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# The Hijazi inequalities on complete Riemannian Spin<sup>c</sup> manifolds

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## Abstract

In this paper, we extend the Hijazi inequality, involving the Energy-Momentum tensor, for the eigenvalues of the Dirac operator on complete Riemannian Spin<sup>c</sup> manifolds of finite volume and without boundary. Under some additional assumptions and using the refined Kato inequality, we prove the Hijazi inequality for elements of the essential spectrum. The limiting cases are also studied.

**Key words:** Spin<sup>c</sup> structures, Dirac operator, eigenvalues, Energy-Momentum tensor, perturbed Yamabe operator, conformal geometry, refined Kato inequality.

## 1 Introduction

On a compact Riemannian Spin<sup>c</sup> manifold  $(M^n, g)$  of dimension  $n \geq 2$ , any eigenvalue  $\lambda$  of the Dirac operator satisfies a Friedrich type inequality [14]

$$\lambda^2 \geq \lambda_1^2 := \frac{n}{4(n-1)} \inf_M (S - c_n |\Omega|), \quad (1)$$

where  $S$  denotes the scalar curvature of  $M$ ,  $c_n = 2[\frac{n}{2}]^{\frac{1}{2}}$  and  $i\Omega$  is the curvature form of the connection on the line bundle given by the Spin<sup>c</sup> structure. Equality is achieved if and only if the eigenspinor  $\psi$  associated with the first eigenvalue  $\lambda_1$  is a Killing Spin<sup>c</sup> spinor and  $\Omega \cdot \psi = i\frac{c_n}{2} |\Omega| \psi$ , i.e., for every  $X \in \Gamma(TM)$  the eigenspinor  $\psi$  satisfies

$$\begin{cases} \nabla_X \psi = -\frac{\lambda_1}{n} X \cdot \psi, \\ \Omega \cdot \psi = i\frac{c_n}{2} |\Omega| \psi. \end{cases} \quad (2)$$

Here  $X \cdot \psi$  denotes the Clifford multiplication and  $\nabla$  the spinorial Levi-Civita connection [12]. In [15], it is shown that on a compact Riemannian  $\text{Spin}^c$  manifold any eigenvalue  $\lambda$  of the Dirac operator to which is attached an eigenspinor  $\psi$  satisfies

$$\lambda^2 \geq \inf_M \left( \frac{1}{4}S - \frac{c_n}{4}|\Omega| + |\ell^\psi|^2 \right), \quad (3)$$

where  $\ell^\psi$  is the field of symmetric endomorphisms associated with the field of quadratic forms denoted by  $T^\psi$  and called the Energy-Momentum tensor. It is defined on the complement set of zeroes of the eigenspinor  $\psi$ , for any vector field  $X$  by

$$T^\psi(X) = \text{Re} \left\langle X \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2} \right\rangle.$$

The limiting case of (3) is characterized by the existence of a spinor field  $\psi$  satisfying for all  $X \in \Gamma(TM)$ ,

$$\begin{cases} \nabla_X \psi = -\ell^\psi(X) \cdot \psi, \\ \Omega \cdot \psi = i \frac{c_n}{2} |\Omega| \psi. \end{cases} \quad (4)$$

The trace of  $\ell^\psi$  being equal to  $\lambda$ , Inequality (3) improves Inequality (1) since by the Cauchy-Schwarz inequality,  $|\ell^\psi|^2 \geq \frac{(\text{tr}(\ell^\psi))^2}{n}$ , where  $\text{tr}$  denotes the trace of  $\ell^\psi$ . It is also shown that the sphere equipped with a special  $\text{Spin}^c$  structure is a limiting manifold for (3) but equality in (1) cannot occur.

In the same spirit as in [10], A. Moroianu and M. Herzlich (see [14]) generalized the result in [10] to the case of compact  $\text{Spin}^c$  manifolds of dimension  $n \geq 3$ , by proving that any eigenvalue of the Dirac operator satisfies

$$\lambda^2 \geq \lambda_1^2 := \frac{n}{4(n-1)} \mu_1, \quad (5)$$

where  $\mu_1$  is the first eigenvalue of the perturbed Yamabe operator defined by

$$L^\Omega = 4 \frac{n-1}{n-2} \Delta + S - c_n |\Omega|_g.$$

The limiting case of (5) is equivalent to the limiting case in (1). In terms of the Energy-Momentum tensor the author [15] proved that any eigenvalue of the Dirac operator to which is attached an eigenspinor  $\psi$  satisfies

$$\lambda^2 \geq \frac{1}{4} \mu_1 + \inf_M |\ell^\psi|^2. \quad (6)$$

In this paper we examine these lower bounds on open manifolds, and especially on complete Riemannian  $\text{Spin}^c$  manifolds. We prove the following:

**Theorem 1.1.** *Let  $(M^n, g)$  be a complete Riemannian  $\text{Spin}^c$  manifold of finite volume. Let  $\lambda$  be an eigenvalue of the Dirac operator to which is attached an eigenspinor  $\psi$ , then*

$$\lambda^2 \geq \inf_M \left( \frac{1}{4} S - \frac{c_n}{4} |\Omega| + |\ell^\psi|^2 \right). \quad (7)$$

*Equality in (7) holds if and only if the eigenspinor associated with the first eigenvalue  $\lambda_1$  satisfies Equation (4).*

A Friedrich type inequality is derived:

**Corollary 1.2.** *Under the same conditions as Theorem 1.1, any eigenvalue of the Dirac operator satisfies*

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M (S - c_n |\Omega|). \quad (8)$$

*Equality in (8) holds if and only if the eigenspinor associated with the first eigenvalue  $\lambda_1$  satisfies Equation (2).*

This was proven by N. Grosse in [7] and [9] for complete spin manifolds of finite volume. Using the conformal covariance of the Dirac operator we prove

**Theorem 1.3.** *Let  $(M^n, g)$  be a complete Riemannian  $\text{Spin}^c$  manifold of finite volume and dimension  $n > 2$ . Moreover, let  $\lambda$  be an eigenvalue of the Dirac operator  $D$  associated with an eigenspinor  $\psi$ . Then the following inequality holds:*

$$\lambda^2 \geq \frac{1}{4} \mu_1 + \inf_M |\ell^\psi|^2, \quad (9)$$

where  $\mu_1$  is the infimum of the spectrum of the perturbed Yamabe operator.

$$\text{Equality in (9) holds} \iff \begin{cases} \bar{\nabla}_X \bar{\varphi} = -\ell^{\bar{\varphi}}(X) \cdot \bar{\varphi}, \\ \Omega \cdot \psi = i \frac{c_n}{2} |\Omega|_g \psi, \end{cases} \quad (10)$$

for every  $X \in \Gamma(TM)$ , where  $\bar{\varphi} = e^{-\frac{n-1}{2}u} \bar{\psi}$  and  $\bar{\psi}$  is the image of  $\psi$  under the isometry between the spinor bundles of  $(M^n, g)$  and  $(M^n, \bar{g} = e^{2u}g)$ .

From this Theorem, a Hijazi type inequality can be derived:

**Corollary 1.4.** *Under the same conditions as Theorem 1.3, any eigenvalue of the Dirac operator satisfies*

$$\lambda^2 \geq \frac{n}{4(n-1)} \mu_1. \quad (11)$$

*Equality in (11) holds if and only if the eigenspinor associated with the first eigenvalue  $\lambda_1$  satisfies Equation (2).*

This was also proven by N. Grosse in [7] and [9] for complete spin manifolds. On complete manifolds, the Dirac operator is essentially self-adjoint and, in general, its spectrum consists of eigenvalues and the essential spectrum. For elements of the essential spectrum, we also extend to  $\text{Spin}^c$  manifolds a Hijazi-type inequality obtained by N. Grosse in [9] on spin manifolds:

**Theorem 1.5.** *Let  $(M^n, g)$  be a complete Riemannian  $\text{Spin}^c$  manifold of dimension  $n \geq 5$  with finite volume. Furthermore, let  $S - c_n|\Omega|$  be bounded from below. If  $\lambda$  is in the essential spectrum of the Dirac operator  $\sigma_{\text{ess}}(D)$ , then*

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu_1,$$

where  $\mu_1$  is the infimum of the spectrum of the perturbed Yamabe operator.

For the 2-dimensional case, N. Grosse proved in [7] that for any Riemannian spin surface of finite area, homeomorphic to  $\mathbb{R}^2$  we have

$$\lambda^+ \geq \frac{4\pi}{\text{Area}(M^2, g)}, \quad (12)$$

Where  $\lambda^+ = \inf_{\varphi \in C_c^\infty(M)} \frac{(D^2\varphi, \varphi)}{(\varphi, \varphi)}$  (in the compact case,  $\lambda^+$  coincides with the first eigenvalue of the square of the Dirac operator). Recently, in [2] C. Bär showed the same inequality for any connected 2-dimensional Riemannian manifold of genus 0, with finite area and equipped with a spin structure which is *bounding at infinity*. A spin structure on  $M$  is said to be *bounding at infinity* if  $M$  can be embedded into  $\mathbb{S}^2$  in such a way that the spin structure extends to the unique spin structure of  $\mathbb{S}^2$ . On  $\text{Spin}^c$  manifolds, this question will be considered on a forthcoming work.

Even though the number  $\inf_M |\ell^\psi|^2$  is not easily computable, it appears naturally in many situations. For example, on hypersurfaces of certain limiting  $\text{Spin}^c$  manifolds it is easy to see, with the help of the  $\text{Spin}^c$  Gauss formula, that it is precisely the second fundamental form. Also, when deforming the Riemannian metric in the direction of the Energy-Momentum tensor, the eigenvalues of the Dirac operator on a  $\text{Spin}^c$  manifold are then critical (see [16]). The author would like to thank Oussama Hijazi for his support and encouragements.

## 2 Preliminaries

In this section we briefly introduce basic notions concerning  $\text{Spin}^c$  manifolds, the Dirac operator and its conformal covariance. Then we recall the refined Kato inequality which is crucial for the proof.

### The Dirac operator on $\text{Spin}^c$ manifolds

Let  $(M^n, g)$  be a connected and oriented Riemannian manifold of dimension  $n \geq 2$  without boundary. Furthermore, let  $SOM$  be the  $SO_n$ -principal bundle over  $M$  of

positively oriented orthonormal frames. A  $\text{Spin}^c$  structure of  $M$  is a  $\text{Spin}_n^c$ -principal bundle  $(\text{Spin}^c M, \pi, M)$  and an  $\mathbb{S}^1$ -principal bundle  $(\mathbb{S}^1 M, \pi, M)$  together with a double covering given by  $\theta : \text{Spin}^c M \longrightarrow \text{SOM} \times_M \mathbb{S}^1 M$  such that

$$\theta(ua) = \theta(u)\xi(a),$$

for every  $u \in \text{Spin}^c M$  and  $a \in \text{Spin}_n^c$ , where  $\xi$  is the 2-fold covering of  $\text{Spin}_n^c$  over  $\text{SO}_n \times \mathbb{S}^1$ . A Riemannian manifold that admits a  $\text{Spin}^c$  structure is called a Riemannian  $\text{Spin}^c$  manifold.

Let  $\Sigma M := \text{Spin}^c M \times_{\rho_n} \Sigma_n$  be the associated spinor bundle where  $\Sigma_n = \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$  and  $\rho_n : \text{Spin}_n^c \longrightarrow \text{End}(\Sigma_n)$  the complex spinor representation. A section of  $\Sigma M$  will be called a spinor and the set of all spinors will be denoted by  $\Gamma(\Sigma M)$  and those of compactly supported smooth spinors by  $\Gamma_c(\Sigma M)$ . The spinor bundle  $\Sigma M$  is equipped with a natural Hermitian scalar product, denoted by  $\langle \cdot, \cdot \rangle$  and satisfies

$$\langle X \cdot \psi, \varphi \rangle = - \langle \psi, X \cdot \varphi \rangle \quad \text{for every } X \in \Gamma(TM) \text{ and } \psi, \varphi \in \Gamma(\Sigma M),$$

where  $X \cdot \psi$  denotes the Clifford multiplication of  $X$  and  $\psi$ . With this Hermitian scalar product we define an  $L^2$ -scalar product

$$(\psi, \varphi) = \int_M \langle \psi, \varphi \rangle \mathbf{v}_g,$$

for any spinors  $\psi$  and  $\varphi$  in  $\Gamma_c(\Sigma M)$ . Additionally, given a connection 1-form  $A$  on  $\mathbb{S}^1 M$ ,  $A : T(\mathbb{S}^1 M) \longrightarrow i\mathbb{R}$  and the connection 1-form  $\omega^M$  on  $\text{SOM}$  for the Levi-Civita connection  $\nabla^M$ , we consider the associated connection on the principal bundle  $\text{SOM} \times_M \mathbb{S}^1 M$ , and hence a covariant derivative  $\nabla$  on  $\Gamma(\Sigma M)$  [6].

The curvature of  $A$  is an imaginary valued 2-form denoted by  $F_A = dA$ , i.e.,  $F_A = i\Omega$ , where  $\Omega$  is a real valued 2-form on  $\mathbb{S}^1 M$ . We know that  $\Omega$  can be viewed as a real valued 2-form on  $M$  [6]. In this case  $i\Omega$  is the curvature form of the associated line bundle  $L$ . It is the complex line bundle associated with the  $\mathbb{S}^1$ -principal bundle via the standard representation of the unit circle. For any spinor  $\psi$  and any real 2-form  $\Omega$  we have [14]

$$\langle i\Omega \cdot \psi, \psi \rangle \geq -\frac{c_n}{2} |\Omega|_g |\psi|^2, \quad (13)$$

where  $|\Omega|_g$  is the norm of  $\Omega$  with respect to  $g$ . Moreover, if equality holds in (13), then

$$\Omega \cdot \psi = i\frac{c_n}{2} |\Omega|_g \psi. \quad (14)$$

For every spinor  $\psi$ , the Dirac operator is locally defined by

$$D\psi = \sum_{i=1}^n e_i \cdot \nabla_{e_i} \psi.$$

It is an elliptic and formally self-adjoint operator with respect to the  $L^2$ -scalar product, i.e., for all spinors  $\psi, \varphi$  at least one of which is compactly supported on  $M$  we have  $(D\psi, \varphi) = (\psi, D\varphi)$ . An important tool when examining the Dirac operator is the Schrödinger-Lichnerowicz formula

$$D^2 = \nabla^* \nabla + \frac{1}{4} S \text{Id}_{\Gamma(\Sigma M)} + \frac{i}{2} \Omega \cdot, \quad (15)$$

where  $S$  is the scalar curvature,  $\nabla^*$  is the adjoint of  $\nabla$  and  $\Omega \cdot$  is the extension of the Clifford multiplication to differential forms given by  $(e_i^* \wedge e_j^*) \cdot \psi = e_i \cdot e_j \cdot \psi$ . For the Friedrich connection  $\nabla_X^f \psi = \nabla_X \psi + \frac{f}{n} X \cdot \psi$  where  $f$  is real valued function one gets a Schrödinger-Lichnerowicz type formula similar to the one obtained by Friedrich in [5]:

$$(D - f)^2 \psi = \Delta^f \psi + \left( \frac{S}{4} + \frac{n-1}{n} f^2 \right) \psi + \frac{i}{2} \Omega \cdot \psi - \frac{n-1}{n} (2fD\psi + \nabla f \cdot \psi), \quad (16)$$

where  $\Delta^f$  is the spinorial Laplacian associated with the connection  $\nabla^f$ .

A complex number  $\lambda$  is an eigenvalue of  $D$  if there exists a nonzero eigenspinor  $\psi \in \Gamma(\Sigma M) \cap L^2(\Sigma M)$  with  $D\psi = \lambda\psi$ . The set of all eigenvalues is denoted by  $\sigma_p(D)$ , the point spectrum. We know that if  $M$  is closed, the Dirac operator has a pure point spectrum but on open manifolds, the spectrum might have a continuous part. In general the spectrum of the Dirac operator  $\sigma(D)$  is composed of the point, the continuous and the residual spectrum. For complete manifolds, the residual spectrum is empty and  $\sigma(D) \subset \mathbb{R}$ . Thus, for complete manifolds, the spectrum can be divided into point and continuous spectrum. But often another decomposition of the spectrum is used: the one into discrete spectrum  $\sigma_d(D)$  and essential spectrum  $\sigma_{ess}(D)$ .

A complex number  $\lambda$  lies in the essential spectrum of  $D$  if there exists a sequence of smooth compactly supported spinors  $\psi_i$  which are orthonormal with respect to the  $L^2$ -product and

$$\|(D - \lambda)\psi_i\|_{L^2} \longrightarrow 0.$$

The essential spectrum contains all eigenvalues of infinite multiplicity. In contrast, the discrete spectrum  $\sigma_d(D) := \sigma_p(D) \setminus \sigma_{ess}(D)$  consists of all eigenvalues of finite multiplicity. The proof of the next property can be found in [7]:

On a  $\text{Spin}^c$  complete Riemannian manifold, 0 is in the essential spectrum of  $D - \lambda$  if and only if 0 is in the essential spectrum of  $(D - \lambda)^2$  and in this case, there is a normalized sequence  $\psi_i \in \Gamma_c(\Sigma M)$  such that  $\psi_i$  converges  $L^2$ -weakly to 0 with  $\|(D - \lambda)\psi_i\|_{L^2} \longrightarrow 0$  and  $\|(D - \lambda)^2\psi_i\|_{L^2} \longrightarrow 0$ .

### Spinor bundles associated with conformally related metrics

The conformal class of  $g$  is the set of metrics  $\bar{g} = e^{2u}g$ , for a real function  $u$  on  $M$ . At a given point  $x$  of  $M$ , we consider a  $g$ -orthonormal basis  $\{e_1, \dots, e_n\}$  of  $T_x M$ . The corresponding  $\bar{g}$ -orthonormal basis is denoted by  $\{\bar{e}_1 = e^{-u}e_1, \dots, \bar{e}_n = e^{-u}e_n\}$ .

This correspondence extends to the  $\text{Spin}^c$  level to give an isometry between the associated spinor bundles. We put a “ $-$ ” above every object which is naturally associated with the metric  $\bar{g}$ . Then, for any spinor fields  $\psi$  and  $\varphi$ , one has  $\langle \bar{\psi}, \bar{\varphi} \rangle = \langle \psi, \varphi \rangle$ , where  $\langle \cdot, \cdot \rangle$  denotes the natural Hermitian scalar products on  $\Gamma(\Sigma M)$ , and on  $\Gamma(\Sigma \bar{M})$ . The corresponding Dirac operators satisfy

$$\bar{D} ( e^{-\frac{n-1}{2}u} \bar{\psi} ) = e^{-\frac{n+1}{2}u} \bar{D}\bar{\psi}. \quad (17)$$

The norms of any real 2-form  $\Omega$  with respect to  $g$  and  $\bar{g}$  are related by

$$|\Omega|_{\bar{g}} = e^{-2u} |\Omega|_g. \quad (18)$$

O. Hijazi [11] showed that on a spin manifold the Energy-Momentum tensor verifies

$$|\ell^{\bar{\varphi}}|_{\bar{g}}^2 = e^{-2u} |\ell^{\varphi}|_g^2 = e^{-2u} |\ell^{\psi}|_g^2, \quad (19)$$

where  $\varphi = e^{-\frac{(n-1)}{2}u}\psi$ . We extend the result to a  $\text{Spin}^c$  manifold and get the same relation.

### Refined Kato inequalities

The Kato inequality states that for any section  $\psi$  of a Riemannian or Hermitian vector bundle  $E$  endowed with a metric connection  $\nabla$  on a Riemannian manifold  $(M, g)$  we have, away from the zeros of  $\psi$

$$|d|\varphi|| \leq |\nabla\varphi|. \quad (20)$$

This could be seen as follows  $2|\varphi||d|\varphi|| = |d|\varphi|^2| = 2|\langle \nabla\varphi, \varphi \rangle| \leq 2|\varphi||\nabla\varphi|$ . In [4], refined Kato inequalities were obtained for sections in the kernel of first order elliptic differential operators  $P$ . They are of the form

$$|d|\varphi|| \leq k_P |\nabla\varphi|,$$

where  $k_P$  is a constant depending on the operator  $P$ . Without the assumption that  $\varphi \in \ker P$ , we get away from the zero set of  $\varphi$

$$|d|\varphi|| \leq |P\varphi| + k_P |\nabla\varphi|. \quad (21)$$

A proof of (21) can be found in [4], [7], [3] or [9]. In [4] the constant  $k_P$  is determined in terms of the conformal weights of the differential operator  $P$ . For the Dirac operator  $D$  and for  $D - \lambda$ , where  $\lambda \in \mathbb{R}$ , we have  $k_D = k_{D-\lambda} = \sqrt{\frac{n-1}{n}}$ .

## 3 Proof of the Hijazi inequalities

First, we follow the main idea of the proof of the original Hijazi inequality in the compact case ([11], [10]), and its proof on spin noncompact case obtained by N. Grosse [9]. We choose the conformal factor with the help of an eigenspinor and we use cut-off functions near its zero-set and near infinity to obtain compactly supported test functions.

**Proof of Theorem 1.3.** Let  $\psi \in C^\infty(M, S) \cap L^2(M, S)$  be a normalised eigen-spinor, i.e.,  $D\psi = \lambda\psi$  and  $\|\psi\| = 1$ . Its zero-set  $\Upsilon$  is closed and contained in a closed countable union of smooth  $(n-2)$ -dimensional submanifolds which has locally finite  $(n-2)$ -dimensional Hausdorff measure [1]. Fix a point  $p \in M$ . Since  $M$  is complete, there exists a cut-off function  $\eta_i : M \rightarrow [0, 1]$  which is zero on  $M \setminus B_{2l}(p)$  and equal 1 on  $B_l(p)$ , where  $B_l(p)$  is the ball of center  $p$  and radius  $l$ . In between, the function is chosen such that  $|\nabla\eta_i| \leq \frac{4}{l}$  and  $\eta_i \in C_c^\infty(M)$ . While  $\eta_i$  cuts off  $\psi$  at infinity, we define another cut-off near the zeros of  $\psi$ . For this purpose, we can assume without loss of generality that  $\Upsilon$  is itself a countable union of  $(n-2)$ -submanifolds described above. Let  $\rho_{a,\epsilon}$  be defined as

$$\rho_{a,\epsilon}(x) = \begin{cases} 0 & \text{for } r < a\epsilon \\ 1 - \delta \ln \frac{\epsilon}{r} & \text{for } a\epsilon \leq r \leq \epsilon \\ 1 & \text{for } \epsilon < r \end{cases}$$

where  $r = d(x, \Upsilon)$  is the distance from  $x$  to  $\Upsilon$ . The constant  $0 < a < 1$  is chosen such that  $\rho_{a,\epsilon}(a\epsilon) = 0$ , i.e.,  $a = e^{-\frac{1}{\delta}}$ . Then  $\rho_{a,\epsilon}$  is continuous, constant outside a compact set and Lipschitz. Hence, for  $\varphi \in \Gamma(\Sigma M)$  the spinor  $\rho_{a,\epsilon}\varphi$  is an element in  $H_1^r(\Sigma M)$  for all  $1 \leq r \leq \infty$ .

Now, consider  $\Psi := \eta_i \rho_{a,\epsilon} \psi \in H_1^r(\Sigma M)$ . These spinors are compactly supported on  $M \setminus \Upsilon$ . Furthermore,  $\bar{g} = e^{2u}g = h^{\frac{4}{n-2}}g$  with  $h = |\psi|^{\frac{n-2}{n-1}}$  is a metric on  $M \setminus \Upsilon$ . Setting  $\bar{\Phi} := e^{-\frac{n-1}{2}u}\Psi$  ( $\varphi = e^{-\frac{n-1}{2}u}\psi$ ), Equations (13), (18), (19) and the Schrödinger-Lichnerowicz-type formula [15, (18)] imply

$$\begin{aligned} \|\nabla^{\ell^{\bar{\Phi}}}\bar{\Phi}\|_{\bar{g}}^2 &= \|\bar{D}\bar{\Phi}\|_{\bar{g}}^2 - \frac{1}{4} \int_{M \setminus \Upsilon} \bar{S}|\bar{\Phi}|^2_{\bar{v}_{\bar{g}}} - \int_{M \setminus \Upsilon} |\ell^{\bar{\Phi}}|^2 |\bar{\Phi}|^2_{\bar{v}_{\bar{g}}} \\ &\quad - \int_{M \setminus \Upsilon} \langle \frac{i}{2} \Omega \cdot \bar{\Phi}, \bar{\Phi} \rangle_{\bar{v}_{\bar{g}}} \\ &\leq \|\bar{D}\bar{\Phi}\|_{\bar{g}}^2 - \frac{1}{4} \int_M \bar{S}|\Psi|^2 e^u_{v_g} - \int_M |\ell^{\bar{\Phi}}|^2 |\Psi|^2 e^u_{v_g} \\ &\quad + \frac{c_n}{4} \int_M |\Omega|_{\bar{g}} |\Psi|^2 e^u_{v_{\bar{g}}} \\ &= \|\bar{D}\bar{\Phi}\|_{\bar{g}}^2 - \frac{1}{4} \int_M (\bar{S}e^{2u} - c_n |\Omega|_g) |\Psi|^2 e^{-u}_{v_g} - \int_M |\ell^{\Psi}|^2 |\Psi|^2 e^{-u}_{v_g} \\ &= \|\bar{D}\bar{\Phi}\|_{\bar{g}}^2 - \frac{1}{4} \int_M (h^{-1}L^\Omega h) |\Psi|^2 e^{-u}_{v_g} - \int_M |\ell^{\Psi}|^2 |\Psi|^2 e^{-u}_{v_g}, \end{aligned}$$

where  $\nabla_X^{\ell^\varphi} \varphi$  is the spinor field defined in [11] by  $\nabla_X^{\ell^\varphi} \varphi := \nabla_X \varphi + \ell^\varphi(X) \cdot \varphi$  and where we used that  $|\bar{\Phi}|^2_{\bar{v}_{\bar{g}}} = e^u |\Psi|^2_{v_g}$  and  $\bar{S}e^{2u} - c_n |\Omega|_g = h^{-1}L^\Omega h$  (see [15]).

Using  $\overline{D}\overline{\varphi} = \lambda e^{-u}\overline{\varphi}$  and  $\langle \nabla(\eta_i \rho_{a,\epsilon}) \cdot \overline{\varphi}, \overline{\varphi} \rangle \in C^\infty(M, \mathbb{R})$ , we calculate

$$\begin{aligned} \|\overline{D}\overline{\Phi}\|_{\overline{g}}^2 &= \int_M |\overline{D}(\eta_i \rho_{a,\epsilon} \overline{\varphi})|^2_{\overline{v}_g} \\ &= \int_M |\nabla(\eta_i \rho_{a,\epsilon}) \cdot \overline{\varphi}|^2_{\overline{v}_g} + \int_M \lambda^2 \eta_i^2 \rho_{a,\epsilon}^2 e^{-2u} |\varphi|^2_{\overline{v}_g} \\ &= \|\nabla(\eta_i \rho_{a,\epsilon}) \cdot \overline{\varphi}\|_{\overline{g}}^2 + \lambda^2 \int_M \eta_i^2 \rho_{a,\epsilon}^2 e^{-(n+2)u} |\varphi|^2_{\mathbf{v}_g}. \end{aligned} \quad (22)$$

Using (22) and  $\|\overline{\nabla}^{\ell^{\overline{\Phi}}}\overline{\Phi}\|_{\overline{g}}^2 \geq 0$  in the equation above, we get

$$\begin{aligned} \|\nabla(\eta_i \rho_{a,\epsilon}) \cdot \overline{\varphi}\|_{\overline{g}}^2 &\geq \frac{1}{4} \int_M (h^{-1} L^\Omega h) |\Psi|^2 e^{-u} \mathbf{v}_g + \int_M |\ell^\Psi|^2 |\Psi|^2 e^{-u} \mathbf{v}_g \\ &\quad - \lambda^2 \int_M \eta_i^2 \rho_{a,\epsilon}^2 |\varphi|^2 e^{-(n+2)u} \mathbf{v}_g. \end{aligned}$$

Moreover, we have

$$\|\nabla(\eta_i \rho_{a,\epsilon}) \cdot \overline{\varphi}\|_{\overline{g}}^2 = \int_M |e^{-u} \overline{\nabla(\eta_i \rho_{a,\epsilon}) \cdot \varphi}|^2_{\overline{v}_g} = \int_M |\nabla(\eta_i \rho_{a,\epsilon}) \cdot \psi|^2 e^{-u} \mathbf{v}_g.$$

Thus, with  $e^u = |\psi|^{\frac{2}{n-1}}$  the above inequality reads

$$\begin{aligned} \int_M |\nabla(\eta_i \rho_{a,\epsilon})|^2 |\psi|^{\frac{2n-2}{n-1}} \mathbf{v}_g &\geq \frac{1}{4} \int_M \eta_i \rho_{a,\epsilon} |\psi|^{\frac{n-2}{n-1}} L^\Omega (\eta_i \rho_{a,\epsilon} |\psi|^{\frac{n-2}{n-1}}) \mathbf{v}_g \\ &\quad - \frac{n-1}{n-2} \int_M |\nabla(\eta_i \rho_{a,\epsilon})|^2 |\psi|^{\frac{2n-2}{n-1}} \mathbf{v}_g + \int_M |\ell^\psi|^2 |\psi|^{\frac{2n-2}{n-1}} \eta_i^2 \rho_{a,\epsilon}^2 \mathbf{v}_g \\ &\quad - \lambda^2 \int_M \eta_i^2 \rho_{a,\epsilon}^2 |\psi|^{\frac{2n-2}{n-1}} \mathbf{v}_g. \end{aligned}$$

Hence, we obtain

$$\frac{2n-3}{n-2} \int_M |\nabla(\eta_i \rho_{a,\epsilon})|^2 |\psi|^{\frac{2n-2}{n-1}} \mathbf{v}_g \geq \left( \frac{\mu_1}{4} + \inf_M |\ell^\psi|^2 - \lambda^2 \right) \int_M \eta_i^2 \rho_{a,\epsilon}^2 |\psi|^{\frac{2n-2}{n-1}} \mathbf{v}_g,$$

where  $\mu_1$  is the infimum of the spectrum of the perturbed conformal Laplacian. With  $|\eta_i \nabla \rho_{a,\epsilon} + \rho_{a,\epsilon} \nabla \eta_i|^2 \leq 2\eta_i^2 |\nabla \rho_{a,\epsilon}|^2 + 2\rho_{a,\epsilon}^2 |\nabla \eta_i|^2$  we have

$$k \int_M (\eta_i^2 |\nabla \rho_{a,\epsilon}|^2 + \rho_{a,\epsilon}^2 |\nabla \eta_i|^2) |\psi|^{\frac{2n-2}{n-1}} \mathbf{v}_g \geq \left( \frac{\mu_1}{4} + \inf_M |\ell^\psi|^2 - \lambda^2 \right) \|\eta_i \rho_{a,\epsilon} |\psi|^{\frac{n-2}{n-1}}\|^2$$

where  $k = 2\frac{2n-3}{n-2}$ . Next, we examine the limits when  $a$  goes to zero. Recall that  $\Upsilon \cap \overline{B_{2i}(p)}$  is bounded, closed,  $(n-2)$ - $C^\infty$ -rectifiable and has still locally finite  $(n-2)$ -dimensional Hausdorff measure. For fixed  $i$  we estimate

$$\int_M |\nabla \rho_{a,\epsilon}|^2 \eta_i^2 |\psi|^{\frac{2n-2}{n-1}} \mathbf{v}_g \leq \sup_{B_{2i}(p)} |\psi|^{\frac{2n-2}{n-1}} \int_{B_{2i}(p)} |\nabla \rho_{a,\epsilon}|^2 \mathbf{v}_g.$$

Furthermore, we set  $\mathcal{B}_{\epsilon,p} := \{x \in B_\epsilon \mid d(x,p) = d(x,\Upsilon)\}$  with  $B_\epsilon := \{x \in M \mid d(x,\Upsilon) \leq \epsilon\}$ . For  $\epsilon$  sufficiently small each  $\mathcal{B}_{\epsilon,p}$  is star shaped. Moreover, there is an inclusion  $\mathcal{B}_{\epsilon,p} \hookrightarrow B_\epsilon(0) \subset \mathbb{R}^2$  via the normal exponential map. Then we can calculate

$$\begin{aligned} \int_{B_\epsilon \cap B_{2i}(p)} |\nabla \rho_{a,\epsilon}|^2 v_g &\leq \text{vol}_{n-2}(\Upsilon \cap B_{2i}(p)) \sup_{x \in \Upsilon \cap B_{2i}(p)} \int_{\mathcal{B}_{\epsilon,x} \setminus B_{a\epsilon,x}} |\nabla \rho_{a,\epsilon}|^2 v_{g'} \\ &\leq c \text{vol}_{n-2}(\Upsilon \cap B_{2i}(p)) \int_{B_\epsilon(0) \setminus B_{a\epsilon}(0)} |\nabla \rho_{a,\epsilon}|^2 v_{g_E} \\ &\leq c' \int_{a\epsilon}^\epsilon \frac{\delta^2}{r} dr = -c' \delta^2 \ln a = c' \delta \rightarrow 0 \quad \text{for } a \rightarrow 0 \end{aligned}$$

where  $\text{vol}_{n-2}$  denotes the  $(n-2)$ -dimensional volume and  $g' = g|_{\mathcal{B}_{\epsilon,p}}$ . The positive constants  $c$  and  $c'$  arise from  $\text{vol}_{n-2}(\Upsilon \cap B_{2i}(p))$  and the comparison of  $v_{g'}$  with the volume element of the Euclidean metric. Furthermore, for any compact set  $K \subset M$  and any positive function  $f$  it holds  $\rho_{a,\epsilon}^2 f \nearrow f$  and thus by the monotone convergence theorem, we obtain when  $a \rightarrow 0$ ,

$$\int_K \rho_{a,\epsilon}^2 f v_g \rightarrow \int_K f v_g.$$

When applied to the functions  $\rho_{a,\epsilon}^2 |\nabla \eta_i|^2 |\psi|^{\frac{n-2}{n-1}}$ , with  $K = B_{2i}(p)$  we get

$$\int_{B_{2i}(p)} \rho_{a,\epsilon}^2 |\nabla \eta_i|^2 |\psi|^{\frac{n-2}{n-1}} v_g \rightarrow \int_{B_{2i}(p)} |\nabla \eta_i|^2 |\psi|^{\frac{n-2}{n-1}} v_g$$

as  $a \rightarrow 0$  and thus,

$$k \int_M |\nabla \eta_i|^2 |\psi|^{\frac{n-2}{n-1}} v_g \geq \left( \frac{\mu_1}{4} + \inf_M |\ell^\psi|^2 - \lambda^2 \right) \int_M \eta_i^2 |\psi|^{\frac{n-2}{n-1}} v_g.$$

Next we want to study the limit when  $i \rightarrow \infty$ : Since  $M$  has finite volume and  $\|\psi\| = 1$ , the Hölder inequality ensures that  $\int_M |\psi|^{\frac{n-2}{n-1}} v_g$  is bounded. With  $|\nabla \eta_i| \leq \frac{4}{i}$  we get

$$\lambda^2 \geq \frac{1}{4} \mu_1 + \inf_M |\ell^\psi|^2.$$

Equality is attained if and only if  $\|\overline{\nabla}^{\ell^\Phi} \overline{\Phi}\|_{\overline{g}}^2 \rightarrow 0$  for  $i \rightarrow \infty$ ,  $a \rightarrow 0$  and  $\Omega \cdot \psi = i \frac{c_n}{2} |\Omega|_g \psi$ . But we have

$$\begin{aligned} 0 &\leftarrow \|\overline{\nabla}^{\ell^\Phi} \overline{\Phi}\|_{\overline{g}}^2 = \|\eta_i \rho_{a,\epsilon} \overline{\nabla}^{\ell^\Phi} \overline{\varphi} + \nabla(\eta_i \rho_{a,\epsilon})^\cdot \overline{\varphi}\|_{\overline{g}} \\ &\geq \|\eta_i \rho_{a,\epsilon} \overline{\nabla}^{\ell^\Phi} \overline{\varphi}\|_{\overline{g}} - \|\nabla(\eta_i \rho_{a,\epsilon})^\cdot \overline{\varphi}\|_{\overline{g}}. \end{aligned}$$

With  $\|\nabla(\eta_i \rho_{a,\epsilon})^\cdot \overline{\varphi}\|_{\overline{g}} \rightarrow 0$ , see above,  $\overline{\nabla}^{\ell^\Phi} \overline{\varphi}$  has to vanish on  $M \setminus \Upsilon$ .  $\square$

**Proof of Corollary 1.4.** By the Cauchy-Schwarz inequality, we have

$$|\ell^\psi|^2 \geq \frac{(\text{tr}(\ell^\psi))^2}{n} = \frac{\lambda^2}{n}, \quad (23)$$

where  $\text{tr}$  denotes the trace of  $\ell^\psi$ . Hence the lower bound (11) is satisfied. If equality is satisfied in (11), it is satisfied also in (9). So,  $\lambda^2 = \frac{n}{4(n-1)}\mu_1 = \frac{1}{4}\mu_1 + |\ell^\psi|^2$  and equality in (23) is satisfied. Hence it is easy to check that

$$T^\psi(e_i, e_j) = 0 \text{ for } i \neq j \text{ and } T^\psi(e_i, e_i) = \pm \frac{\lambda}{n}.$$

Finally,  $\ell^\psi(X) = \pm \frac{\lambda}{n}X$  and  $\ell^{\bar{\varphi}}(X) = e^{-u}\ell^\psi(X) = \pm \frac{\lambda}{n}e^{-u}X$ . By (10) we get that  $\bar{\varphi}$  is a generalized Killing spinor and hence a Killing spinor for  $n \geq 4$  ([14, Theorem 1.1]). The function  $e^{-u}$  is then constant and  $\psi$  is a Killing spinor. For  $n = 3$ , we follow the same proof as in [14]. First we suppose that  $\lambda_1 \neq 0$ , because if  $\lambda_1 = 0$ , the result is trivial. We consider the Killing vector  $\bar{\xi}$  defined by

$$i\bar{g}(\bar{\xi}, X) = \langle X \cdot \bar{\varphi}, \bar{\varphi} \rangle_{\bar{g}} \quad \text{for every } X \in \Gamma(TM).$$

It is shown in [14] that:

$$\begin{aligned} d\bar{\xi} &= 2\lambda_1 e^{-u} * \bar{\xi}, \\ \nabla |\bar{\xi}|^2 &= 0, \\ \bar{\xi} \cdot \bar{\varphi} &= i|\bar{\xi}|^2 \bar{\varphi}. \end{aligned}$$

Since  $*\bar{\xi}(\bar{\xi}, \cdot) = 0$ , the 2-form  $\Omega$  can be written  $\Omega = F * \bar{\xi} + \bar{\xi} \wedge \alpha$ , where  $\alpha$  is a real 1-form and  $F$  a function. We have [14]

$$\Omega(\bar{\xi}, \cdot) = |\bar{\xi}|^2 \alpha(\cdot) = -4\lambda_1 d(e^{-u})(\cdot). \quad (24)$$

$$\Omega \cdot \bar{\varphi} = -iF\bar{\varphi} - i|\bar{\xi}|^2 \alpha \cdot \bar{\varphi} \quad (25)$$

But equality in (11) is achieved so  $\Omega \cdot \bar{\varphi} = i\frac{e^u}{2}|\Omega|_{\bar{g}}\bar{\varphi}$ , which imply that  $\Omega \cdot \bar{\varphi}$  is collinear to  $\bar{\varphi}$  and hence  $\alpha \cdot \bar{\varphi}$  is collinear to  $\bar{\varphi}$ . But  $d(e^{-u})(\bar{\xi}) = -\frac{1}{4\lambda_1}\Omega(\bar{\xi}, \bar{\xi}) = 0$  so  $\alpha(\bar{\xi}) = 0$ . It is easy to check that  $\langle \alpha \cdot \bar{\varphi}, \bar{\varphi} \rangle_{\bar{g}} = 0$  which gives that  $\alpha \cdot \bar{\varphi} \perp \bar{\varphi}$ . Because of  $\alpha \cdot \bar{\varphi} \perp \bar{\varphi}$  and  $\alpha \cdot \bar{\varphi}$  is collinear to  $\bar{\varphi}$ , we have  $\alpha \cdot \bar{\varphi} = 0$  and finally  $\alpha = 0$ . Using (24), we obtain  $d(e^{-u}) = 0$ , i.e.,  $e^{-u}$  is constant, hence  $\bar{\varphi}$  is a Killing spinor and finally  $\psi$  is also a Killing spinor.  $\square$

**Proof of Theorem 1.1 and Corollary 1.2.** The proof of Theorem 1.1 is similar to Theorem 1.3. It suffices to take  $\bar{g} = g$ , i.e.,  $e^u = 1$ . Corollary 1.2 is obtained from Corollary 1.4.  $\square$

Next, we want to prove Theorem 1.5 using the refined Kato inequality:

**Proof of Theorem 1.5.** We may assume  $\text{vol}(M, g) = 1$ . If  $\lambda$  is in the essential spectrum of  $D$ , then 0 is in the essential spectrum of  $D - \lambda$  and of  $(D - \lambda)^2$ . Thus, there is a sequence  $\psi_i \in \Gamma_c(\Sigma M)$  such that  $\|(D - \lambda)^2 \psi_i\| \rightarrow 0$  and  $\|(D - \lambda) \psi_i\| \rightarrow 0$  while  $\|\psi_i\| = 1$ . We may assume that  $|\psi_i| \in C_c^\infty(M)$ . That can always be achieved by a small perturbation. Now let  $\frac{1}{2} \leq \beta \leq 1$ . Then  $|\psi_i|^\beta \in H_1^2(M)$ . First, we will show that the sequence  $\|d|\psi_i|^\beta\|$  is bounded: By the Hölder inequality we have

$$\begin{aligned} 0 &\leftarrow \|\psi_i\|^{2\beta-1} \|(D - \lambda)^2 \psi_i\| \geq \| |\psi_i|^{2\beta-1} \|_{\{|\psi_i| \neq 0\}} \|(D - \lambda)^2 \psi_i\| \\ &\geq \left| \int_{|\psi_i| \neq 0} |\psi_i|^{2\beta-2} \langle (D - \lambda)^2 \psi_i, \psi_i \rangle \mathbf{v}_g \right|. \end{aligned}$$

Using (13) and the Schrödinger-Lichnerowicz type formula (16), we obtain

$$\begin{aligned} \|(D - \lambda)^2 \psi_i\| &\geq \left| \int_{|\psi_i| \neq 0} |\psi_i|^{2\beta-2} \langle \Delta^\lambda \psi_i, \psi_i \rangle \mathbf{v}_g + \int \left( \frac{S}{4} - \frac{c_n}{4} |\Omega|_g - \frac{n-1}{n} \lambda^2 \right) |\psi_i|^{2\beta} \mathbf{v}_g \right. \\ &\quad \left. - 2 \frac{n-1}{n} \int_{|\psi_i| \neq 0} |\psi_i|^{2\beta-2} \langle (D - \lambda) \psi_i, \lambda \psi_i \rangle \mathbf{v}_g \right| \\ &\geq \int_{|\psi_i| \neq 0} |\psi_i|^{2\beta-2} |\nabla^\lambda \psi_i|^2 \mathbf{v}_g + 2(\beta - 1) \int_{|\psi_i| \neq 0} |\psi_i|^{2\beta-3} \langle d|\psi_i| \cdot \psi_i, \nabla^\lambda \psi_i \rangle \mathbf{v}_g \\ &\quad + \int \left( \frac{S}{4} - \frac{c_n}{4} |\Omega| - \frac{n-1}{n} \lambda^2 \right) |\psi_i|^{2\beta} \mathbf{v}_g \\ &\quad - 2 \frac{n-1}{n} \lambda \| |\psi_i|^{2\beta-1} \|_{\{|\psi_i| \neq 0\}} \|(D - \lambda) \psi_i\|. \end{aligned}$$

With the Hölder inequality (recall that  $\beta \leq 1$ ) and the refined Kato inequality (21) for the connection  $\nabla^\lambda$ , we have

$$\begin{aligned} \|(D - \lambda)^2 \psi_i\| &\geq (2\beta - 1) \int_{|\psi_i| \neq 0} |\psi_i|^{2\beta-2} |d|\psi_i|| |\nabla^\lambda \psi_i| \mathbf{v}_g \\ &\quad + \int \left( \frac{S}{4} - \frac{c_n}{4} |\Omega| - \frac{n-1}{n} \lambda^2 \right) |\psi_i|^{2\beta} \mathbf{v}_g - 2 \frac{n-1}{n} \lambda \| |\psi_i|^{2\beta-1} \| \|(D - \lambda) \psi_i\| \\ &\geq (2\beta - 1) \int_{|\psi_i| \neq 0} |\psi_i|^{2\beta-2} |d|\psi_i||^2 \mathbf{v}_g + \int \left( \frac{S}{4} - \frac{c_n}{4} |\Omega| - \frac{n-1}{n} \lambda^2 \right) |\psi_i|^{2\beta} \mathbf{v}_g \\ &\quad - 2 \frac{n-1}{n} \lambda \|(D - \lambda) \psi_i\| \\ &\geq (2\beta - 1) \frac{1}{\beta^2} \int_{|\psi_i| \neq 0} |d|\psi_i|^\beta|^2 \mathbf{v}_g + \int \left( \frac{S}{4} - \frac{c_n}{4} |\Omega| - \frac{n-1}{n} \lambda^2 \right) |\psi_i|^{2\beta} \mathbf{v}_g \\ &\quad - 2 \frac{n-1}{n} \lambda \|(D - \lambda) \psi_i\|. \end{aligned}$$

Since  $S - c_n|\Omega|$  is bounded from below,  $\int (S - c_n|\Omega|)|\psi_i|^{2\beta} \mathbf{v}_g \geq \inf(S - c_n|\Omega|) \|\psi_i\|_{2\beta}^{2\beta} \geq \min\{\inf(S - c_n|\Omega|), 0\}$  is also bounded. Thus, with  $\|(D - \lambda)\psi_i\| \rightarrow 0$  we see that  $\|d|\psi_i|^\beta\|$  is also bounded. Next we fix  $\alpha = \frac{n-2}{n-1}$  and obtain

$$\begin{aligned} \frac{\mu_1}{4} - \frac{n-1}{n}\lambda^2 &\leq \left(\frac{\mu_1}{4} - \frac{n-1}{n}\lambda^2\right) \|\psi_i\|^\alpha \|\psi_i\|^\alpha \\ &\leq \frac{1}{4} \int |\psi_i|^\alpha L^\Omega |\psi_i|^\alpha \mathbf{v}_g - \frac{n-1}{n}\lambda^2 \|\psi_i\|^\alpha \|\psi_i\|^\alpha \\ &= \int |\psi_i|^{2\frac{n-2}{n-1}-2} \left[ \left(\frac{n}{n-1}|d|\psi_i||^2 + \frac{1}{2}d^*d|\psi_i|^2\right) \right. \\ &\quad \left. + \left(\frac{s}{4} - \frac{c_n}{4}|\Omega| - \frac{n-1}{n}\lambda^2\right) |\psi_i|^2 \right] \mathbf{v}_g, \end{aligned}$$

where we used the definition of  $\mu_1$  as infimum of the spectrum of  $L^\Omega$  and the following

$$|\psi_i|^\alpha d^*d|\psi_i|^\alpha = \frac{\alpha}{2} |\psi_i|^{2\alpha-2} d^*d|\psi_i|^2 - \alpha(\alpha-2) |\psi_i|^{2\alpha-2} |d|\psi_i||^2.$$

Next, using

$$\begin{aligned} \frac{1}{2}d^*d \langle \psi_i, \psi_i \rangle &= \langle \nabla^* \nabla \psi_i, \psi_i \rangle - |\nabla \psi_i|^2 \\ &= \langle D^2 \psi_i, \psi_i \rangle - \frac{S}{4} |\psi_i|^2 - \frac{i}{2} \langle \Omega \cdot \psi_i, \psi_i \rangle - |\nabla \psi_i|^2 \\ &\leq \langle D^2 \psi_i, \psi_i \rangle - \frac{1}{4}(S - c_n|\Omega|) |\psi_i|^2 - |\nabla \psi_i|^2, \end{aligned}$$

and

$$|\nabla^\lambda \psi_i|^2 = |\nabla \psi_i|^2 - 2\operatorname{Re} \frac{\lambda}{n} \langle (D - \lambda)\psi_i, \psi_i \rangle - \frac{\lambda^2}{n} |\psi_i|^2,$$

we have

$$\begin{aligned} \frac{\mu_1}{4} - \frac{n-1}{n}\lambda^2 &\leq \int |\psi_i|^{2\frac{n-2}{n-1}-2} \left( \frac{n}{n-1} |d|\psi_i||^2 - |\nabla^\lambda \psi_i|^2 \right) \mathbf{v}_g \\ &\quad + \int |\psi_i|^{2\frac{n-2}{n-1}-2} \langle (D^2 - \lambda^2)\psi_i, \psi_i \rangle \mathbf{v}_g \\ &\quad - \int 2|\psi_i|^{2\frac{n-2}{n-1}-2} \operatorname{Re} \frac{\lambda}{n} \langle (D - \lambda)\psi_i, \psi_i \rangle \mathbf{v}_g \\ &\leq \int |\psi_i|^{2\frac{n-2}{n-1}-2} \left( \frac{n}{n-1} |d|\psi_i||^2 - |\nabla^\lambda \psi_i|^2 \right) \mathbf{v}_g \\ &\quad + \int |\psi_i|^{2\frac{n-2}{n-1}-2} \langle (D - \lambda)^2 \psi_i, \psi_i \rangle \mathbf{v}_g \\ &\quad + \int 2 \left(1 - \frac{1}{n}\right) \lambda |\psi_i|^{2\frac{n-2}{n-1}-2} \operatorname{Re} \langle (D - \lambda)\psi_i, \psi_i \rangle \mathbf{v}_g. \end{aligned}$$

The last two summands vanish in the limit since

$$\left| \int |\psi_i|^{2\frac{n-2}{n-1}-2} \langle (D - \lambda)^2 \psi_i, \psi_i \rangle_{\mathbf{v}_g} \right| \leq \|(D - \lambda)^2 \psi_i\| \|\psi_i\|^{\frac{n-3}{n-1}} \rightarrow 0,$$

$$\left| \int |\psi_i|^{2\frac{n-2}{n-1}-2} \operatorname{Re} \langle (D - \lambda) \psi_i, \psi_i \rangle_{\mathbf{v}_g} \right| \leq \|(D - \lambda) \psi_i\| \|\psi_i\|^{\frac{n-3}{n-1}} \rightarrow 0.$$

For the other summand we use the Kato-type inequality (21)

$$|d\psi| \leq |(D - \lambda)\psi| + k|\nabla^\lambda \psi|,$$

which holds outside the zero set of  $\psi$  and where  $k = \sqrt{\frac{n-1}{n}}$ . Thus, for  $n \geq 5$  we can estimate

$$\begin{aligned} & \int |\psi_i|^{2\frac{n-2}{n-1}-2} \left( \frac{n}{n-1} |d\psi_i|^2 - |\nabla^\lambda \psi_i|^2 \right)_{\mathbf{v}_g} \\ &= \int |\psi_i|^{2\frac{n-2}{n-1}-2} (k^{-1} |d\psi_i| - |\nabla^\lambda \psi_i|) (k^{-1} |d\psi_i| + |\nabla^\lambda \psi_i|)_{\mathbf{v}_g} \\ &\leq k^{-1} \int_{\{|d\psi_i| \geq k|\nabla^\lambda \psi_i|\}} |\psi_i|^{2\frac{n-2}{n-1}-2} |(D - \lambda)\psi_i| (k^{-1} |d\psi_i| + |\nabla^\lambda \psi_i|)_{\mathbf{v}_g} \\ &\leq 2k^{-2} \int_{\{|d\psi_i| \geq k|\nabla^\lambda \psi_i|\}} |\psi_i|^{2\frac{n-2}{n-1}-2} |(D - \lambda)\psi_i| |d\psi_i|_{\mathbf{v}_g} \\ &\leq 2k^{-2} \int \left( 2\frac{n-2}{n-1} - 1 \right)^{-1} |(D - \lambda)\psi_i| |d\psi_i|^{2\frac{n-2}{n-1}-1}_{\mathbf{v}_g} \\ &\leq 2k^{-2} \frac{n-1}{n-3} \|(D - \lambda)\psi_i\| \|d\psi_i\|^{\frac{n-3}{n-1}}. \end{aligned}$$

For  $n \geq 5$  we have  $1 \geq \frac{n-3}{n-1} \geq \frac{1}{2}$  and, thus,  $\|d\psi_i\|^{\frac{n-3}{n-1}}$  is bounded. Together with  $\|(D - \lambda)\psi_i\| \rightarrow 0$  we obtain the following: For all  $\epsilon > 0$  there is an  $i_0$  such that for all  $i \geq i_0$  we have

$$\int |\psi_i|^{2\frac{n-2}{n-1}-2} \left( \frac{n}{n-1} |d\psi_i|^2 - |\nabla^\lambda \psi_i|^2 \right)_{\mathbf{v}_g} \leq \epsilon.$$

Hence, we have  $\frac{\mu_1}{4} \leq \frac{n-1}{n} \lambda^2$ . □

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