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# The Energy-Momentum Tensor on $\text{Spin}^c$ Manifolds

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## Abstract

On  $\text{Spin}^c$  manifolds, we study the Energy-Momentum tensor associated with a spinor field. First, we give a spinorial Gauss type formula for oriented hypersurfaces of a  $\text{Spin}^c$  manifold. Using the notion of generalized cylinders, we derive the variationnal formula for the Dirac operator under metric deformation and point out that the Energy-Momentum tensor appears naturally as the second fundamental form of an isometric immersion. Finally, we show that generalized  $\text{Spin}^c$  Killing spinors for Codazzi Energy-Momentum tensor are restrictions of parallel spinors.

**Keywords:**  $\text{Spin}^c$  structures;  $\text{Spin}^c$  Gauss formula; metric variation formula for the Dirac operator; Energy-Momentum tensor; generalized cylinder; generalized Killing spinors.

## 1 Introduction

In [14], O. Hijazi proved that on a compact Riemannian spin manifold  $(M^n, g)$  any eigenvalue  $\lambda$  of the Dirac operator to which is attached an eigenspinor  $\psi$  satisfies

$$\lambda^2 \geq \inf_M \left( \frac{1}{4} \text{Scal}^M + |\ell^\psi|^2 \right), \quad (1)$$

where  $\text{Scal}^M$  is the scalar curvature of the manifold  $M$  and  $\ell^\psi$  is the field of symmetric endomorphisms associated with the field of quadratic forms  $T^\psi$  called the Energy-Momentum tensor. It is defined on the complement set of zeroes of the eigenspinor  $\psi$ , for any vector  $X \in \Gamma(TM)$  by

$$T^\psi(X) = \text{Re} \langle X \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2} \rangle.$$

Here  $\nabla$  denotes the Levi-Civita connection on the spinor bundle of  $M$  and “ $\cdot$ ” the Clifford multiplication. The limiting case of (1) is characterized by the existence of a spinor field  $\psi$  satisfying for all  $X \in \Gamma(TM)$ ,

$$\nabla_X \psi = -\ell^\psi(X) \cdot \psi. \quad (2)$$

For  $\text{Spin}^c$  structures, the complex line bundle  $L^M$  is endowed with an arbitrary connection and hence an arbitrary curvature  $i\Omega^M$  which is an imaginary 2-form on the manifold. In terms of the Energy-Momentum tensor the author proved in [25] that on a compact Riemannian  $\text{Spin}^c$  manifold any eigenvalue  $\lambda$  of the Dirac operator to which is attached an eigenspinor  $\psi$  satisfies

$$\lambda^2 \geq \inf_M \left( \frac{1}{4} \text{Scal}^M - \frac{c_n}{4} |\Omega^M| + |\ell^\psi|^2 \right), \quad (3)$$

where  $c_n = 2[\frac{n}{2}]^{\frac{1}{2}}$ . The limiting case of (3) is characterized by the existence of a spinor field  $\psi$  satisfying for every  $X \in \Gamma(TM)$ ,

$$\begin{cases} \nabla_X^{\Sigma M} \psi = -\ell^\psi(X) \cdot \psi, \\ \Omega^M \cdot \psi = i \frac{c_n}{2} |\Omega^M| \psi. \end{cases} \quad (4)$$

Here  $\nabla^{\Sigma M}$  denotes the Levi-Civita connection on the  $\text{Spin}^c$  spinor bundle and “ $\cdot$ ” the  $\text{Spin}^c$  Clifford multiplication. In [25], the author showed also that the sphere with a special  $\text{Spin}^c$  structure is a limiting manifold for (3).

Studying the Energy-Momentum tensor on a Riemannian or semi-Riemannian spin manifolds has been done by many authors, since it is related to several geometric constructions (see [12], [2], [24] and [6] for results in this topic). In this paper we study the Energy-Momentum tensor on Riemannian and semi-Riemannian  $\text{Spin}^c$  manifolds. First, we prove that the Energy-Momentum tensor appears in the study of the variations of the spectrum of the Dirac operator:

**Proposition 1.1** *Let  $(M^n, g)$  be a  $\text{Spin}^c$  Riemannian manifold and  $g_t = g + tk$  a smooth 1-parameter family of metrics. For any spinor field  $\psi \in \Gamma(\Sigma M)$ , we have*

$$\left. \frac{d}{dt} \right|_{t=0} (D^{M_t} \tau_0^t \psi, \tau_0^t \psi)_{g_t} = -\frac{1}{2} \int_M \langle k, T_\psi \rangle dv_g, \quad (5)$$

where  $(.,.) = \int_M \text{Re} \langle ., . \rangle dv_g$ , the Dirac operator  $D^{M_t}$  is the Dirac operator associated with  $M_t = (M, g_t)$  and  $\tau_0^t \psi$  is the image of  $\psi$  under the isometry  $\tau_0^t$  between the spinor bundles of  $(M, g)$  and  $(M, g_t)$ . Here  $T_\psi$  is defined by  $T_\psi = |\psi|^2 T^\psi$  and  $T^\psi$  is the symmetric bilinear form associated with the Energy-Momentum tensor, i.e. it is given for every  $X, Y \in \Gamma(TM)$  by  $T^\psi(X, Y) = \frac{1}{2} \text{Re} \left\langle X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2} \right\rangle$ .

This was proven in [4] by J. P. Bourguignon and P. Gauduchon for spin manifolds. Using this, we extend to  $\text{Spin}^c$  manifolds a result by Th. Friedrich and E. C. Kim in [8] on spin manifolds:

**Theorem 1.2** *Let  $M$  be a  $\text{Spin}^c$  Riemannian manifold. A pair  $(g_0, \psi_0)$  is a critical point of the Lagrange functional*

$$\mathcal{W}(g, \psi) = \int_U \left( \text{Scal}_g^M + \varepsilon \lambda |\psi|_g^2 - \varepsilon \text{Re} \langle D_g \psi, \psi \rangle_g \right) dv_g,$$

$(\lambda, \varepsilon \in \mathbb{R})$  for all open subsets  $U$  of  $M$  if and only if  $(g_0, \psi_0)$  is a solution of the following system

$$\begin{cases} D_g \psi = \lambda \psi, \\ \text{ric}_g^M - \frac{\text{Scal}_g^M}{2} g = \frac{\varepsilon}{2} T_\psi, \end{cases}$$

where  $\text{ric}_g^M$  denotes the Ricci curvature of  $M$  considered as a symmetric bilinear form.

Now, we interpret the Energy-Momentum tensor as the second fundamental form of a hypersurface. In fact, we prove the following:

**Proposition 1.3** *Let  $M^n \hookrightarrow (\mathcal{Z}, g)$  be any compact oriented hypersurface isometrically immersed in an oriented Riemannian  $\text{Spin}^c$  manifold  $(\mathcal{Z}, g)$ , of mean curvature  $H$  and Weingarten map  $W$ . Assume that  $\mathcal{Z}$  admits a parallel spinor field  $\psi$ , then the Energy-Momentum tensor associated with  $\varphi =: \psi|_M$  satisfies*

$$2\ell^\varphi = -W.$$

Moreover, if the mean curvature  $H$  is constant, the hypersurface  $M$  satisfies the equality case in (3) if and only if

$$\text{Scal}^{\mathcal{Z}} - 2 \text{ric}^{\mathcal{Z}}(\nu, \nu) - c_n |\Omega^M| = 0. \quad (6)$$

This was proven by Morel in [24] for a compact oriented hypersurface of a spin manifold carrying parallel spinor but in this case the hypersurface  $M$  is directly a limiting manifold for (1) without the condition (6).

Finally, we study generalized Killing spinors on  $\text{Spin}^c$  manifolds. They are characterized by the identity, for any tangent vector field  $X$  on  $M$ ,

$$\nabla_X^{\Sigma M} \psi = \frac{1}{2} F(X) \cdot \psi, \quad (7)$$

where  $F$  is a given symmetric endomorphism on the tangent bundle. It is straightforward to see that

$$2T^\psi(X, Y) = -\langle F(X), Y \rangle.$$

These spinors are closely related to the so-called  $T$ -Killing spinors studied by Friedrich and Kim in [9] on spin manifolds. It is natural to ask whether the tensor  $F$  can be realized as the Weingarten tensor of some isometric embedding of  $M$  in a manifold  $\mathcal{Z}^{n+1}$  carrying parallel spinors. Morel studied this problem in the case of spin manifolds where the tensor  $F$  is parallel and in [2], the authors studied the problem in the case of semi-Riemannian spin manifolds where the tensor  $F$  is a Codazzi-Mainardi tensor. We establish the corresponding result for semi-Riemannian  $\text{Spin}^c$  manifolds:

**Theorem 1.4** *Let  $(M^n, g)$  be a semi-Riemannian  $\text{Spin}^c$  manifold carrying a generalized  $\text{Spin}^c$  Killing spinor  $\varphi$  with a Codazzi-Mainardi tensor  $F$ . Then the generalized cylinder  $\mathcal{Z} := I \times M$  with the metric  $dt^2 + g_t$ , where  $g_t(X, Y) = g((\text{Id} - tF)^2 X, Y)$ , equipped with the  $\text{Spin}^c$  structure arising from the given one on  $M$  has a parallel spinor whose restriction to  $M$  is just  $\varphi$ .*

A characterisation of limiting 3-dimensional manifolds for (3), having generalized  $\text{Spin}^c$  Killing spinors with Codazzi tensor is then given.

The paper is organised as follows: In Section 2, we collect basic material on spinors and the Dirac operator on semi-Riemannian  $\text{Spin}^c$  manifolds. In Section 3, we study hypersurfaces of  $\text{Spin}^c$  manifolds. We derive a spinorial Gauss formula after identifying the restriction of the  $\text{Spin}^c$  spinor bundle of the ambient manifold with the  $\text{Spin}^c$  spinor bundle of the hypersurface. In Section 4, we define the generalized cylinder of a  $\text{Spin}^c$  manifold  $M$  and we collect formulas relating the curvature of a generalized cylinder to geometric data on  $M$ . In section 5, we compare the Dirac operators for two different semi-Riemannian metrics, then one first has to identify the spinor bundles using parallel transport. In the last section, we interpret the Energy-Momentum tensor as the second fundamental form of a hypersurface and we study generalized  $\text{Spin}^c$  Killing spinors. The author would like to thank Oussama Hijazi for his support and encouragements.

## 2 The Dirac operator on semi-Riemannian $\text{Spin}^c$ manifolds

In this section, we collect some algebraic and geometric preliminaries concerning the Dirac operator on semi-Riemannian  $\text{Spin}^c$  manifolds. Details can be found in [3] and [2]. Let  $r + s = n$  and consider on  $\mathbb{R}^n$  the nondegenerate symmetric bilinear form of signature  $(r, s)$  given by

$$\langle v, w \rangle := \sum_{j=1}^r v_j w_j - \sum_{j=r+1}^n v_j w_j,$$

for any  $v, w \in \mathbb{R}^n$ . We denote by  $\text{Cl}_{r,s}$  the real Clifford algebra corresponding to  $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ , this is the unitary algebra generated by  $\mathbb{R}^n$  subject to the relations

$$e_j \cdot e_k + e_k \cdot e_j = \begin{cases} -2\delta_{jk} & \text{if } j \leq r, \\ 2\delta_{jk} & \text{if } j > r, \end{cases}$$

where  $(e_j)_{1 \leq j \leq n}$  is an orthonormal basis of  $\mathbb{R}^n$  of signature  $(r, s)$ , i.e.,  $\langle e_j, e_k \rangle = \varepsilon_j \delta_{jk}$  and  $\varepsilon_j = \pm 1$ . The complex Clifford algebra  $\mathbb{C}\text{Cl}_{r,s}$  is the complexification of  $\text{Cl}_{r,s}$  and it decomposes into even and odd elements  $\mathbb{C}\text{Cl}_{r,s} = \mathbb{C}\text{Cl}_{r,s}^0 \oplus \mathbb{C}\text{Cl}_{r,s}^1$ . The real spin group is defined by

$$\text{Spin}(r, s) := \{v_1 \cdot \dots \cdot v_{2k} \in \text{Cl}_{r,s} \mid v_j \in \mathbb{R}^n \text{ such that } \langle v_j, v_j \rangle = \pm 1\}.$$

The spin group  $\text{Spin}(r, s)$  is the double cover of  $\text{SO}(r, s)$ , in fact the following sequence is exact

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \text{Spin}(r, s) \xrightarrow{\xi} \text{SO}(r, s) \longrightarrow 1,$$

where  $\xi = \text{Ad}|_{\text{Spin}(r,s)}$  and  $\text{Ad}$  is defined by

$$\begin{aligned} \text{Ad} : \text{Cl}_{r,s}^* &\longrightarrow \text{End}(\mathbb{R}^n) \\ w &\longrightarrow \text{Ad}_w : v \longrightarrow \text{Ad}_w(v) = w \cdot v \cdot w^{-1}. \end{aligned}$$

Here  $\text{Cl}_{r,s}^*$  denotes the group of units of  $\text{Cl}_{r,s}$ . Since  $\mathbb{S}^1 \cap \text{Spin}(r, s) = \{\pm 1\}$ , we define the complex spin group by

$$\text{Spin}^c(r, s) = \text{Spin}(r, s) \times_{\mathbb{Z}_2} \mathbb{S}^1.$$

The complex spin group is the double cover of  $\text{SO}(r, s) \times \mathbb{S}^1$ , this yields to the exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}^c(r, s) \xrightarrow{\xi^c} \text{SO}(r, s) \times \mathbb{S}^1 \longrightarrow 1,$$

where  $\xi^c = (\xi, \text{Id}^2)$ . When  $n = 2m$  is even,  $\mathbb{C}\text{Cl}_{r,s}$  has a unique irreducible complex representation  $\chi_{2m}$  of complex dimension  $2^m$ ,  $\chi_{2m} : \mathbb{C}\text{Cl}_{r,s} \longrightarrow \text{End}(\Sigma_{r,s})$ . If  $n =$

$2m + 1$  is odd,  $\mathbb{C}l_{r,s}$  has two inequivalent irreducible representations both of complex dimension  $2^m$ ,  $\chi_{2m+1}^j : \mathbb{C}l_{r,s} \longrightarrow \text{End}(\Sigma_{r,s}^j)$ , for  $j = 0$  or  $1$ , where  $\Sigma_{r,s}^j = \{\sigma \in \Sigma_{r,s}, \chi_{2m+1}^j(\omega_{r,s})\sigma = (-1)^j \sigma\}$  and  $\omega_{r,s}$  is the complex volume element

$$\omega_{r,s} = \begin{cases} i^{m-s} & e_1 \cdots e_n & \text{if } n = 2m, \\ i^{m-1+s} & e_1 \cdots e_n & \text{if } n = 2m + 1. \end{cases}$$

We define the complex spinorial representation  $\rho_n$  by the restriction of an irreducible representation of  $\mathbb{C}l_{r,s}$  to  $\text{Spin}^c(r, s)$ :

$$\rho_n := \begin{cases} \chi_{2m}^0|_{\text{Spin}^c(r,s)} & \text{if } n = 2m, \\ \chi_{2m+1}^0|_{\text{Spin}^c(r,s)} & \text{if } n = 2m + 1, \end{cases}.$$

When  $n = 2m$  is even,  $\rho_n$  decomposes into two inequivalent irreducible representations  $\rho_n^+$  and  $\rho_n^-$ , i.e.,  $\rho_n = \rho_n^+ + \rho_n^- : \text{Spin}^c(r, s) \rightarrow \text{Aut}(\Sigma_{r,s})$ . The space  $\Sigma_{r,s}$  decomposes into  $\Sigma_{r,s} = \Sigma_{r,s}^+ \oplus \Sigma_{r,s}^-$ , where  $\omega_{r,s}$  acts on  $\Sigma_{r,s}^+$  as the identity and minus the identity on  $\Sigma_{r,s}^-$ . If  $n = r + s$  is odd and when restricted to  $\text{Spin}^c(r, s)$ , the two representations  $\chi_{2m+1}^0|_{\text{Spin}^c(r,s)}$  and  $\chi_{2m+1}^1|_{\text{Spin}^c(r,s)}$  are equivalent and we simply choose  $\Sigma_{r,s} := \Sigma_{r,s}^0$ . The complex spinor bundle  $\Sigma_{r,s}$  carries a Hermitian symmetric bilinear  $\text{Spin}^c(r, s)$ -invariant form  $\langle \cdot, \cdot \rangle$ , such that

$$\langle v \cdot \sigma_1, \sigma_2 \rangle = (-1)^{s+1} \langle \sigma_1, v \cdot \sigma_2 \rangle \text{ for all } \sigma_1, \sigma_2 \in \Sigma_{r,s} \text{ and } v \in \mathbb{R}^n.$$

Now, we give the following isomorphism  $\alpha$ , which is of particular importance for the identification of the  $\text{Spin}^c$  bundles in the context of immersions of hypersurfaces:

$$\begin{aligned} \alpha : \mathbb{C}l_{r,s} &\longrightarrow \mathbb{C}l_{r+1,s}^0 \\ e_j &\longrightarrow \nu \cdot e_j, \end{aligned} \tag{8}$$

where we look at an embedding of  $\mathbb{R}^n$  onto  $\mathbb{R}^{n+1}$  such that  $(\mathbb{R}^n)^\perp$  is spacelike and spanned by a spacelike unit vector  $\nu$ .

Let  $N^n$  be an oriented semi-Riemannian manifold of signature  $(r, s)$  and let  $P_{\text{SO}}N$  be the  $\text{SO}(r, s)$ -principal bundle of positively space and time oriented orthonormal tangent frames. A complex  $\text{Spin}^c$  structure on  $N$  is a  $\text{Spin}^c(r, s)$ -principal bundle  $P_{\text{Spin}^c}N$  over  $N$ , an  $\mathbb{S}^1$ -principal bundle  $P_{\mathbb{S}^1}N$  over  $N$  together with a twofold covering map  $\Theta : P_{\text{Spin}^c}N \longrightarrow P_{\text{SO}}N \times_N P_{\mathbb{S}^1}N$  such that

$$\Theta(ua) = \Theta(u)\xi^c(a),$$

for every  $u \in P_{\text{Spin}^c}N$  and  $a \in \text{Spin}^c(r, s)$ , i.e.,  $N$  has a  $\text{Spin}^c$  structure if and only if there exists an  $\mathbb{S}^1$ -principal bundle  $P_{\mathbb{S}^1}N$  over  $N$  such that the transition functions  $g_{\alpha\beta} \times l_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \text{SO}(r, s) \times \mathbb{S}^1$  of the

$\mathrm{SO}(r, s) \times \mathbb{S}^1$ -principal bundle  $P_{\mathrm{SO}}N \times_N P_{\mathbb{S}^1}N$  admit lifts to  $\mathrm{Spin}^c(r, s)$  denoted by  $\widetilde{g}_{\alpha\beta} \times \widetilde{l}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Spin}^c(r, s)$ , such that  $\xi^c \circ (\widetilde{g}_{\alpha\beta} \times \widetilde{l}_{\alpha\beta}) = g_{\alpha\beta} \times l_{\alpha\beta}$ . This, anyhow, is equivalent to the second Stiefel-Whitney class  $w_2(N)$  being equal, modulo 2, to the first chern class  $c_1(L^N)$  of the complex line bundle  $L^N$ . It is the complex line bundle associated with the  $\mathbb{S}^1$ -principal fibre bundle via the standard representation of the unit circle.

Let  $\Sigma N := P_{\mathrm{Spin}^c}N \times_{\rho_n} \Sigma_{r,s}$  be the spinor bundle associated with the spinor representation. A section of  $\Sigma N$  will be called a spinor field. Using the cocycle condition of the transition functions of the two principal fibre bundles  $P_{\mathrm{Spin}^c}N$  and  $P_{\mathrm{SO}}N \times_N P_{\mathbb{S}^1}N$ , we can prove that

$$\Sigma N = \Sigma' N \otimes (L^N)^{\frac{1}{2}},$$

where  $\Sigma' N$  is the locally defined spin bundle and  $(L^N)^{\frac{1}{2}}$  is locally defined too but  $\Sigma N$  is globally defined. The tangent bundle  $TN = P_{\mathrm{SO}}N \times_{\rho_0} \mathbb{R}^n$  where  $\rho_0$  stands for the standard matrix representation of  $\mathrm{SO}(r, s)$  on  $\mathbb{R}^n$ , can be seen as the associated vector bundle  $TN \simeq P_{\mathrm{Spin}^c}N \times_{pr_1 \circ \xi^c \circ \rho_0} \mathbb{R}^n$  where  $pr_1$  is the first projection. One defines the Clifford multiplication at every point  $p \in N$ :

$$\begin{aligned} T_p N \otimes \Sigma_p N &\longrightarrow \Sigma_p N \\ [b, v] \otimes [b, \sigma] &\longrightarrow [b, v] \cdot [b, \sigma] := [b, v \cdot \sigma = \chi_n(v)\sigma], \end{aligned}$$

where  $b \in P_{\mathrm{Spin}^c}N$ ,  $v \in \mathbb{R}^n$ ,  $\sigma \in \Sigma_{r,s}$  and  $\chi_n = \chi_{2m}$  if  $n$  is even and  $\chi_n = \chi_{2m+1}^0$  if  $n$  is odd. The Clifford multiplication can be extended to differential forms. Clifford multiplication inherits the relations of the Clifford algebra, i.e., for  $X, Y \in T_p N$  and  $\varphi \in \Sigma_p N$  we have  $X \cdot Y \cdot \varphi + Y \cdot X \cdot \varphi = -2 \langle X, Y \rangle \varphi$ . In even dimensions the spinor bundle splits into  $\Sigma N = \Sigma^+ N \oplus \Sigma^- N$ , where  $\Sigma^\pm N = P_{\mathrm{Spin}^c}N \times_{\rho_n^\pm} \Sigma_{r,s}^\pm$ . Clifford multiplication by a non-vanishing tangent vector interchanges  $\Sigma^+ N$  and  $\Sigma^- N$ . The  $\mathrm{Spin}^c(r, s)$ -invariant nondegenerate symmetric sesquilinear form on  $\Sigma_{r,s}$  and  $\Sigma_{r,s}^\pm$  induces inner products on  $\Sigma N$  and  $\Sigma^\pm N$  which we again denote by  $\langle \cdot, \cdot \rangle$  and it satisfies

$$\langle X \cdot \psi, \varphi \rangle = (-1)^{s+1} \langle \psi, X \cdot \varphi \rangle,$$

for every  $X \in \Gamma(TN)$  and  $\psi, \varphi \in \Gamma(\Sigma N)$ . Additionally, given a connection 1-form  $A^N$  on  $P_{\mathbb{S}^1}N$ ,  $A^N : T(P_{\mathbb{S}^1}N) \longrightarrow i\mathbb{R}$  and the connection 1-form  $\omega^N$  on  $P_{\mathrm{SO}}N$  for the Levi-Civita connection  $\nabla^N$ , we can define the connection

$$\omega^N \times A^N : T(P_{\mathrm{SO}}N \times_N P_{\mathbb{S}^1}N) \longrightarrow \mathfrak{so}_n \oplus i\mathbb{R} = \mathfrak{spin}_n^{\mathbb{C}}$$

on the principal fibre bundle  $P_{\mathrm{SO}}N \times_N P_{\mathbb{S}^1}N$  and hence a covariant derivative  $\nabla^{\Sigma N}$  on  $\Sigma N$  [7] given locally by

$$\begin{aligned} \nabla_{e_k}^{\Sigma N} \varphi &= \left[ \widetilde{b \times s}, e_k(\sigma) + \frac{1}{4} \sum_{j=1}^n \varepsilon_j e_j \cdot \nabla_{e_k}^N e_j \cdot \sigma + \frac{1}{2} A^N(s_*(e_k))\sigma \right] \\ &= e_k(\varphi) + \frac{1}{4} \sum_{j=1}^n \varepsilon_j e_j \cdot \nabla_{e_k}^N e_j \cdot \varphi + \frac{1}{2} A^N(s_*(e_k))\varphi, \end{aligned} \quad (9)$$



where  $\varphi = [\widetilde{b \times s}, \sigma]$  is a locally defined spinor field,  $b = (e_1, \dots, e_n)$  is a local space and time oriented orthonormal tangent frame,  $s : U \rightarrow P_{\mathbb{S}^1}N$  is a local section of  $P_{\mathbb{S}^1}N$  and  $\widetilde{b \times s}$  is the lift of the local section  $b \times s : U \rightarrow P_{\text{SO}}N \times_N P_{\mathbb{S}^1}N$  to the 2-fold covering  $\Theta : P_{\text{Spin}^c}N \rightarrow P_{\text{SO}}N \times_N P_{\mathbb{S}^1}N$ . The curvature of  $A^N$  is an imaginary valued 2-form denoted by  $F_{A^N} = dA^N$ , i.e.,  $F_{A^N} = i\Omega^N$ , where  $\Omega^N$  is a real valued 2-form on  $P_{\mathbb{S}^1}N$ . We know that  $\Omega^N$  can be viewed as a real valued 2-form on  $N$  [7]. In this case  $i\Omega^N$  is the curvature form of the associated line bundle  $L^N$ . The curvature tensor  $\mathcal{R}^{\Sigma N}$  of  $\nabla^{\Sigma N}$  is given by

$$\mathcal{R}^{\Sigma N}(X, Y)\varphi = \frac{1}{4} \sum_{j,k=1}^n \varepsilon_j \varepsilon_k \langle R^N(X, Y)e_j, e_k \rangle e_j \cdot e_k \cdot \varphi + \frac{i}{2} \Omega^N(X, Y)\varphi, \quad (10)$$

where  $R^N$  is the curvature tensor of the Levi-Civita connection  $\nabla^N$ . In the  $\text{Spin}^c$  case, the Ricci identity translates, for every  $X \in \Gamma(TN)$ , to

$$\sum_{k=1}^n \varepsilon_k e_k \cdot \mathcal{R}^{\Sigma N}(e_k, X)\varphi = \frac{1}{2} \text{Ric}^N(X) \cdot \varphi - \frac{i}{2} (X \lrcorner \Omega^N) \cdot \varphi. \quad (11)$$

Here  $\text{Ric}^N$  denotes the Ricci curvature considered as a field of endomorphism on  $TN$ . The Ricci curvature considered as a symmetric bilinear form will be written  $\text{ric}^N(Y, Z) = \langle \text{Ric}^N(Y), Z \rangle$ . The Dirac operator maps spinor fields to spinor fields and is locally defined by

$$D^N \varphi = i^s \sum_{j=1}^n \varepsilon_j e_j \cdot \nabla_{e_j}^{\Sigma N} \varphi,$$

for every spinor field  $\varphi$ . The Dirac operator is an elliptic operator, formally selfadjoint, i.e. if  $\psi$  or  $\varphi$  has compact support, then  $\int_N \langle D^N \varphi, \psi \rangle dv_g = \int_N \langle \varphi, D^N \psi \rangle dv_g$ .

### 3 Semi-Riemannian $\text{Spin}^c$ hypersurfaces and the Gauss formula

In this section, we study  $\text{Spin}^c$  structures of hypersurfaces, such as the restriction of a  $\text{Spin}^c$  bundle of an ambient semi-Riemannian manifold and the complex spinorial Gauss formula.

Let  $\mathcal{Z}$  be an oriented  $(n + 1)$ -dimensional semi-Riemannian  $\text{Spin}^c$  manifold and  $M \subset \mathcal{Z}$  a semi-Riemannian hypersurface with trivial spacelike normal bundle. This means that there is a vector field  $\nu$  on  $\mathcal{Z}$  along  $M$  satisfying  $\langle \nu, \nu \rangle = +1$  and  $\langle \nu, TM \rangle = 0$ . Hence if the signature of  $M$  is  $(r, s)$ , then the signature of  $\mathcal{Z}$  is  $(r + 1, s)$ .

**Proposition 3.1** *The hypersurface  $M$  inherits a  $\text{Spin}^c$  structure from that on  $\mathcal{Z}$ , and we have*

$$\begin{cases} \Sigma\mathcal{Z}|_M \simeq \Sigma M & \text{if } n \text{ is even,} \\ \Sigma^+\mathcal{Z}|_M \simeq \Sigma M & \text{if } n \text{ is odd.} \end{cases}$$

Moreover Clifford multiplication by a vector field  $X$ , tangent to  $M$ , is given by

$$X \bullet \varphi = (\nu \cdot X \cdot \psi)|_M, \quad (12)$$

where  $\psi \in \Gamma(\Sigma\mathcal{Z})$  (or  $\psi \in \Gamma(\Sigma^+\mathcal{Z})$  if  $n$  is odd),  $\varphi$  is the restriction of  $\psi$  to  $M$ , “ $\cdot$ ” is the Clifford multiplication on  $\mathcal{Z}$ , and “ $\bullet$ ” that on  $M$ .

**Proof:** The bundle of space and time oriented orthonormal frames of  $M$  can be embedded into the bundle of space and time oriented orthonormal frames of  $\mathcal{Z}$  restricted to  $M$ , by

$$\begin{aligned} \Phi : \quad P_{\text{SO}}M &\longrightarrow P_{\text{SO}}\mathcal{Z}|_M \\ (e_1, \dots, e_n) &\longrightarrow (\nu, e_1, \dots, e_n). \end{aligned} \quad (13)$$

The isomorphism  $\alpha$ , defined in (8) yields the following commutative diagram:

$$\begin{array}{ccc} \text{Spin}^c(r, s) & \hookrightarrow & \text{Spin}^c(r+1, s) \\ \downarrow \xi^c & & \downarrow \xi^c \\ \text{SO}(r, s) \times \mathbb{S}^1 & \hookrightarrow & \text{SO}(r+1, s) \times \mathbb{S}^1 \end{array}$$

where the inclusion of  $\text{SO}(r, s)$  in  $\text{SO}(r+1, s)$  is that which fixes the first basis vector under the action of  $\text{SO}(r+1, s)$  on  $\mathbb{R}^{n+1}$ . This allows to pull back via  $\Phi$  the principal bundle  $P_{\text{Spin}^c}\mathcal{Z}|_M$  as a  $\text{Spin}^c$  structure for  $M$ , denoted by  $P_{\text{Spin}^c}M$ . Thus, we have the following commutative diagram:

$$\begin{array}{ccc} P_{\text{Spin}^c}M & \longrightarrow & P_{\text{Spin}^c}\mathcal{Z}|_M \\ \downarrow \Theta & & \downarrow \Theta \\ P_{\text{SO}}M \times_M P_{\mathbb{S}^1}\mathcal{Z}|_M & \longrightarrow & P_{\text{SO}}\mathcal{Z}|_M \times_M P_{\mathbb{S}^1}\mathcal{Z}|_M \end{array}$$

The  $\text{Spin}^c(r, s)$ -principal bundle  $(P_{\text{Spin}^c}M, \pi, M)$  and the  $\mathbb{S}^1$ -principal bundle  $(P_{\mathbb{S}^1}M =: P_{\mathbb{S}^1}\mathcal{Z}|_M, \pi, M)$  define a  $\text{Spin}^c$  structure on  $M$ . Let  $\Sigma\mathcal{Z}$  be the spinor bundle on  $\mathcal{Z}$ ,

$$\Sigma\mathcal{Z} = P_{\text{Spin}^c}\mathcal{Z} \times_{\rho_{n+1}} \Sigma_{r+1, s},$$

where  $\rho_{n+1}$  stands for the spinorial representation of  $\text{Spin}^c(r+1, s)$ . Moreover, for any spinor  $\psi = [\widetilde{b \times s}, \sigma] \in \Sigma\mathcal{Z}$  we can always assume that  $pr_1 \circ \Theta(\widetilde{b \times s}) = b$  is a local section of  $P_{\text{SO}}\mathcal{Z}$  with  $\nu$  for first basis vector where  $pr_1$  is the projection into  $P_{\text{SO}}\mathcal{Z}$ . Then we have

$$\psi|_M = [\widetilde{b \times s|_{U \cap M}}, \sigma|_{U \cap M}],$$

where the equivalence class is reduced to elements of  $\text{Spin}^c(r, s)$ . It follows that one can realise the restriction to  $M$  of the spinor bundle  $\Sigma\mathcal{Z}$  as

$$\Sigma\mathcal{Z}|_M = P_{\text{Spin}^c} M \times_{\rho_{n+1} \circ \alpha} \Sigma_{r+1, s}.$$

If  $n = 2m$  is even, it is easy to check that  $\chi_{2m+1}^0 \circ \alpha = \chi_{2m+1}^0|_{\mathbb{C}l_{r+1, s}^0}$ . Hence  $\chi_{2m+1}^0 \circ \alpha$  is an irreducible representation of  $\mathbb{C}l_{r, s}$  of dimension  $2^m$ , as  $\chi_{2m+1}^0|_{\mathbb{C}l_{r+1, s}^0}$ , and finally  $\chi_{2m+1}^0 \circ \alpha \cong \chi_{2m}$ . We conclude that

$$\rho_{2m+1} \circ \alpha \cong \rho_{2m}, \quad \text{and} \quad \Sigma\mathcal{Z}|_M \cong \Sigma M.$$

If  $n = 2m + 1$  is odd, we know that  $\chi_{2m+1}^0$  is the unique irreducible representation of  $\mathbb{C}l_{r, s}$  of dimension  $2^m$  for which the action of the complex volume form is the identity. Since  $n+1 = 2m+2$  is even,  $\Sigma\mathcal{Z}$  decomposes into positive and negative parts,  $\Sigma^\pm \mathcal{Z} = P_{\text{Spin}^c} \mathcal{Z} \times_{\rho_{2m+1}^\pm} \Sigma_{r+1, s}^\pm$ . It is easy to show that  $\chi_{2m+2} \circ \alpha = \chi_{2m+2}|_{\mathbb{C}l_{r+1, s}^0}$ , but  $\chi_{2m+2} \circ \alpha$  can be written as the direct sum of two irreducible inequivalent representations, as  $\chi_{2m+2}|_{\mathbb{C}l_{r+1, s}^0}$ . Hence, we have

$$\chi_{2m+2} \circ \alpha = (\chi_{2m+2} \circ \alpha)^+ \oplus (\chi_{2m+2} \circ \alpha)^-,$$

where  $(\chi_{2m+2} \circ \alpha)^\pm(\omega_{r, s}) = \pm \text{Id}_{\Sigma_{r, s}}$ . The representation  $\chi_{2m+1}^0$  being the unique representation of  $\mathbb{C}l_{r, s}$  of dimension  $2^m$  for which the action of the volume form is the identity, we get  $(\chi_{2m+2} \circ \alpha)^+ \cong \chi_{2m+1}^0$ . Finally,

$$\rho_{2m+2}^+ \circ \alpha \cong \rho_{2m+1} \quad \text{and} \quad \Sigma^+ \mathcal{Z}|_M \cong \Sigma M.$$

Now, Equation (12) follows directly from the above identification.

**Remarks 3.2** 1. The algebraic remarks in the previous section show that if  $n$  is odd we can also get  $\Sigma^- \mathcal{Z}|_M \simeq \Sigma M$ , where the Clifford multiplication by a vector field tangent to  $M$  is given by  $X \bullet \varphi = -(\nu \cdot X \cdot \psi)|_M$ .

2. The connection 1-form defined on the restricted  $\mathbb{S}^1$ -principal bundle  $(P_{\mathbb{S}^1} M =: P_{\mathbb{S}^1} \mathcal{Z}|_M, \pi, M)$ , is given by

$$A^M = A^{\mathcal{Z}}|_M : T(P_{\mathbb{S}^1} M) = T(P_{\mathbb{S}^1} \mathcal{Z})|_M \longrightarrow i\mathbb{R}.$$

Then the curvature 2-form  $i\Omega^M$  on the  $\mathbb{S}^1$ -principal bundle  $P_{\mathbb{S}^1} M$  is given by  $i\Omega^M = i\Omega^{\mathcal{Z}}|_M$ , which can be viewed as an imaginary 2-form on  $M$  and hence as the curvature form of the line bundle  $L^M$ , the restriction of the line bundle  $L^{\mathcal{Z}}$  to  $M$ .

3. For every  $\psi \in \Gamma(\Sigma\mathcal{Z})$  ( $\psi \in \Gamma(\Sigma^+ \mathcal{Z})$  if  $n$  is odd), the real 2-forms  $\Omega^M$  and  $\Omega^{\mathcal{Z}}$  are related by the following formulas:

$$|\Omega^{\mathcal{Z}}|^2 = |\Omega^M|^2 + |\nu \lrcorner \Omega^{\mathcal{Z}}|^2, \quad (14)$$

$$(\Omega^{\mathcal{Z}} \cdot \psi)|_M = \Omega^M \bullet \varphi + (\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \varphi. \quad (15)$$

In fact, we can write

$$\Omega^{\mathcal{Z}} = \sum_{i=1}^n \Omega^{\mathcal{Z}}(\nu, e_i) \nu \wedge e_i + \sum_{i < j}^n \Omega^{\mathcal{Z}}(e_i, e_j) e_i \wedge e_j = -(\nu \lrcorner \Omega^{\mathcal{Z}}) \wedge \nu + \Omega^M,$$

which is (14). When restricting the Clifford multiplication of  $\Omega^{\mathcal{Z}}$  by  $\psi$  to the hypersurface  $M$  we obtain

$$(\Omega^{\mathcal{Z}} \cdot \psi)|_M = (\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi)|_M + (\Omega^M \cdot \psi)|_M = (\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \varphi + \Omega^M \bullet \varphi. \quad (16)$$

**Proposition 3.3 (The spinorial Gauss formula)** We denote by  $\nabla^{\Sigma \mathcal{Z}}$  the spinorial Levi-Civita connection on  $\Sigma \mathcal{Z}$  and by  $\nabla^{\Sigma M}$  that on  $\Sigma M$ . For all  $X \in \Gamma(TM)$  and for every spinor field  $\psi \in \Gamma(\Sigma \mathcal{Z})$ , then

$$(\nabla_X^{\Sigma \mathcal{Z}} \psi)|_M = \nabla_X^{\Sigma M} \varphi - \frac{1}{2} W(X) \bullet \varphi, \quad (17)$$

where  $W$  denotes the Weingarten map with respect to  $\nu$  and  $\varphi = \psi|_M$ . Moreover, let  $D^{\mathcal{Z}}$  and  $D^M$  be the Dirac operators on  $\mathcal{Z}$  and  $M$ . Denoting by the same symbol any spinor and it's restriction to  $M$ , we have

$$\nu \cdot D^{\mathcal{Z}} \varphi = \tilde{D} \varphi + \frac{i^s n}{2} H \varphi - i^s \nabla_{\nu}^{\Sigma \mathcal{Z}} \varphi, \quad (18)$$

where  $H = \frac{1}{n} \text{tr}(W)$  denotes the mean curvature and  $\tilde{D} = D^M$  if  $n$  is even and  $\tilde{D} = D^M \oplus (-D^M)$  if  $n$  is odd.

**Proof:** The Riemannian Gauss formula is given, for every vector fields  $X$  and  $Y$  on  $M$ , by

$$\nabla_X^{\mathcal{Z}} Y = \nabla_X^M Y + \langle W(X), Y \rangle \nu. \quad (19)$$

Let  $(e_1, e_2, \dots, e_n)$  a local space and time oriented orthonormal frame of  $M$ , such that  $b = (e_0 = \nu, e_1, e_2, \dots, e_n)$  is that of  $\mathcal{Z}$ . We consider  $\psi$  a local section of  $\Sigma \mathcal{Z}$ ,  $\psi = [b \times s, \sigma]$  where  $s$  is a local section of  $P_{\mathbb{S}^1} \mathcal{Z}$ . Using (9), (19) and the fact that  $X(\psi)|_M = X(\varphi)$  for  $X \in \Gamma(TM)$ , we compute for  $j = 1, \dots, n$

$$\begin{aligned} \left( \nabla_{e_j}^{\Sigma \mathcal{Z}} \psi \right)|_M &= e_j(\varphi) + \frac{1}{4} \sum_{k=0}^n \varepsilon_k (e_k \cdot \nabla_{e_j}^{\mathcal{Z}} e_k \cdot \psi)|_M + \frac{1}{2} A^{\mathcal{Z}}(s_*(e_j)) \varphi \\ &= e_j(\varphi) + \frac{1}{4} \sum_{k=1}^n \varepsilon_k (e_k \cdot \nabla_{e_j}^{\mathcal{Z}} e_k \cdot \psi)|_M + \frac{1}{4} (\nu \cdot \nabla_{e_j}^{\mathcal{Z}} \nu \cdot \psi)|_M + \frac{1}{2} A^M(s_*(e_j)) \varphi \\ &= \nabla_{e_j}^{\Sigma M} \varphi + \frac{1}{4} \sum_{k=1}^n \varepsilon_k \langle W(e_j), e_k \rangle (e_k \cdot \nu \cdot \psi)|_M - \frac{1}{4} (\nu \cdot W(e_j) \cdot \psi)|_M \\ &= \nabla_{e_j}^{\Sigma M} \varphi - \frac{1}{2} (\nu \cdot W(e_j) \cdot \psi)|_M \\ &= \nabla_{e_j}^{\Sigma M} \varphi - \frac{1}{2} W(e_j) \bullet \varphi. \end{aligned}$$

Moreover  $(D^{\mathcal{Z}}\psi)|_M = i^s \sum_{j=1}^n \varepsilon_j (e_j \cdot \nabla_{e_j}^{\Sigma \mathcal{Z}} \psi)|_M + i^s (\nu \cdot \nabla_{\nu}^{\Sigma \mathcal{Z}} \psi)|_M$ , and by (17),

$$\begin{aligned} i^s \sum_{j=1}^n \varepsilon_j (e_j \cdot \nabla_{e_j}^{\Sigma \mathcal{Z}} \psi)|_M &= i^s \sum_{j=1}^n \varepsilon_j (e_j \cdot \nabla_{e_j}^{\Sigma M} \varphi) - i^s \frac{1}{2} \sum_{j=1}^n \varepsilon_j (e_j \cdot \nu \cdot W(e_j) \cdot \psi)|_M \\ &= -i^s \nu \cdot \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \nabla_{e_j}^{\Sigma M} \varphi + i^s \frac{1}{2} \sum_{j=1}^n \varepsilon_j (\nu \cdot e_j \cdot W(e_j) \cdot \psi)|_M \\ &= -\nu \cdot \tilde{D}\varphi - \frac{i^s}{2} \text{tr}(W)(\nu \cdot \psi)|_M. \end{aligned}$$

**Proposition 3.4** *Let  $\mathcal{Z}$  be an  $(n+1)$ -dimensional semi-Riemannian  $\text{Spin}^c$  manifold. Assume that  $\mathcal{Z}$  carries a semi-Riemannian foliation by hypersurfaces with trivial spacelike normal bundle, i.e., the leaves  $M$  are semi-Riemannian hypersurfaces and there exists a vector field  $\nu$  on  $\mathcal{Z}$  perpendicular to the leaves such that  $\langle \nu, \nu \rangle = 1$  and  $\nabla_{\nu}^{\mathcal{Z}} \nu = 0$ . Then the commutator of the leafwise Dirac operator and the normal derivative is given by*

$$i^{-s} [\nabla_{\nu}^{\Sigma \mathcal{Z}}, \tilde{D}] \varphi = \mathfrak{D}^W \varphi - \frac{n}{2} \nu \cdot \text{grad}^M(H) \cdot \varphi + \frac{1}{2} \nu \cdot \text{div}^M(W) \cdot \varphi + \frac{i}{2} \nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi.$$

Here  $\text{grad}^M$  denotes the leafwise gradient,  $\text{div}^M(W) = \sum_{i=1}^n \varepsilon_i (\nabla_{e_i}^M W)(e_i)$  denotes the leafwise divergence of the endomorphism field  $W$  and  $\mathfrak{D}^W \varphi = \sum_{i=1}^n \varepsilon_i \nu \cdot e_i \cdot \nabla_{W(e_i)}^{\Sigma M} \varphi$ .

**Proof:** We choose a local oriented orthonormal tangent frame  $(e_1, \dots, e_n)$  for the leaves and we may assume for simplicity that  $\nabla_{\nu}^{\mathcal{Z}} e_j = 0$ . Now, we compute

$$\begin{aligned} i^{-s} [\nabla_{\nu}^{\Sigma \mathcal{Z}}, \tilde{D}] \varphi &= \sum_{j=1}^n \varepsilon_j \left( \nabla_{\nu}^{\Sigma \mathcal{Z}} (\nu \cdot e_j \cdot \nabla_{e_j}^{\Sigma M} \varphi) - \nu \cdot e_j \cdot \nabla_{e_j}^{\Sigma M} \nabla_{\nu}^{\Sigma \mathcal{Z}} \varphi \right) \\ &= \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \left( \nabla_{\nu}^{\Sigma \mathcal{Z}} \nabla_{e_j}^{\Sigma M} \varphi - \nabla_{e_j}^{\Sigma M} \nabla_{\nu}^{\Sigma \mathcal{Z}} \varphi \right) \\ &\stackrel{(17)}{=} \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \left[ \nabla_{\nu}^{\Sigma \mathcal{Z}} (\nabla_{e_j}^{\Sigma \mathcal{Z}} + \frac{1}{2} \nu \cdot W(e_j)) \right. \\ &\quad \left. - (\nabla_{e_j}^{\Sigma \mathcal{Z}} + \frac{1}{2} \nu \cdot W(e_j)) \nabla_{\nu}^{\Sigma \mathcal{Z}} \right] \varphi \\ &= \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \left( \mathcal{R}^{\Sigma \mathcal{Z}}(\nu, e_j) + \nabla_{[\nu, e_j]}^{\Sigma \mathcal{Z}} + \frac{1}{2} \nu \cdot (\nabla_{\nu}^{\mathcal{Z}} W)(e_j) \right) \varphi \\ &\stackrel{(11)}{=} -\frac{1}{2} \nu \cdot \text{Ric}^{\mathcal{Z}}(\nu) \cdot \varphi + \frac{i}{2} \nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi \\ &\quad + \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \left( \nabla_{W(e_j)}^{\Sigma \mathcal{Z}} + \frac{1}{2} \nu \cdot (\nabla_{\nu}^{\mathcal{Z}} W)(e_j) \right) \varphi \end{aligned}$$

$$\begin{aligned}
&\stackrel{(17)}{=} -\frac{1}{2}\nu \cdot \text{Ric}^{\mathcal{Z}}(\nu) \cdot \varphi + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi \\
&\quad + \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \left( \nabla_{W(e_j)}^{\Sigma M} - \frac{1}{2}\nu \cdot W^2(e_j) + \frac{1}{2}\nu \cdot (\nabla_{\nu}^{\mathcal{Z}} W)(e_j) \right) \varphi \\
&= -\frac{1}{2}\nu \cdot \text{Ric}^{\mathcal{Z}}(\nu) \cdot \varphi + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi + \mathfrak{D}^W \varphi \\
&\quad + \frac{1}{2} \sum_{j=1}^n \varepsilon_j e_j \cdot \left( -W^2(e_j) + (\nabla_{\nu}^{\mathcal{Z}} W)(e_j) \right) \varphi.
\end{aligned}$$

The Riccati equation for the Weingarten map  $(\nabla_{\nu}^{\mathcal{Z}} W)(X) = R^{\mathcal{Z}}(X, \nu)\nu + W^2(X)$  yields

$$\begin{aligned}
i^{-s}[\nabla_{\nu}^{\Sigma \mathcal{Z}}, \tilde{D}] \varphi &= -\frac{1}{2}\nu \cdot \text{Ric}^{\mathcal{Z}}(\nu) \cdot \varphi + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi + \mathfrak{D}^W \varphi \\
&\quad + \frac{1}{2} \sum_{j=1}^n \varepsilon_j e_j \cdot (R^{\mathcal{Z}}(e_j, \nu)\nu) \cdot \varphi \\
&= -\frac{1}{2}\nu \cdot \text{Ric}^{\mathcal{Z}}(\nu) \cdot \varphi + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi + \mathfrak{D}^W \varphi + \frac{1}{2}\text{ric}^{\mathcal{Z}}(\nu, \nu)\varphi \\
&= \mathfrak{D}^W \varphi - \frac{1}{2} \sum_{j=1}^n \varepsilon_j \text{ric}^{\mathcal{Z}}(\nu, e_j) \nu \cdot e_j \cdot \varphi + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi. \quad (20)
\end{aligned}$$

The Codazzi-Mainardi equation for  $X, Y, V \in TM$  is given by  $\langle R^{\mathcal{Z}}(X, Y)V, \nu \rangle = \langle (\nabla_X^M W)(Y), V \rangle - \langle (\nabla_Y^M W)(X), V \rangle$ . Thus,

$$\begin{aligned}
\text{ric}^{\mathcal{Z}}(\nu, X) &= \sum_{j=1}^n \varepsilon_j \langle R^{\mathcal{Z}}(X, e_j)e_j, \nu \rangle \\
&= \sum_{j=1}^n \varepsilon_j \left( \langle (\nabla_X^M W)(e_j), e_j \rangle - \langle (\nabla_{e_j}^M W)(X), e_j \rangle \right) \\
&= \text{tr}(\nabla_X^M W) - \langle \text{div}^M(W), X \rangle.
\end{aligned}$$

Plugging this into (20) we get

$$\begin{aligned}
i^{-s}[\nabla_{\nu}^{\Sigma \mathcal{Z}}, \tilde{D}] \varphi &= \mathfrak{D}^W \varphi - \frac{1}{2} \sum_{j=1}^n \varepsilon_j \left( \text{tr}(\nabla_{e_j}^M W) - \langle \text{div}^M(W), e_j \rangle \right) \nu \cdot e_j \cdot \varphi \\
&\quad + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi. \\
&= \mathfrak{D}^W \varphi - \frac{1}{2} \sum_{j=1}^n \varepsilon_j e_j (\text{tr}(W)) \nu \cdot e_j \cdot \varphi + \frac{1}{2}\nu \cdot \text{div}^M(W) \cdot \varphi \\
&\quad + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi. \\
&= \mathfrak{D}^W \varphi - \frac{n}{2}\nu \cdot \text{grad}^M(H) \cdot \varphi + \frac{1}{2}\nu \cdot \text{div}^M(W) \cdot \varphi + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi.
\end{aligned}$$

## 4 The generalized cylinder on semi-Riemannian $\text{Spin}^c$ manifolds

Let  $M$  be an  $n$ -dimensional smooth manifold and  $g_t$  a smooth 1-parameter family of semi-Riemannian metrics on  $M$ ,  $t \in I$  where  $I \subset \mathbb{R}$  is an interval. We define the generalized cylinder by

$$\mathcal{Z} := I \times M,$$

with semi-Riemannian metric  $g_{\mathcal{Z}} := \langle \cdot, \cdot \rangle = dt^2 + g_t$ . The generalized cylinder is an  $(n+1)$ -dimensional semi-Riemannian manifold of signature  $(r+1, s)$  if the signature of  $g_t$  is  $(r, s)$ .

**Proposition 4.1** *There is a 1-1-correspondence between the  $\text{Spin}^c$  structures on  $M$  and that on  $\mathcal{Z}$ .*

**Proof:** As explained in Section 3,  $\text{Spin}^c$  structures on  $\mathcal{Z}$  can be restricted to  $\text{Spin}^c$  structures on  $M$ . Conversely, given a  $\text{Spin}^c$  structure on  $M$  it can be pulled back to  $I \times M$  via the projection  $pr_2 : I \times M \longrightarrow M$  yields a  $\text{Spin}^c$  structure on  $\mathcal{Z}$ . In fact, the pull back of the  $\text{Spin}^c(r, s)$ -principal bundle  $P_{\text{Spin}^c}M$  on  $M$  gives rise to a  $\text{Spin}^c(r, s)$ -principal bundle on  $\mathcal{Z}$  denoted by  $P_{\text{Spin}^c}\mathcal{Z}$

$$\begin{array}{ccc} P_{\text{Spin}^c}\mathcal{Z} & \longrightarrow & P_{\text{Spin}^c}M \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{Z} = I \times M & \longrightarrow & M \end{array}$$

Enlarging the structure group via the embedding  $\text{Spin}^c(r, s) \hookrightarrow \text{Spin}^c(r+1, s)$ , which covers the standard embedding

$$\begin{aligned} \text{SO}(r, s) \times \mathbb{S}^1 &\hookrightarrow \text{SO}(r+1, s) \times \mathbb{S}^1 \\ (a, z) &\mapsto \left( \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, z \right), \end{aligned}$$

gives a  $\text{Spin}^c(r+1, s)$ -principal fibre bundle on  $\mathcal{Z}$ , denoted also by  $P_{\text{Spin}^c}\mathcal{Z}$ . The pull back of the line bundle  $L^M$  on  $M$  defining the  $\text{Spin}^c$  structure on  $M$ , gives a line bundle  $L^{\mathcal{Z}}$  on  $\mathcal{Z}$  such that the following diagram commutes

$$\begin{array}{ccc} L^{\mathcal{Z}} = pr_2^*(L^M) & \longrightarrow & L^M \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{Z} = I \times M & \longrightarrow & M \end{array}.$$

The line bundle  $L^{\mathcal{Z}}$  on  $\mathcal{Z}$  and the  $\text{Spin}^c(r+1, s)$ -principal fibre bundle  $P_{\text{Spin}^c}\mathcal{Z}$  on  $\mathcal{Z}$  yields the  $\text{Spin}^c$  structure on  $\mathcal{Z}$  which restricts to the given  $\text{Spin}^c$  structure on  $M$ .

**Remark 4.2** If  $M$  is a  $\text{Spin}^c$  Riemannian manifold and if we denote by  $i\Omega^M$  the imaginary valued curvature on the line bundle  $L^M$ , we know that there exists a unique curvature 2-form, denoted by  $i\Omega^Z$ , on the line bundle  $L^Z = pr_2^*(L^M)$ , defined by  $i\Omega^Z = pr_2^*(i\Omega^M)$ . Thus we have

$$\Omega^Z(X, Y) = \Omega^M(X, Y) \quad \text{and} \quad \Omega^Z(\nu, Y) = 0 \quad \text{for any } X, Y \in \Gamma(TM).$$

**Proposition 4.3** [2] On a generalized cylinder  $Z = I \times M$  with semi-Riemannian metric  $g^Z = \langle \cdot, \cdot \rangle = dt^2 + g_t$  we define, in every  $p \in M$  and  $X, Y \in T_p M$ , the first and second derivatives of  $g_t$  by

$$\dot{g}_t(X, Y) := \frac{d}{dt}(g_t(X, Y)) \quad \text{and} \quad \ddot{g}_t(X, Y) := \frac{d^2}{dt^2}(g_t(X, Y)).$$

Hence the following formulas hold:

$$\langle W(X), Y \rangle = -\frac{1}{2}\dot{g}_t(X, Y), \tag{21}$$

$$\begin{aligned} \langle R^Z(U, V)X, Y \rangle &= \langle R^{M_t}(U, V)X, Y \rangle \\ &\quad + \frac{1}{4}(\dot{g}_t(U, X)\dot{g}_t(V, Y) - \dot{g}_t(U, Y)\dot{g}_t(V, X)), \end{aligned} \tag{22}$$

$$\langle R^Z(X, Y)U, \nu \rangle = \frac{1}{2}((\nabla_Y^{M_t}\dot{g}_t)(X, U) - (\nabla_X^{M_t}\dot{g}_t)(Y, U)), \tag{23}$$

$$\langle R^Z(X, \nu)\nu, Y \rangle = -\frac{1}{2}(\ddot{g}_t(X, Y) + \dot{g}_t(W(X), Y)), \tag{24}$$

where  $X, Y, U, V \in T_p M$ ,  $p \in M$ .

## 5 The variation formula for the Dirac operator on $\text{Spin}^c$ manifolds

First we give some facts about parallel transport on  $\text{Spin}^c$  manifolds along a curve  $c$ . We consider a Riemannian  $\text{Spin}^c$  manifold  $N$ , we know that there exists a unique correspondence which associates to a spinor field  $\psi(t) = \psi(c(t))$  along a curve  $c : I \rightarrow N$  another spinor field  $\frac{D}{dt}\psi$  along  $c$ , called the covariant derivative of  $\psi$  along  $c$ , such that

$$\frac{D}{dt}(\psi + \varphi) = \frac{D}{dt}\psi + \frac{D}{dt}\varphi, \quad \text{for any } \psi \text{ and } \varphi \text{ along the curve } c,$$

$$\frac{D}{dt}(f\psi) = f\frac{D}{dt}\psi + \left(\frac{d}{dt}f\right)\psi, \quad \text{where } f \text{ is a differentiable function on } I,$$

$$\nabla_{\dot{c}(t)}^{\Sigma N}\psi = \frac{D}{dt}\varphi, \quad \text{where } \varphi(t) = \psi(c(t)).$$



A spinor field  $\psi$  along a curve  $c$  is called parallel when  $\frac{D}{dt}\psi(t) = 0$  for all  $t \in I$ . Now, if  $\psi_0$  is a spinor at the point  $c(t_0)$ ,  $t_0 \in I$ , ( $\psi_0 \in \Sigma_{c(t_0)}N$ ) then there exists a unique parallel spinor  $\varphi$  along  $c$ , such that  $\psi_0 = \varphi(t_0)$ . The linear isometry  $\tau_{t_0}^{t_1}$  defined by

$$\begin{aligned}\tau_{t_0}^{t_1} : \Sigma_{c(t_0)}N &\longrightarrow \Sigma_{c(t_1)}N \\ \psi_0 &\longrightarrow \varphi(t_1),\end{aligned}$$

is called the parallel transport along the curve  $c$  from  $c(t_0)$  to  $c(t_1)$ . The basic property of the parallel transport on a  $\text{Spin}^c$  manifold is the following: Let  $\psi$  be a spinor field on a Riemannian  $\text{Spin}^c$  manifold  $N$ ,  $X \in \Gamma(TN)$ ,  $p \in N$  and  $c : I \longrightarrow N$  an integral curve through  $p$ , i.e.,  $c(t_0) = p$  and  $\frac{d}{dt}c(t) = X(c(t))$ , we have

$$(\nabla_X^{\Sigma N} \psi)_p = \frac{d}{dt} \left( \tau_{t_0}^{t_1}(\psi(t)) \right) \Big|_{t=t_0}. \quad (25)$$

Now, we consider  $g_t$  a smooth 1-parameter family of semi-Riemannian metrics on a  $\text{Spin}^c$  manifold  $M$  and the generalized cylinder  $\mathcal{Z} = I \times M$  with semi-Riemannian metric  $g^{\mathcal{Z}} = \langle \cdot, \cdot \rangle = dt^2 + g_t$ . For  $t \in I$  we denote by  $M_t$  the manifold  $(M, g_t)$ . Let us write “ $\cdot$ ” for the Clifford multiplication on  $\mathcal{Z}$  and “ $\bullet_t$ ” for that on  $M_t$ . Recall from Section 4 that  $\text{Spin}^c$  structures on  $M$  and  $\mathcal{Z}$  are in 1-1-correspondence and  $\Sigma \mathcal{Z}|_{M_t} = \Sigma M_t$  as hermitian vector bundles if  $n = r + s$  is even and  $\Sigma^+ \mathcal{Z}|_{M_t} = \Sigma M_t$  if  $n$  is odd. For a given  $x \in M$  and  $t_0, t_1 \in I$ , parallel transport  $\tau_{t_0}^{t_1}$  on the generalized cylinder  $\mathcal{Z}$  along the curve  $c : I \rightarrow I \times M, t \rightarrow (t, x)$  is given by

$$\tau_{t_0}^{t_1} : \Sigma_{c(t_0)}\mathcal{Z} \simeq \Sigma_x M_{t_0} \longrightarrow \Sigma_{c(t_1)}\mathcal{Z} \simeq \Sigma_x M_{t_1}.$$

This isomorphism satisfies

$$\begin{aligned}\tau_{t_0}^{t_1}(X \bullet_{t_0} \varphi) &= (\zeta_{t_0}^{t_1} X) \bullet_{t_1} (\tau_{t_0}^{t_1} \varphi), \\ < \tau_{t_0}^{t_1} \psi, \tau_{t_0}^{t_1} \varphi > &= < \psi, \varphi >, \end{aligned}$$

where  $\zeta_{t_0}^{t_1} : T_{(x, t_0)}\mathcal{Z} \simeq T_x M_{t_0} \rightarrow T_{(x, t_1)}\mathcal{Z} \simeq T_x M_{t_1}$  is the parallel transport on  $\mathcal{Z}$  along the same curve  $c$ ,  $X \in T_x M_{t_0}$  and  $\psi, \varphi \in \Sigma_x M_{t_0}$ .

**Theorem 5.1** *On a  $\text{Spin}^c$  manifold  $M$ , let  $g_t$  be a smooth 1-parameter family of semi-Riemannian metrics. Denote by  $D^{M_t}$  the Dirac operator of  $M_t$ , and  $\mathfrak{D}^{\dot{g}_t} = \sum_{i,j=1}^n \varepsilon_i \varepsilon_j \dot{g}_t(e_i, e_j) e_i \bullet_t \nabla_{e_j}^{\Sigma M_t}$ . Then for any smooth spinor field  $\psi$  on  $M_{t_0}$  we have*

$$\frac{d}{dt} \Big|_{t=t_0} \tau_t^{t_0} D^{M_t} \tau_{t_0}^t \psi = -\frac{1}{2} \mathfrak{D}^{\dot{g}_{t_0}} \psi + \frac{1}{4} \text{grad}^{M_{t_0}}(\text{tr}_{g_{t_0}}(\dot{g}_{t_0})) \bullet_{t_0} \psi - \frac{1}{4} \text{div}^{M_{t_0}}(\dot{g}_{t_0}) \bullet_{t_0} \psi.$$

**Proof:** The vector field  $\nu := \frac{\partial}{\partial t}$  is spacelike of unit length and orthogonal to the hypersurfaces  $M_t := \{t\} \times M$ . Denote by  $W_t$  the Weingarten map of  $M_t$  with respect to

$\nu$  and by  $H_t$  the mean curvature. If  $X$  is a local coordinate field on  $M$ , then  $\langle X, \nu \rangle = 0$  and  $[X, \nu] = 0$ . Thus

$$\begin{aligned} 0 &= d_\nu \langle X, \nu \rangle = \langle \nabla_\nu^\mathbb{Z} X, \nu \rangle + \langle X, \nabla_\nu^\mathbb{Z} \nu \rangle = \langle \nabla_X^\mathbb{Z} \nu, \nu \rangle + \langle X, \nabla_\nu^\mathbb{Z} \nu \rangle \\ &= -\langle W_t(X), \nu \rangle + \langle X, \nabla_\nu^\mathbb{Z} \nu \rangle = \langle X, \nabla_\nu^\mathbb{Z} \nu \rangle \end{aligned}$$

and differentiating  $\langle \nu, \nu \rangle = 1$  yields  $\langle \nu, \nabla_\nu^\mathbb{Z} \nu \rangle = 0$ . Hence  $\nabla_\nu^\mathbb{Z} \nu = 0$ , i.e., for  $x \in M$  the curves  $t \mapsto (t, x)$  are geodesics parametrized by arclength. So the assumptions of Proposition 3.4 are satisfied for the foliation  $(M_t)_{t \in I}$ . By Remark 4.2, the commutator formula of Proposition 3.4 gives for a section  $\varphi$  of  $\Sigma M_t$ , (or  $\Sigma^+ M_t$  if  $n$  is odd)

$$i^{-s}[\nabla_\nu^{\Sigma\mathbb{Z}}, D^{M_t}] \varphi = \mathfrak{D}^{W_t} \varphi - \frac{n}{2} \text{grad}^{M_t}(H_t) \bullet_t \varphi + \frac{1}{2} \text{div}^{M_t}(W_t) \bullet_t \varphi. \quad (26)$$

From Proposition 4.3 we deduce

$$\text{div}^{M_t}(W_t) = -\frac{1}{2} \text{div}^{M_t}(\dot{g}_t), \quad H_t = -\frac{1}{2n} \text{tr}_{g_t}(\dot{g}_t) \quad \text{and} \quad \mathfrak{D}^{W_t} = -\frac{1}{2} \mathfrak{D}^{\dot{g}_t}.$$

Thus (26) can be rewritten as

$$i^{-s}[\nabla_\nu^{\Sigma\mathbb{Z}}, D^{M_t}] \varphi = -\frac{1}{2} \mathfrak{D}^{\dot{g}_t} \varphi + \frac{1}{4} \text{grad}^{M_t}(\text{tr}_{g_t}(\dot{g}_t)) \bullet_t \varphi - \frac{1}{4} \text{div}^{M_t}(\dot{g}_t) \bullet_t \varphi. \quad (27)$$

Now if  $\varphi$  is parallel along the curves  $t \mapsto (t, x)$ , i.e., it is of the form  $\varphi(t, x) = \tau_{t_0}^t \psi(t_0, x)$  for some spinor field  $\psi$  on  $M_{t_0}$ , then using (25) at  $t = t_0$ , the left hand side of (27) could be written as

$$\begin{aligned} i^{-s}[\nabla_\nu^{\Sigma\mathbb{Z}}, D^{M_t}] \varphi &= i^{-s} \nabla_\nu^{\Sigma\mathbb{Z}} D^{M_t} \varphi = i^{-s} \frac{d}{dt} \Big|_{t=t_0} \tau_t^{t_0} D^{M_t} \varphi \\ &= i^{-s} \frac{d}{dt} \Big|_{t=t_0} \tau_t^{t_0} D^{M_t} \tau_{t_0}^t \psi, \end{aligned} \quad (28)$$

which gives the variation formula for the Dirac operator.

**Corollary 5.2** *Let  $(M^n, g)$  be a  $\text{Spin}^c$  Riemannian manifold, if we consider the family of metrics defined by  $g_t = g + tk$ , where  $k$  is a symmetric  $(0, 2)$ -tensor, we have*

$$\frac{d}{dt} \Big|_{t=0} \tau_t^0 D^{M_t} \tau_0^t \psi = -\frac{1}{2} \mathfrak{D}^k \psi + \frac{1}{4} \text{grad}^M(\text{tr}_g(k)) \cdot \psi - \frac{1}{4} \text{div}^M(k) \cdot \psi, \quad (29)$$

where “ $\cdot = \bullet_{t_0=0}$ ” is the Clifford multiplication on  $M$ .

This formula has been proved in [4], Theorem 21 for spin Riemannian manifolds and in [2] for spin semi-Riemannian manifolds.

## 6 Energy-Momentum tensor on $\text{Spin}^c$ manifolds

In this section we study the Energy-Momentum tensor on  $\text{Spin}^c$  Riemannian manifolds from a geometric point of view. We begin by giving the proofs of Proposition 1.1, Theorem 1.2 and Proposition 1.3.

**Proof of Proposition 1.1 :** Using Equation (29) we calculate

$$\begin{aligned}
 \left. \frac{d}{dt} \right|_{t=0} (\tau_t^0 D^{M_t} \tau_0^t \psi, \psi)_{g_t} &= \left. \frac{d}{dt} \right|_{t=0} (D^{M_t} \tau_0^t \psi, \tau_0^t \psi)_{g_t} \\
 &= -\frac{1}{2} (\mathfrak{D}^k \psi, \psi)_g \\
 &= -\frac{1}{2} \sum_{i,j} k(e_i, e_j) (e_i \cdot \nabla_{e_j}^{\Sigma M} \psi, \psi) \\
 &= -\frac{1}{2} \int_M \langle k, T_\psi \rangle dv_g.
 \end{aligned}$$

**Proof of Theorem 1.2 :** The Proof of this Theorem will be omitted since it is similar to the one given by Friedrich and Kim in [8] for spin manifolds.

**Proof of Proposition 1.3 :** Let  $\psi$  be any parallel spinor field on  $\mathcal{Z}$ . Then Equation (17) yields

$$\nabla_X^{\Sigma M} \varphi = \frac{1}{2} W(X) \bullet \varphi. \quad (30)$$

Let  $(e_1, \dots, e_n)$  be a positively oriented local orthonormal basis of  $TM$ . For  $j = 1, \dots, n$  we have

$$\nabla_{e_j}^{\Sigma M} \varphi = \frac{1}{2} \sum_{k=1}^n W_{jk} e_k \bullet \varphi.$$

Taking Clifford multiplication by  $e_i$  and the scalar product with  $\varphi$ , we get

$$\text{Re} \langle e_i \bullet \nabla_{e_j}^{\Sigma M} \varphi, \varphi \rangle = \frac{1}{2} \sum_{k=1}^n W_{jk} \text{Re} \langle e_i \bullet e_k \bullet \varphi, \varphi \rangle.$$

Since  $\text{Re} \langle e_i \bullet e_k \bullet \varphi, \varphi \rangle = -\delta_{ik} |\varphi|^2$ , it follows, by the symmetry of  $W$

$$\text{Re} \langle e_i \bullet \nabla_{e_j}^{\Sigma M} \varphi + e_j \bullet \nabla_{e_i}^{\Sigma M} \varphi, \varphi \rangle = -W_{ij} |\varphi|^2.$$

Therefore,  $2\ell^\varphi = -W$ . Using Equation (18) it is easy to see that  $\varphi$  is an eigenspinor associated with the eigenvalue  $-\frac{n}{2}H$  of  $\tilde{D}$ . Since  $\text{Scal}^{\mathcal{Z}} = \text{Scal}^M + 2 \text{ric}^{\mathcal{Z}}(\nu, \nu) - n^2 H^2 + |W|^2$  we get

$$\frac{1}{4} (\text{Scal}^M - c_n |\Omega^M|) + |T^\varphi|^2 = \frac{1}{4} (\text{Scal}^{\mathcal{Z}} - 2 \text{ric}^{\mathcal{Z}}(\nu, \nu) - c_n |\Omega^M|) + n^2 \frac{H^2}{4},$$

hence  $M$  satisfies the equality case in (3) if and only if (6) holds.

**Corollary 6.1** *Under the same conditions as Proposition 1.3, if  $n = 2$  or  $3$ , the hypersurface  $M$  satisfies the equality case in (3) if  $\text{Ric}^{\mathcal{Z}}(\nu) = 0$  and  $\text{Scal}^{\mathcal{Z}} \geq 0$ .*

**Proof:** Since  $\mathcal{Z}$  has a parallel spinor, we have (see [7])

$$|\text{Ric}^{\mathcal{Z}}(\nu)| = |\nu \lrcorner \Omega^{\mathcal{Z}}|, \quad (31)$$

$$i(Y \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi = \text{Ric}^{\mathcal{Z}}(Y) \cdot \psi \text{ for every } Y \in \Gamma(\Sigma \mathcal{Z}). \quad (32)$$

For  $Y = e_j$  in Equation (32) then taking Clifford multiplication by  $e_j$  and summing from  $j = 1, \dots, n+1$ , we get

$$i \sum_{j=1}^{n+1} e_j \cdot (e_j \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi = \sum_{j=1}^{n+1} e_j \cdot \text{Ric}^{\mathcal{Z}}(e_j) \cdot \psi = -\text{Scal}^{\mathcal{Z}} \psi.$$

But  $2 \Omega^{\mathcal{Z}} \cdot \psi = \sum_{j=1}^{n+1} e_j \cdot (e_j \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi$ , hence we deduce that  $\Omega^{\mathcal{Z}} \cdot \psi = i \frac{\text{Scal}^{\mathcal{Z}}}{2} \psi$ . By (31) and (15) we obtain  $\Omega^M \bullet \varphi = i \frac{\text{Scal}^{\mathcal{Z}}}{2} \varphi$ . Since  $n = 2$  or  $3$  we have  $|\Omega^M| = \frac{\text{Scal}^{\mathcal{Z}}}{2}$  and Equation (6) is satisfied.

**Corollary 6.2** *Under the same conditions as Proposition 1.3, if the restriction of the complex line bundle  $L^{\mathcal{Z}}$  is flat, i.e.,  $L^M$  is a flat complex line bundle ( $\Omega^M = 0$ ), the hypersurface  $M$  is a limiting manifold for (3).*

**Proof:** Since  $\Omega^M = 0$ , Equation (15) yields  $i \frac{\text{Scal}^{\mathcal{Z}}}{2} \varphi = \Omega^{\mathcal{Z}} \cdot \psi|_M = (\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \varphi$ . But,

$$\begin{aligned} i(\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \varphi &= i(\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi)|_M = (\nu \cdot \text{Ric}^{\mathcal{Z}}(\nu) \cdot \psi)|_M \\ &= -\text{ric}^{\mathcal{Z}}(\nu, \nu) \varphi + \sum_{j=1}^n \text{ric}^{\mathcal{Z}}(\nu, e_j) e_j \bullet \varphi. \end{aligned} \quad (33)$$

Taking the real part of the scalar product of Equation (33) with  $\varphi$  yields  $\frac{\text{Scal}^{\mathcal{Z}}}{2} = \text{ric}^{\mathcal{Z}}(\nu, \nu)$ , hence Equation (6) is satisfied.

Now, let  $M$  be a  $\text{Spin}^c$  Riemannian manifold having a generalized Killing spinor field  $\varphi$  with a symmetric endomorphism  $F$  on the tangent bundle  $TM$ . As mentioned in the introduction, it is straightforward to see that  $2T^\varphi(X, Y) = -\langle F(X), Y \rangle$ . We will study these generalized Killing spinors when the tensor  $F$  is a Codazzi-Mainardi tensor, i.e.,  $F$  satisfies

$$(\nabla_X^M F)(Y) = (\nabla_Y^M F)(X) \text{ for } X, Y \in \Gamma(TM). \quad (34)$$

For this, we give the following lemma whose proof will be omitted since it is similar to Lemma 7.3 in [2].

**Lemma 6.3** [2] *Let  $g_t$  be a smooth 1-parameter family of semi-Riemannian metrics on a  $\text{Spin}^c$  manifold  $(M^n, g = g_0)$  and let  $F$  be a field of symmetric endomorphisms of  $TM$ . We consider the metric  $g_{\mathcal{Z}} = \langle \cdot, \cdot \rangle = dt^2 + g_t$  on  $\mathcal{Z}$  such that  $g_t(X, Y) = g((\text{Id} - tF)^2 X, Y)$  for all vector fields  $X, Y$  on  $M$ . We have  $\langle R^{\mathcal{Z}}(U, \nu)\nu, V \rangle = 0$  for all vector fields  $U, V$  tangent to  $M$  and if  $F$  satisfies the Codazzi-Mainardi equation then  $\langle R^{\mathcal{Z}}(U, V)W, \nu \rangle = 0$  for all  $U, V$  and  $W$  on  $\mathcal{Z}$ .*

**Proof of Theorem 1.4:** We define  $\psi_{(0,x)} := \varphi_x$  via the identification  $\Sigma_x M \cong \Sigma_{(0,x)} \mathcal{Z}$  (resp.  $\Sigma_{(0,x)}^+ \mathcal{Z}$  for  $n$  odd) and  $\psi_{(t,x)} = \tau_0^t \psi_{(0,x)}$ . By Equation (21), the endomorphism  $F$  is the Weingarten tensor of the immersion of  $\{0\} \times M$  in  $\mathcal{Z}$  and hence by construction we have for all  $X \in \Gamma(TM)$

$$\nabla_X^{\Sigma \mathcal{Z}} \psi|_{\{0\} \times M} = 0 \quad \text{and} \quad \nabla_\nu^{\Sigma \mathcal{Z}} \psi \equiv 0. \quad (35)$$

Since the tensor  $F$  satisfies the Codazzi-Mainardi equation, Lemma 6.3 yields  $g_{\mathcal{Z}}(R^{\mathcal{Z}}(U, V)W, \nu) = 0$  for all  $U, V$  and  $W \in \Gamma(\mathcal{Z})$  and  $g_{\mathcal{Z}}(R^{\mathcal{Z}}(X, \nu)\nu, Y) = 0$  for all  $X$  and  $Y$  tangent to  $M$ . Hence  $R^{\mathcal{Z}}(\nu, X) = 0$  for all  $X \in \Gamma(TM)$ . Let  $X$  be a fixed arbitrary tangent vector field on  $M$ . Using (10) and (35) we get

$$\nabla_\nu^{\Sigma \mathcal{Z}} \nabla_X^{\Sigma \mathcal{Z}} \psi = \mathcal{R}^{\Sigma \mathcal{Z}}(\nu, X)\psi = \frac{1}{2}R^{\mathcal{Z}}(X, \nu) \cdot \psi + \frac{i}{2}\Omega^{\mathcal{Z}}(X, \nu)\psi = 0.$$

Thus showing that the spinor field  $\nabla_X^{\Sigma \mathcal{Z}} \psi$  is parallel along the geodesics  $\mathbb{R} \times \{x\}$ . Now (35) shows that this spinor vanishes for  $t = 0$ , hence it is zero everywhere on  $\mathcal{Z}$ . Since  $X$  is arbitrary, this shows that  $\psi$  is parallel on  $\mathcal{Z}$ .

**Corollary 6.4** *Let  $(M^3, g)$  be a compact, oriented Riemannian manifold and  $\varphi$  an eigenspinor associated with the first eigenvalue  $\lambda_1$  of the Dirac operator such that the Energy-Momentum tensor associated with  $\varphi$  is a Codazzi tensor.  $M$  is a limiting manifold for (3) if and only if the generalized cylinder  $\mathcal{Z}^4$ , equipped with the  $\text{Spin}^c$  structure arising from the given one on  $M$ , is Kähler of positive scalar curvature and the immersion of  $M$  in  $\mathcal{Z}$  has constant mean curvature  $H$ .*

**Proof:** First, we should point out that every 3-dimensional compact, oriented, smooth manifold has a  $\text{Spin}^c$  structure. Now, if  $M^3$  is a limiting manifold for (3), by Theorem 1.4, the generalized cylinder has a parallel spinor whose restriction to  $M$  is  $\varphi$ . Since  $\mathcal{Z}$  is a 4-dimensional  $\text{Spin}^c$  manifold having parallel spinor,  $\mathcal{Z}$  is Kähler [1]. Moreover, using (15), we have

$$\Omega^M \bullet \varphi = i \frac{\text{Scal}^{\mathcal{Z}}}{2} \varphi = i \frac{c_n}{2} |\Omega^M| \varphi,$$

so  $\text{Scal}^{\mathcal{Z}} \geq 0$ . Finally  $H = \frac{1}{n} \text{tr}(W) = \frac{1}{n} \text{tr}(-2T^\varphi) = -\frac{2}{n} \lambda_1$ , which is a constant. Now if the generalized cylinder is Kähler and  $M$  is a compact hypersurface of constant mean curvature  $H$ , thus  $M$  is compact hypersurface immersed in a  $\text{Spin}^c$  manifold having parallel spinor with constant mean curvature. Since  $\text{Scal}^{\mathcal{Z}} \geq 0$  and  $\nu \lrcorner \Omega^{\mathcal{Z}} = \text{Ric}^{\mathcal{Z}}(\nu) = 0$ , Corollary 6.1 gives the result.

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