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Stability properties of steady-states for a network of ferromagnetic nanowires

Stéphane Labbé* Yannick Privat† Emmanuel Trélat‡

Abstract

We investigate the problem of describing the possible stationary configurations of the magnetic moment in a network of ferromagnetic nanowires with length L connected by semiconductor devices, or equivalently, of its possible L-periodic stationary configurations in an infinite nanowire. The dynamical model that we use is based on the one-dimensional Landau-Lifshitz equation of micromagnetism. We compute all L-periodic steady-states of that system, define an associated energy functional, and these steady-states share a quantification property in the sense that their energy can only take some precise discrete values. Then, based on a precise spectral study of the linearized system, we investigate the stability properties of the steady-states.

Keywords: Landau-Lifshitz equation, steady-states, elliptic functions, spectral theory, stability.

1 Introduction

Ferromagnetic materials are nowadays in the heart of innovating technological applications. A concrete example of current use concerns magnetic storage for hard disks, magnetic memories MRAMs or mobile phones. In particular, the ferromagnetic nanowires are objects that establish themselves in the domain of nanoelectronics and in the conception of the memories of the future. Indeed, the storage of magnetic bits all along nanowires seems to be a promising option not only in terms of footprint but also in terms of speed access to the informations (see [23, 24]). The conception of three dimensional memories based on the use of spin injection permits to hope access millions times shorter than the one observed nowadays in hard disks. In view of such potential application issues to rapid magnetic recording, it is of interest to be able to describes all possible stationary configurations of the magnetic moment and to investigate their natural stability properties; this is also a first step towards potential control issues, where the control may be for instance

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an external magnetic field, or an electric current crossing the magnetic domain, in order to act on the configuration of the magnetic moment.

The most common model used to describe the static behavior of ferromagnetic materials was introduced by W.-F. Brown in the 60's (see [4]). From this point of view, the equilibrium states of the magnetization are seen as the minimizers of a given functional energy, consisting of several components. When we consider a ferromagnetic material occupying a domain $\Omega \subset \mathbb{R}^3$, characterized by the presence of a spontaneous magnetization m almost everywhere, of norm 1 in Ω, the associated energy $E(m)$ takes the form (see [13])

$$E(m) = A \int_\Omega |\nabla m|^2 \, \mathrm{d}x - \int_\Omega H_a \cdot m \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3} |H_d(m)|^2 \mathrm{d}x, \tag{1}$$

and other relevant terms can be added for a more accurate physical model (e.g. anisotropic behavior of the crystal composing the ferromagnetic material) but these terms already explain a wide variety of phenomena. The first term is usually called "exchange term", and $A > 0$ is the exchange constant. The second term is the external energy, resulting from the possible presence of an external magnetic field H_a and the last term is the so-called "demagnetizing-field", which reflects the energy of the stray-field $H_d(m)$ induced by the distribution m and is obtained by solving

$$\begin{cases} \operatorname{div}(H_d + m) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^3), \\ \operatorname{curl}(H_d) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^3), \end{cases} \tag{2}$$

where m is extended to $\mathbb{R}^3$ by 0 outside Ω, and $\mathcal{D}'(\mathbb{R}^3)$ denotes the space of distributions on $\mathbb{R}^3$.

The dynamical aspects of micromagnetism are usually described by the *Landau-Lifshitz equation* introduced in the 30's in [21], written as

$$\frac{\partial m}{\partial t} = -m \wedge H_e(m) - m \wedge (m \wedge H_e(m)), \tag{3}$$

where $m(t,x)$ is the magnetic moment of the ferromagnetic material at time t, and $H_e = 2A\Delta u + H_d(u) + H_a$ is called the effective field. The existence of global weak solutions of that equation has been studied in [3, 28]. Results on strong solutions locally in time and initial data have been derived in [9]. For more details about modelization, stability and homogenization properties, we refer the reader to [10, 11, 12, 13, 14, 15, 26, 27, 28] and references therein. Numerical aspects have been investigated e.g. in [1, 11, 20], and control issues using such models have been addresses in [2, 7, 8] for particular magnetic domains.

Notice that, given a solution m of (3), there holds

$$\frac{d}{dt}(E(m(t,\cdot)) = -\int_\Omega \|H_e(m(t,x)) - \langle H_e(m(t,x)), m(t,x)\rangle m(t,x)\|^2 dx,$$

and thus this energy functional is naturally nonincreasing along a solution of (3). Every steady-state of (3) must satisfy $m \wedge H_e(m) = 0$ since both terms appearing in the right-hand side of (3) are orthogonal, and as expected the set of steady-states coincides with extremal points of the energy functional (1).

In this article, we consider a one-dimensional model of a ferromagnetic nanowire, for which Γ convergence arguments permit to derive the one-dimensional version of the Landau-Lifshitz equation

$$\frac{\partial u}{\partial t} = -u \wedge h(u) - u \wedge (u \wedge h(u)), \tag{4}$$

(see [26], see also [6] for arguments concerning a finite length nanowire) where $u(t,x) \in \mathbb{R}^3$ denotes the magnetization vector, for every $x \in \mathbb{R}$ and every time t (recall that it is a unit vector), and where $h(u) = \frac{\partial^2 u}{\partial x^2} - u_2 e_2 - u_3 e_3$ (assuming without loss of generality $A = 1/2$). Here, (e_1, e_2, e_3) denotes the canonical basis of $\mathbb{R}^3$ and the nanowire coincides with the real axis $\mathbb{R}e_1$.

Given a positive real number L, our aim is to obtain a complete description of the L-periodic steady-states of (4) and to investigate their stability properties. The motivation of this question is double. First, the equation above, combined with L-periodic conditions on u and $\frac{\partial u}{\partial x}$, is the limit model for a straightline network of ferromagnetic nanowires of length L, connected by semiconductor devices. In that case, the period L is imposed by the physical setting. Second, our study will provide a description of all possible periodic steady-states of an infinite length one-dimensional ferromagnetic nanowire, which can be seen as the limit case of L-periodic steady-states in a finite length nanowire where L is very small compared with the length of the nanowire. Note that the authors of [5] have studied particular steady-states called travelling walls for straight ferromagnetic nanowires of infinite length. In [6], the stability of one particular steady-state is investigated in a finite length nanowire with Neumann boundary conditions.

The article is organized as follows. We compute all possible L-periodic steady-states of (4) in Section 2 and prove that they share an energy quantification property, in the sense that their energy can only take isolated values. The stability properties of these steady-states are investigated in details in Section 3, based on a spectral study of the linearized system. Section 4 is devoted to the proof of our main result on quantification.

2 Computation of all periodic steady states

In what follows, the prime stands for the derivation with respect to the space variable x, and $\mathbb{S}^2$ denotes the unit sphere of $\mathbb{R}^3$ centered at the origin.

Definition 1. *A L-periodic steady-state of (4) is a function $u \in C^2(\mathbb{R}, \mathbb{S}^2)$ such that*

$$\begin{array}{l} u \wedge h(u) = 0 \quad \text{on } (0,L), \\ u(0) = u(L), \ u'(0) = u'(L). \end{array} \tag{5}$$

Denoting as previously (e_1, e_2, e_3) the canonical basis of $\mathbb{R}^3$, with the agreement that the nanowire coincides with the axis $\mathbb{R}e_1$, every steady-state can be written as $u = u_1 e_1 +$

$u_2e_2 + u_3e_3$, and (5) yields

$$\begin{array}{ll} u_1u_3'' - u_1''u_3 - u_1u_3 = 0 & \text{on } (0,L), \\ u_2u_3'' - u_3u_2'' = 0 & \text{on } (0,L), \\ u_1u_2'' - u_1''u_2 - u_1u_2 = 0 & \text{on } (0,L), \\ u_1^2 + u_2^2 + u_3^2 = 1 & \text{on } (0,L), \\ u(0) = u(L), \quad u'(0) = u'(L). & \end{array} \tag{6}$$

The integration of the second equation of (6) yields the existence of a real number α such that $u_2u_3' - u_2'u_3 = \alpha$ on $[0,L]$. Moreover, since u takes its values in $\mathbb{S}^2$, we set

$$\begin{aligned} u_1(x) &= \cos\theta_\alpha(x), \\ u_2(x) &= \cos\omega_\alpha(x)\sin\theta_\alpha(x), \\ u_3(x) &= \sin\omega_\alpha(x)\sin\theta_\alpha(x), \end{aligned} \tag{7}$$

for every $x \in \mathbb{R}$. Then, we infer from (6) that

$$\begin{aligned} &2\theta_\alpha''\sin\omega_\alpha + \omega_\alpha''\cos\omega_\alpha\sin(2\theta_\alpha) - (\omega_\alpha'^2+1)\sin\omega_\alpha\sin(2\theta_\alpha) + 4\omega_\alpha'\theta_\alpha'\cos\omega_\alpha\cos^2\theta_\alpha = 0, \\ &2\theta_\alpha''\cos\omega_\alpha - \omega_\alpha''\sin\omega_\alpha\sin(2\theta_\alpha) - (\omega_\alpha'^2+1)\cos\omega_\alpha\sin(2\theta_\alpha) - 4\omega_\alpha'\theta_\alpha'\sin\omega_\alpha\cos^2\theta_\alpha = 0, \\ &\omega_\alpha'\sin^2\theta_\alpha = \alpha, \\ &\theta_\alpha(0) = \theta_\alpha(L) \bmod 2\pi,\ \theta_\alpha'(0) = \theta_\alpha'(L), \\ &\omega_\alpha(0) = \omega_\alpha(L) \bmod 2\pi,\ \omega_\alpha'(0) = \omega_\alpha'(L). \end{aligned} \tag{8}$$

Multiplying the first equation by $\sin\omega_\alpha$, the second one by $\cos\omega_\alpha$ and adding these two equalities, it follows that $(\theta_\alpha, \omega_\alpha)$ is solution of

$$\begin{aligned} &\omega_\alpha'\sin^2\theta_\alpha = \alpha, \\ &-\theta_\alpha'' + \frac{1}{2}\left(\omega_\alpha'^2+1\right)\sin(2\theta_\alpha) = 0, \\ &\theta_\alpha(0) = \theta_\alpha(L) \bmod 2\pi,\ \theta_\alpha'(0) = \theta_\alpha'(L), \\ &\omega_\alpha(0) = \omega_\alpha(L) \bmod 2\pi,\ \omega_\alpha'(0) = \omega_\alpha'(L). \end{aligned} \tag{9}$$

At this step, the parameter α plays a particular role. First of all, observe that, if there exists $x_0 \in [0,L]$ such that $\sin^2\theta_\alpha(x_0) = 0$, then there must hold $\alpha = 0$. In that case, ω_0 is constant, and θ_0 satisfies the pendulum equation

$$\theta_0'' - \frac{1}{2}\sin(2\theta_0) = 0, \tag{10}$$

with periodic boundary conditions

$$\theta_0(0) = \theta_0(L) \bmod 2\pi,\ \theta_0'(0) = \theta_0'(L). \tag{11}$$

The case $\alpha \neq 0$ can only occur provided $\sin^2\theta_\alpha(x) > 0$, for every $x \in [0,L]$. In that case, we infer from (9) that θ_α satisfies the equation

$$\theta_\alpha'' - \frac{1}{2}\left(\frac{\alpha^2}{\sin^4\theta_\alpha} + 1\right)\sin(2\theta_\alpha) = 0, \tag{12}$$

with periodic boundary conditions

$$\theta_\alpha(0) = \theta_\alpha(L) \bmod 2\pi,\ \theta'_\alpha(0) = \theta'_\alpha(L). \tag{13}$$

Remark 1. Note that, for every solution θ_α of (12), the function

$$x \mapsto \theta'_\alpha(x)^2 + \frac{\alpha^2}{\sin^2\theta_\alpha(x)} + \cos^2\theta_\alpha(x)$$

is constant, and we define the functional

$$\mathcal{E}_\alpha(\theta_\alpha) = \theta'^2_\alpha + \frac{\alpha^2}{\sin^2\theta_\alpha} + \cos^2\theta_\alpha. \tag{14}$$

It is related to the energy defined by (1) in the following way. Let u be a steady-state, associated with $(\theta_\alpha, \omega_\alpha)$ by the formula (7), where θ_α and ω_α are solutions of (9). Then the energy $E(u)$ defined by (1) is given by

$$E(u) = \frac{1}{2}\int_0^L \left(\theta'_\alpha(x)^2 + \frac{\alpha^2}{\sin^2\theta_\alpha(x)} + \cos^2\theta_\alpha(x)\right) dx = \frac{L}{2}\mathcal{E}_\alpha(\theta_\alpha). \tag{15}$$

In Section 4, we prove the following result.

Theorem 1. *The set of real numbers α for which there exists a steady-state $(\theta_\alpha, \omega_\alpha)$ consists of isolated values, and contains in particular $\alpha = 0$. Furthermore, if α denotes any of these isolated values, there exists a family $(E_n)_{n\in\mathbb{N}^*}$ such that $\mathcal{E}_\alpha(\theta_\alpha) \in \{E_n\}_{n\in\mathbb{N}^*}$.*

The proof of that result is quite long and technical, and is postponed to Section 4. Notice that, using Remark 1, the energy of any steady-state u_α with $\alpha \neq 0$ is greater than the energy of any steady-state u_0 with $\alpha = 0$, that is,

$$E(u_\alpha) > E(u_0).$$

This property makes steady-states with $\alpha = 0$ of particular interest, and in the sequel we focus on them. We next provide a precise description of all steady-states with $\alpha = 0$. In that case, θ_0 is solution of the pendulum equation (10), the solutions of which are well known in terms of elliptic functions (see [22]), as recalled next.

First of all, recall that, for every solution θ_0 of (10), the function $x \mapsto \theta'_0(x)^2 + \cos^2\theta_0(x)$ is constant, and the value of the constant is $\mathcal{E}_0(\theta_0)$.

Recall that, given $k \in (0,1)$, $\tilde{k} = \sqrt{1-k^2}$ and $\eta \in [0,1]$, the Jacobi elliptic functions cn, sn and dn are defined from their inverse functions with respect to the first variable,

$$\begin{aligned}
\mathrm{cn}^{-1} &: (\eta, k) \longmapsto \int_\eta^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(\tilde{k}^2 + k^2t^2)}} \\
\mathrm{sn}^{-1} &: (\eta, k) \longmapsto \int_0^\eta \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}} \\
\mathrm{dn}^{-1} &: (\eta, k) \longmapsto \int_\eta^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(t^2+k^2-1)}} \ (\eta \geqslant \sqrt{1-k^2} \text{ in that case})
\end{aligned}$$

and the complete integral of the first kind is defined by

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

The functions cn and sn are periodic with period $4K(k)$ while dn is periodic with period $2K(k)$.

Using these elliptic functions, solutions of (10) can be integrated as follows, depending on the value of the energy $\mathcal{E}_0(\theta_0)$.

If $\mathcal{E}_0(\theta_0) = 0$, then $\theta_0(x) = \frac{\pi}{2}$ for every $x \in [0, L]$.

If $0 < \mathcal{E}_0(\theta_0) < 1$, then

$$\theta_0'(x) = k \ \mathrm{cn}\left(x + \mathrm{sn}^{-1}\left(\frac{1}{k}\cos\theta(0), k\right), k\right), \tag{16}$$

$$\cos\theta_0(x) = k \ \mathrm{sn}\left(x + \mathrm{sn}^{-1}\left(\frac{1}{k}\cos\theta(0), k\right), k\right), \tag{17}$$

for every $x \in [0, L]$, with $\mathcal{E}_0(\theta_0) = k^2$. The period of θ_0 is $T = 4K(k) = 4K(\sqrt{\mathcal{E}_0(\theta_0)})$. This case corresponds to the closed curves of Figure 1.

If $\mathcal{E}_0(\theta_0) = 1$, then

$$\theta_0'(x) = 1/\cosh\left(x + \mathrm{argth}^{-1}\left(\cos\theta(0)\right)\right), \tag{18}$$

$$\cos\theta_0(x) = \tanh\left(x + \mathrm{argth}^{-1}\left(\cos\theta(0)\right)\right). \tag{19}$$

This case corresponds to the separatrices (in bold) of the phase portrait drawn on Figure 1.

If $\mathcal{E}_0(\theta_0) > 1$, then

$$\theta_0'(x) = \frac{1}{k}\mathrm{dn}\left(\frac{x}{k} + \mathrm{sn}^{-1}\left(\cos\theta(0), k\right), k\right), \tag{20}$$

$$\cos\theta_0(x) = \mathrm{sn}\left(\frac{x}{k} + \mathrm{sn}^{-1}\left(\cos\theta(0), k\right), k\right), \tag{21}$$

for every $x \in [0, L]$, with $\mathcal{E}_0(\theta_0) = 1/k^2$. Moreover, $\theta_0(x + T) = \theta_0(x) + 2\pi$ for every $x \in [0, L]$ with $T = 2kK(k) = 2K(1/\sqrt{\mathcal{E}_0(\theta_0)})/\sqrt{\mathcal{E}_0(\theta_0)}$. This case corresponds to the curves located above and under the separatrices of Figure 1.

Every steady-state must moreover satisfy the boundary conditions (11), with the period L. These boundary conditions appear as an additional constraint to be satisfied by the solutions above, which turns into a quantification property, as explained in the next result, that makes the conclusion of Theorem 1 more precise.

Theorem 2. (Case $\alpha = 0$) *Set $N_0 = \left[\frac{L}{2\pi}\right]$, where the bracket notation stands for the integer part. Then, there exists a family $(E_n)_{1 \leqslant n \leqslant N_0}$ of elements of $(0, 1)$ and a countable family $(\widetilde{E}_n)_{n \in \mathbb{N}^*}$ of elements of $(1, +\infty)$ such that, for every steady-state,*

- *if $0 \leqslant \mathcal{E}_0(\theta_0) < 1$, then $\mathcal{E}_0(\theta_0) \in \{E_1, \ldots, E_{N_0}\}$;*

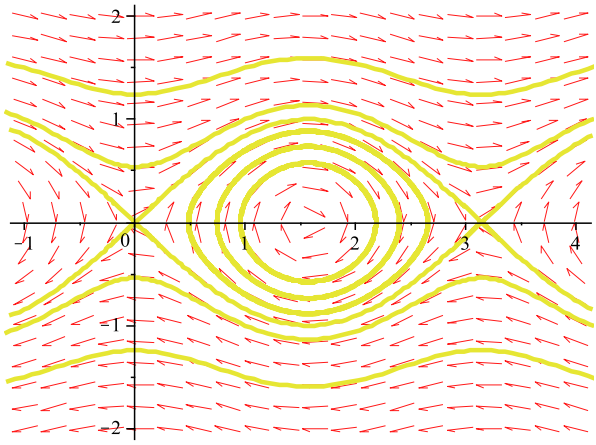

Figure 1: Phase portrait of (10) (in the plane (θ, θ'))

- *if* $\mathcal{E}_0(\theta_0) > 1$, *then* $\mathcal{E}_0(\theta_0) \in \{\tilde{E}_n \mid n \in \mathbb{N}^*\}$.

Furthermore, there are steady states corresponding to the energy level $\mathcal{E}_0(\theta_0) = 1$.

Remark 2. Note that, if $L < 2\pi$, there is no solution satisfying $\mathcal{E}_0(\theta_0) < 1$.

Remark 3. Using (15), this theorem turns into a quantification property of the physical energies of steady-states.

Proof. To take into account the boundary conditions (11), we have to impose that L is equal to an integer multiple of the period T of θ_0. The expression of T using the elliptic function K has been given previously, depending on the energy $\mathcal{E}_0(\theta_0)$. Recall that K is an increasing function from $[0, 1)$ into $[\pi/2, +\infty)$. The graph of the period T as a function of $\mathcal{E}_0(\theta_0)$ is given on Figure 2. The conclusion follows easily. □

Remark 4. If L tends to $+\infty$ then the steady-state tends to one of the separatrices of Figure (1). Analytically, this means that θ tends to the solution of (18)-(19). This corresponds to the case of an infinite length nanowire and to the steady-state studied in [5, 7].

3 Stability properties of the steady-states with $\alpha = 0$

In order to investigate the stability properties of the steady-states such that $\alpha = 0$, we compute the linearized system around a given steady-state and study its spectral properties. In what follows, define the spaces

$$\begin{aligned} H^1_{\text{per}}(0, L; \mathbb{R}^3) &= \{u \in H^1(0, L; \mathbb{R}^3) \mid u(0) = u(L)\}, \\ H^2_{\text{per}}(0, L; \mathbb{R}^3) &= \{u \in H^2(0, L; \mathbb{R}^3) \mid u(0) = u(L) \text{ and } u'(0) = u'(L)\}. \end{aligned}$$

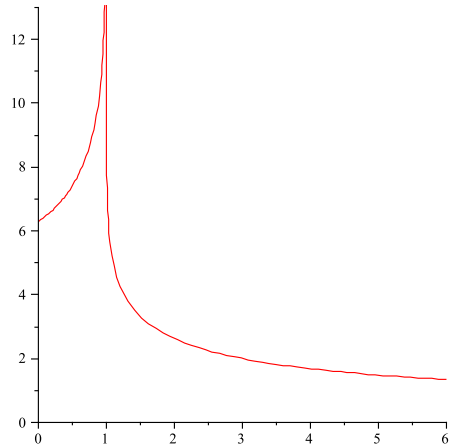

Figure 2: Graph of the period T in function of $\mathcal{E}_0(\theta_0)$ (case $\alpha = 0$)

Endowed respectively with the usual H^1 and H^2 inner product, these are Hilbertian spaces.

Let M_0 be a steady-state with $\alpha = 0$. The results of the previous section show that, in the spherical coordinates (θ, ω) that have been used, the component ω is constant. Clearly, the equation (4) is invariant with respect to rotations around the axis $\mathbb{R}e_1$. Then, up to a rotation of angle ω around the axis $\mathbb{R}e_1$, we assume that

$$M_0(x) = \begin{pmatrix} \cos\theta(x) \\ \sin\theta(x) \\ 0 \end{pmatrix},$$

where θ is solution of (10), (11) as described in Section 2. In Section 3.1, we compute the linearized system around this steady-state. The operator underlying this linearized system is a matrix of one-dimensional operators, one of which, denoted A, plays an important role. We study in details the spectral properties of A in Section 3.2. Based on this preliminary study, we investigate in Section 3.3 the stability properties of the steady-state M_0. Notice that the linearized system is as well invariant with respect to rotations around the axis $\mathbb{R}e_1$, and hence these results hold for every L-periodic steady-state. Finally, Section **??** is devoted to prove that the eigenvalues of the linearized system are simple except for certain discrete values of L.

3.1 Linearization of (4) around a steady-state

Let u be a solution of (4). As in [5], we complete M_0 into the mobile frame $(M_0(x), M_1(x), M_2)$, where M_1 and M_2 are defined by

$$M_1(x) = \begin{pmatrix} -\sin\theta(x) \\ \cos\theta(x) \\ 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Considering u as a perturbation of the steady-state M_0, since $|u(t,x)| = 1$ pointwisely, we decompose $u : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{S}^2 \subset \mathbb{R}^3$ in the mobile frame as

$$u(t,x) = \sqrt{1 - r_1^2(t,x) - r_2^2(t,x)}M_0(x) + r_1(t,x)M_1(x) + r_2(t,x)M_2. \tag{22}$$

Easy but lengthy computations show that u is solution of (4) if and only if $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ satisfies

$$\frac{\partial r}{\partial t} = \mathcal{L}r + R(x, r, r_x, r_{xx}), \tag{23}$$

where

$$R(x, r, r_x, r_{xx}) \quad = \quad G(r)r_{xx} + H_1(x,r)r_x + H_2(r)(r_x, r_x),$$

and

- $\mathcal{L} = \begin{pmatrix} A + \mathrm{Id} & A + \mathcal{E}_0(\theta)\mathrm{Id} \\ -(A + \mathrm{Id}) & A + \mathcal{E}_0(\theta)\mathrm{Id} \end{pmatrix}$ with $A = \partial_{xx}^2 - 2\cos^2\theta\ \mathrm{Id}$ defined on the domain $D(A) = H^2_{\mathrm{per}}(0, L)$,
- $G(r)$ is the matrix defined by

$$G(r) = \begin{pmatrix} \frac{r_1 r_2}{\sqrt{1-|r|^2}} & \frac{r_2^2}{\sqrt{1-|r|^2}} + \sqrt{1-|r|^2} - 1 \\ -\frac{r_1^2}{\sqrt{1-|r|^2}} - \sqrt{1-|r|^2} + 1 & -\frac{r_1 r_2}{\sqrt{1-|r|^2}} \end{pmatrix},$$

- $H_1(x,r)$ is the matrix defined by

$$H_1(x,r) = \frac{2\theta'(x)}{\sqrt{1-|r|^2}} \begin{pmatrix} r_2\sqrt{1-|r|^2} - r_1 r_2^2 & -r_2(1-r_1^2) \\ r_2(1-r_2^2) & \sqrt{1-|r|^2}r_2 + r_1 r_2^2 \end{pmatrix},$$

- $H_2(r)$ is the quadratic form on $\mathbb{R}^2$ defined by

$$H_2(r)(X,X) = \frac{(1-|r|^2)X^\top X + (r^\top X)^2}{(1-|r|)^{3/2}} \begin{pmatrix} \sqrt{1-|r|^2}r_1 + r_2 \\ \sqrt{1-|r|^2}r_2 - r_1 \end{pmatrix},$$

with the estimates

$$G(r) = O(|r|^2), \quad H_1(r) = O(|r|), \quad H_2(r) = O(|r|).$$

It is not difficult to prove that there exists a constant $C > 0$ such that, if $|r|^2 \leqslant \frac{1}{2}$, then, there holds for every $x \in \mathbb{R}$, for every $(p,q) \in (\mathbb{R}^2)^2$,

$$|R(x, r, p, q)| \leqslant C(|r|^2|q| + |r||p| + |r||p|^2).$$

This a priori estimate shows that $R(x, r, r_x, r_{xx})$ is a remainder term in (23).

3.2 Spectral study of the operator $A = \partial^2_{xx} - 2\cos^2\theta$ Id

In this section, we derive spectral properties of the operator A appearing in the expression of the linearized operator $\mathcal{L}$, which will be useful for the stability analysis of Section 3.3. The domain of A is $H^2_{\text{per}}(0, L; \mathbb{R}^3)$, but of course it is equivalent to study A on the domain $D(A) = H^2_{\text{per}}(0, L; \mathbb{R})$ (denoted shortly $H^2_{\text{per}}(0, L)$).

Every eigenpair (λ, u) of A must satisfy

$$\begin{aligned} &u'' - 2\cos^2\theta\ u = \lambda u, \\ &u(0) = u(L),\ u'(0) = u'(L). \end{aligned}$$

This is a particular case of Sturm-Liouville type problems with *real coupled self-adjoint boundary conditions* (see [17, 18, 19]). The following result provides some spectral properties of A.

Proposition 1. *The operator A, defined on $D(A) = H^2_{\text{per}}(0, L)$, is selfadjoint in $L^2(0, L)$ and there exists a hilbertian basis $(e_k)_{k\in\mathbb{N}}$ of $L^2(0, L)$, consisting of eigenfunctions of A, associated with real eigenvalues λ_k that are at most double, with*

$$-\infty < \cdots \leqslant \lambda_k \leqslant \cdots \leqslant \lambda_1 \leqslant \lambda_0, \tag{24}$$

and $\lambda_k \to -\infty$ as $k \to +\infty$. Moreover,

- *the eigenvalue λ_0 is simple, and its associated eigenfunction e_0 vanishes 0 or 1 time on $[0, L]$;*
- *the eigenfunction e_k vanishes $k-1$ or k or $k+1$ times on $[0, L]$.*

Remark 5. A simple computation shows that

$$\begin{aligned} A\sin\theta &= -\mathcal{E}_0(\theta)\sin\theta, \\ A\theta' &= -\theta', \\ A\cos\theta &= -(1 + \mathcal{E}_0(\theta))\cos\theta. \end{aligned}$$

Hence, $\sin\theta$, θ' and $\cos\theta$ are eigenfunctions of A associated respectively with the eigenvalues $-\mathcal{E}_0(\theta), -1, -(1 + \mathcal{E}_0(\theta))$. We are not able to exhibit nor compute explicitly some other eigenelements of A.

Note that, if the steady-state under consideration satisfies $\mathcal{E}_0(\theta) > 1$ (that is, the corresponding trajectory on the phase portrait of Figure 1 is outside the separatrices), then the function θ' does not vanish, and it follows from Proposition 1 that $\lambda_0 = -1$, that is, -1 is the largest eigenvalue of A, and $e_0 = \theta'$. Indeed, according to Proposition 1, the function e_1 could vanish 0, 1 or 2 times. Nevertheless, this is not the case since the inner product between e_0 and e_1 must be zero, which indicates that e_1 vanishes at least one time.

If the steady-state under consideration satisfies $\mathcal{E}_0(\theta) < 1$ (that is, the corresponding trajectory on the phase portrait of Figure 1 is inside the separatrices), then the function

$\sin\theta$ does not vanish, and it follows from Proposition 1 that $\lambda_0 = -\mathcal{E}_0(\theta)$, that is, $-\mathcal{E}_0(\theta)$ is the largest eigenvalue of A, and $e_0 = \sin\theta$.

In the particular case $\theta = \pi/2$ (corresponding to $\mathcal{E}_0(\theta) = 0$), one has $\theta' = 0$ and $\cos\theta = 0$ and thus they are not eigenfunctions. In that case, $\lambda_0 = 0$, and $e_0 = 1$. By the way, all eigenvalues can be easily computed as $\lambda_k = -\left(\frac{2k\pi}{L}\right)^2$, and they are all double except for $k = 0$.

Proof. The proof follows standard arguments. However, we include it from the convenience of the reader. We first prove that the operator A is diagonalisable. Consider the ordinary differential equation with boundary conditions

$$\begin{aligned} -u'' + (2\cos^2\theta + 1)u &= f, \\ u(0) = u(L),\ u'(0) &= u'(L). \end{aligned} \tag{25}$$

This problem is equivalent to the problem of determining $u \in H^2_{\text{per}}(0,L)$ such that $b(u,v) = g(v)$ for every $v \in H^1_{\text{per}}(0,L)$, where the bilinear form b and the linear form g are defined by

$$\begin{aligned} b(u,v) &= \int_0^L u'(x)v'(x)\mathrm{d}x + \int_0^L (2\cos^2\theta(x) + 1)u(x)v(x)\mathrm{d}x, \\ g(v) &= \int_0^L f(x)v(x)\mathrm{d}x. \end{aligned}$$

Moreover, it is clear that

$$\begin{aligned} \|u\|^2_{H^1(0,L)} &\leqslant b(u,u), \\ |b(u,v)| &\leqslant 4\|u\|_{H^1(0,L)}\|v\|_{H^1(0,L)}, \\ |g(v)| &\leqslant \|f\|_{L^2(0,L)}\|v\|_{H^1(0,L)}, \end{aligned}$$

for all $u, v \in H^1_{\text{per}}(0,L)$. This implies that b is continuous and coercive, and g is continuous. Lax-Milgram's Theorem then implies the existence of a unique weak solution in $H^1_{\text{per}}(0,L)$, and it is easy to prove that this solution is strong and belongs to $H^2_{\text{per}}(0,L)$, using a standard bootstrap argument. It is then possible to define the linear operator

$$\begin{array}{rccc} F: & L^2(0,L) & \longrightarrow & L^2(0,L) \\ & f & \longmapsto & u \end{array}$$

where u is the unique solution of (25). The operator F is compact. Indeed, let $u = Ff$, for $f \in L^2(0,L)$. Then,

$$\|u\|^2_{H^1(0,L)} \leqslant b(u,u) \leqslant \|f\|_{L^2(0,L)}\|u\|_{H^1(0,L)},$$

and hence $\|u\|_{H^1(0,L)} = \|Ff\|_{H^1(0,L)} \leqslant \|f\|_{L^2(0,L)}$. Since the imbedding of $H^1(0,L)$ into $L^2(0,L)$ is compact, it follows that the operator F is compact. For $f_1, f_2 \in L^2(0,L)$, denoting $u_1 = Ff_1$ and $u_2 = Ff_2$, one has

$$\langle Ff_1, f_2\rangle_{L^2(0,L)} = \langle u_1, f_2\rangle_{L^2(0,L)} = b(u_1,u_2) = \langle f_1, Ff_2\rangle_{L^2(0,L)},$$

and hence, since F is bounded on $L^2(0,L)$, F is selfadjoint. Since F is compact and selfadjoint, it follows that the operator A is diagonalisable with real eigenvalues satisfying (24). The eigenvalues λ_k are at most double because the associated eigenfunctions are solutions of a linear ordinary differential equation of order two. There cannot be two successive equalities in (24) because the eigenproblem associated to λ_n has exactly two linearly independent solutions. The assertions concerning the zero properties of the eigenfunctions follow from [19]. □

3.3 Stability properties of the steady-states

Consider the linear system

$$\begin{aligned} &\frac{\partial z}{\partial t} = \mathcal{L} z \\ &z(t,0) = z(t,L),\ z'(t,0) = z'(t,L), \end{aligned} \tag{26}$$

obtained in Section 3.1 by linearizing the Landau-Lifshitz equation (4) around the steady state M_0. As stated in Lemma 1, since $(e_k)_{k\geqslant 0}$ is a hilbertian basis of $L^2(0,L)$ whose elements are eigenfunctions of the operator A, we can write

$$z(t,x) = \begin{pmatrix} z^1(t,x) \\ z^2(t,x) \end{pmatrix}$$

for almost every $(t,x) \in \mathbb{R}^+ \times (0,L)$, where

$$z^i(t,x) = \sum_{k=0}^{+\infty} z_k^i(t) e_k(x)$$

for $i = 1,2$, with $z_k^i(t) = \langle z^i(t,\cdot), e_k \rangle_{L^2(0,L)}$ for every $k \in \mathbb{N}$. Then, it is easy to see that (26) is equivalent to the series of 2×2 linear systems

$$\begin{aligned} &\frac{\partial z_k}{\partial t} = \mathcal{L}_k z, \\ &z_k(0) = z_k(L),\ z_k'(0) = z_k'(L), \end{aligned}$$

for every $k \in \mathbb{N}$, where

$$\mathcal{L}_k = \begin{pmatrix} \lambda_k + 1 & \lambda_k + \mathcal{E}_0(\theta) \\ -(\lambda_k + 1) & \lambda_k + \mathcal{E}_0(\theta) \end{pmatrix}.$$

Recall that a matrix is said Hurwitzian whenever all its eigenvalues have their real part lower than 0. One has the following result.

Lemma 1. *For every $k \in \mathbb{N}$, the matrix $\mathcal{L}_k$ is Hurwitzian if and only if $\lambda_k < \min(-1, -\mathcal{E}_0(\theta))$.*

Proof. Set $m = \min(-1, -\mathcal{E}_0(\theta))$ and $M = \max(-1, -\mathcal{E}_0(\theta))$. The matrix $\mathcal{L}_k$ is Hurwitzian if and only if its determinant is positive and its trace is negative, that is, if and only if $(\lambda_k + 1)(\lambda_k + \mathcal{E}_0(\theta)) > 0$ and $2\lambda_k + 1 + \mathcal{E}_0(\theta) < 0$. The trace condition yields $\lambda_k < \frac{m+M}{2}$, and the determinant condition yields $\lambda_k < m$ or $\lambda_k > M$. The conclusion follows. □

To establish spectral properties of the steady-states, we distinguish between four cases, depending on value of the energy $\mathcal{E}_0(\theta)$ of the steady-state under consideration.

3.3.1 Case $\mathcal{E}_0(\theta) = 0$

In this case, there holds $\theta = \pi/2$ and $\theta' = 0$. Hence, $A = \partial^2_{xx}$, and in that case all eigenvalues of A are explicitly computed as $\lambda_k = -\left(\frac{2k\pi}{L}\right)^2$, for $k \in \mathbb{N}$. Unstable modes correspond to the eigenvalues λ_k satisfying $\lambda_k > -1$, and hence there are exactly $\left[\frac{L}{2\pi}\right] + 1$ unstable modes whenever $\frac{L}{2\pi}$ is not integer, and $\frac{L}{2\pi}$ whenever it is an integer. In particular, there is always at least one unstable mode, corresponding to the eigenvalue 0 and the eigenfunction 1.

3.3.2 Case $\mathcal{E}_0(\theta) \in (0, 1)$

This case corresponds to periodic trajectories of the pendulum phase portrait (see Figure 1) that are inside the separatrices.

Lemma 2. *The operator $A + \mathcal{E}_0(\theta)\mathrm{Id}$ admits the factorization*

$$A + \mathcal{E}_0(\theta)\mathrm{Id} = -\ell^*\ell,$$

where the operator ℓ is defined by $\ell = \partial_x - \theta'\mathrm{cotan}\theta\,\mathrm{Id}$ on the domain $\mathcal{D}(\ell) = H^1_{\mathrm{per}}(0, L)$. As a consequence, the largest eigenvalue of A is $\lambda_0 = -\mathcal{E}_0(\theta)$.

Proof. First of all, note that $\sin\theta(x) \neq 0$ for every $x \in [0, L]$. Indeed, the identity $\theta'^2(x) + \cos^2\theta(x) = \mathcal{E}_0(\theta) < 1$ yields $\cos^2\theta(x) < 1$ for every $x \in [0, L]$ and hence $\sin\theta(x) \neq 0$. Defining ℓ as in the statement of Lemma 2, there holds $\ell^* = -\partial_x - \theta'\mathrm{cotan}\,\theta\,\mathrm{Id}$, with $\mathcal{D}(\ell^*) = \mathcal{D}(\ell) = H^1_{\mathrm{per}}(0, L)$. One has $H^2_{\mathrm{per}}(0, L) = \mathcal{D}(A + \mathcal{E}_0(\theta)\mathrm{Id}) \subset \mathcal{D}(\ell)$ and $\ell(\mathcal{D}(A + \mathcal{E}_0(\theta)\mathrm{Id})) \subset \mathcal{D}(\ell^*)$, and one computes

$$\begin{aligned}
-\ell^*\ell &= -(-\partial_x - \theta'\mathrm{cotan}\theta\,\mathrm{Id}) \circ (\partial_x - \theta'\,\mathrm{cotan}\,\theta\,\mathrm{Id}) \\
&= \partial^2_{xx} - \theta''\,\mathrm{cotan}\theta\,\mathrm{Id} + \frac{\theta'^2}{\sin^2\theta}\,\mathrm{Id} - \theta'\mathrm{cotan}\theta\,\partial_x \\
&\quad + \theta'\mathrm{cotan}\theta\,\partial_x + \theta'^2\mathrm{cotan}^{\,2}\theta\,\mathrm{Id} \\
&= \partial^2_{xx} + (\mathcal{E}_0(\theta) - 2\cos^2\theta)\mathrm{Id},
\end{aligned}$$

since $\theta'^2 = \mathcal{E}_0(\theta) - \cos^2\theta$. It follows from this factorization that the operator $A + \mathcal{E}_0(\theta)\mathrm{Id}$ is nonpositive, and hence, since $-\mathcal{E}_0(\theta)$ is an eigenvalue of A, $\lambda_0 = -\mathcal{E}_0(\theta)$. □

From Lemma 1, the matrix $\mathcal{L}_k$ is Hurwitzian if and only if $\lambda_k < -1$. Then, there is always a finite number of unstable modes, corresponding to the eigenvalues λ_k such that $-1 < \lambda_k \leqslant -\mathcal{E}_0(\theta)$. In particular, using Remark 5, $e_0 = \sin\theta$ is an unstable mode associated with $\lambda_0 = -\mathcal{E}_0(\theta)$. Moreover, if L is large, then, when solving $T = L/n$ as in the proof of Theorem 2, the steady-state may be such that the integer n may be large (note that $n \in \{1, \ldots, N_0\}$ with $N_0 = [\frac{L}{2\pi}]$). On the phase portrait of the pendulum (Figure 1), this means that, for this situation, the corresponding trajectory turns n times around the center point $\theta = \pi/2$, $\theta' = 0$ on the interval $[0, L]$, and hence θ' vanishes $2n$ times; it then follows from Proposition 1 that θ' is the kth eigenfunction, with $k \in \{2n - 1, 2n, 2n + 1\}$.

Therefore, in that situation, since the eigenvalue -1 is at most double, there exist at least $2n-1$ and at most $2n+1$ unstable modes.

The eigenvalue -1 (associated at least with the eigenfunction θ', from Remark 5), corresponds to a central manifold for the nonlinear system (4) around the steady-state M_0.

All other eigenvalues λ_k, such that $\lambda_k < -1$, correspond to stable modes (in infinite number).

Notice that, since $n \leqslant N_0$, for every L-periodic steady-state such that $\mathcal{E}_0(\theta) \in (0,1)$, there are at most $2[\frac{L}{2\pi}]+1$ unstable modes.

3.3.3 Case $\mathcal{E}_0(\theta) = 1$

In this case, there must hold either $\theta = \theta' = 0$, or $\theta = \pi$ and $\theta' = 0$. Hence, $\cos\theta$ is constant, equal to 1 or -1. Since it does not vanish, it follows from Proposition 1 and Remark 5 that $\lambda_0 = -2$. Actually, in that case, one has $A = \partial_{xx} - 2\mathrm{Id}$, and all eigenvalues can be easily computed. The corresponding steady-state is $M_0 = (1,0,0)^T$, or $M_0 = (-1,0,0)^T$ (the resulting magnetic field is constant, tangent to the nanowire). It is locally asymptotically stable for the system (4).

3.3.4 Case $\mathcal{E}_0(\theta) > 1$

This case corresponds to periodic trajectories of the pendulum phase portrait (see Figure 1) that are outside the separatrices.

Note that, in that case, the factorization of Lemma 2 does not hold. This is due to the fact that $\sin\theta$ vanishes.

From Lemma 1, the matrix $\mathcal{L}_k$ is Hurwitzian if and only if $\lambda_k < -\mathcal{E}_0(\theta)$. The situation is similar to the case $\mathcal{E}_0(\theta) \in (0,1)$, except that the roles of -1 and $-\mathcal{E}_0(\theta)$ are exchanged. More precisely, there is always a finite number of unstable modes, corresponding to the eigenvalues λ_k such that $-\mathcal{E}_0(\theta) < \lambda_k \leqslant -1$. In particular, using Remark 5, $e_0 = \theta'$ is an unstable mode associated with $\lambda_0 = -1$. Moreover, as previously, when solving $T = L/n$ as in the proof of Theorem 2, the steady-state may be such that the integer n may be large (and contrarily to the case $\mathcal{E}_0(\theta) \in (0,1)$, there exist steady-states such that n is arbitrarily large). This means that, for this situation, $\sin\theta$ vanishes a $2n$ times; it then follows from Proposition 1 that $\sin\theta$ is the kth eigenfunction, with $k \in \{2n-1, 2n, 2n+1\}$. Therefore, in that situation, since $-\mathcal{E}_0(\theta)$ is at most double, there exist at least $2n-1$ and at most $2n+1$ unstable modes.

Notice that, for every integer p, there exists a L-periodic steady-state for which $\mathcal{E}_0(\theta) > 1$, such that the corresponding operator A admits at least p unstable modes.

4 Proof of Theorem 1

Consider a L-periodic steady-state in the case $\alpha \neq 0$. Recall that

$$\mathcal{E}_\alpha(\theta_\alpha) = \theta_\alpha'^2 + \frac{\alpha^2}{\sin^2\theta_\alpha} + \cos^2\theta_\alpha$$

is a constant, and that, since $\alpha \neq 0$, there must hold $\sin\theta_\alpha(x) \neq 0$, for every $x \in [0, L]$, and hence $\theta_\alpha(x) \in (p\pi, (p+1)\pi)$, for some $p \in \mathbb{Z}$. The phase portrait of (12), drawn on Figure 3 is then very different of the one of the pendulum studied previously. The vertical lines $\theta = 0 \bmod \pi$ are made of singular points. The region of the phase portrait of the pendulum (Figure 1) inside the separatrices can be seen as a sort of compactification process in which both vertical lines $\theta = 0$ and $\theta = \pi$ would join to form the separatrices. The trajectories that are outside the separatrices of the phase portrait of the pendulum do not exist in the case $\alpha \neq 0$.

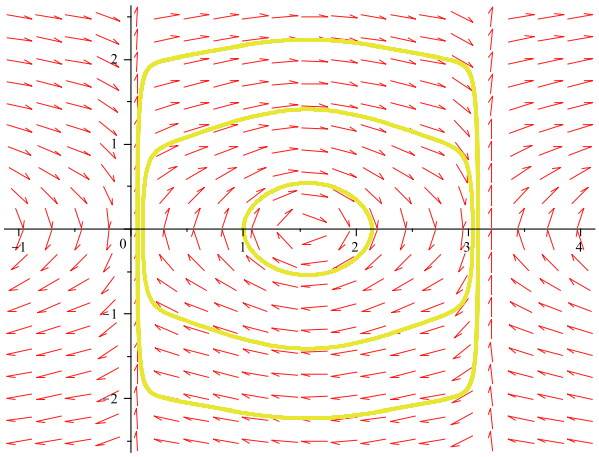

Figure 3: Phase portrait of (12) (case $\alpha \neq 0$)

First of all, note that if $\mathcal{E}_\alpha(\theta_\alpha) = \alpha^2$ then necessarily θ_α is constant, equal to $\frac{\pi}{2} \bmod \pi$; this corresponds to the singular points $\theta = \frac{\pi}{2} \bmod \pi$, $\dot\theta = 0$, of Figure 3. For any other solution there must hold necessarily $\mathcal{E}_\alpha(\theta_\alpha) > \alpha^2$.

Lemma 3. *Every solution θ_α of (12) such that $\mathcal{E}_\alpha(\theta_\alpha) > \alpha^2$ is periodic, with period*

$$T_\alpha = \frac{4\sqrt{2}}{\sqrt{d_\alpha}} K\left(\frac{2\sqrt{\mathcal{E}_\alpha(\theta_\alpha) - \alpha^2}}{d_\alpha}\right), \tag{27}$$

where $d_\alpha = \mathcal{E}_\alpha(\theta_\alpha) + 1 + \sqrt{(1 - \mathcal{E}_\alpha(\theta_\alpha))^2 + 4\alpha^2}$.

Proof. It can be easily seen that every such solution of (12) such that $\mathcal{E}_\alpha(\theta_\alpha) > \alpha^2$ is periodic. We assume that $\theta_\alpha(x) \in (0, \pi)$. Denote by θ_α^- and θ_α^+ the extremal values of $\theta_\alpha(x)$. They are computed by solving the equation

$$\sin^4\theta + (\mathcal{E}_\alpha(\theta_\alpha) - 1)\sin^2\theta - \alpha^2 = 0.$$

This leads to

$$\theta_\alpha^- = \arcsin\sqrt{\frac{1 - \mathcal{E}_\alpha(\theta_\alpha) + \sqrt{(\mathcal{E}_\alpha(\theta_\alpha) - 1)^2 + 4\alpha^2}}{2}}, \quad \theta_\alpha^+ = \pi - \theta_\alpha^-.$$

Notice that the function $x \mapsto \theta(x)$ is monotone between two such successive extremal values. Then,

$$\begin{aligned}
T_\alpha &= 2\int_0^{T_\alpha/2} dt = 2\int_{\theta_\alpha^-}^{\theta_\alpha^+} \frac{d\theta}{\sqrt{\mathcal{E}_\alpha(\theta_\alpha) - \cos^2\theta - \frac{\alpha^2}{\sin^2\theta}}} \\
&= 4\int_{\pi/2}^{\theta_\alpha^+} \frac{d\theta}{\sqrt{\mathcal{E}_\alpha(\theta_\alpha) - \cos^2\theta - \frac{\alpha^2}{\sin^2\theta}}} \\
&= 4\int_{\pi/2}^{\theta_\alpha^+} \frac{\sin\theta d\theta}{\sqrt{\sin^4\theta + (\mathcal{E}_\alpha(\theta_\alpha) - 1)\sin^2\theta - \alpha^2}} \\
&= 4\int_{\pi/2}^{\theta_\alpha^+} \frac{\sin\theta d\theta}{\sqrt{\cos^4\theta - (\mathcal{E}_\alpha(\theta_\alpha) + 1)\cos^2\theta + \mathcal{E}_\alpha(\theta_\alpha) - \alpha^2}} \\
&= 4\int_0^{\cos\theta_\alpha^-} \frac{du}{\sqrt{u^4 - (\mathcal{E}_\alpha(\theta_\alpha) + 1)u^2 + \mathcal{E}_\alpha(\theta_\alpha) - \alpha^2}}.
\end{aligned}$$

Note that

$$(1 - \mathcal{E}_\alpha(\theta_\alpha))^2 + 4\alpha^2 = (1 + \mathcal{E}_\alpha(\theta_\alpha))^2 - 4(\mathcal{E}_\alpha(\theta_\alpha) - \alpha^2),$$

and

$$\cos\theta_\alpha^- = \sqrt{\frac{1 + \mathcal{E}_\alpha(\theta_\alpha) - \sqrt{(1 - \mathcal{E}_\alpha(\theta_\alpha))^2 + 4\alpha^2}}{2}}. \tag{28}$$

Setting $\delta_\alpha = \frac{1}{4}\left((1 - \mathcal{E}_\alpha(\theta_\alpha))^2 + 4\alpha^2\right)$ and $\beta_\alpha = \frac{\mathcal{E}_\alpha(\theta_\alpha)+1}{2\sqrt{\delta_\alpha}}$, one ends up with

$$\begin{aligned}
T_\alpha &= \frac{4}{\sqrt{\delta_\alpha}} \int_0^{\cos\theta_\alpha^-} \frac{du}{\sqrt{\left(\frac{u^2}{\sqrt{\delta_\alpha}} - \frac{\mathcal{E}_\alpha(\theta_\alpha)+1}{2\sqrt{\delta_\alpha}}\right)^2 - 1}} \\
&= \frac{4}{\delta_\alpha^{1/4}} \int_0^{\frac{\cos\theta_\alpha^-}{\delta_\alpha^{1/4}}} \frac{dw}{\sqrt{(w^2 - \beta_\alpha)^2 - 1}}.
\end{aligned}$$

It is known (see [22]) that

$$\int \frac{dw}{\sqrt{(w^2 - \beta)^2 - 1}} = \frac{1}{\sqrt{\beta + 1}} F\left(\frac{w}{\sqrt{\beta - 1}}, \sqrt{\frac{\beta - 1}{\beta + 1}}\right),$$

where

$$F(\sin\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}$$

is the uncomplete elliptic integral of the first kind. Noticing that $\cos\theta_\alpha^- = \delta_\alpha^{1/4}\sqrt{\beta_\alpha - 1}$ and that $\sqrt{\frac{\beta_\alpha - 1}{\beta_\alpha + 1}} = \frac{2\sqrt{\mathcal{E}_\alpha(\theta_\alpha) - \alpha^2}}{d_\alpha}$, we get

$$T_\alpha = \frac{4\sqrt{2}}{\sqrt{d_\alpha}} F\left(1, \frac{2\sqrt{\mathcal{E}_\alpha(\theta_\alpha) - \alpha^2}}{d_\alpha}\right),$$

with $d_\alpha = \mathcal{E}_\alpha(\theta_\alpha) + 1 + \sqrt{(\mathcal{E}_\alpha(\theta_\alpha) - 1)^2 + 4\alpha^2}$, which is the expected result. □

Remark 6. For $\alpha = 0$, we recover the period obtained in the previous section for trajectories that are inside the separatrices. Indeed, taking $\alpha = 0$ in (27) leads to

$$T_0 = \frac{4\sqrt{2}}{\sqrt{\mathcal{E}_0(\theta_0) + 1 + |\mathcal{E}_0(\theta_0) - 1|}} K\left(\frac{2\sqrt{\mathcal{E}_0(\theta_0)}}{\mathcal{E}_0(\theta_0) + 1 + |\mathcal{E}_0(\theta_0) - 1|}\right), \tag{29}$$

and hence

$$T_0 = \begin{cases} 4K(\sqrt{\mathcal{E}_0(\theta_0)}) & \text{if} \quad 0 \leqslant \mathcal{E}_0(\theta_0) < 1, \\ +\infty & \text{if} \quad \mathcal{E}_0(\theta_0) = 1. \end{cases}$$

The function T_0 defined by (29) is also defined for $\mathcal{E}_0(\theta_0) > 1$, however it differs from the period of trajectories of the pendulum phase portrait (see previous section) that are outside the separatrices. This is not surprising, since these trajectories do not exist in the case $\alpha \neq 0$, as explained previously.

For every $\eta > 0$, define $f_1(\eta) = \eta + \alpha^2 + 1 + \sqrt{(\eta + \alpha^2 - 1)^2 + 4\alpha^2}$ and

$$\mathcal{T}_\alpha(\eta) = \frac{4\sqrt{2}}{\sqrt{f_1(\eta)}} K\left(\frac{2\sqrt{\eta}}{f_1(\eta)}\right). \tag{30}$$

The function $\mathcal{T}_\alpha$ is smooth on $(0, +\infty)$, and according to Lemma 3 the period of every solution θ_α of (12) such that $\mathcal{E}_\alpha(\theta_\alpha) > \alpha^2$ is $T_\alpha = \mathcal{T}_\alpha(\mathcal{E}_\alpha(\theta_\alpha) - \alpha^2)$. Note that the function $\mathcal{T}_\alpha$ can be extended as a continuous function on $[0, +\infty)$, with

$$\mathcal{T}_\alpha(0) = \frac{2\pi}{\sqrt{\alpha^2 + 1}}. \tag{31}$$

A lengthy computation shows that

$$\mathcal{T}'_\alpha(\eta) = \frac{4\sqrt{2}}{(f_1(\eta))^{3/2}}\left(\frac{f'_1(\eta)}{2} K\left(\frac{2\sqrt{\eta}}{f_1(\eta)}\right) + \frac{1}{\sqrt{\eta}} K'\left(\frac{2\sqrt{\eta}}{f_1(\eta)}\right) - f'_1(\eta)\widetilde{K}\left(\frac{2\sqrt{\eta}}{f_1(\eta)}\right)\right),$$

for every $\eta > 0$, where $K'(k) = -\frac{1}{k}K(k) + \frac{1}{k}\tilde{K}(k)$, and

$$\tilde{K}(k) = \int_0^{\pi/2} \frac{d\theta}{(1 - k^2 \sin^2\theta)^{3/2}}.$$

Moreover,

$$\mathcal{T}'_\alpha(\eta) \underset{\eta \to 0}{\sim} \frac{\pi(1 - 2\alpha^2)}{2(\alpha^2 + 1)^{5/2}}, \tag{32}$$

and

$$\mathcal{T}'_\alpha(\eta) \underset{\eta \to +\infty}{\sim} -\frac{\pi}{\eta^{3/2}}. \tag{33}$$

A tedious but straightforward study leads to the following result, describing some monotonicity properties of that function.

Lemma 4. • *For every $\alpha \in (0, \frac{\sqrt{2}}{2})$, there exists $\eta^*_\alpha \in (0,1)$ such that the function $\mathcal{T}_\alpha$ is increasing on $(0, \eta^*_\alpha)$ and decreasing on $(\eta^*_\alpha, +\infty)$. Moreover, $\mathcal{T}_\alpha(\eta^*_\alpha) \to +\infty$ and $\eta^*_\alpha \to 1$ whenever $\alpha \to 0$.*

For every $\alpha \geqslant \frac{\sqrt{2}}{2}$, the function $\mathcal{T}_\alpha$ is decreasing on $(0, +\infty)$.

• *For every $\alpha > 0$, $\mathcal{T}_\alpha(\eta) \to 0$ whenever $\eta + \infty$.*

The graph of the function $\mathcal{T}_\alpha$ is given on Figure 4 for different values of α.

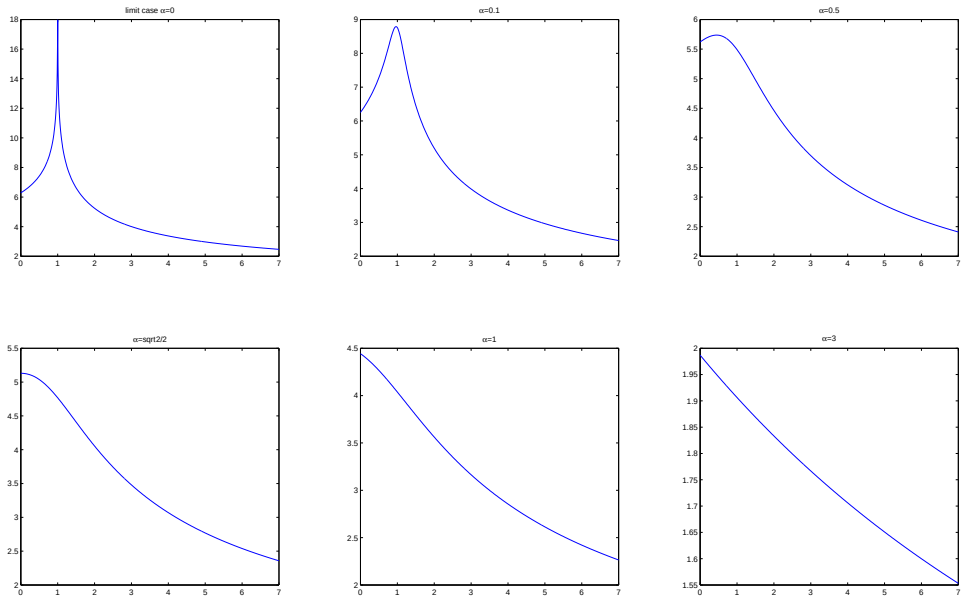

Figure 4: Graph of $\mathcal{T}_\alpha$ for $\alpha \in \{0, 0.1, 0.5, \frac{\sqrt{2}}{2}, 1, 3\}$

Every steady-state must moreover satisfy the boundary conditions (13). As in the previous section, since L is fixed, using Lemma 4, this constraint leads to a quantification property of the energy $\mathcal{E}_\alpha(\theta_\alpha)$. There is however one additional constraint coming from the periodicity of ω_α (see first and last lines of (9)), that results into the constraint

$$\alpha \int_0^L \frac{dx}{\sin^2 \theta_\alpha(x)} = 0 \bmod 2\pi.$$

Since $\alpha \neq 0$, this implies the existence of a nonzero integer k_α such that

$$\alpha \int_0^L \frac{dx}{\sin^2 \theta_\alpha(x)} = 2k_\alpha \pi. \tag{34}$$

This new constraint did not exist in the case $\alpha = 0$ studied in the previous section. Here, for $\alpha \neq 0$, (34) appears as an additional constraint driving to an overdetermined system.

This will imply that such steady-states can only exist for exceptional values of L, as proved below.

Indeed, assume that there exists a steady-state θ_{α_0}, for $\alpha_0 \neq 0$, satisfying this additional constraint (34). It is not restrictive to assume $\alpha_0 > 0$. The positive real number L must be an integer multiple of the period, hence there exists $n \in \mathbb{N}^*$ such that $L = nT_{\alpha_0} = n\mathcal{T}_{\alpha_0}(\mathcal{E}_{\alpha_0}(\theta_{\alpha_0}) - \alpha_0^2)$. We will vary α and follow a path of solutions θ_α satisfying (12) and (13), such that $\theta_\alpha = \theta_{\alpha_0}$ for $\alpha = \alpha_0$, having the same period $T_\alpha = T_{\alpha_0}$, and then use analytic arguments. We stress that, the period T_α of θ_α is kept constant along this homotopy procedure. The existence of such a homotopic path of solutions θ_α with fixed period $T_{\alpha_0} = L/n$, for α close to α_0, follows from the following arguments. It suffices to find a path of initial conditions $\theta_0(\alpha)$ (with $\theta_0'(\alpha) = 0$), with $\theta_0(\alpha_0) = \theta_{\alpha_0}(0)$, for which the corresponding period is exactly L/n. To justify this fact, denote by $\Upsilon_\alpha(\theta_0)$ the period of the solution of (12) with $\theta_\alpha(0) = \theta_0$ and $\theta_\alpha'(0) = 0$, and one has to solve the equation $\Upsilon_\alpha(\theta_0(\alpha)) = L/n$ in a neighborhood of α_0. This follows immediately from an implicit function argument, noticing that $\Upsilon_\alpha(\theta_0(\alpha)) = \mathcal{T}_\alpha(\mathcal{E}_\alpha(\theta_0) - \alpha^2)$, provided that the function $\mathcal{T}_\alpha$ is strictly monotonous at this point and that $\theta_0 \neq \pi/2$ (since then the gradient of the energy is non zero along the corresponding level set).

We argue by contradiction, and assume that α_0 is not an isolated point of the set of real numbers α such that there exists a steady-state $(\theta_\alpha, \omega_\alpha)$. According to the above arguments, there exists locally around α_0 a path of solutions θ_α satisfying (12) and (13), such that $\theta_\alpha = \theta_{\alpha_0}$ for $\alpha = \alpha_0$, whose period is exactly L/n, and such that the additional constraint (34) is satisfied. We distinguish between two cases.

Case $0 < \alpha_0 < \sqrt{2}/2$**.** In the above construction, we decrease α (at least in a neighborhood of α_0) and follow a path of solutions θ_α satisfying (12) and (13), such that $\theta_\alpha = \theta_{\alpha_0}$ for $\alpha = \alpha_0$. Using Lemma 4 and in particular the fact that the maximum $\mathcal{T}_\alpha(\eta_\alpha^*)$ tends to $+\infty$, it is clear that it is possible to make α decrease down to 0 and to follow a path such that $\mathcal{E}_\alpha(\theta_\alpha) < 1$. Moreover, combining the expression of T_0 and the formula (28), it is clear that this path shares the following crucial property: there exists $\varepsilon > 0$ such that, for every $\alpha \in (0, \alpha_0)$, there holds $\varepsilon \leqslant \theta_\alpha(x) \leqslant \pi - \varepsilon$. This implies that there exists $M > 0$ such that, for every $\alpha \in (0, \alpha_0)$,

$$\int_0^L \frac{dx}{\sin^2 \theta_\alpha(x)} \leqslant M. \tag{35}$$

According to (34), and since the function $\alpha \mapsto \alpha \int_0^L \frac{dx}{\sin^2 \theta_\alpha(x)}$ is analytic, this function must be constant on $(0, \alpha_0)$, which raises a contradiction with (35).

Case $\alpha_0 > \sqrt{2}/2$**.** In the above construction, we increase α (at least in a neighborhood of α_0) and follow a path of solutions θ_α satisfying (12) and (13), such that $\theta_\alpha = \theta_{\alpha_0}$ for $\alpha = \alpha_0$. According to the properties of the function $\mathcal{T}_\alpha$ settled in Lemma 4, it is possible to increase α up to a value α_1 satisfying $\mathcal{T}_{\alpha_1}(0) = L/n$, that is,

$$\frac{2\pi}{\sqrt{\alpha_1^2 + 1}} = \frac{L}{n}.$$

On the phase portrait of Figure 3, this corresponds to tracking trajectories shrinking to the center point $\theta = \pi/2$, $\theta' = 0$. For this limit case, passing to the limit in (34) as α tends to α_1 leads to the additional relation $L\alpha_1 = 2k_{\alpha_1}\pi$. It is clear that these two relations are exclusive except for some isolated values of L. It is however possible to refine our argument for these exceptional values, considering first-order expansions, as follows.

As in the previous case, it is clear that there exists $\varepsilon > 0$ such that $\varepsilon \leqslant \theta_\alpha(x) \leqslant \pi - x$ for every $\alpha \in [\alpha_0, \alpha_1]$. Furthermore, this construction implies the existence of a positive integer k such that

$$\alpha \int_0^L \frac{dx}{\sin^2 \theta_\alpha(x)} = 2k\pi,$$

first in a neighborhood of α_0, and by analyticity of $\alpha \mapsto \alpha \int_0^L \frac{dx}{\sin^2 \theta_\alpha(x)}$ on the whole interval $[\alpha_0, \alpha_1]$. For $\alpha < \alpha_1$, α close to α_1, set $\mathcal{E}_\alpha(\theta_\alpha) = \alpha^2 + \eta_\alpha$, with $\eta_\alpha > 0$ small. By construction, $\eta_\alpha = \underset{\alpha\to\alpha_1}{\mathrm{o}}(\alpha - \alpha_1)$. In what follows, we are going to expand asymptotically the solution θ_α as $\alpha \to \alpha_1$ and express, at the first order, the constraint (34). Using (12) and (14), we get,

$$\theta(x) - \frac{\pi}{2} \underset{\alpha\to\alpha_1}{\sim} \sqrt{\frac{\eta_\alpha}{1+\alpha_1^2}} \sin(\sqrt{1+\alpha_1^2}\, x).$$

Using (12), we get

$$\lim_{\alpha\to\alpha_1} \left(\alpha_1 \int_0^L \frac{dx}{\cos^2\left(\sqrt{\frac{\eta_\alpha}{1+\alpha_1^2}} \sin\left(\sqrt{1+\alpha_1^2}\, x\right)\right)} \right) = 2k\pi. \tag{36}$$

A simple asymptotic computation of the integral term of (36) leads to

$$\alpha_1 L + \frac{\alpha_1 \eta_\alpha}{4(1+\alpha_1^2)^{3/2}} (X_1 - \sin X_1) + \underset{\alpha\to\alpha_1}{\mathrm{o}}(\eta_\alpha) = 2k\pi,$$

with $X_1 = 2\sqrt{1+\alpha_1^2}L$. Letting α tend to α_1 yields $\alpha_1 L = 2k\pi$ (as noticed above) and dividing then this equality by η_α and letting α tend to α_1 yields

$$\frac{\alpha_1}{4(1+\alpha_1^2)^{3/2}} (X_1 - \sin X_1) = 0,$$

which is a contradiction.

Finally, the second conclusion of Theorem 1 is a direct consequence of Lemma 4 and of the fact that any energy level $\mathcal{E}_\alpha(\theta_\alpha)$ must satisfy

$$\exists n \in \mathbb{N}^* \mid T_\alpha(\mathcal{E}_\alpha(\theta_\alpha)) = \frac{L}{n}.$$

The proof of Theorem 1 is complete.

References

[1] F. Alouges. A new finite element scheme for Landau-Lifshitz equations. Discrete Contin. Dyn. Syst. Ser. S 1 (2008), no. 2, 187–196.

[2] F. Alouges, K. Beauchard. Magnetization switching on small ferromagnetic ellipsoidal samples. ESAIM Control Optim. Calc. Var. 15 (2009), no. 3, 676–711.

[3] F. Alouges, A. Soyeur. On global weak solutions for Landau-Lifshitz equations: existence and nonuniqueness. Nonlinear Anal. 18 (1992), 1070–1084.

[4] F. Brown. Micromagnetics. Wiley, New York, 1963.

[5] G. Carbou, S. Labbé. Stability for walls in ferromagnetic nanowire. Discrete and Continuous Dynamical Systems Series B, Vol. 6 (2006), no. 2, 273–290.

[6] G. Carbou, S. Labbé. Stabilization of walls for nanowires of finite length. to appear in ESAIM: Control Optim. Calc. Var.

[7] G. Carbou, S. Labbé, E. Trélat. Control of travelling walls in a ferromagnetic nanowire. Discrete Contin. Dyn. Syst. Ser. S 1 (2008), no. 1, 51–59.

[8] G. Carbou, S. Labbé, E. Trélat. Smooth control of nanowires by means of a magnetic field. Comm. Pure Appl. Anal. 8 (2009), no. 3, 871–879.

[9] G. Carbou, P. Fabrie. Regular solutions for Landau-Lifshitz equation in $\mathbb{R}^3$. Commun. Appl. Anal. 5 (2001), no. 1, 17–30.

[10] A. De Simone, H. Knüpfer, F. Otto. $2-d$ stability of the Néel wall. Calc. Var. Partial Differential Equations 27 (2006), 233–253.

[11] A. De Simone, R. Kohn, S. Müller, F. Otto. Two-dimensional modelling of soft ferromagnetic films. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 457 (2001), no. 2016, 2983–2991.

[12] H. Haddar, P. Joly. Stability of thin layer approximation of electromagnetic waves scattering by linear and nonlinear coatings. J. Comput. Appl. Math. 143 (2002), 201–236.

[13] A. Hubert, R. Schäfer. Magnetic domains: the analysis of magnetic microstructures. Springer-Verlag, 2000.

[14] R. Ignat, B. Merlet. Lower bound for the energy of Bloch Walls in micromagnetics. Arch. Ration. Mech. Anal. 199 (2011), 369–406.

[15] J. L. Joly, G. Métivier, J. Rauch. Global solutions to Maxwell equations in a ferromagnetic medium. Ann. Henri Poincaré 1 (2000), no. 2, 307–340.

[16] T. Kato. *Perturbation theory for linear operators.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[17] Q. Kong, A. Zettl. Dependence of eigenvalues of Sturm-Liouville problems on the boundary. J. Diff. Equations 126 (1996), no. 2, 389–407.

[18] Q. Kong, A. Zettl. Eigenvalues of regular Sturm-Liouville problems. J. Diff. Equations 131 (1996), no. 1, 1–19.

[19] Q. Kong, H. Wu, A. Zettl. Dependence of the nth Sturm-Liouville eigenvalue on the problem. J. Diff. Equations 156 (1999), no. 2, 328–354.

[20] S. Labbé. Fast computation for large magnetostatic systems adapted for micromagnetism. SIAM J. Sci. Comput. 26 (2005), no. 6, 2160–2175.

[21] L. Landau and E. Lifshitz. Electrodynamics of continuous media, Course of theoretical Physics, Vol. 8. Pergamon Press, Oxford-London-New York-Paris; Addison-Wesley Publishing Co., Inc., Reading, Mass, 1960. Translated from the russian by J.B. Sykes and J.S. Bell.

[22] D. F. Lawden. Elliptic functions and applications. Springer Verlag, New York, 1989.

[23] S. Parkin et al. Magnetic domain-wall racetrack memory. Science 320 (2008), 190–194.

[24] S. Parkin et al. Magnetically engineered spintronic sensors and memory. Proceedings of IEEE, vol. 91, no. 5 (2003), 661–680.

[25] M. Reed, B. Simon. Methods of modern mathematical physics. IV. Analysis of operators. Academic Press, New York, 1978.

[26] D. Sanchez. Behaviour of the Landau-Lifshitz equation in a periodic thin layer. Asymptot. Anal. 41 (2005), no. 1, 41–69.

[27] K. Santugini-Repiquet, Modelization of a split in a ferromagnetic body by an equivalent boundary condition. Asymptot. Anal. 47 (2006), no. 3-4, 227–259 (Part I), 261–290 (Part II).

[28] A. Visintin. On Landau-Lifshitz equations for ferromagnetism. Japan J. Appl. Math. 2 (1985), no. 1, 69–84.