0&0&0\\1&0&0&1&1&0&1\\1&0&0&0&0&0&1\\0&0&0&0&0&0&0\end{pmatrix},\quad AC^nA=\begin{pmatrix}n+1&1&0&0&0&0&n\\n&1&0&0&0&0&n-1\\n+1&0&0&0&0&0&n+1\\0&0&0&0&0&0&0\\n+1&1&0&0&0&0&n\\1&0&0&0&0&0&1\\0&0&0&0&0&0&0\end{pmatrix},$$

$$AC^nBA=\begin{pmatrix}0&0&0&n+1&n+1&0&0\\0&0&0&n&n&0&0\\1&0&0&n+1&n+1&0&0\\0&0&0&0&0&0&0\\0&0&0&n+1&n+1&0&0\\0&0&0&1&1&0&0\\0&0&0&0&0&0&0\end{pmatrix},\quad AC^nB^2A=\begin{pmatrix}n+2&0&0&0&0&0&n+1\\n+1&0&0&0&0&0&n\\n+1&0&0&1&1&0&n+1\\0&0&0&0&0&0&0\\n+2&0&0&0&0&0&n+1\\1&0&0&0&0&0&1\\0&0&0&0&0&0&0\end{pmatrix},$$

$$AC^nB^3=\begin{pmatrix}n+1&0&n+2&1&0&0&0\\n&0&n+1&1&0&0&0\\n+2&0&n+1&1&1&0&0\\0&0&0&0&0&0&0\\n+1&0&n+2&1&0&0&0\\1&0&1&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\quad AC^nBC=\begin{pmatrix}n+1&0&0&1&1&0&n+1\\n&0&0&1&1&0&n\\n+2&0&0&0&1&0&n+2\\0&0&0&0&0&0&0\\n+1&0&0&1&1&0&n+1\\1&0&0&0&0&0&1\\0&0&0&0&0&0&0\end{pmatrix}$$

$(n \in \mathbb{N})$.

In case w_2w_3 do not contain the digit 0, as seen on the third graph the words w_2 and w_3 belong to $\{1\}\times\{1,2\}^n$ with $n \geq 3$. Hence either $w_2 = 1^4$ or w_2w_3 has a factor in $\{2\}\times\{1\}\times\{1,2\}^3$, so w_2w_3 has a factor $w' \in \mathcal{W}_1 = \{1^4, 21^32, 21^22, 2121, 212^2\}$. This is the list of the matrices $A(w')$ for $w' \in \mathcal{W}_1$, where we omit $A(21^32)$ and $A(2122)$ because $A(1^4)$ and $A(2121)$ are in the list:

$$B^4 = \begin{pmatrix} 1 & 0 & 1 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad CB^2C = \begin{pmatrix} 1 & 0 & 0 & 2 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 3 & 0 & 1 \\ 1 & 0 & 0 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$CBCB = \begin{pmatrix} 1 & 0 & 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

According to Lemma 2.1, for any $X, Y, Z \in \{A, B, C\}$ the product of XYZ by each of these matrices has the property $(\mathcal{P})$, hence $A(w)$ also has. ■

LEMMA 2.3. *There exists an integer $\kappa \geq 7$ such that, for any $w \in \{0,1,2\}^*$ with a factor in $\mathcal{W}^\kappa$, the matrix $A(w)$ has the property (H_2).*

Proof: Let $w = \xi_1 \ldots \xi_n \in \{0,1,2\}^n$ be a word with a factor $w_1 \ldots w_\kappa \in \mathcal{W}^\kappa : w = w' w_1 \ldots w_\kappa w''$. We can apply Lemma 2.2 to the word $m = w_{\kappa-5} \ldots w_\kappa w''$: the non empty sets, among the $c_j(A(m))$ for $j = 1, \ldots, 7$, are equal or they take two values, c for $j \in J$ and c' for $j' \in J'$ with $c \supseteq c' \cup \{1,3,5\}$. There is no problem if all the non-empty $c_j(A(w))$ are equal, so we suppose they take two values and consequently the $c_j(A(m))$ also do. For $j \in J$, the obvious property of the final states in the third graph implies that the values of the nonnull entries in the j^{th} column of $A(w)$ are at least $2^{\kappa-6}$. For $j' \in J'$, $A(\xi_4 \ldots \xi_n)E_{j'}$ is a state of the first graph, otherwise by Lemma 2.1 the $c_j(A(w))$ should be equal for any $j \in J \cup J'$. Consequently $A(\xi_4 \ldots \xi_n)E_{j'}$ has entries at most 2, and $A(w)E_{j'}$ has bounded entries. Choosing κ large enough, $A(w)$ has the property (H_2). ■

It remains to prove that the matrices $A(w)$, for w in some set specified later, satisfy the condition $\left(H^M\right)$. We first notice that $A(w)$ satisfy (H^3) if w is one of the words $010^{4(n+1)}, 1^2 0^{4(n+1)}, 210^{4(n+1)}, 20^{4(n+1)}, 0^2 2^{n+2}, 0^2 102^{n+2}, 10102^{n+2}, 20102^{n+2}, 1^2 02^{n+2}, 2102^{n+2}$, $202^{n+2}, 12^{n+2}$, that we denote by $w_1(n), \ldots, w_{12}(n)$ for any nonnegative integer n: indeed the $A_{w_i(n)}$ are

$$ABA^{4(n+1)} = \begin{pmatrix} n+1 & 0 & 1 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 0 & 1 & 1 & 0 & 0 \\ n+2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 1 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 1 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B^2A^{4(n+1)} = \begin{pmatrix} n+2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n+2 & 0 & 1 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 1 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$CBA^{4(n+1)} = \begin{pmatrix} 2n+2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 1 & 0 & 0 & 0 & 0 \\ n+2 & 0 & 1 & 0 & 0 & 0 & 0 \\ n+1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad CA^{4(n+1)} = \begin{pmatrix} 2n+3 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n+2 & 0 & 0 & 0 & 0 & 0 & 1 \\ n+1 & 0 & 0 & 1 & 1 & 0 & 0 \\ n+1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A^2C^{n+2} = \begin{pmatrix} 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 1 & 0 & 0 & 0 & n+1 & 0 & 1 \end{pmatrix}, \quad A^2BAC^{n+2} = \begin{pmatrix} 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2n+5 & 0 & 1 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+4 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \end{pmatrix},$$

$$BABAC^{n+2} = \begin{pmatrix} 1 & 0 & 0 & 1 & 2n+5 & 0 & 1 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2n+5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad CABAC^{n+2} = \begin{pmatrix} 0 & 0 & 0 & 2 & 2n+7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+4 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B^2AC^{n+2} = \begin{pmatrix} 2 & 0 & 0 & 0 & 2n+4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2n+5 & 0 & 1 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad CBAC^{n+2} = \begin{pmatrix} 0 & 0 & 0 & 2 & 2n+6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2n+5 & 0 & 1 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$CAC^{n+2} = \begin{pmatrix} 2 & 0 & 0 & 0 & 2n+4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad BC^{n+2} = \begin{pmatrix} 1 & 0 & 0 & 1 & 2n+3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & n+3 & 0 & 0 \\ 1 & 0 & 0 & 0 & n+2 & 0 & 1 \\ 1 & 0 & 0 & 0 & n+1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix $A\left((100)^{n+2}\right) = \left(BA^2\right)^{n+2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ also do for any non-negative integer n.

In the following lemma we extend this property to the matrices that are products of a bounded number of matrices $A, B, C, A(w_i(n))$ and $A\left((100)^{n+2}\right)$.

Lemma 2.4. *Let $\mathcal{W}_{k_1,k_2}$ be the set of the words*

$$w = m_0 w_{i_1}(n_1) m_1 w_{i_2}(n_2) \dots w_{i_k}(n_k) m_k \tag{3}$$

with the conditions that $k \le k_1$, that $m_0 m_1 \dots m_k$ is the concatenation of at most k_2 elements of the alphabet $\{0,1,2\} \cup \{(100)^{n+2}\}_{n\in\mathbb{N}}$ and that $n_i \ge 0$. Then for $w \in \mathcal{W}_{k_1,k_2}$, each nonnull entry in the j^{th} column of the matrix $A(w)$ has the form $P\left(n_1, \dots, n_{k(j)}\right)$ where $k(j) \in \{1, \dots, k\}$ and P is a polynomial with positive coefficients and degree 1 in each variable, or $k(j) = 0$ and $P =$ constant. Moreover $A(w)$ satisfies $\left(H^M\right)$ for some constant $M = M(k_1, k_2)$.

Proof: Let $j \in \{1, \dots, 7\}$, one consider the path – in the first graph and then in the second – with initial state j, whose label is the word w read from the right to the left. Let e_ι be the final state of the subpath with label $s_\iota = m_\iota w_{i_{\iota+1}}(n_{\iota+1}) \dots w_{i_k}(n_k) m_k$ (from the right to the left). Notice that only the first column of $A(w_i(n))$, for $i \le 4$, depends on n, and only the fifth do for $i > 4$. Consequently, if $i_\iota \le 4$ and the state e_ι do not contain the letter 1, or if $i_\iota > 4$ and this state do not contain the letter 5, the column $A\left(w_{i_\iota}(n_\iota)\right) A\left(s_\iota\right) E_j$ do not depend on n_ι and $A(w)E_j$ no more do .

Conversely suppose that ι is the greatest integer such that $i_\iota \le 4$ and the state e_ι contains the letter 1, or such that $i_\iota > 4$ and this state contains the letter 5. The nonnull entries of $A\left(w_{i_\iota}(n_\iota) s_\iota\right) E_j$ have the form $a n_\iota + b$ with a and b positive, in particular the entries of indexes $1, 3, 5$ have this form. We multiply this column vector by $A\left(m_{\iota-1}\right)$, $A\left(w_{i_{\iota-1}}(n_{\iota-1})\right)$, $\dots$, $A\left(m_0\right)$ successively; this leads to a column vector whose nonnull entries have the form $a_i n_i + b_i$ for any $i \in \{1, \dots, \iota\}$, with positive a_i and b_i because, in the second graph, each arrow whose initial state contains the digits $1, 3, 5$ ends to such a state.

Clearly, a map $f : \mathbb{R}^\iota \to \mathbb{R}$ which is a polynomial of degree 1 with positive coefficients in each of the variables, is a polynomial with 2^ι positive coefficients that is,

$$f(X_1, \dots, X_\iota) = a_0 + a_1 X_1 + \dots + a_\iota X_\iota + a_{\iota+1} X_1 X_2 + a_{\iota+2} X_1 X_3 + \dots + a_{2^\iota - 1} X_1 \dots X_\iota \ .$$

In our case, the coefficients of the polynomial belong to some finite set because they only depend on the (at most) $k_1 + k_2$ elements of the decomposition (3) of the word w in letters $0, 1, 2$ and words $(001)^n, w_i(n)$. The ratio between two polynomials (with positive coefficients and with the same nonnegative variables) being bounded by the ratio of the greatest coefficient of the first by the lowest of the second, we conclude that the ratio between two nonnull entries of the column $A(w)E_j$ is bounded by some constant that only depends on k_1 and k_2 . ■

Lemma 2.5. *Let $w \in \mathcal{W}^\kappa$ and let s be the suffix of a word of $\mathcal{W}^\kappa$. Then*

(i) $w \in \mathcal{W}_{2\kappa,21\kappa}$,

(ii) 0^2s (if s do not begin by 0) and $2s$ (if s do not begin by 2) belong to $\mathcal{W}_{2\kappa,21\kappa+4}$,

(iii) $ws \in \mathcal{W}_{4\kappa,42\kappa+8}$.

Proof: (i) We consider first a word $w \in \mathcal{W}$ that is, w read from the right to the left is the label of a path of the third graph from the initial state 135 to a final state. We distinguish the four cases: either it has one succession of at least four arrows from $12(356)^27$ to itself, or one succession of at least two arrows from $(134)^25$ to itself, or it has successively the first and the second subpath, or it has no such subpath. In the first case the arrows from $12(356)^27$ to itself are followed by one arrow with label 2, or successively by one arrow with label 1 – without reaching the state $(134)^25$ – and one other arrow; so there is a subpath whose label is the symmetrical of $w_i(n)$, $1 \leq i \leq 4$. In the same way, in the second case there exists a subpath whose label is the symmetrical of $w_i(n)$, $5 \leq i \leq 12$. Now if w do not have such subpaths, it cannot be the concatenation of more than 21 words in $\{0, 1, 2\} \cup \{(100)^{n+2}\}_{n\in\mathbb{N}}$. This proves that $w \in \mathcal{W}_{2,21}$. Of course the concatenation of κ words in $\mathcal{W}_{2,21}$ gives a word in $\mathcal{W}_{2\kappa,21\kappa}$.

(ii) Let s be a suffix of a word $w \in \mathcal{W}^\kappa$; we decompose w in the form (3). Suppose that s do not begin by 0. There is no problem if $s = mw_{i_j}(n_j)m_j \dots w_{i_k}(n_k)m_k$ where m is a suffix of m_{j-1}. If not, s begins by some strict and non empty suffix of some $w_{i_j}(n_j)$, more precisely this suffix is $10^{4(n+1)}$, 2^{n+2} or 102^{n+2} for some $n \geq 0$, or it is 2. Hence 0^2s begins with $0w_1(n)$, $w_5(n)$, $w_6(n)$ or 0^22m_j. In this last case we have $|0^22m_j| = |m_j| + 3$, and in all the cases $0^2s \in \mathcal{W}_{2\kappa,21\kappa+3}$. The proof is similar for $2s$ if s do not begin by 2: if s begins by some strict and non empty suffix of some $w_{i_j}(n_j)$, this suffix is $s' = 10^{4(n+1)}, 0^{4(n+1)}, 02^{n+2}, 102^{n+2}, 0102^{n+2}, 0^3, 0^2$ or 0 and, except in the three last cases, $2s'$ is some of the words $w_i(n)$.

(iii) Let now $w, w' \in \mathcal{W}^\kappa$ and let s be a suffix of w', we decompose w and w' in the form (3) and deduce the following decomposition of ws:

$$ws = m_0 w_{i_1}(n_1) m_1 w_{i_2}(n_2) \dots w_{i_k}(n_k) m_k s' m'_j w_{i'_{j+1}}(n'_{j+1}) \dots w_{i'_{k'}}(n'_{k'}) m'_{k'}$$

where $j \in \{1, \dots, k'\}$ and s' is a – possibily empty of full – suffix of $m'_{j-1} w_{i'_j}(n'_j)$. This suffix has the form (3) except in case it is a strict suffix of $w_{i'_j}(n'_j)$, and in this case it is sufficient to find a decomposition of $w_{i_k}(n_k) m_k s'$ in the form (3). If the word s' has length at most 4, it ends by 0^4 or 2^2 hence it can be completed in order to obtain a factor of $w_{i_k}(n_k) m_k s'$ of the form $w_i(n)$. We distinguish the cases whenever this factor is disjoint or not from the word $w_{i_k}(n_k)$, and the cases whenever it is or not disjoint from its prefix $w_{i_k}(0)$ (which has length at most 6). The decomposition of $w_{i_k}(n_k) m_k s'$ we obtain is the following, where ξ and ξ' belong to $\{\phi, 0, 0^2, 0^3, 2\}$:

$$w_{i_k}(n_k) m_k s' = \begin{cases} w_{i_k}(n_k) m_k s' & (\text{ in case } |s'| \le 3) \\ w_{i_k}(n_k) m w_i(n) \xi & (m \text{ prefix of } m_k) \\ w_{i_k}(n') \xi' w_i(n) \xi & (0 \le n' < n) \\ m w_i(n) \xi & (m \text{ strict prefix of } w_{i_k}(0)). \end{cases}$$

This proves (iii), since $|s'| \le 3$ in the first case, $|m| \le 5$ in the last case, and $|\xi|, |\xi'| \le 3$. ■

Lemma 2.6. *The set of the matrices $A(ws)$ for $w \in \mathcal{W}^\kappa$ and s – possibly empty – suffix of a word in $\mathcal{W}^\kappa$, has the properties (H_1), (H_2) and $\left(H^M\right)$.*

Proof: From Lemma 2.2, $A(ws)$ has the property (H_1) because $ws = w'w''$ with $w' \in \mathcal{W}^6$ and $w'' \in \{0,1,2\}^*$. It has the property (H_2) from Lemma 2.3 and, from Lemmas 2.4 and 2.5(iii), the set of such matrices $A(ws)$ has the property $\left(H^M\right)$. ■

Lemma 2.7. *For any word $w \in \{0,1,2\}^*$ there exist an integer $\alpha \ge 1$, ζ_1 – possibly empty – strict suffix of a word in $\mathcal{W}$ and $\zeta_2, \dots, \zeta_\alpha \in \mathcal{W}$ such that $w = \zeta_1 \dots \zeta_\alpha$.*

Proof: Reading the word w from the right to the left, we may go many times from the initial state to a final state in the third graph by some paths and we call $\zeta_\alpha, \zeta_{\alpha-1}, \dots, \zeta_2$ the symmetricals of their labels; finally we go from the initial state to a non final state by some path and we call ζ_1 the symmetrical of its label. ■

Lemma 2.8. *For any sequence $\omega \in \{0,1,2\}^{\mathbb{N}}$ that is not eventually 0 nor eventually 2 nor eventually 100 there exist ζ_1, – possibly empty – strict suffix of a word in $\mathcal{W}$, and $\zeta_2, \zeta_3, \dots \in \mathcal{W}$ such that*

$$\omega_1 \omega_2 \omega_3 \dots = \zeta_1 \zeta_2 \zeta_3 \dots \ .$$

Proof: By Lemma 2.7,

$$\omega_1 \dots \omega_n = \zeta_1(n) \dots \zeta_{\alpha(n)}(n)$$

where $\zeta_i(n) \in \mathcal{W}$, except $\zeta_1(n)$ which is a possibly empty strict suffix of a word in $\mathcal{W}$. Let us prove – for fixed k – that the word $\zeta_k(n)$ can take only a finite number of values when $n \in \mathbb{N}$. By hypothesis there exists one unique sequence of words $\xi_i \in \mathcal{A}^* = \{0^n\}_{n\in\mathbb{N}} \cup \{1\} \cup \{2^n\}_{n\in\mathbb{N}} \cup \{(100)^n\}_{n\in\mathbb{N}}$ such that

$$\omega_1\omega_2\omega_3\cdots = \xi_1\xi_2\xi_3\ldots \tag{4}$$

and such that for any $i, n, n' \in \mathbb{N}$,

$$\begin{aligned}&\xi_i = 0^n \Rightarrow \xi_{i+1} \neq 0^{n'}, \quad \xi_i = 2^n \Rightarrow \xi_{i+1} \neq 2^{n'}, \quad \xi_i = (100)^n \Rightarrow \xi_{i+1} \neq (100)^{n'},\\ &\xi_i = 1 \Rightarrow \xi_{i+1} \neq 0^{n'+1}.\end{aligned} \tag{5}$$

In the same way each word $\zeta_i(n)$ can be written as a concatenation of words of $\mathcal{A}^*$ that satisfy (5); we see on the third graph that the number of such words is at most 11. We deduce the decomposition of $\zeta_1(n)\ldots\zeta_k(n)$ – grouping together if necessary some suffix of each $\zeta_i(n)$ with some prefix of $\zeta_{i+1}(n)$ – in at most $11k$ words of $\mathcal{A}^*$. Since the decomposition (4) is unique, $\zeta_1(n)\ldots\zeta_k(n)$ is a prefix of $\xi_1\ldots\xi_{11k}$.

For fixed k, the word $\zeta_k(n)$ – for $n \in \mathbb{N}$ – belongs to the finite set of the factors of $\xi_1\ldots\xi_{11k}$ hence it takes infinitely many times the same value. So we can define by induction the sequence of words $\zeta_1, \zeta_2, \ldots$: at the k^{th} step we define ζ_k as a word such that $\zeta_1(n) = \zeta_1$, $\zeta_2(n) = \zeta_2$, … , $\zeta_k(n) = \zeta_k$ for infinitely many n. ■

Proposition 2.9. *If V is a positive column vector, the sequence of the normalized column vectors defined by* $P_n(\omega, V) = \dfrac{A(\omega_1)\ldots A(\omega_n)V}{\|A(\omega_1)\ldots A(\omega_n)V\|}$ *converges uniformly for* $\omega \in \{0,1,2\}^{\mathbb{N}}$. *The set of the indexes of the nonnull entries in the limit vector is* $\{1,2,3,4,5\}$, $\{1,2,3,5,6,7\}$, $\{1,2,3,5,6\}$, $\{1,2,3,5\}$, $\{1,3,4,5\}$, $\{1,3,4\}$ *or* $\{2,3,5,6,7\}$.

Proof: We use Proposition 1.2 that is, given $\omega \in \Omega_{\mathcal{A},V}$ and $\varepsilon > 0$ we prove the existence of $N(\omega,\varepsilon)$ such that

$$n \geq N(\omega,\varepsilon),\ \xi \in [\omega_1\ldots\omega_n],\ r,s \geq n \Rightarrow \|P_r(\xi,V) - P_s(\xi,V)\| \leq \varepsilon. \tag{6}$$

Suppose first that the sequence ω is not eventually 0 nor eventually 2 nor eventually 100. Lemma 2.8 implies there exist ζ_1 , possibly empty strict suffix of a word in $\mathcal{W}^\kappa$, and $\zeta_2, \zeta_3, \dots \in \mathcal{W}^\kappa$ such that

$$\omega_1\omega_2\omega_3\cdots = \zeta_1\zeta_2\zeta_3\ldots\ .$$

According to Lemma 2.6 the matrices $A(\zeta_i)$ satisfy (H_1), (H_2) and $\left(H^M\right)$ for any $i \geq 2$. Since the sequence ω is fixed, one can use the obvious fact that $A(\zeta_1\zeta_2)$ satisfies $\left(H^{M'}\right)$ when M' is the maximum of M and the ratio of the greatest nonnull entry in $A(\zeta_1\zeta_2)$ by the lowest one. It also satisfies (H_1) and (H_2) by Lemmas 2.2 and 2.3.

We do not exactly apply [8, Corollary 1.2] to the product $A(\zeta_1\zeta_2)A(\zeta_3)\dots A(\zeta_k)$ and the column vectors $A(\xi_{n+1})\dots A(\xi_r)V$ and $A(\xi_{n+1})\dots A(\xi_s)V$, $n=|\zeta_1|+\dots+|\zeta_k|$, because the ratio between two nonnull entries in these column vectors is not necessarily bounded for $r\ge n$, for instance in the case $\xi_{n+1}=\xi_{n+2}=\dots=0$. There exist – from Lemma 2.7 – $\nu\ge1$, w_1 – possibly empty – strict suffix of a word in $\mathcal{W}^\kappa$ and $w_2,\dots,w_\nu\in\mathcal{W}^\kappa$ (if $\nu\ne1$) such that

$$\xi_{n+1}\dots\xi_r=w_1\dots w_\nu\ . \tag{7}$$

By Lemma 2.6 $A(\zeta_kw_1)A(w_2)\dots A(w_\nu)$ is a product of matrices which satisfy the conditions (H_1), (H_2) and $\left(H^M\right)$; this product is equal to $A(\xi_{n'+1})\dots A(\xi_r)$ for $n'=|\zeta_1|+\dots+|\zeta_{k-1}|$. The square matrix V' whose all columns are equal to V satisfy obviously (H_1), (H_2) and $\left(H^{M''}\right)$ when M'' is the maximum of M and the ratio of the greatest entry in V by the lowest one. By [8, Lemma 1.3] the matrix $A(\xi_{n'+1})\dots A(\xi_r)V'$ satisfies $\left(H^{2M''d}\right)$, hence the ratio between two nonnull entries of the column vector $A(\xi_{n'+1})\dots A(\xi_r)V$ is bounded for any $\xi\in[\omega_1\dots\omega_n]$ and $r\ge n$. We can use [8, Corollary 1.2] for the product $A(\zeta_1\zeta_2)A(\zeta_3)\dots A(\zeta_{k-1})$ and each of the column vectors $A(\xi_{n'+1})\dots A(\xi_r)V$ and $A(\xi_{n'+1})\dots A(\xi_s)V$. Since – by Lemma 2.1 – the emplacement of the nonnull entries in both columns vectors is the same, we conclude that (6) holds for $n=|\zeta_1|+\dots+|\zeta_k|$ when k is large enough.

Suppose now there exists $n_0\in\mathbb{N}\cup\{0\}$ such that $\omega_{n_0+1}\omega_{n_0+2}\dots=\overline{0}$ or $\overline{2}$ or $\overline{100}$; as previously we consider all the sequences $\xi\in[\omega_1\dots\omega_n]$ and we choose n in order that (6) holds. We treat the first case, the second being similar, and the third trivial because in this case $\|P_r(\xi,V)-P_s(\xi,V)\|=0$ for any $r,s\ge n_0+6$. We use the same decomposition $\xi_{n+1}\dots\xi_r=w_1\dots w_\nu$ as in (7). Let 0^i, $i\ge0$, be the greatest prefix of $\xi_{n+1}\dots\xi_r$ that contains only the letter 0. If $\nu\ge2$, $w_2\in\mathcal{W}^\kappa$ contains some other letters than 0 hence 0^i is a strict prefix of w_1w_2 . We suppose $n\ge n_0+2$ and we make the Euclidean division

$$n+i-n_0-2=4k+k',\ \ k\in\mathbb{N},\ \ k'\in\{0,1,2,3\}, \tag{8}$$

so there exists a suffix s of w_1 (if $\nu=1$) or w_1w_2 (if $\nu\ge2$) that begins by 1 or 2 (if non empty) and such that

$$\xi_{n_0+1}\dots\xi_{n_0+4k}=0^{4k}\ \text{ and }\ \xi_{n_0+4k+1}\dots\xi_r=0^{k'+2}\xi_{n+i+1}\dots\xi_r=\begin{cases}0^{k'+2}s & \text{if }\nu\le2\\ 0^{k'+2}sw_3\dots w_\nu & \text{if }\nu\ge3.\end{cases}$$

From Lemma 2.5(ii) one has $0^{k'+2}s\in\mathcal{W}_{4\kappa,42\kappa+7}$, $0^{k'+2}sw_3\in\mathcal{W}_{6\kappa,63\kappa+7}$ and, from Lemma 2.5(i), $w_i\in\mathcal{W}_{2\kappa,21\kappa}$ for any $i\ge2$. The matrices $A(0^{k'+2}s)$, $A(0^{k'+2}sw_3)$ and $A(w_i)$ for $i\ge2$ satisfy the condition $\left(H^M\right)$ by Lemma 2.4. The second and the third matrix also satisfy (H_1) and (H_2) from Lemmas 2.2 and 2.3; so by [8, Lemma 1.3] the matrix $A\left(\xi_{n_0+4k+1}\right)\dots A(\xi_r)V'$ satisfies $\left(H^{2M''d}\right)$ in all the cases, where V' and M'' are defined as above, and the nonnull entries in the column vector $X = X(\xi, r) := A(\xi_{n_0+4k+1}) \dots A(\xi_r) V$ have bounded ratios independently on $\xi \in [\omega_1 \dots \omega_n]$ and $r \geq n$.

Consequently, denoting by $x_1 = x_1(\xi, r)$ the first entry – obviously nonnull – of X we have

$$A(\xi_{n_0+1} \dots \xi_r)V = A^{4k}X = x_1 \begin{pmatrix} 1 \\ k \\ k \\ 0 \\ k \\ k \\ k \end{pmatrix} + O(x_1).$$

Hence $\|A(\xi_{n_0+1} \dots \xi_r)V\|$, which is equal to $UA(\xi_1 \dots \xi_r)V$, is $x_1(5k+1) + O(x_1)$ and, denoting by σ the shift-map on $\{0,1,2\}^{\mathbb{N}}$,

$$P_{r-n_0}(\sigma^{n_0}\xi, V) = \frac{A(\xi_{n_0+1} \dots \xi_r)V}{\|A(\xi_{n_0+1} \dots \xi_r)V\|} = \frac{1}{5} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + O\left(\frac{1}{k}\right).$$

Since $P_r(\xi, V) = P_{n_0}(\omega, P_{r-n_0}(\sigma^{n_0}\xi, V))$ and since the map $P_{n_0}(\omega, \cdot)$ is continuous and do not vanish at the point $\frac{1}{5} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ – see the first and the second graph – we deduce

$$P_r(\xi, V) = P_{n_0}\left(\omega, \frac{1}{5} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right) + o(1)$$

for $\xi \in [\omega_1 \dots \omega_n]$ and $r \geq n$, where the term $o(1)$ tends to 0 when $n \to \infty$ because, from (8), $4k \geq n - n_0 - 5$. It follows that (6) holds for n large enough.

It remains to specify the form of the limit vector. In case the sequence ω is not eventually 0 nor eventually 2 nor eventually 100 we use the notations of [8, Theorem 1.1]: by [8, Corollary 1.2] the limit vector is V_1 and, by the second and the fourth assertion in [8, Theorem 1.1], $c(V_1) = c(P'_n V)$ for n large enough. Using Lemmas 2.2, 2.8 and the second graph, $c(P'_n V)$ for n large enough is $\{1,2,3,4,5\}$, $\{1,2,3,5,6,7\}$, $\{1,2,3,5,6\}$

or $\{1,3,4,5\}$. Suppose now $\omega = \overline{0},\overline{2}$ or $\overline{100}$; by computation $c\left(\lim_{n\to\infty} P_n(\omega,V)\right) = \{2,3,5,6,7\}, \{1,3,4\}$ or $\{1,2,3,4,5\}$ respectively. Finally if $\sigma^n\omega\overline{0},\overline{2}$ or $\overline{100}$ for some $n\in\mathbb{N}$ but not for $n=0$, then – in view of the first and the second graph – $c\left(\lim_{n\to\infty} P_n(\omega,V)\right) = \{1,2,3,4,5\}, \{1,2,3,5,6,7\}, \{1,2,3,5,6\}, \{1,2,3,5\}$ or $\{1,3,4,5\}$. ■

3. Application to some Bernoulli convolution

The *Parry expansion* [9] of the real number $x\in[0,1[$ in base $\beta>1$ is the unique sequence of integers $(\varepsilon_n(x))_{n\in\mathbb{N}}$ such that

$$\forall n\in\mathbb{N},\ x\in\left[s_n(x), s_n(x)+\frac{1}{\beta^n}\right[,\quad \text{where } s_n(x)=\sum_{k=1}^{n}\frac{\varepsilon_k(x)}{\beta^k}\,. \tag{9}$$

The set $\mathcal{A}dm_\beta$ of the *β-admissible sequences* is constituted by the sequences $(\varepsilon_n(x))_{n\in\mathbb{N}}$ for any $x\in[0,1[$. This set is closed and invariant under the shift $\sigma:(\varepsilon_n)_{n\in\mathbb{N}}\mapsto(\varepsilon_{n+1})_{n\in\mathbb{N}}$. We use the partition of $[0,1[$ by the *β-adic intervals of order n*, defined by

$$I_{\varepsilon_1\dots\varepsilon_n} := \{x\in[0,1[\ ;\ \varepsilon_1(x)=\varepsilon_1,\dots,\varepsilon_n(x)=\varepsilon_n\}\,.$$

We say that a measure η supported by $[0,1]$ has the *weak Gibbs property* in the sense of [12], with respect to the β-adic intervals, if there exists a map $\Phi:\mathcal{A}dm_\beta\to\mathbb{R}$, continuous for the product topology, such that

$$\lim_{n\to\infty}\left(\frac{\eta\left(I_{\varepsilon_1\dots\varepsilon_n}\right)}{\exp\left(\sum_{k=0}^{n-1}\Phi(\sigma^k\varepsilon)\right)}\right)^{1/n} = 1 \quad \text{uniformly on } (\varepsilon_n)_{n\in\mathbb{N}}\in\mathcal{A}dm_\beta. \tag{10}$$

It has the *Gibbs property* if the ratio $\dfrac{\eta\left(I_{\varepsilon_1\dots\varepsilon_n}\right)}{\exp\left(\sum_{k=0}^{n-1}\Phi(\sigma^k\varepsilon)\right)}$ itself is bounded from 0 and ∞.

Let us give a sufficient condition for η to have the weak Gibbs property. For each $\varepsilon\in\mathcal{A}dm_\beta$ we put $\phi_1(\varepsilon)=\log\eta\left(I_{\varepsilon_1}\right)$ and for $n\ge 2$,

$$\phi_n(\varepsilon)=\log\left(\frac{\eta\left(I_{\varepsilon_1\dots\varepsilon_n}\right)}{\eta\left(I_{\varepsilon_2\dots\varepsilon_n}\right)}\right). \tag{11}$$

The continuous map $\phi_n:\mathcal{A}dm_\beta\to\mathbb{R}$ $(n\ge1)$ is the *n-step potential* of η. Assume the existence of the uniform limit $\Phi=\lim_{n\to\infty}\phi_n$; then it is straightforward that for $n\ge1$,

$$\frac{1}{K_n} \leq \frac{\eta\left(I_{\varepsilon_1\ldots\varepsilon_n}\right)}{\exp\left(\sum_{k=0}^{n-1}\Phi(\sigma^k\varepsilon)\right)} \leq K_n \quad \text{with} \quad K_n = \exp\left(\sum_{k=1}^{n}\|\Phi-\phi_n\|_\infty\right). \tag{12}$$

By a well known lemma on the Cesàro sums, $K_1, K_2, \ldots$ form a subexponential sequence of positive real numbers, that is $\lim_{n\to\infty}(K_n)^{1/n} = 1$ and thus, (12) means that η has the weak Gibbs property with respect to the β-adic intervals.

The probability measures defined by Bernoulli convolutions have been studied from the 30th (see [2], and [10] for an overview on the subject). One calls the Bernoulli convolution $\mu_{\beta,p}$ associated with the base $\beta > 1$ and the parameter vector $p = (p_0, \ldots, p_{s-1})$, the infinite product of the sums of Dirac measures $p_0\delta_{\frac{0\cdot\alpha}{\beta^n}} + \cdots + p_{s-1}\delta_{\frac{(s-1)\cdot\alpha}{\beta^n}}$ for $n = 1, 2, \ldots$, where $\alpha = \dfrac{\beta-1}{s-1}$. In other words this measure is defined – for any Borelian $E \subseteq \mathbb{R}$ – by

$$\mu_{\beta,p}(E) := P\left(\left\{\omega \in \{0,\ldots,s-1\}^{\mathbb{N}}\ ;\ \alpha\sum_{n=1}^{\infty}\frac{\omega_n}{\beta^n} \in E\right\}\right).$$

where P be the probability measure defined on $\{0,1\}^{\mathbb{N}}$ by $P[\varepsilon_1\ldots\varepsilon_n] = p_{\varepsilon_1}\ldots p_{\varepsilon_n}$. It is also the unique measure with bounded support that satisfies the self-similarity relation (see[10]):

$$\mu_{\beta,p} = \sum_{i=0}^{s-1} p_i \cdot \mu_{\beta,p} \circ S_i^{-1}, \quad \text{where } S_i(x) := \frac{x+i\alpha}{\beta}.$$

$\mu_{\beta,p}$ is known to be purely singular with respect to the Lebesgue measure when p is uniform and β a Pisot number, that is, the conjuguates of β have modulus less than 1. The problem is to know if it can be purely singular in other cases (see [4]), and if it has the weak Gibbs property. The weak Gibbs property of a probability measure makes possible its multifractal analysis [1]; the one of $\mu_{\beta,p}$ for $\beta = \dfrac{1+\sqrt{5}}{2}$ and $p_0 = p_1 = \dfrac{1}{2}$ is detailed in [3]. Now in case β is a multinacci number and $p = (p_0, p_1)$ and except in one special case, $\mu_{\beta,p}$ has the weak Gibbs property (see [3, 5, 6]). This property seems more complicated to state for the other Pisot numbers of degree at least 3.

From now $\beta \approx 1.755$ is the unique solution greater than 1 of the equation

$$\beta^3 = 2\beta^2 - \beta + 1$$

and μ is the Bernoulli convolution associated with β and $p = \left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ (here $s = 2$ and $\alpha = \beta - 1$).

Lemma 3.1. *A sequence of integers $(\varepsilon_n)_{n\in\mathbb{N}}$ is β-admissible if and only if its terms are* 0 *or* 1*, and each couple of consecutive* 1 *is followed by a couple of consecutive* 0*.*

Proof: The condition (9) is equivalent to $\beta^n(x-s_n(x))=\beta^n(x-s_{n-1}(x))-\varepsilon_n(x)\in[0,1[$, hence $\varepsilon_n(x)=\lfloor\beta^n(x-s_{n-1}(x))\rfloor$ – with the convention that $s_0(x)=0$. This proves that $\varepsilon_n(x)=0$ or 1 because, using (9) at the rank $n-1$, one has $\beta^n(x-s_{n-1}(x))\in[0,\beta[$.

Now (9) also implies

$$\sum_{k=n+1}^{n+4}\frac{\varepsilon_k(x)}{\beta^k}\le x-s_n(x)<\frac{1}{\beta^n}=\frac{1}{\beta^{n+1}}+\frac{1}{\beta^{n+2}}+\frac{1}{\beta^{n+4}}$$

hence if $\varepsilon_{n+1}(x)=\varepsilon_{n+2}(x)=1$, $\varepsilon_{n+3}(x)$ and $\varepsilon_{n+4}(x)$ cannot be 1. Conversely one check easily that a sequence that satisfies the conditions of the lemma, also satisfies (9). ■

Consequently each β-admissible sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ can be decomposed from the left to the right in one sequence of words

$$\varepsilon_1\varepsilon_2\cdots=w_1w_2\ldots\quad\text{where }\forall i,\ w_i\in\mathcal{A}_1:=\{0,10,1100\},\tag{13}$$

and in the sequel we denote by $I_{w_1\ldots w_k}$ the β-adic interval $I_{\varepsilon_1\ldots\varepsilon_n}$ if the word $\varepsilon_1\ldots\varepsilon_n\in\{0,1\}^*$ is a concatenation of $w_1,\ldots,w_k\in\mathcal{A}_1$. In the two following lemma we compute the measure of the β-adic intervals.

Lemma 3.2. *Setting $i_1=0,\ i_2=1,\ i_3=1-(\beta-1)^2,\ i_4=-(\beta-1)^2,\ i_5=\beta-1,$ $i_6=\beta-(\beta-1)^2,\ i_7=\beta(\beta-1)$ one has, for any $w_1\ldots w_k\in{\mathcal{A}_1}^*$*

$$\begin{pmatrix}\mu\left(\frac{1}{\beta}\left(i_1+I_{w_1\ldots w_k}\right)\right)\\ \vdots\\ \mu\left(\frac{1}{\beta}\left(i_7+I_{w_1\ldots w_k}\right)\right)\end{pmatrix}=M(w_1)\ldots M(w_k)V\quad\text{where}\quad V:=\begin{pmatrix}3/5\\2/5\\13/20\\1/5\\3/5\\3/10\\1/5\end{pmatrix}\tag{14}$$

and $M(0):=\dfrac{1}{2}A(0),\ \ M(10):=\dfrac{1}{4}A(1),\ \ M(1100):=\dfrac{1}{16}A(2).$

Proof: For any real γ we evaluate $\mu\left(\dfrac{1}{\beta}\left(\gamma+I_{w_1\ldots w_k}\right)\right)$ in the three cases: $w_1=0,10$ or 1100. In the first, $(\beta-1)\displaystyle\sum_{n=1}^{\infty}\frac{\omega_n}{\beta^n}$ belongs to $\dfrac{1}{\beta}\left(\gamma+I_{w_1\ldots w_k}\right)$ if and only if $(\beta-1)\displaystyle\sum_{n=1}^{\infty}\frac{\omega_{n+1}}{\beta^n}\in$ $\dfrac{1}{\beta}\left(\gamma'+I_{w_1\ldots w_k}\right)$ with $\gamma'=\gamma\beta-\omega_1\beta(\beta-1)$ and, since $\omega_1\in\{0,1\}$,

$$\begin{aligned}&\mu\left(\tfrac{1}{\beta}\left(\gamma+I_{w_1\ldots w_k}\right)\right)=\tfrac{1}{2}\textstyle\sum_{\gamma'\in\Gamma'(\gamma)}\mu\left(\tfrac{1}{\beta}\left(\gamma'+I_{w_1\ldots w_k}\right)\right),\\&\text{where }\Gamma'(\gamma)=\{\gamma\beta-x\beta(\beta-1)\ ;\ x\in\{0,1\}\}.\end{aligned}\tag{15}$$

We proceed in the same way if $w_1=10$:

$$\begin{aligned}&\mu\left(\tfrac{1}{\beta}\left(\gamma+I_{w_1\ldots w_k}\right)\right)=\tfrac{1}{4}\textstyle\sum_{\gamma''\in\Gamma''(\gamma)}\mu\left(\tfrac{1}{\beta}\left(\gamma''+I_{w_1\ldots w_k}\right)\right),\\&\text{where }\Gamma'(\gamma)'=\{\gamma\beta^2+\beta-(x\beta+y)\beta(\beta-1)\ ;\ x,y\in\{0,1\}\}.\end{aligned}\tag{16}$$

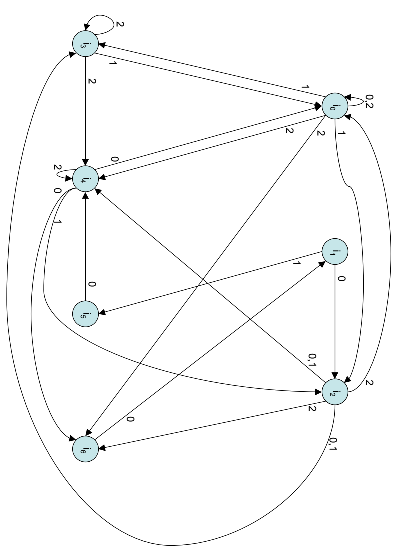
and if $w_1 = 1100$:

(17)

$$\mu\left(\frac{1}{\beta}\left(\gamma + I_{w_1\dots w_k}\right)\right) = \frac{1}{16}\sum_{\gamma'''\in\Gamma'''(\gamma)} \mu\left(\frac{1}{\beta}\left(\gamma''' + I_{w_1\dots w_k}\right)\right),$$

where $\Gamma'''(\gamma) = \{\gamma\beta^4 + \beta^3 + \beta^2 - (x\beta^3 + y\beta^2 + z\beta + t)\beta(\beta - 1)\ ;\ x, y, z, t \in \{0, 1\}\}$.

Since the measure μ has support $[0, 1]$, we can restrict the sums in (15), (16) and (17) to the indexes $\gamma^{(i)}$ such that $\mu\left(\dfrac{1}{\beta}\left(\gamma^{(i)} + [0, 1[\right)\right) \neq 0$ that is, $\gamma^{(i)} \in]-1, \beta[$. The relations $\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2$ defined by $\gamma\mathcal{R}_0\gamma' \Leftrightarrow \gamma' \in \Gamma'(\gamma)$, $\gamma\mathcal{R}_1\gamma'' \Leftrightarrow \gamma'' \in \Gamma''(\gamma)$ and $\gamma\mathcal{R}_2\gamma''' \Leftrightarrow \gamma''' \in \Gamma'''(\gamma)$ respectively, are represented below: each relation $\mathcal{R}_i$ is represented by the edges with label i, and the set of states is the set of elements of $]-1, \beta[$ that can be reached by some path from the initial state 0.



The incidence matrices of the three graphs being $A(0), A(1)$ and $A(2)$, we deduce from (15), (16) and (17) that

$$\begin{pmatrix} \mu\left(\frac{1}{\beta}\left(i_1 + I_{w_1\dots w_k}\right)\right) \\ \vdots \\ \mu\left(\frac{1}{\beta}\left(i_7 + I_{w_1\dots w_k}\right)\right) \end{pmatrix} = M(w_1)\dots M(w_k)V, \quad \text{where } V = \begin{pmatrix} \mu\left(\frac{1}{\beta}(i_1 + [0, 1[)\right) \\ \vdots \\ \left(\frac{1}{\beta}(i_7 + [0, 1[)\right) \end{pmatrix}.$$

We make $k = 1$ in this relation and we sum for $w_1 \in \mathcal{A}_1$. Since – by Lemma 3.1 – the I_{w_1} make a partition of $[0, 1[$, we obtain that V is an eigenvector of $M(0)+M(10)+M(1100)$.

Moreover the sum of the two first entries in V is $\mu\left(\frac{1}{\beta}([0,2[)\right)$ hence it is $\mu([0,1[) = 1$. Computing this eigenvector we obtain the expected value for V. ∎

We can now compute the values of the measure μ on the β-adic intervals:

LEMMA 3.3. *(i) For any $w_1 \dots w_k \in {\mathcal{A}_1}^*$,*

$$\mu\left(I_{w_1\dots w_k}\right) = \begin{cases} {}^tE_1M(w_2)\dots M(w_k)V & \text{if } w_1 = 0 \\ {}^tE_2M(0)M(w_2)\dots M(w_k)V & \text{if } w_1 = 10 \\ {}^tE_2M(10)M(0)M(w_2)\dots M(w_k)V & \text{if } w_1 = 1100. \end{cases} \tag{18}$$

(ii) For any $\varepsilon_1 \dots \varepsilon_n \in \{0,1\}^*$,

(19)
$$\mu\left(I_{\varepsilon_1\dots\varepsilon_n}\right) = \begin{cases} \mu\left(I_{w_1\dots w_k}\right) & \text{if } \varepsilon_1\dots\varepsilon_n = w_1\dots w_k \\ \mu\left(I_{w_1\dots w_k 10}\right) + \mu\left(I_{w_1\dots w_k 1100}\right) & \text{if } \varepsilon_1\dots\varepsilon_n = w_1\dots w_k 1 \\ \mu\left(I_{w_1\dots w_k 1100}\right) & \text{if } \varepsilon_1\dots\varepsilon_n = w_1\dots w_k 11 \text{ or } w_1\dots w_k 110. \end{cases}$$

It remains to prove that μ has the weak-Gibbs property, although we can't use n-step potential of μ because its limit is infinite at the point $1\overline{0}$.

PROPOSITION 3.4. *μ has the weak-Gibbs property with respect to the β-adic intervals.*

Proof: By the Kolmogorov consistency theorem there exists a unique measure μ' on $[0,1]$ such that – for any $w_1 \dots w_k \in {\mathcal{A}_1}^*$

$$\mu'\left(I_{w_1\dots w_k}\right) = \|M(w_1)\dots M(w_k)V\| = UM(w_1)\dots M(w_k)V. \tag{20}$$

We first prove that $\lim_{n\to\infty}\left(\frac{\mu'\left(I_{\varepsilon_1\dots\varepsilon_n}\right)}{\mu\left(I_{\varepsilon_1\dots\varepsilon_n}\right)}\right)^{1/n} = 1$ uniformly in $(\varepsilon_n)_{n\in\mathbb{N}} \in \mathcal{A}dm_\beta$, although the ratio $\frac{\mu'\left(I_{\varepsilon_1\dots\varepsilon_n}\right)}{\mu\left(I_{\varepsilon_1\dots\varepsilon_n}\right)}$ itself tends to ∞ in the cases $\varepsilon = \overline{0}$ and $\varepsilon = \overline{1100}$. Notice that in all cases this ratio is at least 1, in consequence of Lemma 3.3 and the inequalities $UM(0) \geq {}^tE_1$, $UM(10) \geq {}^tE_2M(0)$, $UM(1100) \geq {}^tE_2M(10)M(0)$.

It remains to find some upper bound for $\frac{\mu'\left(I_{\varepsilon_1\dots\varepsilon_n}\right)}{\mu\left(I_{\varepsilon_1\dots\varepsilon_n}\right)}$; it is sufficient to consider only the integers n such that $\varepsilon_1 \dots \varepsilon_n \in {\mathcal{A}_1}^*$, because μ' as well as μ satisfies (19). From now $\varepsilon_1 \dots \varepsilon_n \in {\mathcal{A}_1}^*$ and – except in the case $\varepsilon_1\varepsilon_2 = 10$ – we use the greatest prefix 0^ν or $(1100)^\nu$ of $\varepsilon_1 \dots \varepsilon_n$, with $\nu \in \mathbb{N}$: we have in all cases

$$\varepsilon_1 \dots \varepsilon_n = 0^\nu a w(n) \text{ or } 10w(n) \text{ or } (1100)^\nu b w(n) \tag{21}$$

where $\nu \in \mathbb{N}, w(n) \in {\mathcal{A}_1}^*, aw(n) = \phi$ or $a \in \{10, 1100\}, bw(n) = \phi$ or $b \in \{0, 10\}$. Setting $V_n = \frac{M(w(n))V}{\|M(w(n))V\|}$, or $V_n = \frac{V}{\|V\|}$ if $w(n) = \phi$, we deduce from (21), (20) and (18) that

(22)
$$\frac{\mu'\left(I_{\varepsilon_1\dots\varepsilon_n}\right)}{\mu\left(I_{\varepsilon_1\dots\varepsilon_n}\right)} = \frac{UM(0)^\nu M(a)V_n}{{}^tE_1M(0)^{\nu-1}M(a)V_n} \text{ or } \frac{UM(10)V_n}{{}^tE_2M(0)V_n} \text{ or } \frac{UM(1100)^\nu M(b)V_n}{{}^tE_2M(10)M(0)M(1100)^{\nu-1}M(b)V_n}.$$

By direct computation the entries of $UM(0)^\nu$ are at most $\dfrac{2\nu}{2^\nu}$ and the ones of $UM(1100)^\nu$ at most $\dfrac{3\nu}{16^\nu}$. On the other side ${}^tE_1M(0)^{\nu-1} = \dfrac{1}{2^{\nu-1}}\cdot{}^tE_1$ and ${}^tE_2M(10)M(0)M(1100)^{\nu-1} = \dfrac{1}{8\cdot 16^{\nu-1}}\cdot{}^tE_5$. Finally we obtain

$$\frac{\mu'\left(I_{\varepsilon_1\ldots\varepsilon_n}\right)}{\mu\left(I_{\varepsilon_1\ldots\varepsilon_n}\right)} \le 21\nu\cdot\frac{m_n^+}{m_n^-} \tag{23}$$

where $m_n^+ = \max\limits_i {}^tE_iV_n$ and $m_n^- = \min\left(\left({}^tE_1+{}^tE_7\right)V_n\ ,\ {}^tE_3V_n\ ,\ \min\limits_i\left({}^tE_i\dfrac{V}{\|V\|}\right)\right)$.

Now we use Proposition 2.9 in the following way: the column vector V_n is equal to $P_{n'}(\omega,V)$ for some $\omega\in\{0,1,2\}^{\mathbb{N}}$ and $n'\in\mathbb{N}\cup\{0\}$. The uniform convergence stated in this proposition and the form of the limit vector imply that the sequences

$$\begin{aligned}
&m_n^+(\omega) = \max\nolimits_i {}^tE_iP_n(\omega,V)\\
\text{and}\quad &m_n^-(\omega) = \min\left(\left({}^tE_1+{}^tE_7\right)P_n(\omega,V)\ ,\ {}^tE_3P_n(\omega,V)\ ,\ \min_i\left({}^tE_i\frac{V}{\|V\|}\right)\right)
\end{aligned}$$

converge uniformly and that their limits are two continued maps from the compact set $\{0,1,2\}^{\mathbb{N}}$ to $]0,\infty[$, hence $\dfrac{m_n^+(\omega)}{m_n^-(\omega)}$ is bounded for any $\omega\in\{0,1,2\}^{\mathbb{N}}$ and $n\in\mathbb{N}\cup\{0\}$. So (23), with the inequality $\nu\le n$, imply that $\lim\limits_{n\to\infty}\left(\dfrac{\mu'\left(I_{\varepsilon_1\ldots\varepsilon_n}\right)}{\mu\left(I_{\varepsilon_1\ldots\varepsilon_n}\right)}\right)^{1/n} = 1$.

From the definition of the weak Gibbs property, μ has this property if μ' has. Now, μ' has this property if the exponential of its n-step potential that is, $\dfrac{\mu'[\![\varepsilon_1\ldots\varepsilon_n]\!]}{\mu'[\![\varepsilon_2\ldots\varepsilon_n]\!]}$, converges uniformly in $(\varepsilon_n)_{n\in\mathbb{N}}\in\mathcal{A}dm_\beta$ to a nonnull limit. For the same reason as above it is sufficient to prove this convergence for the integers n such that $\varepsilon_1\ldots\varepsilon_n$ is a concatenation of words in ${\mathcal{A}_1}^*$. We distinguish the cases $\varepsilon_1\ldots\varepsilon_n = 0w'(n)$ or $10w'(n)$ or $1100w'(n)$ with $w'(n)\in{\mathcal{A}_1}^*$ and we obtain

$$\begin{aligned}
\frac{2\mu'[\![\varepsilon_1\ldots\varepsilon_n]\!]}{\mu'[\![\varepsilon_2\ldots\varepsilon_n]\!]} &= \frac{\begin{pmatrix}2&1&1&1&2&0&1\end{pmatrix}S_n}{\begin{pmatrix}1&1&1&1&1&1&1\end{pmatrix}S_n} \text{ or } \frac{\begin{pmatrix}1&0&2&2&1&1&0\end{pmatrix}S_n}{\begin{pmatrix}2&1&1&1&2&0&1\end{pmatrix}S_n}\\
&\quad\text{or } \frac{\begin{pmatrix}2&0&0&1&3&0&2\end{pmatrix}S_n}{\begin{pmatrix}2&0&0&2&3&0&1\end{pmatrix}S_n}
\end{aligned}$$

respectively, where $S_n = \dfrac{M(w'(n))V}{\|M(w'(n))V\|}$. The three ratios converge uniformly to some nonnull limits because, from Proposition 2.9, their numerators and denominators do. ∎

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