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Time Reversed Absorbing Condition: Application to inverse problem

F. Assous*, M. Kray†, F. Nataf‡, E. Turkel‡

Abstract

The aim of this paper is to introduce the time reversed absorbing conditions (*TRAC*) in time reversal methods. They enable to “recreate the past” without knowing the source which has emitted the signals that are back-propagated. We present two applications in inverse problems: the reduction of the size of the computational domain and the determination of the location and volume of an unknown inclusion from boundary measurements. The method does not rely on any *a priori* knowledge of the physical properties of the inclusion. Numerical tests on the wave and Helmholtz equations illustrate the efficiency of the method which proves to be very robust with respect to noise in the data.

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1 Introduction

Since the seminal paper by Fink et al. [FWCM91], time reversal is a subject of very active research. The main idea is to take advantage of the reversibility of wave propagation phenomena such as in acoustics or electromagnetism in a non dissipative unknown medium to back-propagate signals to the sources that emitted them. The initial experiment, see [FWCM91], was to refocus very precisely a recorded signal after passing through a barrier consisting of randomly

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distributed metal rods. The remarkable feature of this experiment is the concrete possibility to focus precisely a signal after it has crossed random barriers and even without knowing its location. There have been numerous applications of this physical principle, see [Fin09] and references therein. The first mathematical analysis can be found in [BF02] and [BPZ02].

An interesting possibility is to “recreate the past” in a medium from time-reversed boundary measurements. As shown experimentally in [dRF02], it is necessary to know the source that emitted the signals to overcome the diffraction limit. The same difficulty was pointed out in [LMF⁺06] when numerically studying the initial instants of an earthquake by sending back long period time-reversed seismograms.

In this paper, we introduce a new method that enables to “recreate the past” without knowing the source which has emitted the signals that will be back-propagated. This is made possible by using time reversed absorbing conditions (*TRAC*) at the expense of removing a small region enclosing the source. This technique has at least two applications in inverse problems:

1. the reduction of the size of the computational domain by redefining the reference surface on which the receivers appear to be located
2. the possible reconstruction of the shape of an unknown inclusion from boundary measurements

The first application is reminiscent of the redatuming method introduced in [Ber79]. In our case, we use the original propagation equation and not a paraxial or parabolic approximation to it. This extends the domain of validity of the redatuming approach. Concerning the second application there is a huge literature that deals with this inverse problem. We mention the MUSIC algorithm [The92] and its application to imaging [LD03], the sampling methods first introduced in [CK96] see the review paper [CCM00] and references therein and the DORT method [PMSF96]. Mathematical analysis of this kind of approach can be found in [CK98]. These methods were developed in the time-harmonic domain for impenetrable objects. The *TRAC* method is designed in both the time-dependent and harmonic domains and does not rely on any *a priori* knowledge of the physical properties of the inclusion. It works both for impenetrable and translucent inclusions.

The outline of the paper is as follows. In subsections 2.1 and 2.2 we introduce the principle of the *TRAC* method both in the time dependent and harmonic domains. We present in § 2.3 two applications of the method in the context of inverse problems. The end of section 2 is devoted to the explicit derivation of the method to the wave and Maxwell equations in both the time dependent and harmonic cases. In section 3 we give numerical applications of the *TRAC* method for the wave equation and then the Helmholtz equation. We propose various criteria for applying our method to inverse problems. We investigate the robustness with respect to the magnitude of the noise in the data and its ability to handle penetrable inclusions.

2 The *TRAC* method and applications

2.1 The *TRAC* method in the time dependent case

We consider an incident wave U^I impinging on an inclusion D characterized by different physical properties from the surrounding medium. We denote by ∂D the boundary of this inclusion. The total field U^T can be decomposed into the incident and scattered field, so $U^T := U^I + U^S$. We consider the problem in d dimensions $d = 1, 2, 3$ and assume that the total field satisfies a linear hyperbolic equation (or system of equations) denoted by \mathcal{L} , that can be written

$$\mathcal{L}(U^T) = 0 \text{ in } \mathbf{R}^d \tag{1}$$

together with zero initial conditions, which will be detailed later. The scattering field U^S has to satisfy a radiation condition at the infinity to ensure the uniqueness of the solution. For the wave equation we have the Sommerfeld radiation condition, or the Silver-Müller radiation condition for the Maxwell equations (see subsections below).

Let Ω denote a bounded domain that surrounds D with Γ_R as its boundary. We assume that the incident wave U^I has compact support in time and space and that after a time T_f the total field U^T vanishes in the bounded domain Ω . Let V be a field that satisfies the wave equation.

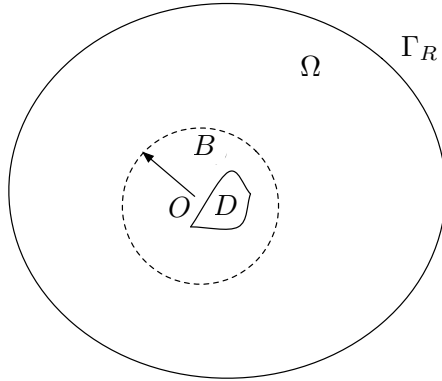


Figure 1: Geometry

We denote by V_R the corresponding time reversed field that also satisfies the same physical equation. For instance, the time reversed solution u_R^T of the wave equation (11) is defined by $u_R^T := u^T(T_f - t, \vec{x})$ see § 2.4. For the Maxwell equation (22), the time reversed electromagnetic field is defined by $(\mathbf{E}_R^T, \mathbf{B}_R^T) := (-\mathbf{E}^T(T_f - t, \vec{x}), \mathbf{B}^T(T_f - t, \vec{x}))$, see § 2.6. Similar definitions will be used for the incident and scattered fields.

Our first aim is to derive a boundary value problem (BVP) whose solution is the time reversed field. For this purpose, we assume that we have recorded the value of the total field U^T on the boundary Γ_R that encloses the domain Ω . However, we don't know the physical properties of the inclusion i.e. we don't know the exact form of the operator \mathcal{L} in the inclusion D . Moreover, we don't know the exact location of the inclusion D . The only things we know are the physical properties of the surrounding medium, in other words the operator \mathcal{L} outside the inclusion D which is assumed to be a constant coefficient operator denoted \mathcal{L}_0 . Thus, U_R^T satisfies the following equation

$$\mathcal{L}_0(U_R^T) = 0 \text{ in } (0, T_f) \times \Omega \setminus D \quad (2)$$

We impose Dirichlet boundary conditions on Γ_R equal to the time reversal of the recorded fields and zero initial conditions. The key point is that we lack a boundary condition on the boundary of the inclusion ∂D in order to define a well-posed BVP on the time reversed field U_R^T in $\Omega \setminus D$. Usually in practical applications, the shape and/or location of the inclusion D is not known nor the boundary condition (hard or soft object) it satisfies.

To overcome all these difficulties, the classical approach consists in solving the problem (2) in the entire domain Ω , assuming that there is no object D . Denote by W_R^T this "approximate" time reversed solution, we have in the entire domain Ω :

$$\mathcal{L}_0(W_R^T) = 0 \text{ in } (0, T_f) \times \Omega \quad (3)$$

with Dirichlet boundary conditions on Γ_R equal to the time reversal of the recorded fields and zero initial conditions. One can readily verify that this approximate time reversed solution W_R^T differs from U_R^T .

Remark 1 *Another possibility is to try to reconstruct the reversed scattered field U_R^S instead of the total reverse field U_R^T . In this case, the classical approach consists in solving*

$$\mathcal{L}_0(W_R^S) = 0 \text{ in } (0, T_f) \times \Omega$$

with Dirichlet boundary conditions on Γ_R equal to the time reversal of the recorded fields minus the time reversed incident field and zero initial conditions. It is easy to check that this approximate time reversed solution W_R^S differs as well from U_R^S .

To write a boundary value problem satisfied by U_R^T without knowing the physical properties of the inclusion D nor its exact location, we introduce B a subdomain enclosing the inclusion D (see Figure 1). Then, we have to determine a specific boundary condition for U_R^T on the boundary ∂B so that the solution to this problem will coincide with U_R^T in the restricted domain $\Omega \setminus B$.

In order to derive this boundary condition, we note that $\mathcal{L}_0(U^I) = 0$ so that the scattered wave U^S satisfies

$$\begin{cases} \mathcal{L}_0(U^S) = 0 \text{ in } \mathbf{R}^d \setminus D \\ U^S \text{ satisfies a radiation condition at } \infty \end{cases} \quad (4)$$

and zero initial conditions. We make use of the property that the surrounding medium $\Omega \setminus D$ is homogeneous. As a first step, we look for a relation satisfied by U^S on ∂B . Absorbing boundary conditions (ABC) e.g. [EM77] and [BT80] construct accurate approximations to a perfectly absorbing boundary condition. We denote by ABC an exact absorbing boundary condition, that can be formally written as

$$\text{ABC}(U^S) = 0 \text{ on } \partial B. \quad (5)$$

Since $U^T = U^I + U^S$, we have $\text{ABC}(U^T - U^I) = 0$ or equivalently $\text{ABC}(U^T) = \text{ABC}(U^I)$. Our main ingredient is to "time reverse" this relation into a relation that we will denote

$$\text{TRAC}(U_R^T) = g(U^I) \text{ on } \partial B \quad (6)$$

where $g(U^I)$ denotes a known function which is related to the time reversal of $\text{ABC}(U^I)$. The way to design TRAC and $g(U^I)$ will be specified in the subsequent sections depending on the considered problem. We shall see that the absorbing boundary condition ABC is different from its time reverse companion TRAC that will be referred to as a *TRAC* (Time Reversed Absorbing Condition). To summarize, the problem satisfied by U_R^T in the restricted domain $\Omega \setminus B$ can be written:

$$\begin{cases} \mathcal{L}_0(U_R^T) = 0 \text{ in } (0, T_f) \times \Omega \setminus B \\ \text{TRAC}(U_R^T) = g(U^I) \text{ on } \partial B \end{cases} \quad (7)$$

together with Dirichlet boundary conditions on Γ_R equal to the time reversal of the recorded fields and zero initial conditions. By solving (7), we are able to reconstruct the total field U^T at any point of the domain $\Omega \setminus B$ and any time in $(0, T_f)$.

We shall illustrate our approach by deriving equation (7) from equation (4) for several classical examples: the wave equation and the Maxwell system. Note, that the same procedure can be applied to the elasticity system and non linear hyperbolic problems before a shock formation.

2.2 The *TRAC* method in the harmonic case

We consider the time-harmonic counterpart of problem (1) and denote by $\widehat{\mathcal{L}}$ the Fourier transform in time of the operator \mathcal{L} . The unknown total field $\mathbf{U}^T(\vec{x})$ is decomposed into the sum of an incident field $\mathbf{U}^I(\vec{x})$ and of a scattered field $\mathbf{U}^S(\vec{x})$. We have:

$$\widehat{\mathcal{L}}(\mathbf{U}^T) = 0 \text{ in } \mathbf{R}^d \tag{8}$$

$$\mathbf{U}^S(\vec{x}) := \mathbf{U}^T(\vec{x}) - \mathbf{U}^I(\vec{x}) \text{ satisfies a Sommerfeld condition at } \infty$$

In this context, the analog to the time reversal method is the phase conjugation technique, see [CM91]. Let \mathbf{V} be a field that satisfies the harmonic equation, we denote by \mathbf{V}_R the corresponding *harmonic time reversed* field that still satisfies the same harmonic equation. For instance, for the Helmholtz equation (19) we proceed in the following way. Let $v(t, \vec{x})$ be a time dependent real valued function solution to the wave equation and $v_R(t, \vec{x}) := v(-t, \vec{x})$ its associated time reversed function. Notice that since we are in the harmonic case, there is no notion of final time T_f as above. The Fourier transform in time of the above definition yields:

$$\widehat{v}_R(\omega, \vec{x}) = \int v(-t, \vec{x}) e^{-i\omega t} dt = \int v(t, \vec{x}) e^{i\omega t} dt = \overline{\int v(t, \vec{x}) e^{-i\omega t} dt} = \overline{\widehat{v}(\omega, \vec{x})}.$$

This identity shows that the phase conjugation is the Fourier transform of the time reversal process in the case of the Helmholtz equation. For the harmonic Maxwell equation (29), we proceed in a slightly different way. Recall that for the Maxwell equation (22), the time reversed electromagnetic field was defined by $(\mathbf{E}_R, \mathbf{B}_R) := (-\mathbf{E}(-t, \vec{x}), \mathbf{B}(-t, \vec{x}))$, see § 2.6. The Fourier transform in time of the above definition yields:

$$(\widehat{\mathbf{E}}_R(\omega, \vec{x}), \widehat{\mathbf{B}}_R(\omega, \vec{x})) = (-\overline{\widehat{\mathbf{E}}(\omega, \vec{x})}, \overline{\widehat{\mathbf{B}}(\omega, \vec{x})})$$

Thus, the classical “time reversal” process in the harmonic regime applied to (8) amounts to conjugate the recorded data (and in the Maxwell case to take the opposite of the electric field) and solve the following harmonic time reversed problem:

$$\begin{aligned} \widehat{\mathcal{L}}_0(\mathbf{W}_R^T) &= 0 \text{ in } \Omega \\ \mathbf{W}_R^T &= \mathbf{U}_R^T \text{ on } \Gamma_R \end{aligned} \tag{9}$$

which is the equivalent to (3) in the harmonic regime. Hence, following the same steps as above for deriving equation (7) from equation (1), we get:

$$\begin{cases} \widehat{\mathcal{L}}_0(\mathbf{U}_R^T) = 0 \text{ in } \Omega \setminus B \\ \text{TRAC}(\mathbf{U}_R^T) = \widehat{g(\mathbf{U}^I)} \text{ on } \partial B \end{cases} \tag{10}$$

together with Dirichlet boundary conditions on Γ_R equal to the harmonic time reverse of the recorded fields.

We shall illustrate our approach by deriving equation (10) for several classical examples: the Helmholtz equation, the time harmonic Maxwell system. The same process can be applied to the time harmonic elasticity system.

2.3 Applications of the *TRAC* method

A first application to inverse problems is a reduction of the size of the computational domain. Since, we are able to compute the total field U^T on an artificial boundary ∂B , it is equivalent to having the boundary Γ_R moved to ∂B . As a consequence, the cost of the forward problems involved in the identification algorithm is reduced.

A second application is to localize the inclusion by a trial and error procedure. At the initial time $t = 0$, the total field U^T is zero. Thus, if B encloses the inclusion D , U_R^T which is exactly the time reversal of U^T is zero at the final time T_f that corresponds to the initial time of the physical problem (1). Conversely, if after solving equation (7), U_R^T is not zero at the final time T_f , it proves that the assumption that D is a subset of B is false. Hence, by playing with the location and size of the subdomain B , it could be possible to determine the location and volume of the inclusion D . This technique will be the core subject of section 3.

2.4 The wave equation

We first consider the case of the three dimensional acoustic wave equation with a propagation speed c which is constant outside an inclusion D . We denote the total field as u^T which is decomposed as above into an incident u^I and a scattered field u^S . With these notations, equation (4) reads

$$\left\{ \begin{array}{l} \frac{\partial^2 u^S}{\partial t^2} - c^2 \Delta u^S = 0 \text{ in } \mathbf{R}^3 \setminus D \\ u^S(t, \vec{x}) \text{ satisfies a Sommerfeld condition at } \infty \\ \text{homogeneous initial conditions} \end{array} \right. \quad (11)$$

This problem is under-determined since we don't specify any boundary condition on ∂D . Thus u^S is not necessarily zero. We define the time reverse total field u_R^T by $u_R^T(t, \cdot) := u^T(T_f - t, \cdot)$. Since, the wave equation involves only second order time derivatives, this definition ensures that the reverse field u_R^T is a solution to the wave equation as well. In order to derive the absorbing boundary condition (5), we consider, for the sake of simplicity, that the subdomain B is a ball of radius ρ centered at the origin denoted B_ρ . Let r be the radial coordinate, we consider the first order Bayliss-Turkel *ABC BT*¹ [BT80] :

$$\frac{\partial u^S}{\partial t} + c \frac{\partial u^S}{\partial r} + c \frac{u^S}{r} = 0 \quad (12)$$

Numerical applications of section 3 will show that even in two dimensions imposing

$$\frac{\partial u^S}{\partial t} + c \frac{\partial u^S}{\partial r} + c \frac{u^S}{2r} = 0$$

is reasonable. For ease of computations, we use an equivalent form of equation (12):

$$\frac{\partial}{\partial t}(r u^S) + c \frac{\partial}{\partial r}(r u^S) = 0. \quad (13)$$

The above equation is the counterpart of equation (5) for the wave equation. We next wish to express (6) explicitly for the time reversed equation (13). The total field $u^T = u^I + u^S$ satisfies

$$\frac{\partial}{\partial t}(r u^T(t, \cdot)) + c \frac{\partial}{\partial r}(r u^T(t, \cdot)) = \frac{\partial}{\partial t}(r u^I(t, \cdot)) + c \frac{\partial}{\partial r}(r u^I(t, \cdot))$$

Using $u_R^T(t, \cdot) = u^T(T_f - t, \cdot)$, we get

$$\left(-\frac{\partial}{\partial t}(r u_R^T) + c \frac{\partial}{\partial r}(r u_R^T)\right)|_{T_f-t} = \frac{\partial}{\partial t}(r u^I(t, \cdot)) + c \frac{\partial}{\partial r}(r u^I(t, \cdot))$$

or equivalently,

$$-\frac{\partial}{\partial t}(r u_R^T(t, \cdot)) + c \frac{\partial}{\partial r}(r u_R^T(t, \cdot)) = \left(\frac{\partial}{\partial t}(r u^I) + c \frac{\partial}{\partial r}(r u^I)\right)|_{T_f-t}$$

Note, that on ∂B_ρ , $\partial / \partial r = -\partial / \partial n$ where n is the outward normal to the restricted domain $\Omega \setminus B_\rho$. Multiplying by -1 , we get

$$\frac{\partial}{\partial t}(r u_R^T(t, \cdot)) + c \frac{\partial}{\partial n}(r u_R^T(t, \cdot)) = -\left(\frac{\partial}{\partial t}(r u^I) - c \frac{\partial}{\partial n}(r u^I)\right)|_{T_f-t} \quad (14)$$

Another way to write this boundary condition is to introduce the time reversed incident field

$$u_R^I(t, \cdot) := u^I(T_f - t, \cdot),$$

so that (14) can be expressed as

$$\frac{\partial}{\partial t}(r u_R^T(t, \cdot)) + c \frac{\partial}{\partial n}(r u_R^T(t, \cdot)) = \frac{\partial}{\partial t}(r u_R^I(t, \cdot)) + c \frac{\partial}{\partial n}(r u_R^I(t, \cdot)). \quad (15)$$

This expression is the counterpart of equation (6). Since $\partial r / \partial n = -1$, relation (15) can be rewritten as:

$$\frac{\partial}{\partial t}(u_R^T(t, \cdot)) + c \frac{\partial}{\partial n}(u_R^T(t, \cdot)) - c \frac{u_R^T(t, \cdot)}{r} = \frac{\partial}{\partial t}(u_R^I(t, \cdot)) + c \frac{\partial}{\partial n}(u_R^I(t, \cdot)) - c \frac{u_R^I(t, \cdot)}{r}. \quad (16)$$

So we define the boundary condition TRAC by:

$$\text{TRAC}(u_R^T) := \frac{\partial}{\partial t}(u_R^T(t, \cdot)) + c \frac{\partial}{\partial n}(u_R^T(t, \cdot)) - c \frac{u_R^T(t, \cdot)}{r}. \quad (17)$$

Note, that due to the minus sign before the term u_R^T/r , the TRAC (16) is *not* the BT^1 absorbing boundary condition. The time reversed problem analog to (7) reads:

$$\left\{ \begin{array}{l} \frac{\partial^2 u_R^T}{\partial t^2} - c^2 \Delta u_R^T = 0 \text{ in } (0, T_f) \times \Omega \setminus B_\rho \\ \text{TRAC}(u_R^T) = \text{TRAC}(u_R^I) \text{ on } \partial B_\rho \\ u_R^T(t, \vec{x}) = u^T(T_f - t, \vec{x}) \text{ on } \Gamma_R \\ \text{zero initial conditions} \end{array} \right. \quad (18)$$

The TRAC is not only not the standard BT^1 ABC but also has an “anti absorbing” term $(-cu_R^T/r)$. A natural concern arises about the well-posedness of BVP (18). Although we have not developed a general theory, we prove an energy estimate for this problem in a special geometry, see [AKNT]. Moreover, in many computations we have never encountered stability problems, see § 3 where a numerical procedure for inclusion identification will be deduced from this formulation.

2.5 The Helmholtz equation

Denote by ω the dual variable of t for the Fourier transform in time. The total field u^T can be decomposed into an incident and scattered field, i.e. $u^T := u^I + u^S$. The Helmholtz equation is derived by taking the Fourier transform in time of the wave equation, yielding:

$$\begin{cases} -\omega^2 u^T - c^2 \Delta u^T = 0 \text{ in } \mathbf{R}^3 \\ (u^T - u^I) \text{ satisfies a Sommerfeld radiation condition at } \infty \end{cases} \quad (19)$$

The frequency is $\omega = 5$ and the sound speed is $c_0 = 1$, so the wavelength is 0.2 i.e. $k = \frac{\omega}{c_0} = 5$. The size of the domain of computation is 5 wavelengths while the radius of the inclusion is 2 wavelengths. The radius of the artificial ball can be as small as one wavelength to avoid difficulties because of the absorbing boundary in two dimensions with a small radius.

Recall that our aim is to write a *BVP* whose solution is the conjugate of u^T . Following § 2.2, it is sufficient to take the Fourier transform of equation (18) to get:

$$\begin{cases} -\omega^2 u_R^T - c^2 \Delta u_R^T = 0 \text{ in } \Omega \setminus B_\rho \\ i\omega u_R^T + c \frac{\partial u_R^T}{\partial n} - c \frac{u_R^T}{r} = i\omega u_R^I + c \frac{\partial u_R^I}{\partial n} - c \frac{u_R^I}{r} \text{ on } \partial B_\rho \\ u_R^T(\vec{x}) = \bar{u}^T(\vec{x}) \text{ on } \Gamma_R \end{cases} \quad (20)$$

We emphasize that the time reversed absorbing condition *TRAC*

$$\text{TRAC}(u_R^T) := i\omega u_R^T + c \frac{\partial u_R^T}{\partial n} - c \frac{u_R^T}{r} \quad (21)$$

is not the *BT*¹ absorbing boundary condition

$$i\omega u^T + c \frac{\partial u^T}{\partial n} + c \frac{u^T}{r}.$$

2.6 The Maxwell equation

As a second application, we consider the case of the three dimensional Maxwell equations. For the sake of simplicity, we assume that outside the inclusion D the medium is linear, homogeneous and isotropic with a constant speed of light denoted by c . Denote by \mathbf{E} the electric field and by \mathbf{B} the magnetic induction. Denote the total field by $(\mathbf{E}^T, \mathbf{B}^T)$ which is decomposed as above into an incident $(\mathbf{E}^I, \mathbf{B}^I)$ and a scattered field $(\mathbf{E}^S, \mathbf{B}^S)$. With these notations, the counterpart of equation (4) reads

$$\begin{cases} \frac{\partial \mathbf{E}^S}{\partial t} - c^2 \nabla \times \mathbf{B}^S = 0, \text{ in } \mathbf{R}^3 \setminus D, \\ \frac{\partial \mathbf{B}^S}{\partial t} + \nabla \times \mathbf{E}^S = 0, \text{ in } \mathbf{R}^3 \setminus D, \\ (\mathbf{E}^S(t, \vec{x}), \mathbf{B}^S(t, \vec{x})) \text{ satisfies satisfies a Silver - Muller radiation condition at } \infty \\ \text{and zero initial conditions.} \end{cases} \quad (22)$$

We introduce the time reversed solution

$$(\mathbf{E}_R^T(t, \vec{x}), \mathbf{B}_R^T(t, \vec{x})) := (-\mathbf{E}(T_f - t, \vec{x}), \mathbf{B}(T_f - t, \vec{x})). \quad (23)$$

Note, that the Maxwell system is a first order hyperbolic system, but is not invariant under a time reversal. For this reason, we multiply the electric field by minus one so that the reversed electromagnetic field $(\mathbf{E}_R^T(t, \vec{x}), \mathbf{B}_R^T(t, \vec{x}))$ solves the to build an absorbing boundary condition analog to (5). We still consider a subdomain B that is a ball of radius ρ centered at the origin and denoted B_ρ . We assume that the following approximate absorbing Silver-Muller boundary condition is reasonable:

$$(\mathbf{E}^S \wedge \nu) \wedge \nu + c \mathbf{B}^S \wedge \nu = 0 \quad (24)$$

where ν is the outward normal to the ball B_ρ . The above equation is the counterpart of equation (5) for the Maxwell equation. The next step is to derive an explicit expression for the time reverse of equation (24). This means that the total field satisfies

$$(\mathbf{E}^T \wedge \nu) \wedge \nu + c \mathbf{B}^T \wedge \nu = (\mathbf{E}^I \wedge \nu) \wedge \nu + c \mathbf{B}^I \wedge \nu \quad (25)$$

Using definition (23), we get

$$(-\mathbf{E}_R^T \wedge \nu) \wedge \nu + c \mathbf{B}_R^T \wedge \nu = \left((\mathbf{E}^I \wedge \nu) \wedge \nu + c \mathbf{B}^I \wedge \nu \right)_{T_f-t} \quad (26)$$

Note, that on ∂B_ρ , $n = -\nu$ and multiplying by -1 , we get

$$(\mathbf{E}_R^T \wedge n) \wedge n + c \mathbf{B}_R^T \wedge n = - \left((\mathbf{E}^I \wedge \nu) \wedge \nu + c \mathbf{B}^I \wedge \nu \right)_{T_f-t} \quad (27)$$

This expression is the counterpart of (6). Finally, the time reversed problem analog to (7) reads:

$$\left\{ \begin{array}{l} \frac{\partial \mathbf{E}_R^T}{\partial t} - c^2 \nabla \times \mathbf{B}_R^T = 0, \text{ in } \Omega \setminus B_\rho, \\ \frac{\partial \mathbf{B}_R^T}{\partial t} + \nabla \times \mathbf{E}_R^T = 0, \text{ in } \Omega \setminus B_\rho, \\ (\mathbf{E}_R^T \wedge n) \wedge n + c \mathbf{B}_R^T \wedge n = - \left((\mathbf{E}^I \wedge \nu) \wedge \nu + c \mathbf{B}^I \wedge \nu \right)_{T_f-t} \text{ on } \partial B_\rho \\ (\mathbf{E}_R^T(t, \vec{x}), \mathbf{B}_R^T(t, \vec{x})) = (-\mathbf{E}(T_f - t, \vec{x}), \mathbf{B}(T_f - t, \vec{x})) \text{ on } \Gamma_R \\ \text{and zero initial conditions} \end{array} \right. \quad (28)$$

The penultimate equation in the above system expresses that we have time reversed the data recorded on Γ_R .

2.7 Time harmonic Maxwell system

We again, denote by ω the dual variable of t for the Fourier transform in time. The total field $(\mathbf{E}^T(\vec{x}), \mathbf{B}^T(\vec{x}))$ can be decomposed into an incident and scattered field, $(\mathbf{E}^T(\vec{x}), \mathbf{B}^T(\vec{x})) := (\mathbf{E}^I(\vec{x}), \mathbf{B}^I(\vec{x})) + (\mathbf{E}^S(\vec{x}), \mathbf{B}^S(\vec{x}))$. The time harmonic Maxwell equation is written by taking the Fourier transform in time of the Maxwell equation:

$$\left\{ \begin{array}{l} i\omega \mathbf{E}^T - c^2 \nabla \times \mathbf{B}^T = 0, \text{ in } \mathbf{R}^3, \\ i\omega \mathbf{B}^T + \nabla \times \mathbf{E}^T = 0, \text{ in } \mathbf{R}^3, \\ (\mathbf{E}^T, \mathbf{B}^T) - (\mathbf{E}^I, \mathbf{B}^I) \text{ satisfies a Silver - Muller radiation condition at } \infty \end{array} \right. \quad (29)$$

Recall that our aim is to write a *BVP* whose solution is the harmonic time reverse of $(\mathbf{E}^T(\vec{x}), \mathbf{B}^T(\vec{x}))$. Following 2.2, it is sufficient to take the Fourier transform of equation (28) to get

$$\left\{ \begin{array}{l} i\omega \mathbf{E}_R^T - c^2 \nabla \times \mathbf{B}_R^T = 0, \text{ in } \Omega \setminus B_\rho, \\ i\omega \mathbf{B}_R^T + \nabla \times \mathbf{E}_R^T = 0, \text{ in } \Omega \setminus B_\rho, \\ (\mathbf{E}_R^T \wedge \mathbf{n}) \wedge \mathbf{n} + c \mathbf{B}_R^T \wedge \mathbf{n} = - \left((\overline{\mathbf{E}}^I \wedge \boldsymbol{\nu}) \wedge \boldsymbol{\nu} + c \overline{\mathbf{B}}^I \wedge \boldsymbol{\nu} \right) \text{ on } \partial B_\rho \\ (\mathbf{E}_R^T(\vec{x}), \mathbf{B}_R^T(\vec{x})) = (-\overline{\mathbf{E}}(\vec{x}), \overline{\mathbf{B}}(\vec{x})) \text{ on } \Gamma_R \end{array} \right. \quad (30)$$

3 Numerical applications of the *TRAC* method

In section 3.1, we describe the *TRAC* method to locate an inclusion in the two-dimensional case. We first consider the time-dependent case based on the wave equation whereas the harmonic case will be studied in 3.2 using the Helmholtz equation. As explained in § 2.1, the method does not rely on any *a priori* knowledge of the physical properties of the inclusion. We will treat, in the same manner, the cases of a hard, soft or translucent object. Moreover, the method will prove to be very robust with respect to the magnitude of the noise in the data. In the harmonic case, we compare the *TRAC* approach with the phase conjugation approach defined in (9).

3.1 The wave equation

We consider an inclusion D surrounded by a homogeneous and isotropic medium with a velocity of sound denoted by c_0 . The inclusion is illuminated by an incident field u^I . Equation (1) reads:

$$\left\{ \begin{array}{l} \frac{\partial^2 u^T}{\partial t^2} - c^2 \Delta u^T = 0 \text{ in } \mathbf{R}^2 \\ (u^T(t, \vec{x}) - u^I(t, \vec{x})) \text{ satisfies a Sommerfeld condition at } \infty \\ \text{zero initial conditions} \end{array} \right. \quad (31)$$

In order to create synthetic data, equation (31) is approximated by the FreeFem++ package [Hec10] with a finite element method in space. The scheme is a second order centered finite difference scheme in time so that it can be time reversed. The computational domain is truncated by using an absorbing boundary condition. The incident wave is simulated by the same procedure with a uniform velocity of sound, c_0 . We introduce a boundary Γ_R where the signal is recorded. The boundary Γ_R encloses a domain denoted Ω , see Fig. 1. The next stage of the method is to introduce a “trial” domain B_ρ and solve the time reversed problem (18) in $\Omega \setminus B_\rho$.

Recall, that the principle of the method (see § 2.3) is that when the ball B_ρ encloses the inclusion D then, at the final time of the reverse simulation, the solution must be equal to the initial condition of the forward problem i.e. zero. Conversely if at the end of the reverse simulation the field is not zero, it shows that the ball B_ρ does not enclose the inclusion D .

As a first test case, we consider an inclusion that is a soft disk and the ball is a disk of variable radius. In figure 2, we have several lines and three columns. Each column corresponds to a numerical time-reversed experiment and each line to a snapshot of the solution at a given time. The top line corresponds to the initial time for the time reversed problem that is $t = T_f$ for the forward problem. The last line is the solution at the final time of the reversed simulation which corresponds to the initial time $t = 0$ of the forward problem. In the left column, we

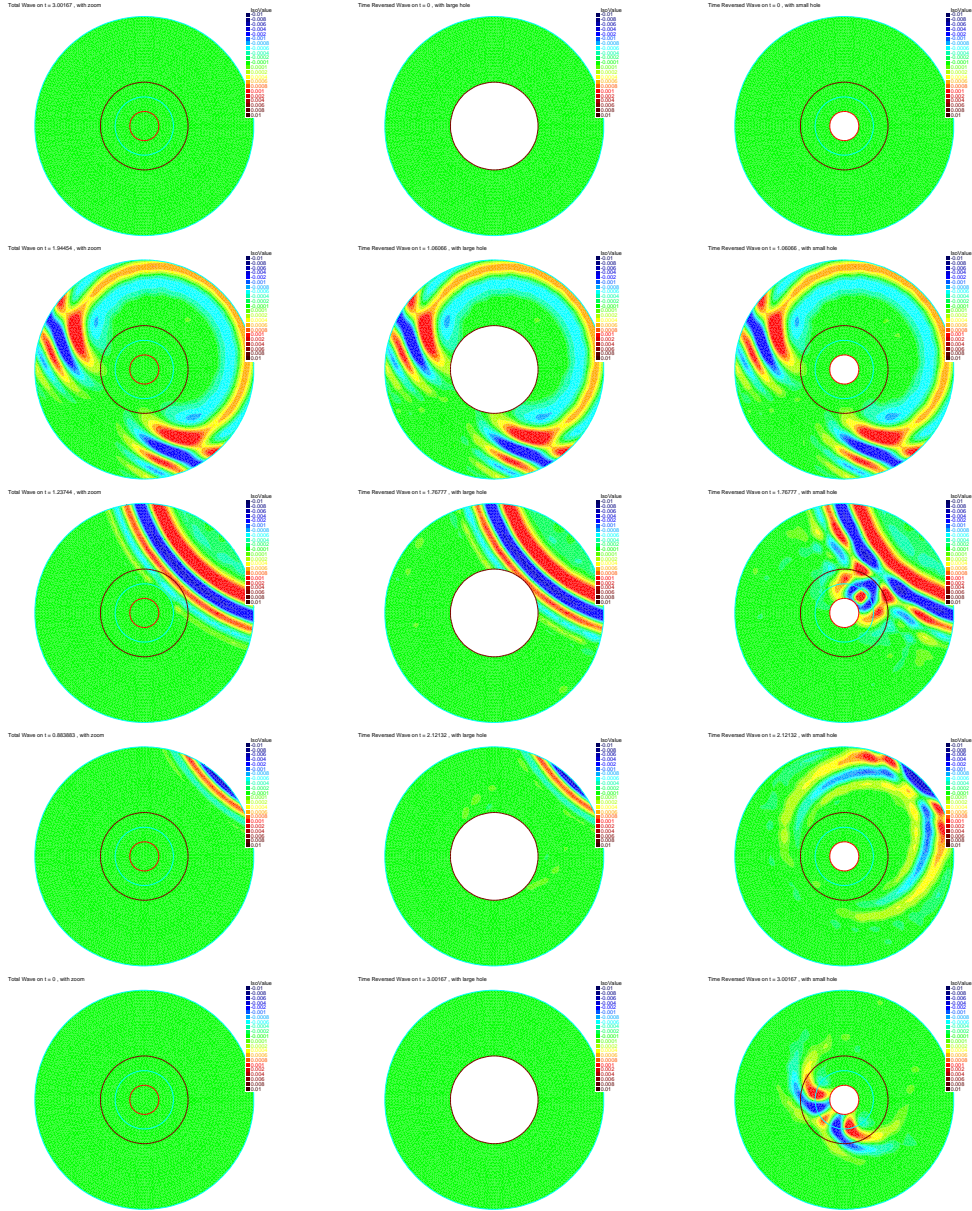


Figure 2: Time reversed BVP results for a soft object: perfect time reversed solution on the left, larger ball in the middle, smaller ball on the right. The incident signal comes from top-right.

display the perfect time reverse solution which is the reverse of the forward problem. In the inverse problem, these data are not known. They are shown here for reference only.

In the middle column, we show the solution of the reversed problem (18) with a ball B_ρ which encloses the inclusion. As expected, the sequence of snapshots is the restriction to the domain $\Omega \setminus B_\rho$ of the left column. This exemplifies one application of the *TRAC* method: if we know that the ball B_ρ encloses the inclusion we are able to reconstruct the signal in region that is closer to the inclusion than the line of receivers Γ_R . Thus, this enables us to reduce the size of the computational domain in an inverse problem. In this respect, the method is related to the redatuming method, see[Ber79]. In the last column we show the solution of the reversed problem (18) with a ball B_ρ smaller than the soft disk. In contrast to the previous case, the

sequence of snapshots differs from the left column.

For the inverse problem of locating the inclusion, we only know that at the final time the solution has to be zero. In the middle column, this criterion is satisfied and we can thus infer that the inclusion is included in the ball B_ρ . On the other hand, when the final solution in the last column is not zero it shows that the inclusion is not included in the ball B_ρ . This observation leads us to introduce an easy to compute criterion which is independent of the size of the domain:

$$J(B_\rho) := \frac{\|u_R^T(T_f, \cdot)\|_{L^\infty(\Omega \setminus B_\rho)}}{\sup_{t \in [0, T_f]} \|u^I(t, \cdot)\|_{L^\infty(\Omega)}} \quad (32)$$

which vanishes when the artificial ball encloses the inclusion. Due to numerical errors and the fact that the *TRAC* is not derived from an exact *ABC*, the criterion is small but not zero, see Fig. 5.

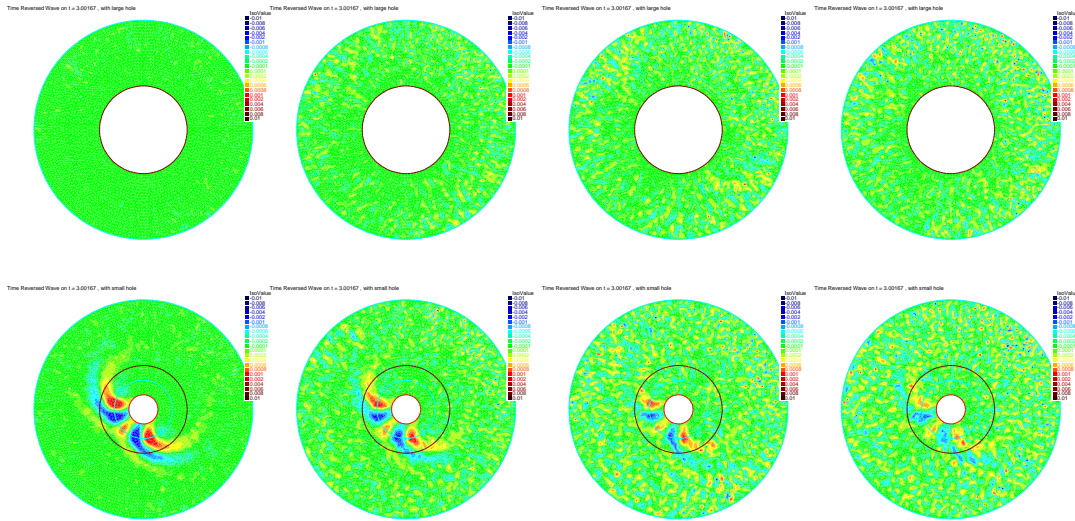


Figure 3: Time reversed solutions at the final time for noise on the recorded data from left to right: 10%, 30%, 45% and 50%, for a ball larger than the inclusion on the top and a smaller ball on the bottom

A crucial question in inverse problems is the robustness with respect to the magnitude of noise in the data. Hence, we add a Gaussian noise by replacing the recorded data u^T on Γ_R by

$$u^T := (1. + Coeff * (-1. + 2. * randn)) * u^T, \quad (33)$$

where *randn* satisfies a centered reduced normal law and *Coeff* is the level of noise. The solutions at the final time are depicted on Fig. 3. The level of noise is between 10% and 50%. On the top line we display the results for a ball larger than the inclusion. Due to the noise, the final solution is no longer zero but is a random signal of size related to the level on noise. In the bottom line we display the result when the ball is smaller than the inclusion. Now, together with a random signal there appears a structured non-zero solution which looks like the right column of Fig. 2 without noise in the data. We can discriminate between the case where the ball contains or not the inclusion D up to 45% of noise. Indeed in this case, $J(B_\rho) = .2$ so the magnitude of the final solution is 20% of the incident field when the ball B_ρ encloses the inclusion whereas $J(B_\rho) = .6$ when the ball does not enclose the inclusion. Thus, the method *TRAC* appears to be robust with respect to noise on the recorded data. We have conducted additional experiments where the inclusion is a triangle or a square with similar results.

In the previous tests, for the sake of simplicity, the artificial balls and the inclusion were concentric. We now consider the case where either the artificial ball and the inclusion do not intersect or the artificial ball crosses the object without including or being included, see Fig. 4. Results are shown in Tables 1 and 2. In Table 1, the inclusion is between the artificial boundaries. In Table 2, since in our tests the source emits in North-East, we place the object respectively in North-East, North-West, South-West and South-East. The radius of the object is 1.35λ , the larger ball is about 3λ and the smaller one is always λ . In all these cases, the criterion $J(B_\rho)$ is discriminative since it takes values smaller than .05 when the artificial ball encloses the inclusion whereas it lies between 0.3 and 1.0 in the other cases. In the latter case, moderate values of the criterion ($J(B_\rho) = 0.3$) correspond to a situation where the inclusion is in the shadow of the ball.

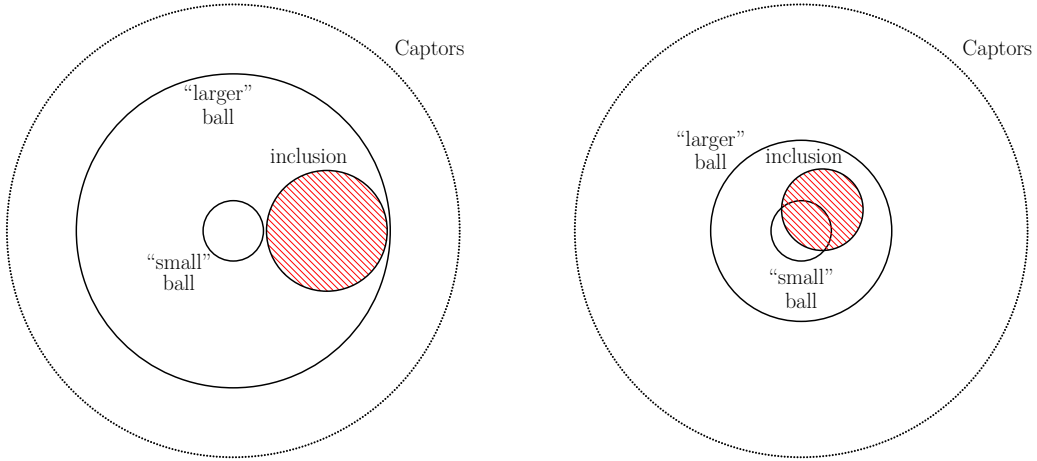


Figure 4: Inclusion between the artificial boundaries – Inclusion crossing the artificial boundaries

Size object	Size large ball	Size small ball	Results for large ball	Results for small ball
2λ	5.2λ	λ	$< 5\%$	100%
λ	5.2λ	λ	$< 5\%$	100%
λ	3.2λ	λ	$< 5\%$	100%
0.5λ	5.2λ	λ	$< 5\%$	50%
0.5λ	2.2λ	λ	$< 5\%$	50%

Table 1: Results for a object with decreasing size, between the artificial balls

Geographical position	Results for large ball	Results for small ball
N-E	$< 5\%$	65%
N-W	$< 5\%$	70%
S-W	$< 5\%$	30%
S-E	$< 5\%$	75%

Table 2: Results for an object crossing the artificial ball

As a second test case we illustrate the *TRAC* method by taking translucent inclusions with

velocities which correspond to medical applications. The values of the velocity in the inclusion are taken from [STX⁺05]: $c = 1.7$ (breast tumor), $c = 1.14$ (fibroadenoma) and $c = 0.93$ (surrounding tissue). We again stress that the method does not rely on any a priori knowledge of these data. For these various inclusions, the ball and the inclusion are concentric disks and we vary the size of the artificial balls. In Fig. 5, we plot the criterion $J(B_\rho)$ as a function of the algebraic distance between the inclusion and the artificial balls for various translucent inclusions. When the abscissa is negative, the ball does not enclose the inclusion. $J(B_\rho)$ is even larger when the ball is smaller than the inclusion. The criterion increases with the distance between the ball and the inclusion. On the other hand, when the ball encloses the inclusion (i.e. the abscissa is positive) the criterion is smaller than 0.1 and nearly flat. Note, that as expected, the larger the contrast between the inclusion and the surrounding medium is the larger is $J(B_\rho)$. As before the results are independent of the shape of the inclusion.

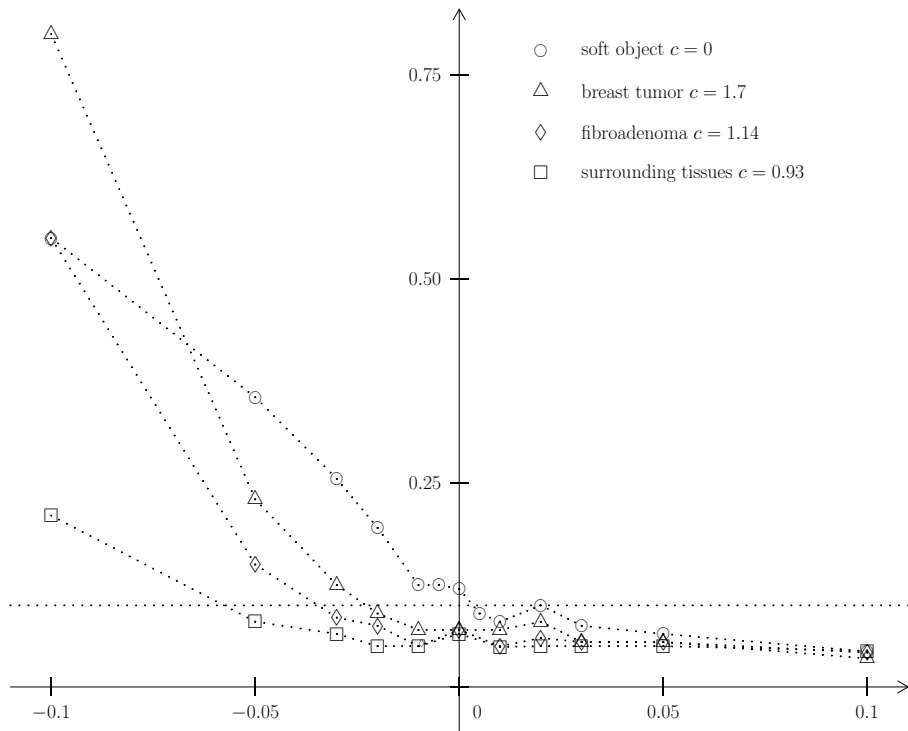


Figure 5: Criterion $J(B_\rho)$ vs. the algebraic distance between the inclusion and the artificial balls for various translucent inclusions

3.2 The Helmholtz equation

For the harmonic case, we again consider an inclusion D surrounded by a homogeneous and isotropic medium with a velocity of sound denoted by c_0 . The inclusion is illuminated by an incident field u^I . Equation (8) becomes:

$$\begin{cases} -\omega^2 u^T - c^2 \Delta u^T = 0 \text{ in } \mathbf{R}^2 \\ (u^T(\vec{x}) - u^I(\vec{x})) \text{ satisfies a Sommerfeld condition at } \infty \end{cases} \quad (34)$$

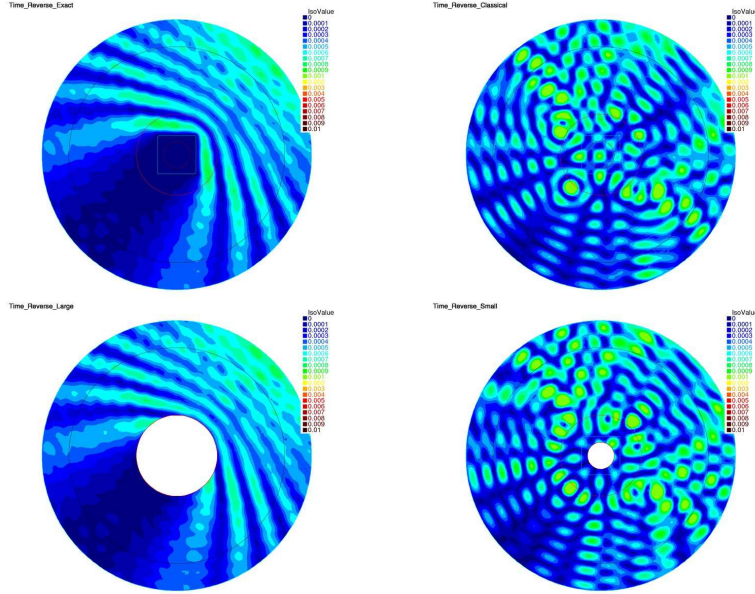


Figure 6: Phase conjugation for a soft square shaped inclusion of length 2λ . From left to right, from top to bottom : perfect, phase conjugation, *TRAC* with a ball enclosing the inclusion, *TRAC* with a ball inside the inclusion.

In order to create synthetic data, equation (34) is approximated by the FreeFem++ package [Hec10] with a finite element method in space. The computational domain is truncated by using an absorbing boundary condition. The incident wave is simulated by the same procedure with a uniform velocity of sound equals to c_0 . We then introduce a boundary Γ_R where the signal is recorded. The boundary Γ_R encloses a domain denoted Ω , see Fig. 1. The next stage of the method is to introduce a “trial” domain B_ρ and solve the phase conjugated problem (20) in $\Omega \setminus B_\rho$. In contrast to the time dependent case, we cannot use the criterion defined in (32) since in the harmonic case, there is no time and thus no final time. We define, later, two new criteria adapted to the harmonic case. We first look at the numerical simulations obtained with artificial balls and a soft square shaped inclusion that are concentric, see Fig. 6. On the top left figure we plot the modulus of the total field $|u^T|$ which coincides with the modulus of its conjugate field $|\bar{u}^T|$. On its right, we display the field obtained by the phase-conjugation method presented in § 2.2, see equation (9). We see that there is big difference between the total field and the field reconstructed by the phase conjugation method. The two bottom figures illustrate the *TRAC* method. On the left figure, the ball encloses the square and the computed field is the restriction of the total field. On the right figure, the ball is inside the square and the computed field is very different from the total field.

In practice, we don’t know the total field and we have to introduce a way to measure if the artificial ball encloses the inclusion or not. For this purpose, we introduce two different criteria. The first one is based on Dirichlet and Neumann data and the second one is based on the *TRAC* method. The latter will prove to be robust with respect to noise without requiring some regularization techniques. For the first criterion, we assume that in addition to the total field u^T we have also recorded the value of the normal derivative $\partial u^T / \partial n$ on the boundary Γ_R . When the ball encloses the inclusion, the normal derivative of the solution to the phase conjugation problem (20) coincides with the conjugation of the corresponding recorded data. Thus, it is

natural to introduce the following first criterion as:

$$J_{DN}(B_\rho) := \frac{\left\| \frac{\partial u_R^T}{\partial n} - \frac{\partial \bar{u}^T}{\partial n} \right\|_{L^\infty(\Gamma_R)}}{\left\| \frac{\partial \bar{u}^T}{\partial n} \right\|_{L^\infty(\Gamma_R)}} \quad (35)$$

The second criterion is deduced from the *TRAC* method itself. Indeed, the basis of the method is that the phase conjugated scattered field

$$\bar{u}^S := \bar{u}^T - \bar{u}^I$$

satisfies

$$\text{TRAC}(\bar{u}^S) = \text{TRAC}(\bar{u}^T - \bar{u}^I) = 0 \quad (36)$$

at any point outside the inclusion. In equation (20) this relation is used on the boundary of the artificial ball B_ρ . Since u_R^T is computed numerically and u^I is given data, that is readily available at any point in $\Omega \setminus B_\rho$. Thus, following the principle of the *TRAC* method, we introduce a new boundary $\Gamma_{J_{TRAC}}$ (see Fig. 7) to design a new criterion:

$$J_{TRAC}(B_\rho, \Gamma_{J_{TRAC}}) := \frac{\left\| \text{TRAC}(u_R^T - \bar{u}^I) \right\|_{L^\infty(\Gamma_{J_{TRAC}})}}{\left\| \text{TRAC}(\bar{u}^I) \right\|_{L^\infty(\Gamma_{J_{TRAC}})}} \quad (37)$$

Notice that this criterion is not based on a direct comparison between numerical data and recorded data in contrast to the first criterion. Hence it does not require any additional recorded data.

Both criteria (35) and (37) are zero when the artificial ball B_ρ encloses the inclusion. Since the *TRAC* is not derived from an exact *ABC*, the above criteria are small but not zero. Since, equation (36) is imposed on ∂B_ρ in order to compute u_R^T (see equation (20), we have to take $\Gamma_{J_{TRAC}}$ different from ∂B_ρ .

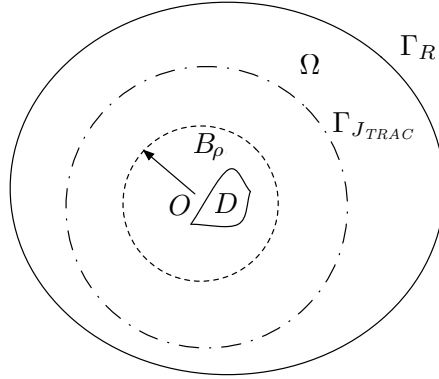


Figure 7: Internal boundary $\Gamma_{J_{TRAC}}$ for the criterion J_{TRAC}

We tested both criteria for a soft circular inclusion of radius two wavelengths (2λ) with various noise magnitudes on the recorded data defined as in (33). The results are summarized in Table 3.2. The results of the enclosing case correspond to a concentric artificial ball B_ρ of radius 3λ and the results of a non enclosing ball correspond to a concentric ball of radius λ . The

first criterion J_{DN} works only up to 5% of noise. However, the use of a filtering technique could improve the domain of validity of this criterion. The second criterion J_{TRAC} enables one to discriminate between the enclosing and non enclosing cases up to 30% of noise even though we have not filtered the data. This robustness can be explained by the fact that the noise comes from boundary measurements while, the Helmholtz equation has regularizing properties so that the computed field u_R^T is much less noisy on the internal boundary $\Gamma_{J_{TRAC}}$ than on the boundary Γ_R .

Noise Magnitude	J_{DN}		J_{TRAC}	
	enclosing	Non enclosing	enclosing	Non enclosing
0%	5.04%	69.20%	6.34%	69.18%
5%	46.38%	93.55%	12.79%	71.62%
10%	103.40%	91.77%	13.55%	70.87%
20%	213.65%	231.30%	35.99%	80.23%
30%	301.46%	269.07%	51.03%	87.80%
40%	532.97%	374.67%	57.99%	84.34%
50%	644.47%	552.57%	63.28%	124.30%

Table 3: Values of the criteria vs. the noise magnitude

In the last experiment, we consider a soft object of arbitrary and non smooth shape. We experiment with the *TRAC* method to detect the location of the inclusion by varying the respective locations of the artificial ball. The results are summarized in Table 4. First we take a large ball B_ρ so that we check that the inclusion D is not too close to the boundary Γ_R . Then, we move B_ρ to the left and the high value of the criterion shows us that the inclusion is not included inside. The third and fourth show other failed attempts to enclose the inclusion. The last column corresponds to a successful location of the inclusion. This shows the possibility by trials and errors to recover the location and volume of the inclusion.

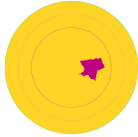
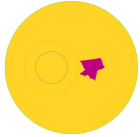
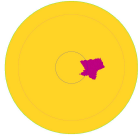
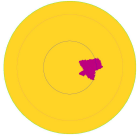
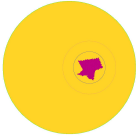
Cases					
J_{TRAC}	8.52%	66.09%	60.28%	28.35%	11.89%

Table 4: Criterion J_{TRAC} for a soft inclusion and various ball locations.

4 Conclusion

The aim of this paper is to introduce the time reversed absorbing conditions (*TRAC*) in time reversal methods. They enable one to “recreate the past” without knowing the source which has emitted the signals that are back-propagated. This is made possible at the expense of removing a small region enclosing the source. We presented two applications in inverse problems: the reduction of the size of the computational domain and the determination of the location and volume of an unknown inclusion from boundary measurements. We stress the fact that in contrast to many methods in inverse problems, the method does not rely on any *a priori*

knowledge of the physical properties of the inclusion. Hard, soft and translucent inclusions are treated in the same way. Feasibility of the method was shown with both time-dependent and harmonic examples. Moreover, the method has proved to be very robust with respect to noise in the data.

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