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# Crossover in the Efimov spectrum

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A filtering method is introduced for solving the zero-range three-boson problem. This scheme permits to solve the original Skorniakov Ter-Martirosian integral equation for an arbitrary large Ultra-Violet cut-off and to avoid the Thomas collapse of the three particles. The method is applied to a more general zero-range model including a finite background two-body scattering length and the effective range. A cross-over in the Efimov spectrum is found in such systems and a specific regime emerges where Efimov states are long-lived.

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In bosonic systems where two-body scattering is resonant, Efimov predicted in the early 70's the existence of shallow trimers with a spectrum characterized by a discrete scaling symmetry and an accumulation point at zero energy [1]. The first clear evidence of such states occurred in ultra-cold atomic systems [2–5], where a fine control of the resonant behavior through Feshbach Resonances (FR) is possible [6]. The few-body problem which was originally deeply linked to Nuclear Physics is now a hot topic in ultra-cold physics [7–9] and plays a significant role also in the search of highly correlated many-body states [10–12]. While experiments unambiguously confirm general properties of Efimov physics [13], deviations from universal predictions are observed showing the interest of using detailed approaches like the one in Ref. [14] or including a microscopic description of the FR mechanism with a coupling between atoms (in the 'open channel') and molecules (in the 'closed channel') as in Refs. [15–17]. In these last references, the finite width of the magnetic FR and the 'background' scattering length (denoted by  $\Delta\mathcal{B}$  and  $a_{\text{bg}}$ , respectively) appear as key parameters for the low energy scattering amplitude which plays a central role in three-body properties [16, 17]. At the two-body level, two-channel models provide the expression of the scattering length (denoted by  $a$ ) as a function of the external magnetic field  $\mathcal{B}$  in the vicinity of a given resonance located at  $\mathcal{B} = \mathcal{B}_0$  with:

$$a = a_{\text{bg}} \left( 1 - \frac{\Delta\mathcal{B}}{\mathcal{B} - \mathcal{B}_0} \right). \quad (1)$$

The magnetic width can be also characterized by a length  $R^*$  referred hereafter as the 'width radius' and defined by  $R^* = \hbar^2 / (ma_{\text{bg}}\delta\mu\Delta\mathcal{B})$ , where  $\delta\mu$  is the difference of magnetic moment for an atomic pair in the open- and in the closed-channel [18]. In the limit of an asymptotically narrow resonance, the width radius takes a large value as compared to the range of the interparticles forces (denoted by  $b$ ) and defines a low energy scale. In this peculiar regime it is possible to determine Efimov physics and deviations from universality (in the intermediate detuning regime) both in terms of  $a$  and  $R^*$  only [18, 19]. Recently, it has been shown that the superposition of an external electric field permits to tune the background

scattering length  $a_{\text{bg}}$  and thus the width radius  $R^*$  [20] which makes possible a study of low energy three-body properties as a function of these quantities.

Following this idea, a zero-range model generalizing the universal theory of broad resonances and the effective range approach of ultra-narrow resonances is solved in what follows. This modeling is especially pertinent in situations where each of the three parameters ( $a$ ,  $a_{\text{bg}}$ ,  $R^*$ ) of the FR are large (in absolute value) with respect to the range  $b$  which is of the order of the van der Waals radius given by  $(mC_6/\hbar^2)^{1/4}$  (where  $C_6$  is the van der Waals coefficient of the interatomic potential). In this model, the *so-called* three-body parameter (denoted by  $\kappa^*$ ) which characterizes the high momentum of the three-body wave function, is introduced by means of a nodal condition. While keeping the simplicity of the zero-range approach, this 'filtering' method of physical solutions permits to avoid the Thomas collapse [21, 22]. In the limit of broad resonances, the nodal condition implemented in the original Skorniakov Ter-Martirosian (STM) equation (Eq.(12) of Ref. [23]) through a simple subtraction scheme permits to recover the results of the 'universal theory'. In the limit of asymptotically narrow resonance ( $R^* \rightarrow \infty$ ), a decoupling occurs between shallow states of the effective range theory [18, 19] and deeper states depending on  $\kappa^*$ . In between the ultra-narrow and the infinitely broad resonance limits, the Efimov spectrum exhibits a crossover involving these two families of states. In the regime of large positive background scattering length and width parameter, the model supports a shallow dimer state. In this situation, Efimov trimers of lowest energy are quasi-bound states and manifest themselves as asymmetric resonance peaks in atom-dimer scattering which are characteristic of Fano resonances [24]. This result confirms the scenario introduced in Nuclear physics for the compound system made of two neutrons interacting with one  $^{18}\text{C}$  nucleus [25].

The zero-range approach used hereafter is based on the expression of the low energy two-body  $s$ -wave scattering phase-shift (denoted by  $\delta(k)$ ) or equivalently of the scattering amplitude (denoted by  $f(k)$ ), obtained from the limit of zero potential range ( $b \rightarrow 0$ ) of realistic models [17, 20, 26] or directly from a zero-range diagrammatic

approach (see Eq. (2) of Ref. [16]). In this limit, the scattering phase shift (related to the scattering amplitude by  $\frac{1}{f(k)} = k \cot \delta(k) - ik$ ) at collisional momentum  $k$  is:

$$\cot \delta(k) = -\frac{1}{ak} - \frac{R^*(1 - \frac{a_{bg}}{a})^2 k}{R^* a_{bg} (1 - \frac{a_{bg}}{a}) k^2 + 1}. \quad (2)$$

Eq. (2) can also be obtained in the small momentum limit ( $kb \ll 1$ ) of the models in Refs. [17, 20], and is thus the basic object which describes the interparticle interaction for low energy process [27]. For a vanishing background scattering length, Eq. (2) coincides with the effective range approximation used in Refs. [18, 19]. Systems studied hereafter are translation invariant, hence the zero-range approach is expressed directly in the momentum representation in the same manner as in Ref. [28, 29]. In the center of mass frame, the wave function of the two interacting particles of mass  $m$  at colliding energy  $E_{col} = \hbar^2 k_{col}^2/m$  satisfies

$$\left( \frac{\hbar^2}{m} k^2 - E_{col} \right) \langle \mathbf{k} | \Psi \rangle = \frac{4\pi \hbar^2 S_{\Psi}}{m}, \quad (3)$$

where  $k$  is the relative momentum. The zero-range interaction appears as a source term in the right hand side of Eq. (3) and the source amplitude (denoted by  $S_{\Psi}$ ) is obtained from a contact condition in the configuration space which reduces to an integral equation in the momentum space

$$\text{Reg}_{\epsilon \rightarrow 0} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{-k^2 \epsilon^2} \langle \mathbf{k} | \Psi \rangle = \frac{k_{col} S_{\Psi}}{\tan \delta(k_{col})}, \quad (4)$$

where the operator  $\text{Reg}_{\epsilon \rightarrow 0}$  extracts the regular part of the integral in the limit  $\epsilon \rightarrow 0$ . In the regime where  $R^*$  is vanishingly small, the condition expressed in Eq. (4) is equivalent to the Bethe-Peierls contact condition introduced in pioneering works on the deuteron [30].

This formalism is now applied to three identical bosons of mass  $m$  at negative energy ( $E$ ) in the center of mass frame. The wave function of the three particles labeled by  $i$  ( $i \in \{1, 2, 3\}$ ) with respective momentum  $\mathbf{k}_i$  factorizes as  $(2\pi)^3 \delta(\sum_i \mathbf{k}_i) \times \psi(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$  and

$$\left( \sum_{i=1}^3 \frac{k_i^2}{2} + q^2 \right) \psi(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = 4\pi \sum_{i=1}^3 \phi(\mathbf{k}_i), \quad (5)$$

where  $q$  is a positive wavenumber defined by  $E = -\hbar^2 q^2/m$  and  $\phi(\mathbf{k}_i)$  is the source amplitude associated with the pair ( $jk$ ) ( $i, j, k$  are distinct labels). The contact condition in Eq. (4) applied for the pair (23) with the colliding energy  $E_{col} = E - \frac{3\hbar^2}{4m} \mathbf{k}_1^2 < 0$  gives the integral equation satisfied by the source amplitude:

$$\frac{\phi(\mathbf{k})}{f(iq_{col})} = 8\pi \int \frac{d^3 \mathbf{u}}{(2\pi)^3} \frac{\phi(\mathbf{u})}{u^2 + k^2 + \mathbf{k} \cdot \mathbf{u} + q^2}, \quad (6)$$

where  $q_{col} = \sqrt{q^2 + \frac{3k^2}{4}}$  is the imaginary colliding momentum. Eq. (6) is a generalization of the STM equation (corresponding to the case  $R^* = a_{bg} = 0$ ). It has

been first derived by Massignan and Stoof (Eqs. (1,2) of Ref. [16]) and follows from the zero-range limit of the generalized STM equation in Ref. [17]. From these two references,  $\phi(k)$  can be interpreted as a dressed (atom  $\otimes$  closed channel molecule) wave function (the molecule in the closed channel is not described by the zero range approach). For  $R^* \neq 0$  and  $a_{bg} = 0$  (*i.e.* the FR is asymptotically narrow), Eq. (6) permits to extract all the low energy three-body properties and it has been solved in Refs. [18, 19]. For  $a_{bg} \neq 0$  or  $R^* = 0$ , Eq. (6) does not constitute a well defined problem in the  $s$ -wave sector of the source amplitude. This can be shown along the same lines as Ref. [31] by considering the high momentum behavior of the eigenfunctions in the  $s$ -wave sector. In this limit, the scattering amplitude coincides with the unitary expression:  $f(iq_{col}) \rightarrow 1/q_{col}$  and Eq. (6) supports a pair of power law solutions of the form  $\phi(k) \sim k^{-2 \pm i s_0}$  where  $s_0 \simeq 1.00624$ . Danilov showed that the STM model is self-adjoint if one filters eigenfunctions by fixing the linear combination of the two conjugate solutions as  $k \rightarrow \infty$  with an asymptotic phase shift. However the spectrum obtained from this filtering procedure is not bounded from below [22]: this is the Thomas collapse [21] analog to the falling of a particle in a  $1/r^2$  attractive potential [1]. In the present work, instead of the Danilov's phase-shift filtering, a nodal condition is used which permits to get a finite minimal energy in the spectrum. This method is based on the expression of the source amplitude at unitarity (*i.e.*  $R^* = 0$  and  $|a| = \infty$ ) which is given, up to a normalizing constant by [19]:

$$\phi(k) = \frac{1}{k \sqrt{q^2 + 3k^2/4}} \sin \left[ s_0 \text{arcsinh} \left( \frac{k\sqrt{3}}{2q} \right) \right]. \quad (7)$$

Eq. (7) supports the discrete scaling symmetry found by Efimov and the binding wavenumbers verify in the zero-range approximation:

$$q_n = \kappa^* e^{n\pi/s_0}, \quad (8)$$

where  $n$  is integer and  $\kappa^*$  is the three-body parameter. At large momentum, one obtains:

$$\phi(k) \underset{k \gg q}{\sim} \frac{1}{k^2} \sin \left[ s_0 \ln \left( \frac{k\sqrt{3}}{\kappa^*} \right) \right], \quad (9)$$

which gives the relation between the three-body parameter and the Danilov's phase shift. For  $R^* = 0$ , the spectrum of the zero-range (or universal) theory is invariant in the transformation  $\kappa^* \rightarrow \kappa^* \exp(\pi/s_0)$  showing that  $\kappa^*$  is not unique. However, for interatomic forces of finite range  $b$ , the binding wavenumber  $q_n$  in Eq. (8) is bounded from below by  $1/b$ . In what follows,  $\kappa^*$  is chosen of the order of  $1/b$  meaning that the scaling law in Eq. (8) is supposed to be valid for  $n \geq 0$ . Unphysical solutions of Eq. (6) are characterized by a large binding wave number ( $qb \gtrsim 1$ ) and are also given by Eq. (7) for the zero-range theory at unitarity. By noticing that as  $k$  increase from

0, the first node in the expression of Eq. (7) is found at a momentum of the order of  $q$ , it is thus possible to filter out the unphysical eigenfunctions of the STM equation by imposing the nodal condition:

$$\phi(k_{\text{reg}}) = 0 \quad \text{with} \quad k_{\text{reg}} = \frac{\kappa^*}{\sqrt{3}} e^{\pi/s_0}. \quad (10)$$

That way the spectrum obtained from the STM equation at unitarity has a minimum energy  $E_0 \simeq -\hbar^2 \kappa^{*2}/m$  (with a relative error given by  $2 \exp(-2\pi/s_0) \simeq 3.8 \times 10^{-3}$ ) and for  $E_n > E_0$ , it coincides asymptotically (*i.e.* for large  $n$ ) with the Efimov spectrum of the 'universal theory' [1, 13].

In the regime where  $1/a \neq 0$  and/or ( $a_{\text{bg}} \neq 0, R^* \neq 0$ ), the three-body parameter  $\kappa^*$  characterizes the high momentum behavior of the eigenstates (*i.e.* the Danilov's phase-shift) given by Eq. (9) which can be considered as a supplementary and necessary contact condition at the three-body level. In this regime of parameters, at least if a trimer exists, the binding wave numbers no longer verify the scaling law Eq. (8). Formally, the fact that  $\kappa^*$  is a parameter which is independent from the low energy two-body properties (and thus from the width radius  $R^*$ ) can be seen as resulting from the high momentum behavior of the solution of the generalized STM equation (for finite  $a_{\text{bg}}$ ) which is of the same type as the universal solutions. In the specific limit where  $R^* \neq 0$  and  $a_{\text{bg}} \rightarrow 0$  in Eq. (2), eigenfunctions are non zero in region of momentum less than a momentum of the order of  $1/R^*$ . In this limit, results are thus independent from  $\kappa^*$  and the spectrum is characterized by another momentum hereafter denoted by  $\kappa^{*'} = 2.65/R^*$  [18, 19] and also called 'three-body parameter' in these last references. In the general case where the parameters in Eq. (2) are finite, the nodal condition in Eq. (10) can again be used in order to prevent from any Thomas collapse without modifying Eq. (6).

In numerical computations there always exists a large but finite high momentum cut-off (denoted hereafter by  $\Lambda_3$ ) in the evaluation of the right hand side of Eq. (6) and one needs a stable procedure to implementing the nodal condition Eq. (10). This can be achieved by subtracting term by term Eq. (6) from its own expression at  $k = k_{\text{reg}}$  where Eq. (10) is also used. One obtains the regularized integral equation:

$$\frac{\phi(k)}{f(iq_{\text{col}})} = \frac{2}{\pi} \int_0^\infty du [\mathcal{K}_q(k, u) - \mathcal{K}_q(k_{\text{reg}}, u)] \phi(u), \quad (11)$$

where the Kernel is given by:

$$\mathcal{K}_q(k, u) = \frac{u}{k} \ln \left( \frac{u^2 + k^2 + q^2 + ku}{u^2 + k^2 + q^2 - ku} \right). \quad (12)$$

One can verify in numerical solutions of Eq. (11), that results are invariant in a change of the Ultra-Violet cut-off  $\Lambda_3$  of the integral term for  $\Lambda_3 \gg \kappa^*$ . As a benchmark, the source amplitude  $\phi_n(k)$  have been computed

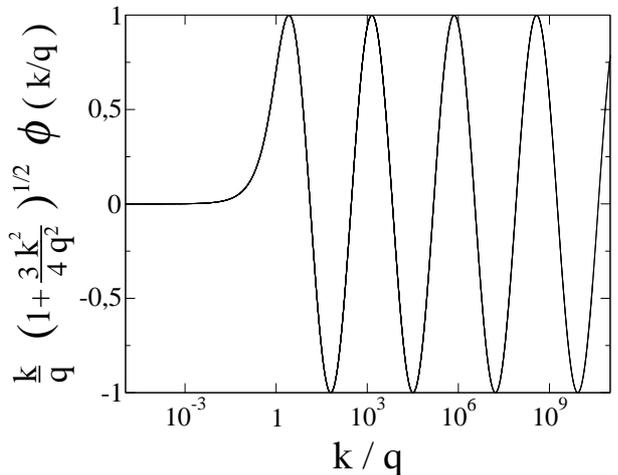


FIG. 1: Functions  $k/q \sqrt{1 + 3k^2/(4q^2)} \phi_n(k/q)$  obtained at unitarity ( $|a| = \infty, R^* = 0$ ) from Eq. (11) for the eight first trimers (including the ground state). The computation has been performed with the UV cut-off  $\Lambda_3 = 5 \times 10^2 \times \kappa^*$ . The function  $\sin \{s_0 \operatorname{arcsinh}[k\sqrt{3}/(2q)]\}$  appearing in the exact solution Eq. (7) has been superimposed and is not distinguishable from the functions obtained by solving Eq. (11).

at unitarity for the eight first trimers with the UV cut-off  $\Lambda_3 = 5 \times 10^2 \times \kappa^*$  and are displayed in Fig. (1). This figure illustrates the high degree of convergence of the method even for the first excited states (which is a consequence of the relatively large value of the scale factor  $e^{\pi/s_0} \sim 22.7$ ). Moreover, the binding wavenumbers of the excited trimers differ from the exact values of Eq. (8) by a relative error lesser than  $10^{-5}$ .

While not equivalent, the present regularization procedure of the STM equation as some similarity with the one used in the Effective Field Theory approach of the three-boson problem [32] where an integral counter-term is also introduced in order to cancel the  $\Lambda_3$  dependence. The present nodal condition has the advantage to make clear that the integral counter-term is just a way to achieve a filtering of the ‘‘physical’’ solutions which are eigenfunctions of the generalized STM equation Eq. (6). In Refs. [33, 34] a subtraction scheme has been also introduced for the atom-dimer scattering equation (see Eq. (15) below). In these last references, the renormalization is done at zero-energy with the atom-dimer scattering length as the input parameter and this regularizing method does not refers to a nodal condition. In Ref. [16] another regularizing technique was used: a supplementary effective range term  $-R_{\text{fit}}^* k$  was introduced in the two-body phase shift of Eq. (2) with a fitting parameter  $R_{\text{fit}}^*$  of the order of  $R^*$ , modifying thus the two-body low energy properties in the regime of a large width parameter [35].

The present model can be used in order to exemplify the crossover in the Efimov spectrum as one varies the two parameters ( $a_{\text{bg}}, R^*$ ) going from an infinitely broad

FR ( $R^* = 0$ ) to an asymptotically narrow FR ( $R^* = \infty$ ). In order to focus the discussion on this issue, in what follows the model is solved at resonance ( $|a| = \infty$ ). For a negative background scattering length ( $a_{\text{bg}} < 0$ ), there is no dimer, and the results are displayed in Fig. (2). This figure shows the crossover between two types of Efimov spectrum supporting each the discrete symmetry of scaling factor  $e^{\pi/s_0}$  but having different ground states. The first one is the spectrum of the effective range approach [18] characterized by a ground state binding wave number  $q_0 = \kappa^{*'} e^{-\pi/s_0}$  where  $\kappa^{*'} \simeq 2.65/R^*$  [18, 19]. The second one is the spectrum of a broad resonance where  $q_0 = \kappa^*$ . For a vanishingly small background scattering length ( $|a_{\text{bg}}| \rightarrow 0$ ) and a large width radius, the spectrum obtained from Eq. (11) coincides with the first one. As  $|a_{\text{bg}}|$  grows at a fixed and large value of the width parameter [Fig. (2-a)], deeper trimers ( $q \gg \kappa^{*'}$ ) of the second spectrum appear and 'narrow type' states evolve gradually towards the 'broad type' states. Interestingly by noticing that the high momentum behavior of the scattering amplitude is given by:

$$\frac{1}{f(k)} \underset{k \rightarrow \infty}{=} -\frac{1}{a_{\text{bg}}} - ik + O(k^{-2}), \quad (13)$$

one can deduce from the universal theory the threshold of apparition of the 'deepest' trimer by using the formal substitution ( $a \leftrightarrow a_{\text{bg}}$ ):  $\kappa^* a_{\text{bg}} = -1.51$  (value of  $\kappa^* a_*$  in Ref. [13, 19, 29]). Second and third thresholds for the apparition of the two others 'deep' trimers in Fig. (2-a)

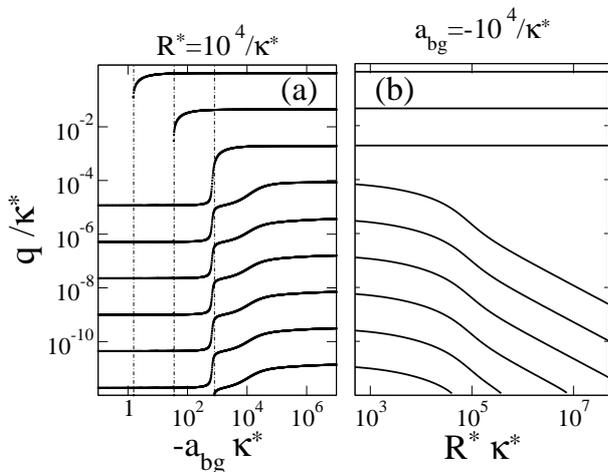


FIG. 2: Trimers' spectrum of Eq. (11) at resonance ( $|a| = \infty$ ) and for  $a_{\text{bg}} < 0$ . (a): Binding wavenumbers  $q$  (continuous lines) as a function of  $a_{\text{bg}}$  for a constant and large value of the width radius  $R^*$ ; vertical dashed lines correspond to the thresholds  $a_{\text{bg}} \kappa^* \simeq -1.51 e^{n\pi/s_0}$ ,  $n = 0, 1, 2$ . (b): Binding wavenumbers as a function of the width radius for a constant and large value of  $|a_{\text{bg}}|/\kappa^*$ . Fig. (a) Crossover from a narrow FR ( $R^* \kappa^* \gg 1$  and  $|a_{\text{bg}}| \ll R^*$ ) to a shape resonance ( $|a_{\text{bg}}|/\kappa^*$  is large). Fig. (b) Crossover from a broad FR (continuation of the left of Fig. (b) where  $R^* \kappa^* \ll 1$ ) to a narrow FR superimposed on a shape resonance.

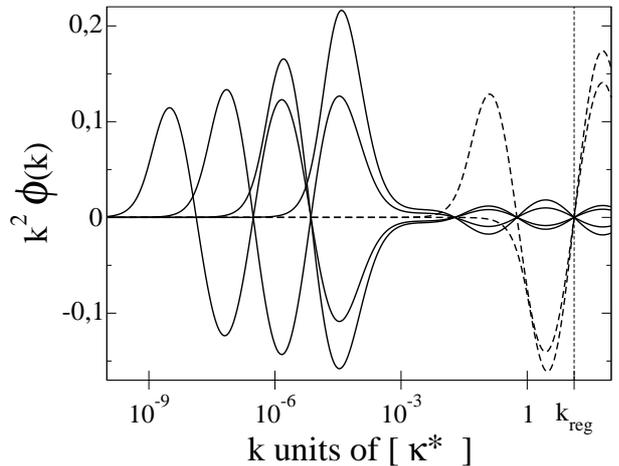


FIG. 3: Eigenfunctions  $k^2 \phi_n(k)$  for the deepest trimers states obtained after normalization from Eq. (11) for  $a_{\text{bg}} \kappa^* = -10^2$  and  $R^* \kappa^* = 10^4$ . Continuous lines: eigenstates with a binding wavenumber approximately given by the spectrum of a narrow resonance [ $q_p = \kappa^{*' } \exp(p\pi/s_0)$ ]. Dashed lines: eigenstates with a binding wavenumber approximately given by the spectrum of a broad resonance [ $q_n = \kappa^* \exp(n\pi/s_0)$ ]. Vertical dotted line: position of the node imposed by Eq. (10).

are deduced from the first threshold value by the discrete symmetry of scaling factor  $e^{n\pi/s_0}$ . As shown in Figs. (3-5) 'broad type' states are located in a momentum region  $k \gtrsim 1/|a_{\text{bg}}|$  (this is a consequence of Eq. (13)) and 'narrow type' states are located in the region  $k \lesssim \kappa^{*'}$ . Therefore, they experience a very small coupling and this explains the tiny avoided crossings in Fig. (2-a).

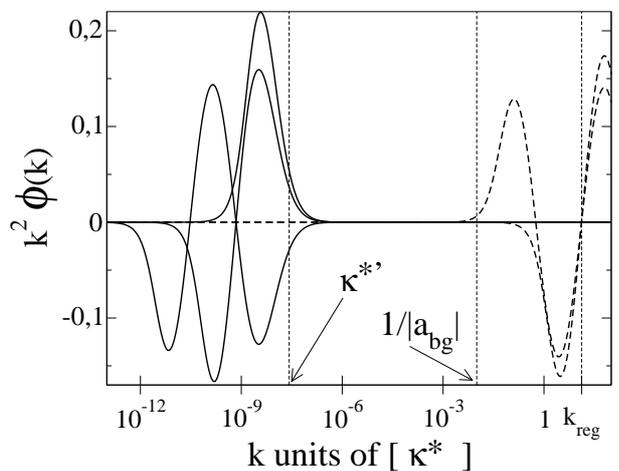


FIG. 4: Same as in Fig. (3) for  $a_{\text{bg}} \kappa^* = -10^2$  and  $R^* \kappa^* = 10^8$ . In this regime characterized by the inequality  $R^* \gg |a_{\text{bg}}|$ , there is a clear separation of scale for the two type of eigenstates: the deepest one of the 'broad type' are located in a momentum region  $k \gtrsim 1/|a_{\text{bg}}|$ , the other one of the 'narrow type' are located in a momentum region  $k \lesssim \kappa^{*'}$ .

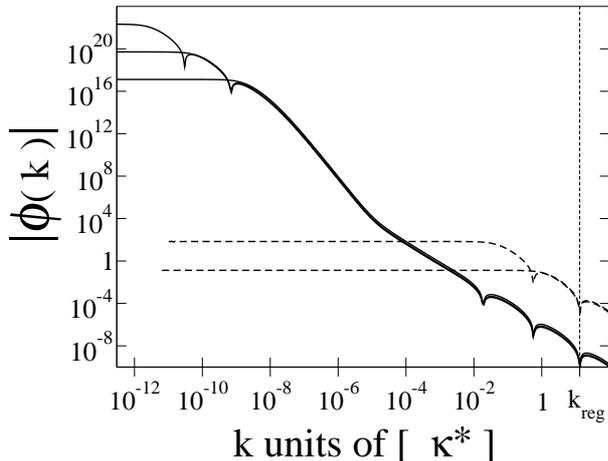


FIG. 5: Absolute value of the eigenfunctions  $|\phi(k)|$  of Fig. (4) ( $a_{\text{bg}}\kappa^* = -10^2$ ,  $R^*\kappa^* = 10^8$ ) in logarithmic scale. This figure, shows that even in this regime where 'broad type' and 'narrow type' eigenfunctions are well separated, eigenfunctions share the same zero at  $k = k_{\text{reg}}$ . 'Narrow type' eigenstates experience a  $k^{-4}$  power law in the region  $1 \ll kR^* \ll \sqrt{\frac{R^*}{|a_{\text{bg}}|}}$ .

In the case of a positive background scattering length  $a_{\text{bg}} > 0$ , the model supports a dimer with a binding wave number  $q_{\text{dim}} = (1 + \sqrt{1 + 4a_{\text{bg}}/R^*})/2a_{\text{bg}}$ . For  $q > q_{\text{dim}}$ , one finds eigenstates of the 'broad type'. For  $q < q_{\text{dim}}$  there is an atom-dimer continuum, and in what follows Eq. (6) is solved for a  $s$ -wave atom-dimer scattering process. For an incoming atom-dimer plane wave of momentum  $\mathbf{k}_0$ , the wavenumber  $q$  is given by  $q = \sqrt{q_{\text{dim}}^2 - 3k_0^2/4}$  and the  $s$ -wave scattering ansatz is:

$$\phi(\mathbf{k}) = 2\pi^2 \frac{\delta(k - k_0)}{k_0^2} + \frac{4\pi g(k)}{k^2 - k_0^2 - i0^+}. \quad (14)$$

One recognizes in Eq. (14) the  $s$ -wave atom-dimer scattering amplitude defined by  $f_{\text{ad}}(k_0) = g(k_0)$ . Injecting Eq. (14) in Eq. (6), one obtains:

$$\frac{\mathcal{K}_q(k, k_0)}{2k_0^2} [1 + ik_0 g(k_0)] + \mathcal{P} \int_0^\infty \frac{du}{\pi} \frac{\mathcal{K}_q(k, u)}{u^2 - k_0^2} g(u) = \frac{3g(k)}{8(q_{\text{dim}} + q_{\text{col}})} \left[ 1 + \frac{(1 + \frac{q_{\text{col}}}{q_{\text{dim}}})(1 - \frac{1}{aq_{\text{dim}}})}{R^* a_{\text{bg}} (1 - \frac{a_{\text{bg}}}{a}) q_{\text{col}}^2 - 1} \right]. \quad (15)$$

In Eq. (15), the integral is performed in the sense of Cauchy principal value and the filtered equation is obtained by making the substitution  $\mathcal{K}_q(k, v) \rightarrow \mathcal{K}_q(k, v) - \mathcal{K}_q(k_{\text{reg}}, v)$ . As shown in Fig. (6), the atom-dimer scattering amplitude obtained after regularization of Eq. (15) supports a series of peaks as a function of the wavenumber  $q$  at momenta  $q_{\text{trim}}$  related by the Efimov scaling factor  $e^{\pi/s_0}$ . In the regime where the peaks are well separated, their profile are typical of Fano resonances which can be parameterized by the fol-

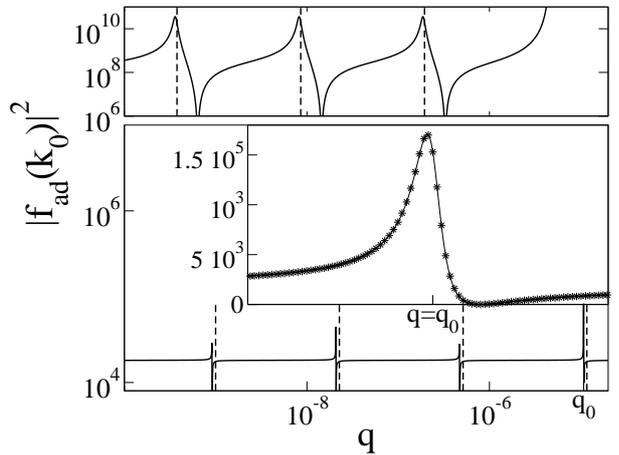


FIG. 6: Square modulus of the  $s$ -wave atom-dimer scattering amplitude in the resonant regime ( $|a| = \infty, a_{\text{bg}} > 0$ ) as a function of the wave number  $q = \sqrt{q_{\text{dim}}^2 - 3k_0^2/4}$ , where  $k_0$  is the collisional momentum and  $q_{\text{dim}}$  is the dimer's binding wave number. The plot is in units of  $\kappa^*$ . (a):  $R^*\kappa^* = 10^4$  and  $a_{\text{bg}}\kappa^* = 5 \times 10^2$ ; vertical dashed lines: 'narrow states' spectrum of Refs. [18, 19]. (b): Details of the resonant peak at  $q = q_0$  in Fig. (a); stars: results of the computation; continuous line: fit using the Fano profile. (c):  $R^*\kappa^* = 10^4$  and  $a_{\text{bg}}\kappa^* = 5 \times 10^6$ ; vertical dashed lines: spectrum of universal theory [1, 13].

lowing expression [24]:

$$|f(k_0)|^2 = |f_0|^2 \frac{(q_F + \epsilon)^2}{1 + \epsilon^2}, \quad (16)$$

where  $q_F$  is the 'profile parameter' which is related to the asymmetry of the peak,  $\epsilon = 2(q^2 - q_{\text{trim}}^2)/\gamma$  is the reduced energy,  $q_{\text{trim}}$  is the binding wave number of the excited Efimov state, and  $\gamma$  is the width of the resonance. Interestingly, for a fixed value of  $(a_{\text{bg}}, R^*)$  the peaks that have been computed in Fig. (6) are characterized approximately by the same ratio  $\gamma/q_{\text{trim}}^2$ . As expected, in the limit where  $R^* \gg a_{\text{bg}} \gg 1/\kappa^*$  excited trimers are decoupled from the high momentum scale  $\kappa^*$ . First, the peaks are located at the positions predicted by the effective range approach and second, their width goes to zero as  $R^*$  increases. This can be interpreted from the existence of a large scale separation between quasi-bound trimers extending in the region  $k \lesssim 1/R^*$  and the dimer of momentum  $q_{\text{dim}} \gg 1/R^*$ . Hence the production of long lived Efimov's trimers seems to be possible in this regime. If one takes only into account spontaneous decay of one the resonant three-body state into one atom and one dimer, the trimer's lifetime is given by:  $\tau_D = m/(\gamma\hbar)$ . As an example, the peak of Fig. (6-b) at  $q_{\text{trim}} = q_0$  has been computed for  $R^*\kappa^* = 10^4 \gg a_{\text{bg}}\kappa^* = 5 \times 10^2 \gg 1$ , and one finds  $\gamma \simeq 2 \times 10^{-3} q_0^2$ . In this case the lifetime of the quasi-bound state is limited by inelastic process with other components in the system. In the opposite limit where  $a_{\text{bg}} \gg R^*$  [see Fig. (6-c)], resonances are large meaning that the three-body quasi-bound state are

not long-lived and peaks are located at the positions predicted by the universal theory. For negative background scattering length  $a_{\text{bg}} < 0$ , the decay rate of 'narrow type' trimers involves deep dimer state which are not described in the present model. In the regime where  $R^* \gg |a_{\text{bg}}| \gg 1/\kappa^*$ , the eigenfunction  $\phi(k)$  experience a  $1/k^4$  decreasing law in the region  $1 \ll kR^* \ll \sqrt{R^*/|a_{\text{bg}}|}$  where  $1/f(ik) \sim R^*k^2$  [see Fig. (5)]. This suggests that the coupling with deep dimers (not supported by the model) is small. One thus expects also large lifetimes in this limit. A possible way to observe long-lived trimers is given by the radio-frequency association technique which has been already implemented in fermionic systems [36].

For finite background scattering length, the present

zero-range model is justified in the limit where  $|a_{\text{bg}}| \gg b$ . When this last condition is not satisfied, more realistic two-channel model as in Refs. [15, 17] should be used in order to evaluate the lifetime of the trimers. Work in this direction is in progress. The main result of this paper, concerns the possible production of long lived Efimov trimers. More generally, this opens the stability issue of resonant many-boson systems in the regime of narrow resonances.

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