



HAL
open science

The Energy-Momentum Tensor on $Spin^c$ Manifolds

Roger Nakad

► **To cite this version:**

| Roger Nakad. The Energy-Momentum Tensor on $Spin^c$ Manifolds. 2010. hal-00492141v1

HAL Id: hal-00492141

<https://hal.science/hal-00492141v1>

Preprint submitted on 15 Jun 2010 (v1), last revised 24 May 2011 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The Energy-Momentum Tensor on Spin^c Manifolds

Roger Nakad

May 26, 2010

Institut Élie Cartan, Université Henri Poincaré, Nancy I, B.P 239
54506 Vandoeuvre-Lès-Nancy Cedex, France.

nakad@iecn.u-nancy.fr

Abstract

On Spin^c manifolds, we study the Energy-Momentum tensor associated with a spinor field. First, we give a spinorial Gauss type formula for oriented hypersurfaces of a Spin^c manifold. Using the notion of generalized cylinders, we derive the variational formula for the Dirac operator under metric deformation and point out that the Energy-Momentum tensor appears naturally as the second fundamental form of an isometric immersion. Finally, we show that generalized Spin^c Killing spinors for Codazzi Energy-Momentum tensor are restrictions of parallel spinors.

Key words: Spin^c structures, Spin^c Gauss formula, metric variation formula for the Dirac operator, Energy-Momentum tensor, generalized cylinder, generalized Killing spinors.

1 Introduction

In [14], O. Hijazi proved that on a compact Riemannian spin manifold (M^n, g) any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies

$$\lambda^2 \geq \inf_M \left(\frac{1}{4} \text{Scal}^M + |\ell^\psi|^2 \right), \quad (1)$$

where Scal^M is the scalar curvature of the manifold M and ℓ^ψ is the field of symmetric endomorphisms associated with the field of quadratic forms T^ψ called the Energy-Momentum tensor and defined on the complement set of zeroes of the eigenspinor ψ , for any vector $X \in \Gamma(TM)$ by

$$T^\psi(X) = \text{Re} \langle X \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2} \rangle.$$

Here ∇ denotes the Levi-Civita connection on the spinor bundle of M and “ \cdot ” the Clifford multiplication. The limiting case of (1) is characterized by the existence of a spinor field ψ satisfying for all $X \in \Gamma(TM)$,

$$\nabla_X \psi = -\ell^\psi(X) \cdot \psi. \quad (2)$$

For Spin^c structures, the complex line bundle L^M is endowed with an arbitrary connection and hence an arbitrary curvature $i\Omega^M$ which is an imaginary 2-form on the manifold. In terms of the Energy-Momentum tensor the author proved in [25] that on a compact Riemannian Spin^c manifold any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies

$$\lambda^2 \geq \inf_M \left(\frac{1}{4} \text{Scal}^M - \frac{c_n}{4} |\Omega^M| + |\ell^\psi|^2 \right), \quad (3)$$

where $c_n = 2[\frac{n}{2}]^{\frac{1}{2}}$. The limiting case of (3) is characterized by the existence of a spinor field ψ satisfying for every $X \in \Gamma(TM)$,

$$\begin{cases} \nabla_X^{\Sigma M} \psi = -\ell^\psi(X) \cdot \psi, \\ \Omega^M \cdot \psi = i \frac{c_n}{2} |\Omega^M| \psi. \end{cases} \quad (4)$$

Here $\nabla^{\Sigma M}$ denotes the Levi-Civita connection on the Spin^c spinor bundle and “ \cdot ” the Spin^c Clifford multiplication. In [25], the author showed also that the sphere with a special Spin^c structure is a limiting manifold for (3).

Studying the Energy-Momentum tensor on a Riemannian or semi-Riemannian spin manifolds has been done by many authors, since it is related to several geometric constructions (see [12], [2], [24] and [6] for results in this topic). In this paper we study the Energy-Momentum tensor on Riemannian and semi-Riemannian Spin^c manifolds. First, we prove that the Energy-Momentum tensor appears in the study of the variations of the spectrum of the Dirac operator:

Proposition 1.1 *Let (M^n, g) be a Spin^c Riemannian manifold and $g_t = g + tk$ a smooth 1-parameter family of metrics. For any spinor field $\psi \in \Gamma(\Sigma M)$, we have*

$$\frac{d}{dt} \Big|_{t=0} (D^{M_t} \tau_0^t \psi, \tau_0^t \psi)_{g_t} = -\frac{1}{2} \int_M \langle k, T_\psi \rangle dv_g, \quad (5)$$

where $(\cdot, \cdot) = \int_M \operatorname{Re} \langle \cdot, \cdot \rangle dv_g$, the Dirac operator D^{M_t} is the Dirac operator associated with $M_t = (M, g_t)$, $T_\psi = |\psi|^2 T^\psi = \operatorname{Re} \langle X \cdot \nabla_X \psi, \psi \rangle$ and $\tau_0^t \psi$ is the image of ψ under the isometry τ_0^t between the spinor bundles of (M, g) and (M, g_t) .

This was proven in [4] by J. P. Bourguignon and P. Gauduchon for spin manifolds. Using this, we extend to Spin^c manifolds a result by Th. Friedrich and E. C. Kim in [8] on spin manifolds:

Theorem 1.2 *Let M be a Spin^c Riemannian manifold. A pair (g_0, ψ_0) is a critical point of the Lagrange functional*

$$\mathcal{W}(g, \psi) = \int_U (\operatorname{Scal}_g^M + \varepsilon \lambda \langle \psi, \psi \rangle_g - \langle D_g \psi, \psi \rangle_g) dv_g,$$

$(\lambda, \varepsilon \in \mathbb{R})$ for all open subsets U of M if and only if (g_0, ψ_0) is a solution of the following system

$$\begin{cases} D_g \psi = \lambda \psi, \\ \operatorname{ric}_g^M - \frac{\operatorname{Scal}_g^M}{2} g = \frac{\varepsilon}{4} T_\psi, \end{cases}$$

where ric_g^M denotes the Ricci curvature of M considered as a symmetric bilinear form.

Now, we interpret the Energy-Momentum tensor as the second fundamental form of a hypersurface. In fact, we prove the following:

Proposition 1.3 *Let $M^n \hookrightarrow (\mathcal{Z}, g)$ be any compact oriented hypersurface isometrically immersed in an oriented Riemannian Spin^c manifold (\mathcal{Z}, g) , of constant mean curvature H and Weingarten map W . We assume that \mathcal{Z} admits a parallel spinor field ψ , then the Energy-Momentum tensor associated with $\varphi =: \psi|_M$ satisfies*

$$2\ell^\varphi = -W.$$

Moreover the hypersurface M satisfies the equality case in (3) if and only if

$$\operatorname{Scal}^{\mathcal{Z}} - 2 \operatorname{ric}^{\mathcal{Z}}(\nu, \nu) - c_n |\Omega^M| = 0. \quad (6)$$

This was proven by Morel in [24] for a compact oriented hypersurface of a spin manifold carrying parallel spinor but in this case the hypersurface M is directly a limiting manifold for (1) without the condition (6).

Finally, we study generalized Killing spinors on Spin^c manifolds. They are characterized by the identity, for any tangent vector field X on M ,

$$\nabla_X^{\Sigma M} \psi = \frac{1}{2} F(X) \cdot \psi, \quad (7)$$

where F is a given symmetric endomorphism on the tangent bundle. It is straightforward to see that

$$2T^\psi(X, Y) = -\langle F(X), Y \rangle.$$

These spinors are closely related to the so-called T -Killing spinors studied by Friedrich and Kim in [9] on spin manifolds. It is natural to ask whether the tensor F can be realized as the Weingarten tensor of some isometric embedding of M in a manifold \mathcal{Z}^{n+1} carrying parallel spinors. Morel studied this problem in the case of spin manifolds where the tensor F is parallel and in [2], the authors studied the problem in the case of semi-Riemannian spin manifolds where the tensor F is a Codazzi-Mainardi tensor. We establish the corresponding result for semi-Riemannian Spin^c manifolds:

Theorem 1.4 *Let (M^n, g) be a semi-Riemannian Spin^c manifold carrying a generalized Spin^c Killing spinor φ with a Codazzi-Mainardi tensor F . Then the generalized cylinder $\mathcal{Z} := I \times M$ with the metric $dt^2 + g_t$, where $g_t(X, Y) = g((\text{Id} - tF)^2 X, Y)$, equipped with the Spin^c structure arising from the given one on M has a parallel spinor whose restriction to M is just φ .*

A characterisation of limiting 3-dimensional manifolds for (3), having generalized Spin^c Killing spinors with Codazzi tensor is then given.

The paper is organised as follows: In Section 2, we collect basic material on spinors and the Dirac operator on semi-Riemannian Spin^c manifolds. In Section 3, we study hypersurfaces of Spin^c manifolds. We derive a spinorial Gauss formula after identifying the restriction of the Spin^c spinor bundle of the ambient manifold with the Spin^c spinor bundle of the hypersurface. In Section 4, we define the generalized cylinder of a Spin^c manifold M and we collect formulas relating the curvature of a generalized cylinder to geometric data on M . In section 5, we compare the Dirac operators for two different semi-Riemannian metrics, then one first has to identify the spinor bundles using parallel transport. In the last section, we interpret the Energy-Momentum tensor as the second fundamental form of a hypersurface and we study generalized Spin^c Killing spinors. The author would like to thank Oussama Hijazi for his support and encouragements.

2 The Dirac operator on semi-Riemannian Spin^c manifolds

In this section, we collect some algebraic and geometric preliminaries concerning the Dirac operator on semi-Riemannian Spin^c manifolds. Details can be found in [3] and [2]. Let $r + s = n$ and consider on \mathbb{R}^n the nondegenerate symmetric bilinear form of

signature (r, s) given by

$$\langle v, w \rangle := \sum_{j=1}^r v_j w_j - \sum_{j=r+1}^n v_j w_j,$$

for any $v, w \in \mathbb{R}^n$. We denote by $\text{Cl}_{r,s}$ the real Clifford algebra corresponding to $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, this is the unitary algebra generated by \mathbb{R}^n subject to the relations

$$e_j \cdot e_k + e_k \cdot e_j = \begin{cases} -2\delta_{jk} & \text{if } j \leq r, \\ 2\delta_{jk} & \text{if } j > r, \end{cases}$$

where $(e_j)_{1 \leq j \leq n}$ is an orthonormal basis of \mathbb{R}^n of signature (r, s) , i.e., $\langle e_j, e_k \rangle = \varepsilon_j \delta_{jk}$ and $\varepsilon_j = \pm 1$. The complex Clifford algebra $\text{Cl}_{r,s}$ is the complexification of $\text{Cl}_{r,s}$ and it decomposes into even and odd elements $\text{Cl}_{r,s} = \text{Cl}_{r,s}^0 \oplus \text{Cl}_{r,s}^1$. The real spin group is defined by

$$\text{Spin}(r, s) := \{v_1 \cdot \dots \cdot v_{2k} \in \text{Cl}_{r,s} \mid v_j \in \mathbb{R}^n \text{ such that } \langle v_j, v_j \rangle = \pm 1\}.$$

The spin group $\text{Spin}(r, s)$ is the double cover of $\text{SO}(r, s)$, in fact the following sequence is exact

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \text{Spin}(r, s) \xrightarrow{\xi} \text{SO}(r, s) \longrightarrow 1,$$

where $\xi = \text{Ad}|_{\text{Spin}(r,s)}$ and Ad is defined by

$$\begin{aligned} \text{Ad} : \text{Cl}_{r,s}^* &\longrightarrow \text{End}(\mathbb{R}^n) \\ w &\longrightarrow \text{Ad}_w : v \longrightarrow \text{Ad}_w(v) = w \cdot v \cdot w^{-1}. \end{aligned}$$

Here $\text{Cl}_{r,s}^*$ denotes the group of units of $\text{Cl}_{r,s}$. Since $\mathbb{S}^1 \cap \text{Spin}(r, s) = \{\pm 1\}$, we define the complex spin group by

$$\text{Spin}^c(r, s) = \text{Spin}(r, s) \times_{\mathbb{Z}_2} \mathbb{S}^1.$$

The complex spin group is the double cover of $\text{SO}(r, s) \times \mathbb{S}^1$, this yields to the exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}^c(r, s) \xrightarrow{\xi^c} \text{SO}(r, s) \times \mathbb{S}^1 \longrightarrow 1,$$

where $\xi^c = (\xi, \text{Id}^2)$. When $n = 2m$ is even, $\text{Cl}_{r,s}$ has a unique irreducible complex representation χ_{2m} of complex dimension 2^m , $\chi_{2m} : \text{Cl}_{r,s} \longrightarrow \text{End}(\Sigma_{r,s})$. If $n = 2m + 1$ is odd, $\text{Cl}_{r,s}$ has two inequivalent irreducible representations both of complex dimension 2^m , $\chi_{2m+1}^j : \text{Cl}_{r,s} \longrightarrow \text{End}(\Sigma_{r,s}^j)$, for $j = 0$ or 1 , where $\Sigma_{r,s}^j = \{\sigma \in \Sigma_{r,s}, \chi_{2m+1}^j(\omega_{r,s})\sigma = (-1)^j \sigma\}$ and $\omega_{r,s}$ is the complex volume element

$$\omega_{r,s} = \begin{cases} i^{m-s} e_1 \cdot \dots \cdot e_n & \text{if } n = 2m, \\ i^{m-1+s} e_1 \cdot \dots \cdot e_n & \text{if } n = 2m + 1. \end{cases}$$

We define the complex spinorial representation ρ_n by the restriction of an irreducible representation of $\mathbb{C}l_{r,s}$ to $\text{Spin}^c(r, s)$:

$$\rho_n := \begin{cases} \chi_{2m|_{\text{Spin}^c(r,s)}} & \text{if } n = 2m, \\ \chi_{2m+1|_{\text{Spin}^c(r,s)}}^0 & \text{if } n = 2m + 1, \end{cases}.$$

When $n = 2m$ is even, ρ_n decomposes into two inequivalent irreducible representations ρ_n^+ and ρ_n^- , i.e., $\rho_n = \rho_n^+ + \rho_n^- : \text{Spin}^c(r, s) \rightarrow \text{Aut}(\Sigma_{r,s})$. The space $\Sigma_{r,s}$ decomposes into $\Sigma_{r,s} = \Sigma_{r,s}^+ \oplus \Sigma_{r,s}^-$, where $\omega_{r,s}$ acts on $\Sigma_{r,s}^+$ as the identity and minus the identity on $\Sigma_{r,s}^-$. If $n = r + s$ is odd and when restricted to $\text{Spin}^c(r, s)$, the two representations $\chi_{2m+1|_{\text{Spin}^c(r,s)}}^0$ and $\chi_{2m+1|_{\text{Spin}^c(r,s)}}^1$ are equivalent and we simply choose $\Sigma_{r,s} := \Sigma_{r,s}^0$. The complex spinor bundle $\Sigma_{r,s}$ carries a Hermitian symmetric bilinear $\text{Spin}^c(r, s)$ -invariant form $\langle \cdot, \cdot \rangle$, such that

$$\langle v \cdot \sigma_1, \sigma_2 \rangle = (-1)^{s+1} \langle \sigma_1, v \cdot \sigma_2 \rangle \text{ for all } \sigma_1, \sigma_2 \in \Sigma_{r,s} \text{ and } v \in \mathbb{R}^n.$$

Now, we give the following isomorphism α , which is of particular importance for the identification of the Spin^c bundles in the context of immersions of hypersurfaces:

$$\begin{aligned} \alpha : \mathbb{C}l_{r,s} &\rightarrow \mathbb{C}l_{r+1,s}^0 \\ e_j &\rightarrow \nu \cdot e_j, \end{aligned} \tag{8}$$

where we look at an embedding of \mathbb{R}^n onto \mathbb{R}^{n+1} such that $(\mathbb{R}^n)^\perp$ is spacelike and spanned by a spacelike unit vector ν .

Let N^n be an oriented semi-Riemannian manifold of signature (r, s) and let $P_{\text{SO}}N$ be the $\text{SO}(r, s)$ -principal bundle of positively space and time oriented orthonormal tangent frames. A complex Spin^c structure on N is a $\text{Spin}^c(r, s)$ -principal bundle $P_{\text{Spin}^c}N$ over N , an \mathbb{S}^1 -principal bundle $P_{\mathbb{S}^1}N$ over N together with a twofold covering map $\Theta : P_{\text{Spin}^c}N \rightarrow P_{\text{SO}}N \times_N P_{\mathbb{S}^1}N$ such that

$$\Theta(ua) = \Theta(u)\xi^c(a),$$

for every $u \in P_{\text{Spin}^c}N$ and $a \in \text{Spin}^c(r, s)$, i.e., N has a Spin^c structure if and only if there exists an \mathbb{S}^1 -principal bundle $P_{\mathbb{S}^1}N$ over N such that the transition functions $g_{\alpha\beta} \times l_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{SO}(r, s) \times \mathbb{S}^1$ of the $\text{SO}(r, s) \times \mathbb{S}^1$ -principal bundle $P_{\text{SO}}N \times_N P_{\mathbb{S}^1}N$ admit lifts to $\text{Spin}^c(r, s)$ denoted by $\tilde{g}_{\alpha\beta} \times \tilde{l}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{Spin}^c(r, s)$, such that $\xi^c \circ (\tilde{g}_{\alpha\beta} \times \tilde{l}_{\alpha\beta}) = g_{\alpha\beta} \times l_{\alpha\beta}$. This, anyhow, is equivalent to the second Stiefel-Whitney class $w_2(N)$ being equal, modulo 2, to the first Chern class $c_1(L^N)$ of the complex line bundle L^N . It is the complex line bundle associated with the \mathbb{S}^1 -principal fibre bundle via the standard representation of the unit circle.

Let $\Sigma N := P_{\text{Spin}^c} N \times_{\rho_n} \Sigma_{r,s}$ be the spinor bundle associated with the spinor representation. A section of ΣN will be called a spinor field. Using the cocycle condition of the transition functions of the two principal fibre bundles $P_{\text{Spin}^c} N$ and $P_{\text{SO}} N \times_N P_{\mathbb{S}^1} N$, we can prove that

$$\Sigma N = \Sigma' N \otimes (L^N)^{\frac{1}{2}},$$

where $\Sigma' N$ is the locally defined spin bundle and $(L^N)^{\frac{1}{2}}$ is locally defined too but ΣN is globally defined. The tangent bundle $TN = P_{\text{SO}} N \times_{\rho_0} \mathbb{R}^n$ where ρ_0 stands for the standard matrix representation of $\text{SO}(r, s)$ on \mathbb{R}^n , can be seen as the associated vector bundle $TN \simeq P_{\text{Spin}^c} N \times_{pr_1 \circ \xi^c \circ \rho_0} \mathbb{R}^n$ where pr_1 is the first projection. One defines the Clifford multiplication at every point $p \in N$:

$$\begin{aligned} T_p N \otimes \Sigma_p N &\longrightarrow \Sigma_p N \\ [b, v] \otimes [b, \sigma] &\longrightarrow [b, v] \cdot [b, \sigma] := [b, v \cdot \sigma = \chi_n(v)\sigma], \end{aligned}$$

where $b \in P_{\text{Spin}^c} N$, $v \in \mathbb{R}^n$, $\sigma \in \Sigma_{r,s}$ and $\chi_n = \chi_{2m}$ if n is even and $\chi_n = \chi_{2m+1}^0$ if n is odd. The Clifford multiplication can be extended to differential forms. Clifford multiplication inherits the relations of the Clifford algebra, i.e., for $X, Y \in T_p N$ and $\varphi \in \Sigma_p N$ we have $X \cdot Y \cdot \varphi + Y \cdot X \cdot \varphi = -2 \langle X, Y \rangle \varphi$. In even dimensions the spinor bundle splits into $\Sigma N = \Sigma^+ N \oplus \Sigma^- N$, where $\Sigma^\pm N = P_{\text{Spin}^c} N \times_{\rho_n^\pm} \Sigma_{r,s}^\pm$. Clifford multiplication by a non-vanishing tangent vector interchanges $\Sigma^+ N$ and $\Sigma^- N$. The $\text{Spin}^c(r, s)$ -invariant nondegenerate symmetric sesquilinear form on $\Sigma_{r,s}$ and $\Sigma_{r,s}^\pm$ induces inner products on ΣN and $\Sigma^\pm N$ which we again denote by $\langle \cdot, \cdot \rangle$ and it satisfies

$$\langle X \cdot \psi, \varphi \rangle = (-1)^{s+1} \langle \psi, X \cdot \varphi \rangle,$$

for every $X \in \Gamma(TN)$ and $\psi, \varphi \in \Gamma(\Sigma N)$. Additionally, given a connection 1-form A^N on $P_{\mathbb{S}^1} N$, $A^N : T(P_{\mathbb{S}^1} N) \longrightarrow i\mathbb{R}$ and the connection 1-form ω^N on $P_{\text{SO}} N$ for the Levi-Civita connection ∇^N , we can define the connection

$$\omega^N \times A^N : T(P_{\text{SO}} N \times_N P_{\mathbb{S}^1} N) \longrightarrow \mathfrak{so}_n \oplus i\mathbb{R} = \mathfrak{spin}_n^{\mathbb{C}}$$

on the principal fibre bundle $P_{\text{SO}} N \times_N P_{\mathbb{S}^1} N$ and hence a covariant derivative $\nabla^{\Sigma N}$ on ΣN [7] given locally by

$$\begin{aligned} \nabla_{e_k}^{\Sigma N} \varphi &= \left[\widetilde{b \times s}, e_k(\sigma) + \frac{1}{4} \sum_{j=1}^n \varepsilon_j e_j \cdot \nabla_{e_k}^N e_j \cdot \sigma + \frac{1}{2} A^N(s_*(e_k))\sigma \right] \\ &= e_k(\varphi) + \frac{1}{4} \sum_{j=1}^n \varepsilon_j e_j \cdot \nabla_{e_k}^N e_j \cdot \varphi + \frac{1}{2} A^N(s_*(e_k))\varphi, \end{aligned} \quad (9)$$

where $\varphi = [\widetilde{b \times s}, \sigma]$ is a locally defined spinor field, $b = (e_1, \dots, e_n)$ is a local space and time oriented orthonormal tangent frame, $s : U \longrightarrow P_{\mathbb{S}^1} N$ is a local section of

$P_{\mathbb{S}^1}N$ and $\widetilde{b \times s}$ is the lift of the local section $b \times s : U \rightarrow P_{\text{SO}}N \times_N P_{\mathbb{S}^1}N$ to the 2-fold covering $\Theta : P_{\text{Spin}^c}N \rightarrow P_{\text{SO}}N \times_N P_{\mathbb{S}^1}N$. The curvature of A^N is an imaginary valued 2-form denoted by $F_{A^N} = dA^N$, i.e., $F_{A^N} = i\Omega^N$, where Ω^N is a real valued 2-form on $P_{\mathbb{S}^1}N$. We know that Ω^N can be viewed as a real valued 2-form on N [7]. In this case $i\Omega^N$ is the curvature form of the associated line bundle L^N . The curvature tensor $\mathcal{R}^{\Sigma N}$ of $\nabla^{\Sigma N}$ is given by

$$\mathcal{R}^{\Sigma N}(X, Y)\varphi = \frac{1}{4} \sum_{j,k=1}^n \varepsilon_j \varepsilon_k \langle R^N(X, Y)e_j, e_k \rangle e_j \cdot e_k \cdot \varphi + \frac{i}{2} \Omega^N(X, Y)\varphi, \quad (10)$$

where R^N is the curvature tensor of the Levi-Civita connection ∇^N . In the Spin^c case, the Ricci identity translates, for every $X \in \Gamma(TN)$, to

$$\sum_{k=1}^n \varepsilon_k e_k \cdot \mathcal{R}^{\Sigma N}(e_k, X)\varphi = \frac{1}{2} \text{Ric}^N(X) \cdot \varphi - \frac{i}{2} (X \lrcorner \Omega^N) \cdot \varphi. \quad (11)$$

Here Ric^N denotes the Ricci curvature considered as a field of endomorphism on TN . The Ricci curvature considered as a symmetric bilinear form will be written $\text{ric}^N(Y, Z) = \langle \text{Ric}^N(Y), Z \rangle$. The Dirac operator maps spinor fields to spinor fields and is locally defined by

$$D^N \varphi = i^s \sum_{j=1}^n \varepsilon_j e_j \cdot \nabla_{e_j}^{\Sigma N} \varphi,$$

for every spinor field φ . The Dirac operator is an elliptic operator, formally selfadjoint, i.e., if ψ or φ has compact support, then $(D^N \varphi, \psi) = (\varphi, D^N \psi)$, where $(\varphi, \psi) = \int_N \langle \varphi, \psi \rangle dv_g$.

3 Semi-Riemannian Spin^c hypersurfaces and the Gauss formula

In this section, we study Spin^c structures of hypersurfaces, such as the restriction of a Spin^c bundle of an ambient semi-Riemannian manifold and the complex spinorial Gauss formula.

Let \mathcal{Z} be an oriented $(n + 1)$ -dimensional semi-Riemannian Spin^c manifold and $M \subset \mathcal{Z}$ a semi-Riemannian hypersurface with trivial spacelike normal bundle. This means that there is a vector field ν on \mathcal{Z} along M satisfying $\langle \nu, \nu \rangle = +1$ and $\langle \nu, TM \rangle = 0$. Hence if the signature of M is (r, s) , then the signature of \mathcal{Z} is $(r + 1, s)$.

Proposition 3.1 *The hypersurface M inherits a Spin^c structure from that on \mathcal{Z} , and we have*

$$\begin{cases} \Sigma\mathcal{Z}|_M \simeq \Sigma M & \text{if } n \text{ is even,} \\ \Sigma^+\mathcal{Z}|_M \simeq \Sigma M & \text{if } n \text{ is odd.} \end{cases}$$

Moreover Clifford multiplication by a vector field X , tangent to M , is given by

$$X \bullet \varphi = (\nu \cdot X \cdot \psi)|_M, \quad (12)$$

where $\psi \in \Gamma(\Sigma\mathcal{Z})$ (or $\psi \in \Gamma(\Sigma^+\mathcal{Z})$ if n is odd), φ is the restriction of ψ to M , “ \cdot ” is the Clifford multiplication on \mathcal{Z} , and “ \bullet ” that on M .

Proof: The bundle of space and time oriented orthonormal frames of M can be embedded into the bundle of space and time oriented orthonormal frames of \mathcal{Z} restricted to M , by

$$\begin{aligned} \Phi : \quad P_{\text{SO}}M &\longrightarrow P_{\text{SO}}\mathcal{Z}|_M \\ (e_1, \dots, e_n) &\longrightarrow (\nu, e_1, \dots, e_n). \end{aligned} \quad (13)$$

The isomorphism α , defined in (8) yields the following commutative diagram:

$$\begin{array}{ccc} \text{Spin}^c(r, s) & \hookrightarrow & \text{Spin}^c(r+1, s) \\ \downarrow \xi^c & & \xi^c \downarrow \\ \text{SO}(r, s) \times \mathbb{S}^1 & \hookrightarrow & \text{SO}(r+1, s) \times \mathbb{S}^1 \end{array}$$

where the inclusion of $\text{SO}(r, s)$ in $\text{SO}(r+1, s)$ is that which fixes the first basis vector under the action of $\text{SO}(r+1, s)$ on \mathbb{R}^{n+1} . This allows to pull back via Φ the principal bundle $P_{\text{Spin}^c}\mathcal{Z}|_M$ as a Spin^c structure for M , denoted by $P_{\text{Spin}^c}M$. Thus, we have the following commutative diagram:

$$\begin{array}{ccc} P_{\text{Spin}^c}M & \longrightarrow & P_{\text{Spin}^c}\mathcal{Z}|_M \\ \downarrow \Theta & & \Theta \downarrow \\ P_{\text{SO}}M \times_M P_{\mathbb{S}^1}\mathcal{Z}|_M & \longrightarrow & P_{\text{SO}}\mathcal{Z}|_M \times_M P_{\mathbb{S}^1}\mathcal{Z}|_M \end{array}$$

The $\text{Spin}^c(r, s)$ -principal bundle $(P_{\text{Spin}^c}M, \pi, M)$ and the \mathbb{S}^1 -principal bundle $(P_{\mathbb{S}^1}M =: P_{\mathbb{S}^1}\mathcal{Z}|_M, \pi, M)$ define a Spin^c structure on M . Let $\Sigma\mathcal{Z}$ be the spinor bundle on \mathcal{Z} ,

$$\Sigma\mathcal{Z} = P_{\text{Spin}^c}\mathcal{Z} \times_{\rho_{n+1}} \Sigma_{r+1, s},$$

where ρ_{n+1} stands for the spinorial representation of $\text{Spin}^c(r+1, s)$. Moreover, for any spinor $\psi = [\widetilde{b \times s}, \sigma] \in \Sigma\mathcal{Z}$ we can always assume that $pr_1 \circ \Theta(\widetilde{b \times s}) = b$ is a local section of $P_{\text{SO}}\mathcal{Z}$ with ν for first basis vector where pr_1 is the projection into $P_{\text{SO}}\mathcal{Z}$. Then we have

$$\psi|_M = [\widetilde{b \times s}|_{U \cap M}, \sigma|_{U \cap M}],$$

where the equivalence class is reduced to elements of $\text{Spin}^c(r, s)$. It follows that one can realise the restriction to M of the spinor bundle $\Sigma\mathcal{Z}$ as

$$\Sigma\mathcal{Z}|_M = P_{\text{Spin}^c}M \times_{\rho_{n+1} \circ \alpha} \Sigma_{r+1, s}.$$

If $n = 2m$ is even, it is easy to check that $\chi_{2m+1}^0 \circ \alpha = \chi_{2m+1}^0|_{\mathbb{C}l_{r+1, s}^0}$. Hence $\chi_{2m+1}^0 \circ \alpha$ is an irreducible representation of $\mathbb{C}l_{r, s}$ of dimension 2^m , as $\chi_{2m+1}^0|_{\mathbb{C}l_{r+1, s}^0}$, and finally $\chi_{2m+1}^0 \circ \alpha \cong \chi_{2m}$. We conclude that

$$\rho_{2m+1} \circ \alpha \cong \rho_{2m}, \quad \text{and} \quad \Sigma\mathcal{Z}|_M \cong \Sigma M.$$

If $n = 2m + 1$ is odd, we know that χ_{2m+1}^0 is the unique irreducible representation of $\mathbb{C}l_{r, s}$ of dimension 2^m for which the action of the complex volume form is the identity. Since $n+1 = 2m+2$ is even, $\Sigma\mathcal{Z}$ decomposes into positive and negative parts, $\Sigma^\pm\mathcal{Z} = P_{\text{Spin}^c}\mathcal{Z} \times_{\rho_{2m+1}^\pm} \Sigma_{r+1, s}^\pm$. It is easy to show that $\chi_{2m+2} \circ \alpha = \chi_{2m+2}|_{\mathbb{C}l_{r+1, s}^0}$, but $\chi_{2m+2} \circ \alpha$ can be written as the direct sum of two irreducible inequivalent representations, as $\chi_{2m+2}|_{\mathbb{C}l_{r+1, s}^0}$. Hence, we have

$$\chi_{2m+2} \circ \alpha = (\chi_{2m+2} \circ \alpha)^+ \oplus (\chi_{2m+2} \circ \alpha)^-,$$

where $(\chi_{2m+2} \circ \alpha)^\pm(\omega_{r, s}) = \pm \text{Id}_{\Sigma_{r, s}}$. The representation χ_{2m+1}^0 being the unique representation of $\mathbb{C}l_{r, s}$ of dimension 2^m for which the action of the volume form is the identity, we get $(\chi_{2m+2} \circ \alpha)^+ \cong \chi_{2m+1}^0$. Finally,

$$\rho_{2m+2}^+ \circ \alpha \cong \rho_{2m+1} \quad \text{and} \quad \Sigma^+\mathcal{Z}|_M \cong \Sigma M.$$

Now, Equation (12) follows directly from the above identification.

Remarks 3.2 1. *The algebraic remarks in the previous section show that if n is odd we can also get $\Sigma^-\mathcal{Z}|_M \simeq \Sigma M$, where the Clifford multiplication by a vector field tangent to M is given by $X \bullet \varphi = -(\nu \cdot X \cdot \psi)|_M$.*

2. *The connection 1-form defined on the restricted \mathbb{S}^1 -principal bundle $(P_{\mathbb{S}^1}M =: P_{\mathbb{S}^1}\mathcal{Z}|_M, \pi, M)$, is given by*

$$A^M = A^{\mathcal{Z}}|_M : T(P_{\mathbb{S}^1}M) = T(P_{\mathbb{S}^1}\mathcal{Z})|_M \longrightarrow i\mathbb{R}.$$

Then the curvature 2-form $i\Omega^M$ on the \mathbb{S}^1 -principal bundle $P_{\mathbb{S}^1}M$ is given by $i\Omega^M = i\Omega^{\mathcal{Z}}|_M$, which can be viewed as an imaginary 2-form on M and hence as the curvature form of the line bundle L^M , the restriction of the line bundle $L^{\mathcal{Z}}$ to M .

3. *For every $\psi \in \Gamma(\Sigma\mathcal{Z})$ ($\psi \in \Gamma(\Sigma^+\mathcal{Z})$ if n is odd), the real 2-forms Ω^M and $\Omega^{\mathcal{Z}}$ are related by the following formulas:*

$$|\Omega^{\mathcal{Z}}|^2 = |\Omega^M|^2 + |\nu \lrcorner \Omega^{\mathcal{Z}}|^2, \quad (14)$$

$$(\Omega^{\mathcal{Z}} \cdot \psi)|_M = \Omega^M \bullet \varphi + (\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \varphi. \quad (15)$$

In fact, we can write

$$\Omega^{\mathcal{Z}} = \sum_{i=1}^n \Omega^{\mathcal{Z}}(\nu, e_i) \nu \wedge e_i + \sum_{i<j}^n \Omega^{\mathcal{Z}}(e_i, e_j) e_i \wedge e_j = -(\nu \lrcorner \Omega^{\mathcal{Z}}) \wedge \nu + \Omega^M,$$

which is (14). When restricting the Clifford multiplication of $\Omega^{\mathcal{Z}}$ by ψ to the hypersurface M we obtain

$$(\Omega^{\mathcal{Z}} \cdot \psi)|_M = (\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi)|_M + (\Omega^M \cdot \psi)|_M = (\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \varphi + \Omega^M \bullet \varphi. \quad (16)$$

Proposition 3.3 (The spinorial Gauss formula) We denote by $\nabla^{\Sigma \mathcal{Z}}$ the spinorial Levi-Civita connection on $\Sigma \mathcal{Z}$ and by $\nabla^{\Sigma M}$ that on ΣM . For all $X \in \Gamma(TM)$ and for every spinor field $\psi \in \Gamma(\Sigma \mathcal{Z})$, then

$$(\nabla_X^{\Sigma \mathcal{Z}} \psi)|_M = \nabla_X^{\Sigma M} \varphi - \frac{1}{2} W(X) \bullet \varphi, \quad (17)$$

where W denotes the Weingarten map with respect to ν and $\varphi = \psi|_M$. Moreover, let $D^{\mathcal{Z}}$ and D^M be the Dirac operators on \mathcal{Z} and M . Denoting by the same symbol any spinor and it's restriction to M , we have

$$\nu \cdot D^{\mathcal{Z}} \varphi = \tilde{D} \varphi + \frac{i^s n}{2} H \varphi - i^s \nabla_{\nu}^{\Sigma \mathcal{Z}} \varphi, \quad (18)$$

where $H = \frac{1}{n} \text{tr}(W)$ denotes the mean curvature and $\tilde{D} = D^M$ if n is even and $\tilde{D} = D^M \oplus (-D^M)$ if n is odd.

Proof: The Riemannian Gauss formula is given, for every vector fields X and Y on M , by

$$\nabla_X^{\mathcal{Z}} Y = \nabla_X^M Y + \langle W(X), Y \rangle \nu. \quad (19)$$

Let (e_1, e_2, \dots, e_n) a local space and time oriented orthonormal frame of M , such that $b = \widetilde{(e_0 = \nu, e_1, e_2, \dots, e_n)}$ is that of \mathcal{Z} . We consider ψ a local section of $\Sigma \mathcal{Z}$, $\psi = [b \times s, \sigma]$ where s is a local section of $P_{\mathbb{S}^1} \mathcal{Z}$. Using (9), (19) and the fact that $X(\psi)|_M = X(\varphi)$ for $X \in \Gamma(TM)$, we compute for $j = 1, \dots, n$

$$\begin{aligned} \left(\nabla_{e_j}^{\Sigma \mathcal{Z}} \psi \right)|_M &= e_j(\varphi) + \frac{1}{4} \sum_{k=0}^n \varepsilon_k (e_k \cdot \nabla_{e_j}^{\mathcal{Z}} e_k \cdot \psi)|_M + \frac{1}{2} A^{\mathcal{Z}}(s_*(e_j)) \varphi \\ &= e_j(\varphi) + \frac{1}{4} \sum_{k=1}^n \varepsilon_k (e_k \cdot \nabla_{e_j}^{\mathcal{Z}} e_k \cdot \psi)|_M + \frac{1}{4} (\nu \cdot \nabla_{e_j}^{\mathcal{Z}} \nu \cdot \psi)|_M + \frac{1}{2} A^M(s_*(e_j)) \varphi \\ &= \nabla_{e_j}^{\Sigma M} \varphi + \frac{1}{4} \sum_{k=1}^n \varepsilon_k \langle W(e_j), e_k \rangle (e_k \cdot \nu \cdot \psi)|_M - \frac{1}{4} (\nu \cdot W(e_j) \cdot \psi)|_M \\ &= \nabla_{e_j}^{\Sigma M} \varphi - \frac{1}{2} (\nu \cdot W(e_j) \cdot \psi)|_M \\ &= \nabla_{e_j}^{\Sigma M} \varphi - \frac{1}{2} W(e_j) \bullet \varphi. \end{aligned}$$

Moreover $(D^{\mathcal{Z}}\psi)|_M = i^s \sum_{j=1}^n \varepsilon_j (e_j \cdot \nabla_{e_j}^{\Sigma^{\mathcal{Z}}}\psi)|_M + i^s (\nu \cdot \nabla_{\nu}^{\Sigma^{\mathcal{Z}}}\psi)|_M$, and by (17),

$$\begin{aligned} i^s \sum_{j=1}^n \varepsilon_j (e_j \cdot \nabla_{e_j}^{\Sigma^{\mathcal{Z}}}\psi)|_M &= i^s \sum_{j=1}^n \varepsilon_j (e_j \cdot \nabla_{e_j}^{\Sigma^M}\varphi) - i^s \frac{1}{2} \sum_{j=1}^n \varepsilon_j (e_j \cdot \nu \cdot W(e_j) \cdot \psi)|_M \\ &= -i^s \nu \cdot \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \nabla_{e_j}^{\Sigma^M}\varphi + i^s \frac{1}{2} \sum_{j=1}^n \varepsilon_j (\nu \cdot e_j \cdot W(e_j) \cdot \psi)|_M \\ &= -\nu \cdot \tilde{D}\varphi - \frac{i^s}{2} \text{tr}(W)(\nu \cdot \psi)|_M. \end{aligned}$$

Proposition 3.4 *Let \mathcal{Z} be an $(n+1)$ -dimensional semi-Riemannian Spin^c manifold. Assume that \mathcal{Z} carries a semi-Riemannian foliation by hypersurfaces with trivial spacelike normal bundle, i.e., the leaves M are semi-Riemannian hypersurfaces and there exists a vector field ν on \mathcal{Z} perpendicular to the leaves such that $\langle \nu, \nu \rangle = 1$ and $\nabla_{\nu}^{\mathcal{Z}}\nu = 0$. Then the commutator of the leafwise Dirac operator and the normal derivative is given by*

$$i^{-s} [\nabla_{\nu}^{\Sigma^{\mathcal{Z}}}, \tilde{D}] \varphi = \mathfrak{D}^W \varphi - \frac{n}{2} \nu \cdot \text{grad}^M(H) \cdot \varphi + \frac{1}{2} \nu \cdot \text{div}^M(W) \cdot \varphi + \frac{i}{2} \nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi.$$

Here grad^M denotes the leafwise gradient, $\text{div}^M(W) = \sum_{i=1}^n \varepsilon_i (\nabla_{e_i}^M W)(e_i)$ denotes the leafwise divergence of the endomorphism field W and $\mathfrak{D}^W \varphi = \sum_{i=1}^n \varepsilon_i \nu \cdot e_i \cdot \nabla_{W(e_i)}^{\Sigma^M} \varphi$.

Proof: We choose a local oriented orthonormal tangent frame (e_1, \dots, e_n) for the leaves and we may assume for simplicity that $\nabla_{\nu}^{\mathcal{Z}} e_j = 0$. Now, we compute

$$\begin{aligned} i^{-s} [\nabla_{\nu}^{\Sigma^{\mathcal{Z}}}, \tilde{D}] \varphi &= \sum_{j=1}^n \varepsilon_j \left(\nabla_{\nu}^{\Sigma^{\mathcal{Z}}} (\nu \cdot e_j \cdot \nabla_{e_j}^{\Sigma^M} \varphi) - \nu \cdot e_j \cdot \nabla_{e_j}^{\Sigma^M} \nabla_{\nu}^{\Sigma^{\mathcal{Z}}} \varphi \right) \\ &= \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \left(\nabla_{\nu}^{\Sigma^{\mathcal{Z}}} \nabla_{e_j}^{\Sigma^M} \varphi - \nabla_{e_j}^{\Sigma^M} \nabla_{\nu}^{\Sigma^{\mathcal{Z}}} \varphi \right) \\ &\stackrel{(17)}{=} \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \left[\nabla_{\nu}^{\Sigma^{\mathcal{Z}}} (\nabla_{e_j}^{\Sigma^{\mathcal{Z}}} + \frac{1}{2} \nu \cdot W(e_j)) \right. \\ &\quad \left. - (\nabla_{e_j}^{\Sigma^{\mathcal{Z}}} + \frac{1}{2} \nu \cdot W(e_j)) \nabla_{\nu}^{\Sigma^{\mathcal{Z}}} \right] \varphi \\ &= \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \left(\mathcal{R}^{\Sigma^{\mathcal{Z}}}(\nu, e_j) + \nabla_{[\nu, e_j]}^{\Sigma^{\mathcal{Z}}} + \frac{1}{2} \nu \cdot (\nabla_{\nu}^{\mathcal{Z}} W)(e_j) \right) \varphi \\ &\stackrel{(11)}{=} -\frac{1}{2} \nu \cdot \text{Ric}^{\mathcal{Z}}(\nu) \cdot \varphi + \frac{i}{2} \nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi \\ &\quad + \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \left(\nabla_{W(e_j)}^{\Sigma^{\mathcal{Z}}} + \frac{1}{2} \nu \cdot (\nabla_{\nu}^{\mathcal{Z}} W)(e_j) \right) \varphi \end{aligned}$$

$$\begin{aligned}
&\stackrel{(17)}{=} -\frac{1}{2}\nu \cdot \text{Ric}^{\mathcal{Z}}(\nu) \cdot \varphi + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi \\
&\quad + \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \left(\nabla_{W(e_j)}^{\Sigma M} - \frac{1}{2}\nu \cdot W^2(e_j) + \frac{1}{2}\nu \cdot (\nabla_{\nu}^{\mathcal{Z}} W)(e_j) \right) \varphi \\
&= -\frac{1}{2}\nu \cdot \text{Ric}^{\mathcal{Z}}(\nu) \cdot \varphi + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi + \mathfrak{D}^W \varphi \\
&\quad + \frac{1}{2} \sum_{j=1}^n \varepsilon_j e_j \cdot \left(-W^2(e_j) + (\nabla_{\nu}^{\mathcal{Z}} W)(e_j) \right) \varphi.
\end{aligned}$$

The Riccati equation for the Weingarten map $(\nabla_{\nu}^{\mathcal{Z}} W)(X) = R^{\mathcal{Z}}(X, \nu)\nu + W^2(X)$ yields

$$\begin{aligned}
i^{-s}[\nabla_{\nu}^{\Sigma \mathcal{Z}}, \tilde{D}] \varphi &= -\frac{1}{2}\nu \cdot \text{Ric}^{\mathcal{Z}}(\nu) \cdot \varphi + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi + \mathfrak{D}^W \varphi \\
&\quad + \frac{1}{2} \sum_{j=1}^n \varepsilon_j e_j \cdot (R^{\mathcal{Z}}(e_j, \nu)\nu) \cdot \varphi \\
&= -\frac{1}{2}\nu \cdot \text{Ric}^{\mathcal{Z}}(\nu) \cdot \varphi + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi + \mathfrak{D}^W \varphi + \frac{1}{2} \text{ric}^{\mathcal{Z}}(\nu, \nu) \varphi \\
&= \mathfrak{D}^W \varphi - \frac{1}{2} \sum_{j=1}^n \varepsilon_j \text{ric}^{\mathcal{Z}}(\nu, e_j) \nu \cdot e_j \cdot \varphi + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi. \quad (20)
\end{aligned}$$

The Codazzi-Mainardi equation for $X, Y, V \in TM$ is given by $\langle R^{\mathcal{Z}}(X, Y)V, \nu \rangle = \langle (\nabla_X^M W)(Y), V \rangle - \langle (\nabla_Y^M W)(X), V \rangle$. Thus,

$$\begin{aligned}
\text{ric}^{\mathcal{Z}}(\nu, X) &= \sum_{j=1}^n \varepsilon_j \langle R^{\mathcal{Z}}(X, e_j)e_j, \nu \rangle \\
&= \sum_{j=1}^n \varepsilon_j \left(\langle (\nabla_X^M W)(e_j), e_j \rangle - \langle (\nabla_{e_j}^M W)(X), e_j \rangle \right) \\
&= \text{tr}(\nabla_X^M W) - \langle \text{div}^M(W), X \rangle.
\end{aligned}$$

Plugging this into (20) we get

$$\begin{aligned}
i^{-s}[\nabla_{\nu}^{\Sigma \mathcal{Z}}, \tilde{D}] \varphi &= \mathfrak{D}^W \varphi - \frac{1}{2} \sum_{j=1}^n \varepsilon_j \left(\text{tr}(\nabla_{e_j}^M W) - \langle \text{div}^M(W), e_j \rangle \right) \nu \cdot e_j \cdot \varphi \\
&\quad + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi. \\
&= \mathfrak{D}^W \varphi - \frac{1}{2} \sum_{j=1}^n \varepsilon_j e_j (\text{tr}(W)) \nu \cdot e_j \cdot \varphi + \frac{1}{2}\nu \cdot \text{div}^M(W) \cdot \varphi \\
&\quad + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi. \\
&= \mathfrak{D}^W \varphi - \frac{n}{2}\nu \cdot \text{grad}^M(H) \cdot \varphi + \frac{1}{2}\nu \cdot \text{div}^M(W) \cdot \varphi + \frac{i}{2}\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \varphi.
\end{aligned}$$

4 The generalized cylinder on semi-Riemannian Spin^c manifolds

Let M be an n -dimensional smooth manifold and g_t a smooth 1-parameter family of semi-Riemannian metrics on M , $t \in I$ where $I \subset \mathbb{R}$ is an interval. We define the generalized cylinder by

$$\mathcal{Z} := I \times M,$$

with semi-Riemannian metric $g_{\mathcal{Z}} := \langle \cdot, \cdot \rangle = dt^2 + g_t$. The generalized cylinder is an $(n+1)$ -dimensional semi-Riemannian manifold of signature $(r+1, s)$ if the signature of g_t is (r, s) .

Proposition 4.1 *There is a 1-1-correspondence between the Spin^c structures on M and that on \mathcal{Z} .*

Proof: As explained in Section 3, Spin^c structures on \mathcal{Z} can be restricted to Spin^c structures on M . Conversely, given a Spin^c structure on M it can be pulled back to $I \times M$ via the projection $pr_2 : I \times M \longrightarrow M$ yields a Spin^c structure on \mathcal{Z} . In fact, the pull back of the $\text{Spin}^c(r, s)$ -principal bundle $P_{\text{Spin}^c}M$ on M gives rise to a $\text{Spin}^c(r, s)$ -principal bundle on \mathcal{Z} denoted by $P_{\text{Spin}^c}\mathcal{Z}$

$$\begin{array}{ccc} P_{\text{Spin}^c}\mathcal{Z} & \longrightarrow & P_{\text{Spin}^c}M \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{Z} = I \times M & \longrightarrow & M \end{array}$$

Enlarging the structure group via the embedding $\text{Spin}^c(r, s) \hookrightarrow \text{Spin}^c(r+1, s)$, which covers the standard embedding

$$\begin{aligned} \text{SO}(r, s) \times \mathbb{S}^1 & \hookrightarrow \text{SO}(r+1, s) \times \mathbb{S}^1 \\ (a, z) & \mapsto \left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, z \right), \end{aligned}$$

gives a $\text{Spin}^c(r+1, s)$ -principal fibre bundle on \mathcal{Z} , denoted also by $P_{\text{Spin}^c}\mathcal{Z}$. The pull back of the line bundle L^M on M defining the Spin^c structure on M , gives a line bundle $L^{\mathcal{Z}}$ on \mathcal{Z} such that the following diagram commutes

$$\begin{array}{ccc} L^{\mathcal{Z}} = pr_2^*(L^M) & \longrightarrow & L^M \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{Z} = I \times M & \longrightarrow & M \end{array} .$$

The line bundle $L^{\mathcal{Z}}$ on \mathcal{Z} and the $\text{Spin}^c(r+1, s)$ -principal fibre bundle $P_{\text{Spin}^c}\mathcal{Z}$ on \mathcal{Z} yields the Spin^c structure on \mathcal{Z} which restricts to the given Spin^c structure on M .

Remark 4.2 If M is a Spin^c Riemannian manifold and if we denote by $i\Omega^M$ the imaginary valued curvature on the line bundle L^M , we know that there exists a unique curvature 2-form, denoted by $i\Omega^{\mathcal{Z}}$, on the line bundle $L^{\mathcal{Z}} = pr_2^*(L^M)$, defined by $i\Omega^{\mathcal{Z}} = pr_2^*(i\Omega^M)$. Thus we have

$$\Omega^{\mathcal{Z}}(X, Y) = \Omega^M(X, Y) \quad \text{and} \quad \Omega^{\mathcal{Z}}(\nu, Y) = 0 \quad \text{for any } X, Y \in \Gamma(TM).$$

Proposition 4.3 [2] On a generalized cylinder $\mathcal{Z} = I \times M$ with semi-Riemannian metric $g^{\mathcal{Z}} = \langle \cdot, \cdot \rangle = dt^2 + g_t$ we define, in every $p \in M$ and $X, Y \in T_pM$, the first and second derivatives of g_t by

$$\dot{g}_t(X, Y) := \frac{d}{dt}(g_t(X, Y)) \quad \text{and} \quad \ddot{g}_t(X, Y) := \frac{d^2}{dt^2}(g_t(X, Y)).$$

Hence the following formulas hold:

$$\langle W(X), Y \rangle = -\frac{1}{2}\dot{g}_t(X, Y), \quad (21)$$

$$\begin{aligned} \langle R^{\mathcal{Z}}(U, V)X, Y \rangle &= \langle R^{M_t}(U, V)X, Y \rangle \\ &+ \frac{1}{4}(\dot{g}_t(U, X)\dot{g}_t(V, Y) - \dot{g}_t(U, Y)\dot{g}_t(V, X)), \end{aligned} \quad (22)$$

$$\langle R^{\mathcal{Z}}(X, Y)U, \nu \rangle = \frac{1}{2}((\nabla_Y^{M_t}\dot{g}_t)(X, U) - (\nabla_X^{M_t}\dot{g}_t)(Y, U)), \quad (23)$$

$$\langle R^{\mathcal{Z}}(X, \nu)\nu, Y \rangle = -\frac{1}{2}(\ddot{g}_t(X, Y) + \dot{g}_t(W(X), Y)), \quad (24)$$

where $X, Y, U, V \in T_pM$, $p \in M$.

5 The variation formula for the Dirac operator on Spin^c manifolds

First we give some facts about parallel transport on Spin^c manifolds along a curve c . We consider a Riemannian Spin^c manifold N , we know that there exists a unique correspondence which associates to a spinor field $\psi(t) = \psi(c(t))$ along a curve $c : I \rightarrow N$ another spinor field $\frac{D}{dt}\psi$ along c , called the covariant derivative of ψ along c , such that

$$\frac{D}{dt}(\psi + \varphi) = \frac{D}{dt}\psi + \frac{D}{dt}\varphi, \quad \text{for any } \psi \text{ and } \varphi \text{ along the curve } c,$$

$$\frac{D}{dt}(f\psi) = f\frac{D}{dt}\psi + \left(\frac{d}{dt}f\right)\psi, \quad \text{where } f \text{ is a differentiable function on } I,$$

$$\nabla_{\dot{c}(t)}^{\Sigma_N}\psi = \frac{D}{dt}\varphi, \quad \text{where } \varphi(t) = \psi(c(t)).$$

A spinor field ψ along a curve c is called parallel when $\frac{D}{dt}\psi(t) = 0$ for all $t \in I$. Now, if ψ_0 is a spinor at the point $c(t_0)$, $t_0 \in I$, ($\psi_0 \in \Sigma_{c(t_0)}N$) then there exists a unique parallel spinor φ along c , such that $\psi_0 = \varphi(t_0)$. The linear isometry $\tau_{t_0}^{t_1}$ defined by

$$\begin{aligned}\tau_{t_0}^{t_1} : \Sigma_{c(t_0)}N &\longrightarrow \Sigma_{c(t_1)}N \\ \psi_0 &\longrightarrow \varphi(t_1),\end{aligned}$$

is called the parallel transport along the curve c from $c(t_0)$ to $c(t_1)$. The basic property of the parallel transport on a Spin^c manifold is the following: Let ψ be a spinor field on a Riemannian Spin^c manifold N , $X \in \Gamma(TN)$, $p \in N$ and $c : I \longrightarrow N$ an integral curve through p , i.e., $c(t_0) = p$ and $\frac{d}{dt}c(t) = X(c(t))$, we have

$$(\nabla_X^{\Sigma N} \psi)_p = \frac{d}{dt} \left(\tau_{t_0}^{t_1}(\psi(t)) \right) \Big|_{t=t_0}. \quad (25)$$

Now, we consider g_t a smooth 1-parameter family of semi-Riemannian metrics on a Spin^c manifold M and the generalized cylinder $\mathcal{Z} = I \times M$ with semi-Riemannian metric $g^{\mathcal{Z}} = \langle \cdot, \cdot \rangle = dt^2 + g_t$. For $t \in I$ we denote by M_t the manifold (M, g_t) . Let us write “ \cdot ” for the Clifford multiplication on \mathcal{Z} and “ \bullet_t ” for that on M_t . Recall from Section 4 that Spin^c structures on M and \mathcal{Z} are in 1-1-correspondence and $\Sigma \mathcal{Z}|_{M_t} = \Sigma M_t$ as hermitian vector bundles if $n = r + s$ is even and $\Sigma^+ \mathcal{Z}|_{M_t} = \Sigma M_t$ if n is odd. For a given $x \in M$ and $t_0, t_1 \in I$, parallel transport $\tau_{t_0}^{t_1}$ on the generalized cylinder \mathcal{Z} along the curve $c : I \rightarrow I \times M, t \rightarrow (t, x)$ is given by

$$\tau_{t_0}^{t_1} : \Sigma_{c(t_0)}\mathcal{Z} \simeq \Sigma_x M_{t_0} \longrightarrow \Sigma_{c(t_1)}\mathcal{Z} \simeq \Sigma_x M_{t_1}.$$

This isomorphism satisfies

$$\begin{aligned}\tau_{t_0}^{t_1}(X \bullet_{t_0} \varphi) &= (\zeta_{t_0}^{t_1} X) \bullet_{t_1} (\tau_{t_0}^{t_1} \varphi), \\ \langle \tau_{t_0}^{t_1} \psi, \tau_{t_0}^{t_1} \varphi \rangle &= \langle \psi, \varphi \rangle,\end{aligned}$$

where $\zeta_{t_0}^{t_1} : T_{(x,t_0)}\mathcal{Z} \simeq T_x M_{t_0} \rightarrow T_{(x,t_1)}\mathcal{Z} \simeq T_x M_{t_1}$ is the parallel transport on \mathcal{Z} along the same curve c , $X \in T_x M_{t_0}$ and $\psi, \varphi \in \Sigma_x M_{t_0}$.

Theorem 5.1 *On a Spin^c manifold M , let g_t be a smooth 1-parameter family of semi-Riemannian metrics. Denote by D^{M_t} the Dirac operator of M_t , and $\mathfrak{D}^{\dot{g}_t} = \sum_{i,j=1}^n \varepsilon_i \varepsilon_j \dot{g}_t(e_i, e_j) e_i \bullet_t \nabla_{e_j}^{\Sigma M_t}$. Then for any smooth spinor field ψ on M_{t_0} we have*

$$\frac{d}{dt} \Big|_{t=t_0} \tau_{t_0}^{t_0} D^{M_t} \tau_{t_0}^t \psi = -\frac{1}{2} \mathfrak{D}^{\dot{g}_{t_0}} \psi + \frac{1}{4} \text{grad}^{M_{t_0}}(\text{tr}_{g_{t_0}}(\dot{g}_{t_0})) \bullet_{t_0} \psi - \frac{1}{4} \text{div}^{M_{t_0}}(\dot{g}_{t_0}) \bullet_{t_0} \psi.$$

Proof: The vector field $\nu := \frac{\partial}{\partial t}$ is spacelike of unit length and orthogonal to the hypersurfaces $M_t := \{t\} \times M$. Denote by W_t the Weingarten map of M_t with respect to

ν and by H_t the mean curvature. If X is a local coordinate field on M , then $\langle X, \nu \rangle = 0$ and $[X, \nu] = 0$. Thus

$$\begin{aligned} 0 &= d_\nu \langle X, \nu \rangle = \langle \nabla_\nu^{\mathbb{Z}} X, \nu \rangle + \langle X, \nabla_\nu^{\mathbb{Z}} \nu \rangle = \langle \nabla_X^{\mathbb{Z}} \nu, \nu \rangle + \langle X, \nabla_\nu^{\mathbb{Z}} \nu \rangle \\ &= -\langle W_t(X), \nu \rangle + \langle X, \nabla_\nu^{\mathbb{Z}} \nu \rangle = \langle X, \nabla_\nu^{\mathbb{Z}} \nu \rangle \end{aligned}$$

and differentiating $\langle \nu, \nu \rangle = 1$ yields $\langle \nu, \nabla_\nu^{\mathbb{Z}} \nu \rangle = 0$. Hence $\nabla_\nu^{\mathbb{Z}} \nu = 0$, i.e., for $x \in M$ the curves $t \mapsto (t, x)$ are geodesics parametrized by arclength. So the assumptions of Proposition 3.4 are satisfied for the foliation $(M_t)_{t \in I}$. By Remark 4.2, the commutator formula of Proposition 3.4 gives for a section φ of ΣM_t , (or $\Sigma^+ M_t$ if n is odd)

$$i^{-s}[\nabla_\nu^{\Sigma \mathbb{Z}}, D^{M_t}] \varphi = \mathfrak{D}^{W_t} \varphi - \frac{n}{2} \text{grad}^{M_t}(H_t) \bullet_t \varphi + \frac{1}{2} \text{div}^{M_t}(W_t) \bullet_t \varphi. \quad (26)$$

From Proposition 4.3 we deduce

$$\text{div}^{M_t}(W_t) = -\frac{1}{2} \text{div}^{M_t}(\dot{g}_t), \quad H_t = -\frac{1}{2n} \text{tr}_{g_t}(\dot{g}_t) \quad \text{and} \quad \mathfrak{D}^{W_t} = -\frac{1}{2} \mathfrak{D}^{\dot{g}_t}.$$

Thus (26) can be rewritten as

$$i^{-s}[\nabla_\nu^{\Sigma \mathbb{Z}}, D^{M_t}] \varphi = -\frac{1}{2} \mathfrak{D}^{\dot{g}_t} \varphi + \frac{1}{4} \text{grad}^{M_t}(\text{tr}_{g_t}(\dot{g}_t)) \bullet_t \varphi - \frac{1}{4} \text{div}^{M_t}(\dot{g}_t) \bullet_t \varphi. \quad (27)$$

Now if φ is parallel along the curves $t \mapsto (t, x)$, i.e., it is of the form $\varphi(t, x) = \tau_{t_0}^t \psi(t_0, x)$ for some spinor field ψ on M_{t_0} , then using (25) at $t = t_0$, the left hand side of (27) could be written as

$$\begin{aligned} i^{-s}[\nabla_\nu^{\Sigma \mathbb{Z}}, D^{M_t}] \varphi &= i^{-s} \nabla_\nu^{\Sigma \mathbb{Z}} D^{M_t} \varphi = i^{-s} \frac{d}{dt} \Big|_{t=t_0} \tau_t^{t_0} D^{M_t} \varphi \\ &= i^{-s} \frac{d}{dt} \Big|_{t=t_0} \tau_t^{t_0} D^{M_t} \tau_{t_0}^t \psi, \end{aligned} \quad (28)$$

which gives the variation formula for the Dirac operator.

Corollary 5.2 *Let (M^n, g) be a Spin^c Riemannian manifold, if we consider the family of metrics defined by $g_t = g + tk$, where k is a symmetric $(0, 2)$ -tensor, we have*

$$\frac{d}{dt} \Big|_{t=0} \tau_t^0 D^{M_t} \tau_0^t \psi = -\frac{1}{2} \mathfrak{D}^k \psi + \frac{1}{4} \text{grad}^M(\text{tr}_g(k)) \cdot \psi - \frac{1}{4} \text{div}^M(k) \cdot \psi, \quad (29)$$

where “ $\cdot = \bullet_{t_0=0}$ ” is the Clifford multiplication on M .

This formula has been proved in [4], Theorem 21 for spin Riemannian manifolds and in [2] for spin semi-Riemannian manifolds.

6 Energy-Momentum tensor on Spin^c manifolds

In this section we study the Energy-Momentum tensor on Spin^c Riemannian manifolds from a geometric point of view. We begin by giving the proofs of Proposition 1.1, Theorem 1.2 and Proposition 1.3.

Proof of Proposition 1.1 : Using Equation (29) we calculate

$$\begin{aligned}
 \left. \frac{d}{dt} \right|_{t=0} (\tau_t^0 D^{M_t} \tau_0^t \psi, \psi)_{g_t} &= \left. \frac{d}{dt} \right|_{t=0} (D^{M_t} \tau_0^t \psi, \tau_0^t \psi)_{g_t} \\
 &= -\frac{1}{2} (\mathfrak{D}^k \psi, \psi)_g \\
 &= -\frac{1}{2} \sum_{i,j} k(e_i, e_j) (e_i \cdot \nabla_{e_j}^{\Sigma M} \psi, \psi) \\
 &= -\frac{1}{2} \int_M \langle k, T_\psi \rangle dv_g.
 \end{aligned}$$

Proof of Theorem 1.2 : The Proof of this Theorem will be omitted since it is similar to the one given by Friedrich and Kim in [8] for spin manifolds.

Proof of Proposition 1.3 : Let ψ be any parallel spinor field on \mathcal{Z} . Then Equation (17) yields

$$\nabla_X^{\Sigma M} \varphi = \frac{1}{2} W(X) \bullet \varphi. \quad (30)$$

Let (e_1, \dots, e_n) be a positively oriented local orthonormal basis of TM . For $j = 1, \dots, n$ we have

$$\nabla_{e_j}^{\Sigma M} \varphi = \frac{1}{2} \sum_{k=1}^n W_{jk} e_k \bullet \varphi.$$

Taking Clifford multiplication by e_i and the scalar product with φ , we get

$$\text{Re}(e_i \bullet \nabla_{e_j}^{\Sigma M} \varphi, \varphi) = \frac{1}{2} \sum_{k=1}^n W_{jk} \text{Re}(e_i \bullet e_k \bullet \varphi, \varphi).$$

Since $\text{Re}(e_i \bullet e_k \bullet \varphi, \varphi) = -\delta_{ik} |\varphi|^2$, it follows, by the symmetry of W

$$\text{Re}(e_i \bullet \nabla_{e_j}^{\Sigma M} \varphi + e_j \bullet \nabla_{e_i}^{\Sigma M} \varphi, \varphi) = -W_{ij} |\varphi|^2.$$

Therefore, $2\ell^\varphi = -W$. Using Equation (18) it is easy to see that φ is an eigenspinor associated with the eigenvalue $-\frac{n}{2}H$ of \tilde{D} . Since $\text{Scal}^{\mathcal{Z}} = \text{Scal}^M + 2 \text{ric}^{\mathcal{Z}}(\nu, \nu) - n^2 H^2 + |W|^2$ we get

$$\begin{aligned}
 \frac{1}{4} (\text{Scal}^M - c_n |\Omega^M|) + |T^\varphi|^2 &= \frac{1}{4} (\text{Scal}^{\mathcal{Z}} - 2 \text{ric}^{\mathcal{Z}}(\nu, \nu) - c_n |\Omega^M|) + n^2 \frac{H^2}{4} \\
 &= n^2 \frac{H^2}{4},
 \end{aligned}$$

hence M satisfies the equality case in (3) if and only if (6) holds.

Corollary 6.1 *Under the same conditions as Proposition 1.3, if $n = 2$ or 3 , the hypersurface M satisfies the equality case in (3) if $\text{Ric}^{\mathcal{Z}}(\nu) = 0$ and $\text{Scal}^{\mathcal{Z}} \geq 0$.*

Proof: Since \mathcal{Z} has a parallel spinor, we have (see [7])

$$|\text{Ric}^{\mathcal{Z}}(\nu)| = |\nu \lrcorner \Omega^{\mathcal{Z}}|, \quad (31)$$

$$i(Y \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi = \text{Ric}^{\mathcal{Z}}(Y) \cdot \psi \text{ for every } Y \in \Gamma(\Sigma \mathcal{Z}). \quad (32)$$

For $Y = e_j$ in Equation (32) then taking Clifford multiplication by e_j and summing from $j = 1, \dots, n+1$, we get

$$i \sum_{j=1}^{n+1} e_j \cdot (e_j \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi = \sum_{j=1}^{n+1} e_j \cdot \text{Ric}^{\mathcal{Z}}(e_j) \cdot \psi = -\text{Scal}^{\mathcal{Z}} \psi.$$

But $2 \Omega^{\mathcal{Z}} \cdot \psi = \sum_{j=1}^{n+1} e_j \cdot (e_j \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi$, hence we deduce that $\Omega^{\mathcal{Z}} \cdot \psi = i \frac{\text{Scal}^{\mathcal{Z}}}{2} \psi$. By (31) and (15) we obtain $\Omega^M \bullet \varphi = i \frac{\text{Scal}^{\mathcal{Z}}}{2} \varphi$. Since $n = 2$ or 3 we have $|\Omega^M| = \frac{\text{Scal}^{\mathcal{Z}}}{2}$ and Equation (6) is satisfied.

Corollary 6.2 *Under the same conditions as Proposition 1.3, if the restriction of the complex line bundle $L^{\mathcal{Z}}$ is flat, i.e., L^M is a flat complex line bundle ($\Omega^M = 0$), the hypersurface M is a limiting manifold for (3).*

Proof: Since $\Omega^M = 0$, Equation (15) yields $i \frac{\text{Scal}^{\mathcal{Z}}}{2} \varphi = \Omega^{\mathcal{Z}} \cdot \psi|_M = (\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \varphi$. But,

$$\begin{aligned} i(\nu \lrcorner \Omega^{\mathcal{Z}}) \bullet \varphi &= i(\nu \cdot (\nu \lrcorner \Omega^{\mathcal{Z}}) \cdot \psi)|_M = (\nu \cdot \text{Ric}^{\mathcal{Z}}(\nu) \cdot \psi)|_M \\ &= -\text{ric}^{\mathcal{Z}}(\nu, \nu) \varphi + \sum_{j=1}^n \text{ric}^{\mathcal{Z}}(\nu, e_j) e_j \bullet \varphi. \end{aligned} \quad (33)$$

Taking the real part of the scalar product of Equation (33) with φ yields $\frac{\text{Scal}^{\mathcal{Z}}}{2} = \text{ric}^{\mathcal{Z}}(\nu, \nu)$, hence Equation (6) is satisfied.

Now, let M be a Spin^c Riemannian manifold having a generalized Killing spinor field φ with a symmetric endomorphism F on the tangent bundle TM . As mentioned in the introduction, it is straightforward to see that $2T^\varphi(X, Y) = -\langle F(X), Y \rangle$. We will study these generalized Killing spinors when the tensor F is a Codazzi-Mainardi tensor, i.e., F satisfies

$$(\nabla_X^M F)(Y) = (\nabla_Y^M F)(X) \text{ for } X, Y \in \Gamma(TM). \quad (34)$$

For this, we give the following lemma whose proof will be omitted since it is similar to Lemma 7.3 in [2].

Lemma 6.3 [2] *Let g_t be a smooth 1-parameter family of semi-Riemannian metrics on a Spin^c manifold $(M^n, g = g_0)$ and let F be a field of symmetric endomorphisms of TM . We consider the metric $g_{\mathcal{Z}} = \langle \cdot, \cdot \rangle = dt^2 + g_t$ on \mathcal{Z} such that $g_t(X, Y) = g((\text{Id} - tF)^2 X, Y)$ for all vector fields X, Y on M . We have $\langle R^{\mathcal{Z}}(U, \nu)\nu, V \rangle = 0$ for all vector fields U, V tangent to M and if F satisfies the Codazzi-Mainardi equation then $\langle R^{\mathcal{Z}}(U, V)W, \nu \rangle = 0$ for all U, V and W on \mathcal{Z} .*

Proof of Theorem 1.4: We define $\psi_{(0,x)} := \varphi_x$ via the identification $\Sigma_x M \cong \Sigma_{(0,x)} \mathcal{Z}$ (resp. $\Sigma_{(0,x)}^+ \mathcal{Z}$ for n odd) and $\psi_{(t,x)} = \tau_0^t \psi_{(0,x)}$. By Equation (21), the endomorphism F is the Weingarten tensor of the immersion of $\{0\} \times M$ in \mathcal{Z} and hence by construction we have for all $X \in \Gamma(TM)$

$$\nabla_X^{\Sigma \mathcal{Z}} \psi|_{\{0\} \times M} = 0 \quad \text{and} \quad \nabla_\nu^{\Sigma \mathcal{Z}} \psi \equiv 0. \quad (35)$$

Since the tensor F satisfies the Codazzi-Mainardi equation, Lemma 6.3 yields $g_{\mathcal{Z}}(R^{\mathcal{Z}}(U, V)W, \nu) = 0$ for all U, V and $W \in \Gamma(\mathcal{Z})$ and $g_{\mathcal{Z}}(R^{\mathcal{Z}}(X, \nu)\nu, Y) = 0$ for all X and Y tangent to M . Hence $R^{\mathcal{Z}}(\nu, X) = 0$ for all $X \in \Gamma(TM)$. Let X be a fixed arbitrary tangent vector field on M . Using (10) and (35) we get

$$\nabla_\nu^{\Sigma \mathcal{Z}} \nabla_X^{\Sigma \mathcal{Z}} \psi = \mathcal{R}^{\Sigma \mathcal{Z}}(\nu, X)\psi = \frac{1}{2}R^{\mathcal{Z}}(X, \nu) \cdot \psi + \frac{i}{2}\Omega^{\mathcal{Z}}(X, \nu)\psi = 0.$$

Thus showing that the spinor field $\nabla_X^{\Sigma \mathcal{Z}} \psi$ is parallel along the geodesics $\mathbb{R} \times \{x\}$. Now (35) shows that this spinor vanishes for $t = 0$, hence it is zero everywhere on \mathcal{Z} . Since X is arbitrary, this shows that ψ is parallel on \mathcal{Z} .

Corollary 6.4 *Let (M^3, g) be a compact, oriented Riemannian manifold and φ an eigenspinor associated with the first eigenvalue λ_1 of the Dirac operator such that the Energy-Momentum tensor associated with φ is a Codazzi tensor. M is a limiting manifold for (3) if and only if the generalized cylinder \mathcal{Z}^4 , equipped with the Spin^c structure arising from the given one on M , is Kähler of positive scalar curvature and the immersion of M in \mathcal{Z} has constant mean curvature H .*

Proof: First, we should point out that every 3-dimensional compact, oriented, smooth manifold has a Spin^c structure. Now, if M^3 is a limiting manifold for (3), by Theorem 1.4, the generalized cylinder has a parallel spinor whose restriction to M is φ . Since \mathcal{Z} is a 4-dimensional Spin^c manifold having parallel spinor, \mathcal{Z} is Kähler [1]. Moreover, using (15), we have

$$\Omega^M \bullet \varphi = i \frac{\text{Scal}^{\mathcal{Z}}}{2} \varphi = i \frac{c_n}{2} |\Omega^M| \varphi,$$

so $\text{Scal}^{\mathcal{Z}} \geq 0$. Finally $H = \frac{1}{n} \text{tr}(W) = \frac{1}{n} \text{tr}(-2T^\varphi) = -\frac{2}{n} \lambda_1$, which is a constant. Now if the generalized cylinder is Kähler and M is a compact hypersurface of constant mean curvature H , thus M is compact hypersurface immersed in a Spin^c manifold having parallel spinor with constant mean curvature. Since $\text{Scal}^{\mathcal{Z}} \geq 0$ and $\nu \lrcorner \Omega^{\mathcal{Z}} = \text{Ric}^{\mathcal{Z}}(\nu) = 0$, Corollary 6.1 gives the result.

References

- [1] M. Atiyah, N. Hitchin and I. Singer *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London A 362 (1978), 425-461.
- [2] C. Bär, P. Gauduchon and A. Moroianu, *Generalized cylinders in semi-Riemannian spin manifolds*, Math. Z. 249, 545-580 (2005).
- [3] H. Baum, *Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten*, B. G. teubner Verlagsgesellschaft, Leipzig, 1981.
- [4] J. -P. Bourguignon and P. Gauduchon, *Spinors, opérateurs de Dirac et variations de métriques*, Commun. Math. Phys. 144 (1992), 581-599.
- [5] J. P. Bourguignon, O. Hijazi, J. L. Milhorat and A. Moroianu, *A spinorial approach to Riemannian and conformal geometry*, Monograph (In preparation).
- [6] T. Friedrich, *On the spinor representation of surfaces in Euclidean 3-spaces*, J. Geom. Phys. 28 (1998), 143-157.
- [7] T. Friedrich, *Dirac operator's in Riemannian Geometry*, Graduate studies in mathematics, Volume 25, American Mathematical Society.
- [8] T. Friedrich and E.C.Kim, *The Einstein-Dirac equation on Riemannian spin manifolds*, Journal of Geometry and Physics, Volume 33, Number 1, March 2000 , pp. 128-172(45).
- [9] T. Friedrich and E.C. Kim, *Some remarks on the Hijazi inequality and generalizations of the Killing equation for spinors*, J. Geom. Phys. 37 (2001), 1-14.
- [10] M.J. Gursky and C. Lebrun, *Yamabe invariants and Spin^c structures*, Indiana University, May 1999.
- [11] N. Ginoux, *The Dirac spectrum*, Lect. Notes in Math. 1976, Springer 2009.
- [12] G. Habib, *Energy-Momentum tensor on foliations*, J. Geom. Phys. 57 (2007), 2234-2248.
- [13] M. Herzlich et A. Moroianu, *Generalized Killing spinors and conformal eigenvalue estimates for Spin^c manifold*, Annals of Global Analysis and Geometry 17: 341-370, 1999.
- [14] O. Hijazi, *Lower bounds for the eigenvalues of the Dirac operator*, Journal of Geometry and Physics, 16 (1995), 27-38.
- [15] O. Hijazi, *Spectral properties of the Dirac operator and geometrical structures*, Proceedings of the summer school on geometric methods in quantum field theory, Villa de Leyva, Colombia, July 12-30, 1999, world Scientific 2001.

- [16] O. Hijazi, S. Montiel and X. Zhang, *Dirac operator on embedded hypersurfaces*, Math. Res. Lett. 8 (2001), 195-208.
- [17] O. Hijazi, S. Montiel and F. Urbano, *Spin^c geometry of Kähler manifolds and the Hodge Laplacian on minimal Lagrangian submanifolds*, Math. Z. 253, Number 4 (2006) 821-853.
- [18] O. Hijazi, S. Montiel and X. Zhang, *Dirac Operator on Embedded Hypersurfaces*, Math. Res. Letters 8, (2001) 195-208.
- [19] M. A. Lawn, Thèse de troisième cycle, *Méthodes spinorielles et géométrie para-complexe et para-quaternionique en théorie des sous-variétés*, 2006.
- [20] H. B. Lawson and M. L. Michelson, *Spin Geometry*, Princeton University press, Princeton, New Jersey, 1989.
- [21] A. Moroianu, *Parallel and Killing spinors on Spin^c manifolds*, Commun. Math. Phys. 187 (1997), 417-428.
- [22] S. Montiel, *Using spinors to study submanifolds*, Roma 2004 - Nancy 2005.
- [23] B. Morel, *Eigenvalue estimates for the Dirac-Schrödinger operators*, Journal of Geometry and Physics. 38 (2001), 1-18.
- [24] B. Morel, *Tenseur d'impulsion d'énergie et géométrie spinorielle extrinsèque*, Ph.D. thesis, Institut Élie Cartan, 2002.
- [25] R. Nakad, *Lower bounds for the eigenvalues of the Dirac operator on Spin^c manifolds*, to appear in Journal of Geometry and Physics.
- [26] B. O'Neill, *Semi-Riemannian geometry*, Academic Press, New York, London, 1983.