Branching random walk in $\mathbb{Z}^4$ with branching at the origin only

by

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Summary. For the critical branching random walk in $\mathbb{Z}^4$ with branching at the origin only we find the asymptotic behavior of the probability of the event that there are particles at the origin at moment $t \to \infty$ and prove a Yaglom type conditional limit theorem for the number of individuals at the origin given that there are particles at the origin.

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1 Introduction

Consider the following modification of a standard branching random walk on $\mathbb{Z}^d$. The population is initiated at time $t = 0$ by a single particle. Being outside the origin the particle performs a continuous time random walk on $\mathbb{Z}^d$ with infinitesimal transition matrix

$$A = |a(\mathbf{x}, \mathbf{y})|_{\mathbf{x},\mathbf{y}\in\mathbb{Z}^d}, \qquad a(\mathbf{0}, \mathbf{0}) < 0,$$

until the moment when it hits the origin (that is, the time which the particle spends at a point $\mathbf{x} \neq \mathbf{0}$ is exponentially distributed with parameter $a := -a(\mathbf{0}, \mathbf{0})$). At the origin it spends an exponentially distributed time with parameter 1 and then either jumps to a point $\mathbf{y} \neq \mathbf{0}$ with probability

$$-(1-\alpha)a(\mathbf{0}, \mathbf{y})a^{-1}(\mathbf{0}, \mathbf{0}) =: (1-\alpha)\pi_{\mathbf{y}}, \tag{1.1}$$

or dies with probability α producing just before the death a random number of children ξ in accordance with offspring generating function

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$$f(s) := \mathbf{E}s^{\xi} = \sum_{k=0}^{\infty} f_k s^k, \qquad 0 \le s \le 1.$$

At the birth moment the newborn particles are located at the origin and from this point they begin their own branching random walks behaving independently and stochastically the same as the parent individual.

This model was investigated in [1, 2, 3, 4, 5] where some basic equations for the probability generating functions of the number of particles $\eta(t;\mathbf{x})$ at point $\mathbf{x} \in \mathbb{Z}^d$ at time t were deduced and, under certain conditions, the asymptotic behavior of the moments of $\eta(t;\mathbf{x})$ as $t \to \infty$ was investigated. For the continuous counterpart of the above model, we refer to [10], [7], [11], [8] and the references therein.

In the present paper we assume that the branching random walk is initiated at time $t = 0$ by a single individual located at the origin and impose the following restrictions on the characteristics of the process:

Hypothesis (I): The underlying random walk on $\mathbb{Z}^d$ is symmetric, irreducible and homogeneous

$$a(\mathbf{x},\mathbf{y}) = a(\mathbf{y},\mathbf{x}), \qquad a(\mathbf{x},\mathbf{y}) = a(\mathbf{0},\mathbf{y}-\mathbf{x}) =: a(\mathbf{y}-\mathbf{x}),$$

where $a(\mathbf{x}) \ge 0$ for $\mathbf{x} \ne \mathbf{0}$ and $a(\mathbf{0}) < 0$; besides we assume that

$$\sum_{\mathbf{x}\in\mathbb{Z}^d} a(\mathbf{x}) = 0, \qquad b^2 := \sum_{\mathbf{x}\in\mathbb{Z}^d} |\mathbf{x}|^2 a(\mathbf{x}) < \infty, \tag{1.2}$$

where $|\mathbf{x}|^2 := \sum_{i=0}^{d} x_i^2$ for $\mathbf{x} = (x_1, ..., x_i, ..., x_d) \in \mathbb{Z}^d$.

Denote h_d the probability that a particle, leaving the origin and performing a random walk on $\mathbb{Z}^d$ satisfying Hypothesis (I), will never come back. Observe that $h_1 = h_2 = 0$, while $h_d \in (0,1)$, $d \ge 3$.

Hypothesis (II): The offspring process is critical

$$\alpha f'(1) + (1-\alpha)(1-h_d) = 1$$

and $f^{(2)}(1) \in (0,\infty)$.

Here and in what follows for $k = 2, 3, \ldots$ we use the notation $f^{(k)}(s) := d^k f(s)/ds^k$.

Clearly, for the dimensions $d = 1, 2$ the introduced criticality of the branching random walk is reduced to the criticality of the offspring generating function at the origin, that is to the condition $f'(1) = 1$. For the dimensions $d \ge 3$ this is not the case.

Let $\mu_0(t)$ denote the number of particles in the process located at time t at the origin, $\mu(t)$ denote the number of particles in the process at time t outside the origin, and let $\eta(t) = \mu_0(t) + \mu(t)$ be the total number of individuals at the process at moment t.

For the case $d = 1$ the authors of papers [13, 14, 15, 16] investigate the asymptotic properties of the probabilities $\{\mu_0(t) > 0\}$ and $\{\mu(t) > 0\}$ and prove conditional limit theorems for a properly scaled vectors $(\mu_0(t), \mu(t))$ by studying an auxiliary branching process. A modification of this model when the initial number of particles at the origin is large was considered for the case $d = 1$ in [12].

The asymptotic behavior of the probability $q(t) := \mathbf{P}(\mu_0(t) > 0)$ for the dimensions $d \neq 4$ (again, by constructing an auxiliary branching process) has been found in [17]. However, the behavior of the probability $q(t)$ for the case $d = 4$ remained an open problem. The present paper fills this gap.

Our main result looks as follows.

Theorem 1.1 *For a branching random walk evolving in $\mathbb{Z}^4$ and satisfying Hypotheses (I) and (II),*

$$q(t) \sim C\frac{\log t}{t}, \qquad t \to \infty, \tag{1.3}$$

with $C = 3a(1-\alpha)\gamma_4 h_4^2/(\alpha f^{(2)}(1)) > 0$ and γ_4 a constant given by Lemma 2.1 below, and

$$\lim_{t\to\infty} \mathbf{P}\left(\frac{\mu_0(t)}{\mathbf{E}\left[\mu_0(t)|\mu_0(t) > 0\right]} \leq x \,|\mu_0(t) > 0\right) = \frac{1}{3} + \frac{2}{3}\left(1 - e^{-2x/3}\right), \ x > 0. \tag{1.4}$$

Remark 1.2 *It will be shown later on that*

$$\mathbf{E}\left[\mu_0(t)|\mu_0(t) > 0\right] = \frac{\mathbf{E}\mu_0(t)}{\mathbf{P}\left(\mu_0(t) > 0\right)} \sim \frac{3}{\alpha f^{(2)}(1) C^2}\,\frac{t}{\log^2 t}, \qquad t \to \infty.$$

To obtain Theorem 1.1, the crucial step is to show the asymptotic for the survival probability $q(t)$, which satisfies some convolution equation (see (2.6) below). It turns out that a first order analysis of this equation only gives a rough upper bound for $q(t)$ (Lemma 3.3), and we need a second order argument to get the exact asymptotic (Proposition 3.4).

The rest of this paper is organized as follows: In Section 2, we recall some known facts and present some basic evolution equations. The proof of Theorem 1.1 will be given in Section 3.

2 Auxiliary results and basic equations

For further references we first list some results obtained in [17]. We forget for a wile that we deal with a branching random walk in $\mathbb{Z}^d, d \geq 3$, and consider only the motion of a particle performing a random walk satisfying Hypothesis (I) without branching.

Denote

$$p(t; \mathbf{x}, \mathbf{y}) = p(t; \mathbf{0}, \mathbf{y} - \mathbf{x}) =: p(t; \mathbf{y} - \mathbf{x})$$

the probability that a particle located at moment $t = 0$ at point $\mathbf{x}$ will be at point $\mathbf{y}$ at moment t.

Lemma 2.1 *(see [17]) Under Hypothesis (I) as* $t \to \infty$

$$p(t;\mathbf{0}) \sim \gamma_d\, t^{-d/2}, \quad \gamma_d > 0.$$

Consider a particle located at the origin at moment $t = 0$ and let τ_0 be the time the particle spends at the origin. Denote by $\tau_{2,d}$ the time needed for a particle which has left the origin to come back to the origin. Let $\theta(\mathbf{y})$ be the time needed for a particle located at point $\mathbf{y} \neq \mathbf{0}$ at time 0 to hit the origin for the first time. Let

$$G_1(t) := \mathbf{P}(\tau_0 \leq t) = 1 - e^{-t},\ G_{2,d}(t) := \mathbf{P}(\tau_{2,d} \leq t),\ G^{\mathbf{y}}(t) := \mathbf{P}\left(\theta(\mathbf{y}) \leq t\right)$$

and $G_{2,d} := \mathbf{P}(\tau_{2,d} < \infty)$. It follows from (1.1) that

$$G_{2,d}(t) = \sum_{\mathbf{y}\neq\mathbf{0}} \pi_{\mathbf{y}} G^{\mathbf{y}}(t).$$

Lemma 2.2 *(see [17]) Under Hypothesis (I),* $\tau_{2,d}, d \geq 3$, *is an improper random variable with*

$$\mathbf{P}(\tau_{2,d} = \infty) = h_d = 1 - G_{2,d}(\infty) = 1 - G_{2,d}.$$

Besides, for $G_2(t) := G_{2,d}(t)/(1-h_d)$ *as* $t \to \infty$

$$1 - G_2(t) \sim \frac{2a\gamma_d h_d^2}{(1-h_d)\,(d-2)} t^{-d/2+1}.$$

Let

$$K_d(t) = K(t) =: \alpha f'(1) G_1(t) + (1-\alpha)(1-h_d)\, G_1 * G_2(t).$$

Lemma 2.3 *(see [17]) Under Hypothesis (I) for* $d \geq 3$ *as* $t \to \infty$

$$1 - K_d(t) \sim (1-\alpha)\left(G_{2,d} - G_{2,d}(t)\right) \sim \frac{2a(1-\alpha)\gamma_d h_d^2}{d-2}\, t^{-d/2+1}, \tag{2.1}$$

$$k_d(t) := K_d'(t) \sim a(1-\alpha)\gamma_d h_d^2\, t^{-d/2}. \tag{2.2}$$

Let

$$F_{\mathbf{y}}(t; s_1, s_2) := \mathbf{E}_{\mathbf{y}} s_1^{\mu_0(t)} s_2^{\mu_+(t)}, \qquad 0 \leq s_1, s_2 \leq 1,$$

be probability generating function for the number of particles at the origin and outside the origin at moment $t \geq 0$ in the branching random walk generated by a single particle located at point $\mathbf{y}$ at moment 0. By the total probability formula we have

$$\begin{aligned} F_{\mathbf{0}}(t; s_1, s_2) &= s_1(1 - G_1(t)) + \int_0^t \alpha f(F_{\mathbf{0}}(t-u; s_1, s_2))\, dG_1(u) \\ &\quad + (1-\alpha) \int_0^t \left(\sum_{\mathbf{y}\neq\mathbf{0}} \pi_{\mathbf{y}} F_{\mathbf{y}}(t-u; s_1, s_2) \right) dG_1(u), \end{aligned}$$

while for $\mathbf{y} \neq \mathbf{0}$

$$F_{\mathbf{y}}(t; s_1, s_2) = s_2(1 - G^{\mathbf{y}}(t)) + \int_0^t F_{\mathbf{0}}(t-u; s_1, s_2)\, dG^{\mathbf{y}}(u).$$

Hence

$$\begin{aligned}
F_{\mathbf{0}}(t; s_1, s_2) &= s_1(1 - G_1(t)) + \int_0^t \alpha f(F_{\mathbf{0}}(t-u; s_1, s_2))\, dG_1(u) \\
&\quad + (1-\alpha) s_2 \int_0^t \left(\sum_{\mathbf{y} \neq \mathbf{0}} \pi_{\mathbf{y}} (1 - G^{\mathbf{y}}(t-u)) \right) dG_1(u) \\
&\quad + (1-\alpha) \int_0^t \left(\sum_{\mathbf{y} \neq \mathbf{0}} \pi_{\mathbf{y}} \int_0^{t-v} F_{\mathbf{0}}(t-u-v; s_1, s_2)\, dG^{\mathbf{y}}(v) \right) dG_1(u) \\
&= s_1(1 - G_1(t)) + \int_0^t \alpha f(F_{\mathbf{0}}(t-u; s_1, s_2))\, dG_1(u) \\
&\quad + (1-\alpha) s_2 \int_0^t (1 - G_{2,d}(t-u)) dG_1(u) \\
&\quad + (1-\alpha) \int_0^t F_{\mathbf{0}}(t-u; s_1, s_2) d\left(G_1 * G_{2,d}(u)\right).
\end{aligned}$$

Using this relation and setting $F(t; s) := F_{\mathbf{0}}(t; s, 1) = \mathbf{E}_{\mathbf{0}} s^{\mu_0(t)}$, we get that for all $0 \leq s \leq 1$,

$$\begin{aligned}
F(t; s) &= s(1 - G_1(t)) + (1-\alpha)(1-h_d)(1 - G_2(\cdot)) * G_1(t) \\
&+ \int_0^t \alpha f(F(t-u; s))\, dG_1(u) + (1-\alpha) h_d G_1(t) \\
&+ \int_0^t (1-\alpha)(1-h_d) F(t-u; s)\, d(G_1 * G_2(u)). \qquad (2.3)
\end{aligned}$$

Hence, letting $q(t; s) := 1 - F(t; s)$,

$$\Phi(x) := f'(1)x - (1 - f(1-x)) =: x\Psi(x) \sim \frac{f^{(2)}(1)}{2} x^2, \qquad x \to 0, \qquad (2.4)$$

we deduce that for all $0 \leq s \leq 1$,

$$q(t; s) = (1-s)(1 - G_1(t)) + q(\cdot; s) * K(t) - \alpha \Phi(q(\cdot; s)) * G_1(t). \qquad (2.5)$$

Define $q(t) := \mathbf{P}\left(\mu_0(t) > 0\right) = q(t; 0)$. We have

$$q(t) = 1 - G_1(t) + q * K(t) - \alpha \Phi(q(\cdot)) * G_1(t), \qquad t \geq 0.$$

Note that

$$k_d(t) = \alpha f'(1) e^{-t} + (1-\alpha)(1-h_d)(G_1 * G_2(t))'.$$

Thus, we have

$$q(t) = 1 - G_1(t) + \int_0^t q(t-u)\left(k_d(u) - \alpha\Psi(q(t-u))e^{-u}\right)du. \tag{2.6}$$

The previous arguments hold for any dimension d. Now we concentrate on the case $d = 4$ and recall that by (2.2)

$$k_4(t) \sim c_4 t^{-2} \tag{2.7}$$

as $t \to \infty$ where $c_4 := a\,(1-\alpha)\,\gamma_4 h_4^2 > 0$. This asymptotic formula allows us to prove the following statement.

Lemma 2.4 *For any fixed $\varepsilon \in (0,1)$ and $p \in (0,1]$, we have for $t \to \infty$*

$$I := \int_0^{t\varepsilon} \frac{\log^p(t-u+1)}{t-u+1} k_4(u)du = \frac{\log^p(t+1)}{t+1} + \frac{c_4 \log^{1+p} t}{t^2}(1+o(1)), \tag{2.8}$$

$$I^* := \int_{t\varepsilon}^{t} \frac{\log^p(t-u+1)}{t-u+1} k_4(u)du = \frac{c_4 \log^{1+p} t}{(1+p)\,t^2} + o\left(\frac{\log^{1+p} t}{t^2}\right), \tag{2.9}$$

$$\int_0^{t} \frac{\log^p(t-u+1)}{t-u+1} k_4(u)du = \frac{\log^p(t+1)}{t+1} + c_4\frac{2+p}{1+p}(1+o(1))\frac{\log^{1+p} t}{t^2}. \tag{2.10}$$

Proof. Clearly (2.10) follows from (2.8) and (2.9). To prove (2.8), we use the expansions (valid for $0 \le u \le \varepsilon t$)

$$\begin{aligned}\log^p(t-u+1) &= \left(\log(t+1) - \frac{u}{t} + O\left(\frac{u^2+u}{t^2}\right)\right)^p \\ &= \log^p(t+1)\left(1 - \frac{pu}{t\log(t+1)} + O\left(\frac{u^2+u}{t^2\log(t+1)}\right)\right)\end{aligned}$$

and

$$\frac{1}{t-u+1} = \frac{1}{t+1}\left(1 + \frac{u}{t} + O\left(\frac{u^2+1}{t^2}\right)\right)$$

implying

$$\begin{aligned}&\frac{\log^p(t-u+1)}{t-u+1} \\ =\ &\frac{\log^p(t+1)}{t+1} + \frac{u\log^p t}{t^2} + O\left(\frac{(u^2+u)\log^p t}{t^3}\right) - \frac{pu}{t^2\log^{1-p}(t+1)} + O\left(\frac{u^2+u}{t^3\log^{1-p}(t+1)}\right).\end{aligned}$$

Hence we have

$$\begin{aligned} I &= \frac{\log^p(t+1)}{t+1}\int_0^{t\varepsilon} k_4(u)du + \frac{\log^p t}{t^2}\int_0^{t\varepsilon} uk_4(u)du \\ &\quad +O\left(\frac{\log^p t}{t^3}\right)\int_0^{t\varepsilon}(u^2+1)k_4(u)du + O\left(\frac{\log^{p-1}(t+1)}{t^2}\right)\int_0^{t\varepsilon} uk_4(u)du \\ &= \frac{\log^p(t+1)}{t+1} + \frac{c_4\log^{1+p}t}{t^2}(1+o(1)) + O\left(\frac{\log^p t}{t^2}\right) \\ &= \frac{\log^p(t+1)}{t+1} + \frac{c_4\log^{1+p}t}{t^2}(1+o(1)), \end{aligned}$$

proving (2.8). It remains to show (2.9). Observe that

$$I^* := c_4(1+o(1))\int_{t\varepsilon}^{t}\frac{\log^p(t-u+1)}{t-u+1}\frac{1}{u^2}du.$$

Further,

$$\begin{aligned} \int_{t\varepsilon}^{t}\frac{\log^p(t-u+1)}{t-u+1}\frac{1}{u^2}du &= \int_0^{t(1-\varepsilon)}\frac{\log^p(u+1)}{u+1}\frac{1}{(t-u)^2}du \\ &= \frac{1}{t^2}\int_0^{t(1-\varepsilon)}\frac{\log^p(u+1)}{u+1}du + O\left(\frac{1}{t^3}\int_0^{t(1-\varepsilon)}\log^p(u+1)\,du\right) \\ &= \left(\frac{1}{1+p}+o(1)\right)\frac{\log^{1+p}t}{t^2}, \end{aligned}$$

proving (2.9) and the Lemma. □

Let

$$I_k(t) := \int_2^{t-2}\frac{(t-u)^{k-1}\,\delta_1(t-u)\delta_2(u)}{\log^{2k}(t-u)}\frac{du}{\log u}, \qquad k=1,2,...,$$

where $\delta_1(t)$ and $\delta_2(t)$ are bounded functions such that $\delta_1(t)\sim 1$ and $\delta_2(t)\sim 1$ as $t\to\infty$. We end this section by the following estimate on $I_k(t)$:

Lemma 2.5 *For any fixed integer $k\geq 1$, we have as $t\to\infty$*

$$I_k(t) = \frac{1+o(1)}{k}\,\frac{t^k}{\log^{2k+1}t}.$$

Proof. For any $\varepsilon\in(0,1)$, if $0\leq u\leq \varepsilon t$ then

$$\log(t-u) = \log t + O(1)$$

while if $t\geq u\geq \varepsilon t$ then

$$\log u = \log t + O(1).$$

Clearly,

$$I_1(t) = \frac{(1+o(1))}{\log^2 t}\int_2^{t/2}\frac{1}{\log u}du + \frac{(1+o(1))}{\log t}\int_0^{t/2}\frac{1}{\log^2 u}du = (1+o(1))\frac{t}{\log^3 t}.$$

Further, for $k \geq 2$ we have

$$I_k(t) = \frac{(1+o(1))}{\log^{2k} t}\int_2^{t\varepsilon}\frac{(t-u)^{k-1}}{\log u}du + \frac{(1+o(1))}{\log t}\int_{t\varepsilon}^{t-2}\frac{(t-u)^{k-1}}{\log^{2k}(t-u)}du.$$

For large t

$$\int_2^{\varepsilon t}\frac{(t-u)^{k-1}}{\log u}du \leq 2t^{k-1}\frac{\varepsilon t}{\log t}$$

while

$$\int_{\varepsilon t}^{t-2}\frac{(t-u)^{k-1}}{\log^{2k}(t-u)}du = \int_2^{t(1-\varepsilon)}\frac{u^{k-1}}{\log^{2k} u}du \sim \frac{1}{k}\frac{(1-\varepsilon)^k t^k}{\log^{2k} t}.$$

Hence letting first $t \to \infty$ and than $\varepsilon \to +0$ the statement follows. □

3 Proofs of the main results

First we study the asymptotic behavior of the moments

$$P_k(t) := \mathbf{E}\mu_0^{[k]}(t),$$

where $x^{[k]} = x(x-1)\cdots(x-k+1)$, assuming that the offspring generating function is infinitely differentiable at point $s = 1$. To this aim we need the classical Faa di Bruno formula for the n-th derivative of the composition of functions $g(h(s))$ (see [6]):

$$\frac{d^n g(h(s))}{ds^n} = g'(h(s))h^{(n)}(s) + \sum_{k=2}^{n} g^{(k)}(h(s)) \sum_{\substack{j_1+j_2+\cdots+j_{n-1}=k \\ j_1+2j_2+\cdots+(n-1)j_{n-1}=n}} \frac{n!}{j_1!j_2!\cdots j_{n-1}!}\left(\frac{h^{(1)}(s)}{1!}\right)^{j_1}\cdots\left(\frac{h^{(n-1)}(s)}{(n-1)!}\right)^{j_{n-1}}.$$

Lemma 3.1 *If $f(s)$ satisfies Hypothesis (II) and is infinitely differentiable at point $s = 1$ then for any fixed $n \geq 1$,*

$$P_n(t) \sim n!\left(\frac{\alpha f^{(2)}(1)}{2}\right)^{n-1}\frac{1}{c_4^{2n-1}}\frac{t^{n-1}}{\log^{2n-1} t}, \qquad t \to \infty. \tag{3.1}$$

Proof. From formula (2.3) by differentiation we have for all $t \geq 0$,

$$P_1(t) = 1 - G_1(t) + \int_0^t \alpha f'(1)P_1(t-u)\,dG_1(u) + \int_0^t (1-\alpha)(1-h_4)P_1(t-u)\,d(G_1 * G_2(u)).$$

It follows that

$$P_1(t) = 1 - G_1(t) + \int_0^t P_1(t-u)\, dK(u) = (1 - G_1) * V_K(t)$$

where

$$V_K(t) = \sum_{j=0}^{\infty} K^{*j}(t)$$

is the renewal function corresponding to the distribution function $K(t)$ (Recalling that $K(t) := \alpha f'(1) G_1(t) + (1-\alpha)(1-h_4) G_1 * G_2(t)$). Since

$$1 - K(t) \sim c_4 t^{-1}$$

we have by Theorem 3 in [9] that

$$P_1(t) \sim \frac{1}{c_4 \log t}, \qquad t \to \infty. \tag{3.2}$$

Note that in view of $G_1(t) = 1 - e^{-t}$ we have

$$\frac{d}{dt}(G_1 * V_K(t)) = (1 - G_1) * V_K(t) = P_1(t). \tag{3.3}$$

Further, by writing $F^{(n)}(t;s) = \frac{\partial^n F(t;s)}{\partial^n s}$ for $n \geq 2$, we have

$$F^{(n)}(t;s) = \int_0^t \alpha \frac{d^n f(F(t-u;s))}{ds^n}\, dG_1(u) + \int_0^t (1-\alpha)(1-h_4) F^{(n)}(t-u;s)\, d(G_1 * G_2(u))$$

or, by the Faa di Bruno formula at $s = 1$,

$$P_n(t) = \alpha \int_0^t H_n(t-u)\, dG_1(u) + \int_0^t P_n(t-u)\, dK(u), \tag{3.4}$$

where

$$H_n(t) := \sum_{k=2}^{n} f^{(k)}(1) \sum_{\substack{j_1+j_2+\cdots+j_{n-1}=k \\ j_1+2j_2+\cdots+(n-1)j_{n-1}=n}} \frac{n!}{j_1! j_2! \cdots j_{n-1}!} \left(\frac{P_1(t)}{1!}\right)^{j_1} \cdots \left(\frac{P_{n-1}(t)}{(n-1)!}\right)^{j_{n-1}}. \tag{3.5}$$

Solving the renewal equation (3.4) with respect to $P_n(t)$ gives

$$P_n(t) = \alpha \int_0^t H_n(t-u)\, d\,(G_1 * V_K(u))$$

or, in view of (3.3)

$$P_n(t) = \alpha \int_0^t H_n(t-u)\, P_1(u) du.$$

For $n = 2$ we have

$$P_2(t) = \alpha f^{(2)}(1) \int_0^t P_1^2(t-u)\, P_1(u) du.$$

On account of Lemma 2.5 this leads to

$$P_2(t) \sim \frac{\alpha f^{(2)}(1)}{c_4^3} I_1(t) \sim \frac{\alpha f^{(2)}(1)}{c_4^3} \frac{t}{\log^3 t} = 2! \frac{\alpha f^{(2)}(1)}{2c_4^3} \frac{t}{\log^3 t}.$$

Now we use induction. Assume that for all $i < n$

$$P_i(t) \sim i! \frac{\left(\alpha f^{(2)}(1)\right)^{i-1}}{2^{i-1} c_4^{2i-1}} \frac{t^{i-1}}{\log^{2i-1} t}.$$

Then for $2 \le k \le n$ as $t \to \infty$

$$\begin{aligned}
& \sum_{\substack{j_1+j_2+\cdots+j_{n-1}=k \\ j_1+2j_2+\cdots+(n-1)j_{n-1}=n}} \frac{n!}{j_1! j_2! \cdots j_{n-1}!} \left(\frac{P_1(t)}{1!}\right)^{j_1} \cdots \left(\frac{P_{n-1}(t)}{(n-1)!}\right)^{j_{n-1}} \\
\sim & \sum_{\substack{j_1+j_2+\cdots+j_{n-1}=k \\ j_1+2j_2+\cdots+(n-1)j_{n-1}=n}} \frac{n!}{j_1! j_2! \cdots j_{n-1}!} \left(\frac{1}{c_4 \log t}\right)^{j_1} \cdots \left(\frac{\left(\alpha f^{(2)}(1)\right)^{n-2}}{2^{n-2} c_4^{2n-3}} \frac{t^{n-2}}{\log^{2n-3} t}\right)^{j_{n-1}} \\
= & \left(\frac{\alpha f^{(2)}(1)}{2}\right)^{n-k} \frac{1}{c_4^{2n-k}} \frac{t^{n-k}}{\log^{2n-k} t} \sum_{\substack{j_1+j_2+\cdots+j_{n-1}=k \\ j_1+2j_2+\cdots+(n-1)j_{n-1}=n}} \frac{n!}{j_1! j_2! \cdots j_{n-1}!}.
\end{aligned}$$

One may check that for any $n \ge 2$

$$\sum_{\substack{j_1+j_2+\cdots+j_{n-1}=2 \\ j_1+2j_2+\cdots+(n-1)j_{n-1}=n}} \frac{1}{j_1! j_2! \cdots j_{n-1}!} = \frac{n-1}{2}.$$

Thus, as $t \to \infty$

$$\begin{aligned}
H_n(t) &\sim f^{(2)}(1) \left(\frac{\alpha f^{(2)}(1)}{2}\right)^{n-2} \frac{1}{c_4^{2n-2}} \frac{t^{n-2}}{\log^{2n-2} t} \sum_{\substack{j_1+j_2+\cdots+j_{n-1}=2 \\ j_1+2j_2+\cdots+(n-1)j_{n-1}=n}} \frac{n!}{j_1! j_2! \cdots j_{n-1}!} \\
&= n! \frac{f^{(2)}(1)}{2} \left(\frac{\alpha f^{(2)}(1)}{2}\right)^{n-2} \frac{(n-1)}{c_4^{2n-2}} \frac{t^{n-2}}{\log^{2n-2} t}.
\end{aligned}$$

Therefore, on account of Lemma 2.5

$$\begin{aligned}
P_n(t) &= \alpha \int_0^t H_n(t-u)\, P_1(u) du \\
&\sim n! \left(\frac{\alpha f^{(2)}(1)}{2}\right)^{n-1} \frac{(n-1)}{c_4^{2n-1}} \int_2^{t-2} \frac{(t-u)^{n-2}}{\log^{2n-2}(t-u)} \frac{1}{\log u} du \\
&\sim n! \left(\frac{\alpha f^{(2)}(1)}{2}\right)^{n-1} \frac{1}{c_4^{2n-1}} \frac{t^{n-1}}{\log^{2n-1} t},
\end{aligned}$$

as desired. □

Corollary 3.2 *If $f(s)$ satisfies Hypothesis (II) then*

$$\liminf_{t\to\infty} \frac{tq(t)}{\log t} \geq \frac{c_4}{\alpha f^{(2)}(1)}. \tag{3.6}$$

Proof. From the proof of Lemma 3.1 it is clear that for the asymptotic representation (3.1) be valid for $n = 1, 2$ it suffices that $f^{(2)}(1) < \infty$. From this and the Lyapunov inequality

$$q(t) = \mathbf{P}\left(\mu_0(t) > 0\right) \geq \frac{\left(\mathbf{E}\mu_0(t)\right)^2}{\mathbf{E}\mu_0^2(t)} \sim \frac{\log t}{t} \frac{c_4}{\alpha f^{(2)}(1)}$$

the needed statement easily follows. □

Before giving the exact asymptotic of $q(t)$, we show at first a rough upper bound for $q(t)$.

Lemma 3.3 *We have*

$$\limsup_{t\to\infty} \frac{tq(t)}{\log t} < \infty.$$

Proof. Fix an arbitrary $u > 0$. Define for $0 \leq x \leq 1$,

$$T(x) := xk_4(u) - \alpha\Phi(x)e^{-u} = \alpha(f'(1)x - \Phi(x))e^{-u} + x(1-\alpha)(1-h_4)(G_1 * G_2)'(u).$$

Recalling (2.4). We have

$$T^{(1)}(x) \;=\; \alpha f'(1-x)e^{-u} + (1-\alpha)(1-h_4)(G_1 * G_2)'(u) > 0, \quad 0 \leq x \leq 1.$$

Hence $T(x)$ is monotone increasing in $x \in (0, 1)$. This fact will be used several times in the sequel.

Let us write a formal representation

$$q(t) = \beta(t)\frac{\log(t+1)}{t+1}, \qquad t \geq 0,$$

and set

$$q(u,t) := \beta(t)\frac{\log(t-u+1)}{t-u+1}, \qquad 0 \leq u \leq t.$$

Assume that the desired statement is not true, that is that

$$\limsup_{t\to\infty} \beta(t) = \infty. \tag{3.7}$$

Under (3.7), there exists a sequence $t_k \to \infty$ as $k \to \infty$ such that $\beta(t_k) = \sup_{u\leq t_k} \beta(u)$ and $\lim_{k\to\infty} \beta(t_k) = \infty$. Then, for sufficiently large $t = t_k$ (we omit the low index for simplicity), $\beta(t) = \sup_{0\leq s\leq t} \beta(s) > 1$. In view of (2.9) with $p = 1$, we have

$$\begin{aligned}
&\left|\int_{t/2}^{t} q(t-u)\left(k_4(u) - \alpha\Psi(q(t-u))e^{-u}\right)du\right| \\
\leq\; &\int_{t/2}^{t} \beta(t-u)\frac{\log(t-u+1)}{t-u+1}k_4(u)du + \alpha\int_{t/2}^{t}\Phi(q(t-u))e^{-u}du \\
\leq\; &\beta(t)\frac{2c_4}{t^2}\log^2 t, \qquad (3.8)
\end{aligned}$$

since $\beta(t) > 1$ and $\alpha \int_{t/2}^{t} \Phi(q(t-u))e^{-u}du \le \alpha f'(1)e^{-t/2} = o(\frac{\log^2 t}{t^2})$.

Let us observe that $q(t) \to 0$ as $t \to \infty$. In fact, we have a rough bound: $q(t) = \mathbf{P}(\mu_0(t) > 0) \le \mathbf{E}\mu_0(t) = P_1(t) \sim \frac{1}{c_4 \log t}$ by (3.2). It follows that for $0 \le u \le t/2$,

$$\begin{aligned} q(t-u) &= \beta(t-u)\frac{\log(t-u+1)}{t-u+1} \le \beta(t)\frac{\log(t-u+1)}{t-u+1} \\ &= q(u,t) \le \beta(t)\frac{\log(t/2)}{t/2} \le 3\beta(t)\frac{\log t}{t} = 3q(t) \le 1, \end{aligned}$$

for all sufficiently large t. Thus, the inequality

$$q(t-u)\left(k_4(u) - \alpha\Psi(q(t-u))e^{-u}\right) \le q(u,t)\left(k_4(u) - \alpha\Psi(q(u,t))e^{-u}\right)$$

is valid for $0 \le u \le t/2$. This, in view of (2.6) and (3.8), implies that

$$\begin{aligned} & \beta(t)\frac{\log(t+1)}{t+1} = q(t) \\ = \;& 1 - G_1(t) + \int_0^t q(t-u)\left(k_4(u) - \alpha\Psi(q(t-u))e^{-u}\right)du \\ \le \;& \beta(t)\frac{3c_4}{t^2}\log^2 t \\ & + \int_0^{t/2} \beta(t-u)\frac{\log(t-u+1)}{t-u+1}\left(k_4(u) - \alpha\Psi\left(\beta(t-u)\frac{\log(t-u+1)}{t-u+1}\right)e^{-u}\right)du \\ \le \;& \beta(t)\frac{3c_4}{t^2}\log^2 t \\ & + \beta(t)\int_0^{t/2} \frac{\log(t-u+1)}{t-u+1}\left(k_4(u) - \alpha\Psi\left(\beta(t)\frac{\log(t-u+1)}{t-u+1}\right)e^{-u}\right)du. \end{aligned}$$

By (2.8), we have that

$$\int_0^{t/2} \frac{\log(t-u+1)}{t-u+1}k_4(u)du \le \frac{\log(t+1)}{t+1} + 2c_4\frac{\log^2 t}{t^2} \tag{3.9}$$

and, for any small $\delta > 0$ and sufficiently large t,

$$\begin{aligned} & \int_0^{t/2} \frac{\log(t-u+1)}{t-u+1}\alpha\Psi\left(\beta(t)\frac{\log(t-u+1)}{t-u+1}\right)e^{-u}du \\ \ge \;& (1-\delta)\beta(t)\frac{\alpha f^{(2)}(1)}{2}\int_0^{t/2}\frac{\log^2(t-u+1)}{(t-u+1)^2}e^{-u}du \ge (1-2\delta)\beta(t)\frac{\alpha f^{(2)}(1)}{2}\frac{\log^2 t}{t^2}. \end{aligned}$$

As a result we get that

$$\beta(t)\frac{\log(t+1)}{t+1} \le \beta(t)\left(\frac{\log(t+1)}{t+1} + 5c_4\frac{\log^2 t}{t^2}\right) - (1-2\delta)\beta^2(t)\frac{\alpha f^{(2)}(1)}{2}\frac{\log^2 t}{t^2}.$$

Hence, after simplification we see that

$$(1-2\delta)\beta^2(t)\frac{\alpha f^{(2)}(1)}{2} \le 5c_4\beta(t),$$

which is impossible if $\beta(t) \to \infty$. Hence $\limsup_{t\to\infty} \beta(t) < \infty$ and the lemma is proved. □

The crucial step in the proof of Theorem 1 is the following lemma:

Proposition 3.4 *In the case $d = 4$,*

$$q(t) = C\frac{\log t}{t}(1+o(1)), \tag{3.10}$$

where

$$C := \frac{3c_4}{\alpha f^{(2)}(1)}. \tag{3.11}$$

Proof. We will use a formal representation

$$q(t) = \frac{\log(t+1)}{t+1}\left(C + \frac{\rho(t)}{\sqrt{\log(t+1)}}\right) = C\frac{\log(t+1)}{t+1} + \frac{\rho(t)\sqrt{\log(t+1)}}{t+1}.$$

Note that by Corollary 3.2

$$\liminf_{t\to\infty} \frac{\rho(t)}{\sqrt{\log t}} = C_0 \ge -\frac{2}{3}C, \tag{3.12}$$

and, by Lemma 3.3

$$\limsup_{t\to\infty} \frac{\rho(t)}{\sqrt{\log t}} \le C_1 < \infty,$$

for some constant $C_1 > 0$. If $\rho(t)$ is bounded then the lemma is proved. Thus, assume that

$$\limsup_{t\to\infty} \rho(t) = \infty \text{ and } \liminf_{t\to\infty} \rho(t) = -\infty.$$

Clearly, if we prove the desired statement for this case then the cases when one of the limits above is finite will follow easily.

Assume first that $\limsup_{t\to\infty} \rho(t) = \infty$ and let

$$\mathcal{A}_+ := \left\{ t \in [0,\infty) : \rho(t) = \sup_{v\le t} \rho(v) \right\}.$$

Then there exists an unbounded sequence $(t_k) \in \mathcal{A}_+$ such that $\rho(t_k) \to \infty$ as $k \to \infty$. Our subsequent arguments are for large $t \in \mathcal{A}_+$. Then a priori, $\rho(t) > 1$. For $0 \le u \le t$, set

$$Q(u,t) := C\frac{\log(t-u+1)}{t-u+1} + \frac{\rho(t)\sqrt{\log(t-u+1)}}{t-u+1} \ge q(t-u).$$

Fix a small $\varepsilon > 0$. Clearly, for $0 \le u \le t\varepsilon$ and sufficiently large t,

$$Q(u,t) \le C\frac{\log(t+1)}{t(1-\varepsilon)} + \frac{\rho(t)\sqrt{\log(t+1)}}{t(1-\varepsilon)} \le \frac{1}{1-\varepsilon}q(t) < 1,$$

since $q(t) \to 0$. Using again the monotonicity of the function: $x(\in (0,1)) \to xk_4(u) - \alpha\Phi(x)e^{-u}$ (just like in Lemma 3.3), we get the inequality

$$q(t-u)\left(k_4(u) - \alpha\Psi\left(q(t-u)\right)e^{-u}\right) \le Q(u,t)\left(k_4(u) - \alpha\Psi\left(Q(u,t)\right)e^{-u}\right).$$

It follows that

$$\begin{aligned} q(t) &= \int_{t\varepsilon}^{t} q(t-u)k_4(u)du \\ &\quad + \int_0^{t\varepsilon} q(t-u)\left(k_4(u) - \alpha\Psi\left(q(t-u)\right)e^{-u}\right)du + o\left(\frac{\log^2 t}{t^2}\right) \\ &\le \int_{t\varepsilon}^{t} Q(u,t)k_4(u)du + \int_0^{t\varepsilon} Q(u,t)\left(k_4(u) - \alpha\Psi\left(Q(u,t)\right)e^{-u}\right)du + o\left(\frac{\log^2 t}{t^2}\right) \\ &= \int_0^{t} Q(u,t)k_4(u)du - \alpha\int_0^{t\varepsilon}\Phi\left(Q(u,t)\right)e^{-u}du + o\left(\frac{\log^2 t}{t^2}\right). \end{aligned} \tag{3.13}$$

We now evaluate the integrals in (3.13). Recalling (2.10) with $p = 1$ and $p = 1/2$, we have

$$\begin{aligned} \int_0^t Q(u,t)k_4(u)du &= C\int_0^t \frac{\log(t-u+1)}{t-u+1}k_4(u)du + \rho(t)\int_0^t \frac{\sqrt{\log(t-u+1)}}{t-u+1}k_4(u)du \\ &= C\frac{\log(t+1)}{t+1} + \frac{3Cc_4\log^2 t}{2t^2}(1+o(1)) + \frac{\rho(t)\sqrt{\log(t+1)}}{t+1} + \frac{5c_4\rho(t)\log^{3/2} t}{3t^2}(1+o(1)). \end{aligned}$$

Since all the moments of the distribution with density e^{-u} are finite, and $u^k e^{-u}$ decays rapidly at infinity for any $k \ge 0$, we have

$$\begin{aligned} -\alpha\int_0^{t\varepsilon}\Phi\left(Q(u,t)\right)e^{-u}du &= -\alpha\Phi\left(q(t)\right)(1+o(1)) = -\alpha\frac{f^{(2)}(1)}{2}q^2(t)(1+o(1)) \\ &= -\alpha\frac{f^{(2)}(1)}{2}\left(C^2\frac{\log^2 t}{t^2} + 2C\frac{\rho(t)\log^{3/2} t}{t^2} + \frac{\rho^2(t)\log t}{t^2}\right)(1+o(1)) \\ &= -\alpha\frac{f^{(2)}(1)}{2}\left(C^2\frac{\log^2 t}{t^2} + 2C\frac{\rho(t)\log^{3/2} t}{t^2} + \frac{\rho^2(t)\log t}{t^2}\right) + o\left(\frac{\log^2 t}{t^2}\right). \end{aligned} \tag{3.14}$$

Substituting this in (3.13) we get that for any small $\delta > 0$,

$$\begin{aligned} C\frac{\log(t+1)}{t+1} + \frac{\rho(t)\sqrt{\log(t+1)}}{t+1} &= q(t) \\ &\le \delta\frac{\log^2 t}{t^2} + C\frac{\log(t+1)}{t+1} + \frac{3Cc_4\log^2 t}{2t^2} + \frac{\rho(t)\sqrt{\log(t+1)}}{t+1} + \frac{5c_4\rho(t)\log^{3/2} t}{3t^2} \\ &\quad -\alpha\frac{f^{(2)}(1)}{2}\left(C^2\frac{\log^2 t}{t^2} + 2C\frac{\rho(t)\log^{3/2} t}{t^2} + \frac{\rho^2(t)\log t}{t^2}\right) \end{aligned}$$

which, on account the definition (3.11) leads, after natural transformations, to

$$\frac{\alpha f^{(2)}(1)\rho(t)}{2\sqrt{\log t}}\left(\frac{8}{9}C+\frac{\rho(t)}{\sqrt{\log t}}\right)\le\delta.$$

Recall that $\rho(t)>1$. We get that $\frac{\alpha f^{(2)}(1)\rho(t)}{2\sqrt{\log t}}\le\frac{9\delta}{8}$ and hence

$$\lim\sup_{t\to\infty,t\in\mathcal{A}_+}\frac{\rho(t)}{\sqrt{\log t}}=0.$$

Let

$$t_+(u)=\sup\{t\le u: t\in\mathcal{A}_+\}.$$

Clearly,

$$\rho(u)\le\rho(t_+(u)).$$

Now

$$\begin{aligned}\lim\sup_{u\to\infty}\frac{uq(u)}{\log u} &= C+\lim\sup_{u\to\infty}\left(\frac{\rho(u)}{\sqrt{\log u}}\right)\le C+\lim\sup_{u\to\infty}\left(\frac{\rho(t_+(u))}{\sqrt{\log t_+(u)}}\right)\\ &= C+\lim\sup_{t\to\infty,t\in\mathcal{A}_+}\left(\frac{\rho(t)}{\sqrt{\log t}}\right)=C. \qquad (3.15)\end{aligned}$$

To get estimate from below we assume that

$$\lim\inf_{t\to\infty}\rho(t)=-\infty$$

(otherwise, we are done) and let

$$\mathcal{A}_-:=\left\{t\in[0,\infty):\rho(t)=\inf_{u\le t}\rho(v)\right\}.$$

Our subsequent arguments are for large $t\in\mathcal{A}_-$. For $0\le u\le t$

$$Q(u,t)=C\frac{\log(t-u+1)}{t-u+1}+\frac{\rho(t)\sqrt{\log(t-u+1)}}{t-u+1}\le q(t-u)\le 1.$$

Fix a small $\varepsilon>0$ and consider $0\le u\le t\varepsilon$. In view of (3.12), we get that

$$Q(u,t)\ge C\frac{\log(t+1)}{t+1}+\frac{\rho(t)\sqrt{\log t(1-\varepsilon)}}{t(1-\varepsilon)}\ge\frac{1}{4}C\frac{\log(t+1)}{t+1}>0.$$

Just like in (3.13), we use again the monotonicity in the reverse order and get that

$$q(t)\ge\int_0^t Q(u,t)k_4(u)du-\alpha\int_0^{t\varepsilon}\Phi\left(Q(u,t)\right)e^{-u}du+o\Big(\frac{\log^2 t}{t^2}\Big).$$

By (2.10) with $p=1$ and $p=1/2$ (and using the fact that $|\rho(t)|=O(\sqrt{\log t})$), we get as before

$$\begin{aligned}\int_0^t Q(u,t)k_4(u)du &= C\frac{\log(t+1)}{t+1}+\frac{3Cc_4\log^2 t}{2t^2}\\ &\quad+\frac{\rho(t)\sqrt{\log(t+1)}}{t+1}+\frac{5c_4\rho(t)\log^{3/2}t}{3t^2}+o\left(\frac{\log^2 t}{t^2}\right).\end{aligned}$$

This together with (3.14) implies that for any small $\delta>0$,

$$\begin{aligned}&C\frac{\log(t+1)}{t+1}+\frac{\rho(t)\sqrt{\log(t+1)}}{t+1}=q(t)\\ &\geq C\frac{\log(t+1)}{t+1}+\frac{3Cc_4\log^2 t}{2t^2}+\frac{\rho(t)\sqrt{\log(t+1)}}{t+1}+\frac{5c_4\rho(t)\log^{3/2}t}{3t^2}\\ &\quad-\alpha\frac{f^{(2)}(1)}{2}\left(C^2\frac{\log^2 t}{t^2}+2C\frac{\rho(t)\log^{3/2}t}{t^2}+\frac{\rho^2(t)\log t}{t^2}\right)-\delta\frac{\log^2 t}{t^2}\end{aligned}$$

which, similarly to the previous case gives after simplifications

$$\delta\geq-\frac{\rho(t)\alpha f^{(2)}(1)}{2\sqrt{\log t}}\left(\frac{8}{9}C+\frac{\rho(t)}{\sqrt{\log t}}\right).$$

This, on account of (3.12) gives that (t being large)

$$\delta\geq-\frac{\rho(t)\alpha f^{(2)}(1)}{2\sqrt{\log t}}\frac{1}{9}C.$$

Since δ is arbitrary, we get that

$$\lim\sup_{t\to\infty,t\in\mathcal{A}_-}\frac{|\rho(t)|}{\sqrt{\log t}}=0.$$

Let

$$t_-(u)=\sup\{t\leq u: t\in\mathcal{A}_-\}.$$

Clearly,

$$|\rho(u)|\leq|\rho(t_-(u))|.$$

Now

$$\begin{aligned}\lim\inf_{u\to\infty}\frac{uq(u)}{\sqrt{\log u}} &= C-\lim\sup_{u\to\infty}\left(\frac{|\rho(u)|}{\sqrt{\log u}}\right)\geq C-\lim\sup_{u\to\infty}\left(\frac{|\rho(t_-(u))|}{\sqrt{\log t_-(u)}}\right)\\ &= C-\lim\sup_{t\to\infty,t\in\mathcal{A}_-}\frac{|\rho(t)|}{\sqrt{\log t}}=C. \qquad (3.16)\end{aligned}$$

Combining (3.15) and (3.16) gives

$$\lim_{t\to\infty}\frac{tq(t)}{\log t}=C,$$

proving the Proposition. □

Theorem 3.5 *Assume that $f(s)$ is infinitely differentiable at point $s = 1$ and satisfies*

$$\alpha f'(1) + (1-\alpha)(1-h_4) = 1.$$

Then

$$\lim_{t\to\infty} \mathbf{P}\left(\frac{\mu_0(t)q(t)}{P_1(t)} \le x \,|\mu_0(t) > 0\right) = \frac{1}{3} + \frac{2}{3}\left(1 - e^{-2x/3}\right), x > 0,$$

or, what is the same

$$\lim_{t\to\infty} \mathbf{E}\left[e^{-\lambda\mu_0(t)q(t)/P_1(t)}|\mu_0(t) > 0\right] = \frac{1}{3} + \frac{2}{3}\frac{2}{2+3\lambda}, \lambda \ge 0.$$

Proof. It follows from Lemma 3.1 that

$$\begin{aligned}
\mathbf{E}\mu_0^n(t) &\sim n!\left(\frac{\alpha f^{(2)}(1)}{2}\right)^{n-1} \frac{1}{c_4^{2n-1}} \frac{t^{n-1}}{\log^{2n-1} t} \\
&= \left(\frac{3}{2}\right)^{n-1} n!\left(\frac{\alpha f^{(2)}(1)}{3c_4}\frac{t}{\log t}\right)^{n-1}\left(\frac{1}{c_4 \log t}\right)^n \\
&\sim \left(\frac{3}{2}\right)^{n-1} n!\frac{P_1^n(t)}{q^{n-1}(t)} = q(t)\left(\frac{3}{2}\right)^{n-1} n!\left(\frac{P_1(t)}{q(t)}\right)^n.
\end{aligned}$$

Therefore, as $t \to \infty$

$$\mathbf{E}\left[\left(\frac{\mu_0(t)q(t)}{P_1(t)}\right)^n |\mu_0(t) > 0\right] = \frac{1}{q(t)}\left(\frac{q(t)}{P_1(t)}\right)^n \mathbf{E}\mu_0^n(t) \to \frac{2}{3}\left(\frac{3}{2}\right)^n n!. \tag{3.17}$$

Thus, for any $n \ge 1$ the n-th moment of the conditional distribution converges to the $n-$th moment of the mixture (with probabilities 2/3 and 1/3, respectively) of the exponential distribution with parameter 2/3 (which is uniquely defined by its moments) and the distribution having the unit atom at zero. Hence the statement of the theorem follows. □

Our next step is to generalize Theorem 3.5 to the case of arbitrary probability generating function $f(s)$ with finite second moment. To this aim we need an approximation lemma.

Lemma 3.6 *Let $f(s)$ be an arbitrary probability generating function with $f'(1) > 0$ and $f^{(2)}(1) \in (0, \infty)$. For any $\varepsilon \in (0,1)$, there exist two polynomial probability generating functions $f_-(s)$, $f_+(s)$ and some constant $s_0 = s_0(f_-, f_+, \varepsilon) < 1$ such that*

$$f_-(s) \le f(s) \le f_+(s), \qquad \forall s \in (s_0, 1], \tag{3.18}$$

and

$$f'_-(1) = f'_+(1) = f'(1),$$

and

$$\frac{f_+^{(2)}(1)}{1+\varepsilon} \leq f^{(2)}(1) \leq \frac{f_-^{(2)}(1)}{1-\varepsilon}. \tag{3.19}$$

Remark that necessarily, $f_+^{(2)}(1) \geq f^{(2)}(1) \geq f_-^{(2)}(1)$.

Proof. Let N be an integer valued random variable with generating function f. We only need to consider the unbounded N case, otherwise there is nothing to prove.

Let $\varepsilon > 0$ be small. Assume for the moment that there exist two integer valued and bounded random variables N_1 and N_2, such that

$$\mathbf{E}(N_1) = \mathbf{E}(N_2) = \mathbf{E}(N), \qquad \mathrm{v}ar(N_1) < \mathrm{v}ar(N) < \mathrm{v}ar(N_2), \tag{3.20}$$

and

$$\mathrm{v}ar(N_2) - \varepsilon < \mathrm{v}ar(N) < \mathrm{v}ar(N_1) + \varepsilon. \tag{3.21}$$

Define $f_-(s) := \mathbf{E}(s^{N_1})$, $f_+(s) := \mathbf{E}(s^{N_2})$ for $0 \leq s \leq 1$. Then (3.18) follows from (3.20) by developing the three generating functions at 1, whereas (3.19) follows from (3.21) since ε is arbitrary.

To construct N_1 and N_2 satisfying (3.20) and (3.21), we fix k an integer sufficiently large such that $0 < \mathbf{E}(N(N-k)^+) \leq \varepsilon/3$ and $\mathbf{E}((N-k)^-) \geq 1$ (where $x^+ := \max(x, 0)$ and $x^- := \max(-x, 0)$ for any real x). Since $N = N \wedge k + (N-k)^+$, elementary computation shows that

$$\mathrm{v}ar(N) = \mathrm{v}ar(N \wedge k) + \mathrm{v}ar((N-k)^+) + 2\mathbf{E}((N-k)^+)\,\mathbf{E}((N-k)^-).$$

Therefore,

$$2\mathbf{E}((N-k)^+) \leq \mathrm{v}ar(N) - \mathrm{v}ar(N \wedge k) \leq 2\mathbf{E}(N(N-k)^+) \leq \frac{2\varepsilon}{3}. \tag{3.22}$$

Let $r = \mathbf{E}((N-k)^+) > 0$. Plainly $r \leq \frac{1}{k}\mathbf{E}(N(N-k)^+) \leq \frac{\varepsilon}{3k} < 1$. Choose a Bernoulli variable B with $\mathbf{P}(B = 1) = r = 1 - \mathbf{P}(B = 0)$, independent of N. Define $N_1 := N \wedge k + B$. Hence $\mathbf{E}(N_1) = \mathbf{E}(N)$. On the other hand, it follows from (3.22) that $\mathrm{v}ar(N) \leq \mathrm{v}ar(N \wedge k) + 2\varepsilon/3 < \mathrm{v}ar(N_1) + 2\varepsilon/3$, and $\mathrm{v}ar(N_1) = \mathrm{v}ar(N \wedge k) + r(1-r) \leq \mathrm{v}ar(N \wedge k) + r \leq \mathrm{v}ar(N) - r$ since $\mathbf{E}((N-k)^+) = r$. Then N_1 fulfills the conditions in (3.20) and (3.21).

To construct N_2, we choose $\ell := \lfloor 3\frac{\mathbf{E}(N(N-k)^+))}{\mathbf{E}((N-k)^+)} \rfloor$ and $b := \frac{1}{\ell}\mathbf{E}((N-k)^+)$. Let $\widetilde{B}$ be a Bernoulli variable with $\mathbf{P}(\widetilde{B} = 1) = b = 1 - \mathbf{P}(\widetilde{B} = 0)$, independent of N. Define $N_2 := N \wedge k + \ell\,\widetilde{B}$. Plainly, $\mathbf{E}(N_2) = \mathbf{E}(N)$ and

$$\mathrm{v}ar(N_2) = \mathrm{v}ar(N \wedge k) + \ell^2 b(1-b) < \mathrm{v}ar(N \wedge k) + 3\mathbf{E}(N(N-k)^+)) \leq \mathrm{v}ar(N) + \varepsilon.$$

Note that $\ell^2 b(1-b) \geq \frac{5}{2}\,\ell b\,\frac{\mathbf{E}(N(N-k)^+))}{\mathbf{E}((N-k)^+)} = \frac{5}{2}\,\mathbf{E}(N(N-k)^+))$. It follows that $\mathrm{v}ar(N_2) \geq \mathrm{v}ar(N \wedge k) + \frac{5}{2}\,\mathbf{E}(N(N-k)^+)) \geq \mathrm{v}ar(N) + \frac{1}{2}\,\mathbf{E}(N(N-k)^+)) > \mathrm{v}ar(N)$. This shows that N_2 also fulfills the conditions in (3.20) and (3.21) and completes the proof of lemma. □

Lemma 3.7 *If there are two branching random walks with branching at the origin only whose offspring generating functions $f_1(s)$ and $f_2(s)$ are such that*

$$\alpha f_i'(1) + (1-\alpha)(1-h_4) = 1, \quad i = 1, 2,$$

and for some constant $0 \leq s_0 < 1$,

$$f_1(s) \leq f_2(s), \qquad \forall\, s_0 < s \leq 1,$$

then the respective generating functions $F^1(t;s)$ and $F^2(t;s)$ for the number of particles at the origin at moment t meet the inequality

$$F^1(t;s) \leq F^2(t;s) \tag{3.23}$$

for all $s \in (s_0, 1]$.

Proof. Let $0 \leq s \leq 1$ and $t \geq 0$. Introduce the notation

$$\begin{aligned} L\,(f,F)\,(t;s) \;&:=\; s(1-G_1(t)) + (1-\alpha)(1-h_4)(1-G_2(\cdot)) * G_1(t) \\ &+\; \int_0^t \alpha f(F(t-u;s))\,dG_1(u) + (1-\alpha)h_4 G_1(t) \\ &+\; \int_0^t (1-\alpha)(1-h_4)F(t-u;s)\,d(G_1 * G_2(u)), \end{aligned} \tag{3.24}$$

and for $i = 1, 2$, set

$$F_0^i(t;s) = s, \quad F_{n+1}^i(t;s) := L\left(f_i, F_n^i\right)(t;s)\,.$$

Let us show by induction on n that

$$s \leq F_n^i(t;s) \leq F_{n+1}^i(t;s), \qquad \forall 0 \leq s \leq 1.$$

Indeed, the expression

$$R_i(s) := \alpha f_i(s) + (1-\alpha)(1-h_4)s + (1-\alpha)h_4$$

is a probability generating function with $R_i'(1) = \alpha f_i'(1) + (1-\alpha)(1-h_4) = 1$. Hence $R_i(s) \geq s$ for all $s \in [0,1]$. Using this fact, we have

$$\begin{aligned}
& F_1^i(t;s) = L\left(f_i, F_0^i\right)(t;s) \\
= \;& s(1-G_1(t)) + (1-\alpha)(1-h_4)(1-G_2(\cdot)) * G_1(t) \\
& + \int_0^t \alpha f_i(s)\, dG_1(u) + (1-\alpha)h_4 G_1(t) + \int_0^t (1-\alpha)(1-h_4)s\, d(G_1 * G_2(u)) \\
= \;& s(1-G_1(t)) + (1-\alpha)(1-h_4)(1-G_2(\cdot)) * G_1(t) + (1-\alpha)h_4 G_1(t) \\
& + \alpha f_i(s) G_1(t) + (1-\alpha)(1-h_4)sG_1 * G_2(t) \\
= \;& s(1-G_1(t)) + (1-s)(1-\alpha)(1-h_4)(1-G_2(\cdot)) * G_1(t) + R_i(s)G_1(t) \\
\geq \;& s(1-G_1(t)) + sG_1(t) = s.
\end{aligned}$$

And if this is true for some n then, by monotonicity

$$F_{n+2}^i(t;s) = L\left(f_i, F_{n+1}^i\right)(t;s) \geq L\left(f_i, F_n^i\right)(t;s) = F_{n+1}^i(t;s) \geq s. \tag{3.25}$$

Next we claim that if $s \in (s_0, 1]$ then

$$F_n^1(t;s) \leq F_n^2(t;s), \qquad \forall n \geq 0. \tag{3.26}$$

Indeed, this is true for $n = 0$ and if this is true for some n then in view of (3.25) for $s \in (s_0, 1]$

$$F_{n+1}^1(t;s) = L\left(f_1, F_n^1\right)(t;s) \leq L\left(f_2, F_n^1\right)(t;s) \leq L\left(f_2, F_n^2\right)(t;s) = F_{n+1}^2(t;s).$$

Now on account of (3.26) we may pass to the limit as $n \to \infty$ to get (3.23). The lemma is proved. □

Proof of Theorem 1.1. The first part of the theorem is simply Proposition 3.4.

To prove the second part assume that $f(s)$ is not a polynomial probability generating function (otherwise Theorem 3.5 gives the desired statement). Let, for a fixed $\varepsilon > 0$, $f_-(s)$ and $f_+(s)$ be the polynomial probability generating functions satisfying the conditions of Lemma 3.6 and let $F^-(t;s)$ and $F^+(t;s)$ be the probability generating functions corresponding to the branching processes in $\mathbb{Z}^4$ with branching at the origin only and the reproduction laws specified by $f_-(s)$ and $f_+(s)$ respectively. Let $q^\pm(t) := 1 - F^\pm(t;0)$. Remark that the asymptotic of $q^\pm(t)$ is given by (3.10) with corresponding constants related to $f_\pm^{(2)}(1)$. Let $\varepsilon > 0$ be small. By (3.19), we have that for all sufficiently large t

$$\frac{1}{1+2\varepsilon} \leq \frac{q^+(t)}{q(t)} \text{ and } \frac{q^-(t)}{q(t)} \leq \frac{1}{1-2\varepsilon}$$

while by (3.10), (3.2) and (3.19), we have that for all large t,

$$(1-2\varepsilon)\frac{q^-(t)}{P_1^-(t)} \le \frac{q(t)}{P_1(t)} \le (1+2\varepsilon)\frac{q^-(t)}{P_1^-(t)}, \quad (1-2\varepsilon)\frac{q^+(t)}{P_1^+(t)} \le \frac{q(t)}{P_1(t)} \le (1+2\varepsilon)\frac{q^+(t)}{P_1^+(t)},$$

where $P_1^{\pm}(t)$ are defined in the obvious way. Clearly, for any $\lambda > 0$

$$\begin{aligned} \mathbf{E}\left[e^{-\lambda\mu_0(t)q(t)/P_1(t)}|\mu_0(t)>0\right] &= \frac{F\left(t;e^{-\lambda q(t)/P_1(t)}\right)-F\left(t;0\right)}{1-F\left(t;0\right)} \\ &= 1-\frac{1-F\left(t;e^{-\lambda q(t)/P_1(t)}\right)}{q(t)}. \end{aligned} \tag{3.27}$$

Further, $e^{-\lambda q(t)/P_1(t)} > s_0$ for all large t and we deduce from Lemma 3.7 that

$$\begin{aligned} \frac{q^+(t)}{q(t)}\frac{1-F^+\left(t;e^{-\lambda q(t)/P_1(t)}\right)}{q^+(t)} &\le \frac{1-F\left(t;e^{-\lambda q(t)/P_1(t)}\right)}{q(t)} \\ &\le \frac{q^-(t)}{q(t)}\frac{1-F^-\left(t;e^{-\lambda q(t)/P_1(t)}\right)}{q^-(t)}. \end{aligned}$$

On the other hand, by monotonicity, $F^+\left(t;e^{-\lambda q(t)/P_1(t)}\right) \le F^+\left(t;e^{-\lambda(1+2\varepsilon)q^+(t)/P_1^+(t)}\right)$ and $F^-\left(t;e^{-\lambda q(t)/P_1(t)}\right) \ge F^-\left(t;e^{-\lambda(1-2\varepsilon)q^-(t)/P_1^-(t)}\right)$. Hence, letting $t\to\infty$ we get on account of Theorem 3.5 that

$$\begin{aligned} \frac{1}{1+2\varepsilon}\left(\frac{2}{3}-\frac{2}{3}\frac{2}{2+3(1+2\varepsilon)\lambda}\right) &\le \liminf_{t\to\infty}\frac{1-F\left(t;e^{-\lambda q(t)/P_1(t)}\right)}{q(t)} \\ &\le \limsup_{t\to\infty}\frac{1-F\left(t;e^{-\lambda q(t)/P_1(t)}\right)}{q(t)} \\ &\le \frac{1}{1-2\varepsilon}\left(\frac{2}{3}-\frac{2}{3}\frac{2}{2+3(1-2\varepsilon)\lambda}\right). \end{aligned}$$

Letting now $\varepsilon\to+0$ we see by (3.27) that

$$\lim_{t\to\infty}\mathbf{E}\left[e^{-\lambda\mu_0(t)q(t)/P_1(t)}|\mu_0(t)>0\right] = \frac{1}{3}+\frac{2}{3}\frac{2}{2+3\lambda},$$

as desired. □

Remark 3.8 *It follows from (3.2) and Proposition 3.4 that the scaling in Theorem 1.1 has the following asymptotic behavior*

$$\mathbf{E}\left[\mu_0(t)|\mu_0(t)>0\right] = \frac{\mathbf{E}\mu_0(t)}{\mathbf{P}\left(\mu_0(t)>0\right)} \sim \frac{3}{\alpha f^{(2)}(1)C^2}\,\frac{t}{\log^2 t}, \qquad t\to\infty.$$

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