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# Exact and high order discretization schemes for Wishart processes and their affine extensions

Abdelkoddousse Ahdida and Aurélien Alfonsi\*

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## Abstract

This work deals with the simulation of Wishart processes and affine diffusions on positive semidefinite matrices. To do so, we focus on the splitting of the infinitesimal generator, in order to use composition techniques as Ninomiya and Victoir [20] or Alfonsi [1]. Doing so, we have found a remarkable splitting for Wishart processes that enables us to sample exactly Wishart distributions, without any restriction on the parameters. It is related but extends existing exact simulation methods based on Bartlett's decomposition. Moreover, we can construct high-order discretization schemes for Wishart processes and second-order schemes for general affine diffusions. These schemes are in practice faster than the exact simulation to sample entire paths. Numerical results on their convergence are given.

*Key words:* Wishart processes, affine processes, exact simulation, discretization schemes, weak error, Bartlett's decomposition.

*AMS Class 2010:* 65C30, 60H35, 91B70.

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## Introduction and first definitions

This paper focuses on simulation methods for Wishart processes and more generally for affine diffusions on positive semidefinite matrices. Before explaining our motivations and our main results, we start with a short introduction to these processes. Even though we use rather standard notations for matrices, they are recalled in Appendix A, and we invite the reader to give first a quick look at it. Wishart processes have been initially introduced by Bru [3, 4]. They are also named because their marginal laws follow Wishart distributions. Very recently, Cuchiero et al. [6] have introduced a general framework for affine processes on positive semidefinite matrices  $\mathcal{S}_d^+(\mathbb{R})$  that embeds Wishart processes and includes possible jumps. In this paper, we only consider continuous processes of this kind. Such processes solve the following SDE:

$$X_t^x = x + \int_0^t (\bar{\alpha} + B(X_s^x)) ds + \int_0^t \left( \sqrt{X_s^x} dW_s a + a^T dW_s^T \sqrt{X_s^x} \right). \quad (1)$$

Here, and throughout the paper,  $(W_t, t \geq 0)$  denotes a  $d$ -by- $d$  square matrix made of independent standard Brownian motions,

$$x, \bar{\alpha} \in \mathcal{S}_d^+(\mathbb{R}), a \in \mathcal{M}_d(\mathbb{R}) \text{ and } B \in \mathcal{L}(\mathcal{S}_d(\mathbb{R})) \quad (2)$$

is a linear mapping on  $\mathcal{S}_d(\mathbb{R})$ . Wishart processes correspond to the case where

$$\exists \alpha \geq 0, \bar{\alpha} = \alpha a^T a \text{ and } \exists b \in \mathcal{M}_d(\mathbb{R}), \forall x \in \mathcal{S}_d(\mathbb{R}), B(x) = bx + xb^T. \quad (3)$$

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When  $d = 1$ , (1) is simply the SDE of the Cox-Ingersoll-Ross process that has been broadly studied, and we will implicitly assume that  $d \geq 2$  throughout the paper. Weak and strong uniqueness of SDE (1) has been studied by Bru [4], Cuchiero et al. [6] and Mayerhofer et al. [19]. We sum up here their results.

**Theorem 1** — *If  $x \in \mathcal{S}_d^+(\mathbb{R})$ ,  $\bar{\alpha} - (d-1)a^T a \in \mathcal{S}_d^+(\mathbb{R})$  and  $B$  satisfies the following condition*

$$\forall x_1, x_2 \in \mathcal{S}_d^+(\mathbb{R}), \quad \text{Tr}(x_1 x_2) = 0 \implies \text{Tr}(B(x_1)x_2) \geq 0, \quad (4)$$

*there is a unique weak solution to the SDE (1). We denote by  $AFF_d(x, \bar{\alpha}, B, a)$  the law of  $(X_t^x)_{t \geq 0}$  and  $AFF_d(x, \bar{\alpha}, B, a; t)$  the marginal law of  $X_t^x$ . If we assume moreover that  $\bar{\alpha} - (d+1)a^T a \in \mathcal{S}_d^+(\mathbb{R})$  and  $x \in \mathcal{S}_d^{+,*}(\mathbb{R})$ , there is a unique strong solution to the SDE (1).*

*Under the parametrization of Wishart processes (3), condition (4) is satisfied and weak uniqueness holds as soon as  $\alpha \geq d-1$ . In that case, we denote by  $WIS_d(x, \alpha, b, a)$  the law of the Wishart process  $(X_t^x)_{t \geq 0}$  and  $WIS_d(x, \alpha, b, a; t)$  the law of  $X_t^x$ .*

Throughout the paper, when we use the notation  $AFF_d(x, \bar{\alpha}, B, a)$  or  $AFF_d(x, \bar{\alpha}, B, a; t)$  (resp.  $WIS_d(x, \alpha, b, a)$  or  $WIS_d(x, \alpha, b, a; t)$ ), we implicitly assume that  $\bar{\alpha} - (d-1)a^T a \in \mathcal{S}_d^+(\mathbb{R})$  (resp.  $\alpha \geq d-1$ ) and  $B$  satisfies (4) so that weak uniqueness holds.

In her Ph.D. thesis [3], Bru has introduced Wishart processes and used them in biology to study perturbed experimental data. Recently, a great attention has been paid to Wishart processes for applications in finance. Namely, Gourieroux and Sufana [13] and Da Fonseca et al. [7] have suggested to use these processes to model the instantaneous covariance matrix of  $d$  assets. It naturally extends stochastic volatility models for only one asset like the Heston model [15]. Obviously, processes on positive semidefinite matrices are really interesting to model the evolution of a dependence structure because they can describe a covariance matrix. However, when dealing with applications, it is in general crucial to be able to sample paths of such processes and make Monte-Carlo algorithms.

To the best of our knowledge, there is few literature on simulation methods for Wishart and general affine processes (1). Wishart distributions have been intensively studied in statistics when  $\alpha \in \mathbb{N}$ . In this case, exact simulation methods have proposed by Odell and Feiveson [21], Smith and Hocking [22] and Gleser [11] to mention a few. Concerning discretization schemes, the usual Euler-Maruyama scheme is not well-defined because of the square-root. This is what already happens for the Cox-Ingersoll-Ross process which corresponds to the case  $d = 1$ . One has then to find specific schemes. Recently, Benabid et al. [2] and Gauthier and Possamaï [9] have proposed numerical approximations for Wishart processes that are well defined under some restrictions on the parameters. However, there is no result on the accuracy of their methods. Currently, Teichmann [25] is working on dedicated schemes for general affine processes by approximating their characteristic functions. Our study here is only dedicated to the diffusion (1).

Initially, our goal was to find high order discretization schemes for Wishart processes by splitting operators and using scheme compositions. Indeed, this approach has already proved to be very efficient for other affine diffusions, see [1]. Doing so, we incidentally have found a remarkable splitting for some canonical Wishart processes which enables us to sample exactly Wishart processes. In particular, our result extends the Bartlett's decomposition that is commonly used to sample central Wishart distributions when  $\alpha \in \mathbb{N}$ . This splitting also enables us to get high order discretization schemes for Wishart processes. Then, by using scheme composition, we also get a second order scheme for any affine diffusion (1). Contrary to the exact scheme, one has then to study the discretization error, and we provide a rigorous analysis of the weak error.

This paper is structured as follows. First, we present some general results on affine diffusions. We calculate their infinitesimal generator and obtain interesting identities in law that are intensively used next for the different simulation methods. Section 2 is devoted to the exact simulation of Wishart processes. It exhibits a remarkable splitting of the infinitesimal generator and shows how it can be used to sample exactly any Wishart distribution. Section 3 deals with high order schemes for affine diffusions. Thanks to the remarkable splitting, we are able to construct a third order scheme for Wishart processes and second order schemes for affine diffusions. Last, we give numerical illustrations of our convergence results in Section 4. We compare the time required by each method and also give a possible application of our results in finance.

# 1 General properties of affine processes on positive semidefinite matrices

## 1.1 The infinitesimal generator on $\mathcal{M}_d(\mathbb{R})$ and $\mathcal{S}_d(\mathbb{R})$

We start this subsection with a simple Lemma, which is useful to calculate the infinitesimal generator of affine processes. Its proof is basic and is left in Appendix B.1.

**Lemma 2** — Let  $(\mathcal{F}_t)_{t \geq 0}$  denote the filtration generated by  $(W_t, t \geq 0)$ . We consider a process  $(Y_t)_{t \geq 0}$  valued in  $\mathcal{S}_d(\mathbb{R})$ , and we assume that there exist continuous  $(\mathcal{F}_t)$ -adapted processes  $(A_t)_{t \geq 0}$ ,  $(B_t)_{t \geq 0}$  and  $(C_t)_{t \geq 0}$  that are respectively valued in  $\mathcal{M}_d(\mathbb{R})$ ,  $\mathcal{M}_d(\mathbb{R})$  and  $\mathcal{S}_d(\mathbb{R})$  so that  $Y_t$  admits the following semimartingale decomposition:

$$dY_t = C_t dt + B_t dW_t A_t + A_t^T dW_t^T B_t^T \quad (5)$$

Then, for  $i, j, m, n \in \{1, \dots, d\}$ , the quadratic covariation of  $(Y_t)_{i,j}$  and  $(Y_t)_{m,n}$  is given by:

$$d\langle (Y_t)_{i,j}, (Y_t)_{m,n} \rangle = (B_t B_t^T)_{i,m} (A_t^T A_t)_{j,n} + (B_t B_t^T)_{i,n} (A_t^T A_t)_{j,m} + (B_t B_t^T)_{j,m} (A_t^T A_t)_{i,n} + (B_t B_t^T)_{j,n} (A_t^T A_t)_{i,m}. \quad (6)$$

It is worth to notice that the quadratic covariation given by (5) depends on  $A_t$  and  $B_t$  only through the matrices  $A_t^T A_t$  and  $B_t B_t^T$ . Lemma 2 enables us to calculate easily the infinitesimal generator for the affine process (1) which is defined by:

$$x \in \mathcal{S}_d^+(\mathbb{R}), L^{\mathcal{M}} f(x) = \lim_{t \rightarrow 0^+} \frac{\mathbb{E}[f(X_t^x)] - f(x)}{t} \text{ for } f \in \mathcal{C}^2(\mathcal{M}_d(\mathbb{R}), \mathbb{R}) \text{ with bounded derivatives.}$$

**Proposition 3 — Infinitesimal Generator on  $\mathcal{M}_d(\mathbb{R})$ .** Let  $(X_t^x)_{t \geq 0} \sim AFF_d(x, \bar{\alpha}, B, a)$  be an affine process. Its infinitesimal generator on  $\mathcal{M}_d(\mathbb{R})$  is given by:

$$L^{\mathcal{M}} = \text{Tr}([\bar{\alpha} + B(x)]D^{\mathcal{M}}) + \frac{1}{2} \{2\text{Tr}(xD^{\mathcal{M}} a^T a D^{\mathcal{M}}) + \text{Tr}(x(D^{\mathcal{M}})^T a^T a D^{\mathcal{M}}) + \text{Tr}(xD^{\mathcal{M}} a^T a (D^{\mathcal{M}})^T)\}, \quad (7)$$

where  $D^{\mathcal{M}} = (\partial_{i,j})_{1 \leq i, j \leq d}$ .

*Proof:* On the one hand, the drift part of the operator is given by  $\sum_{k,l=1}^d \bar{\alpha}_{k,l} \partial_{k,l} + \sum_{k,l=1}^d (B(x))_{k,l} \partial_{k,l} = \text{Tr}((\bar{\alpha} + B(x))D^{\mathcal{M}})$ . On the other hand, we get from (6) that the diffusion part of the operator is:

$$\begin{aligned} & \frac{1}{2} \sum_{n,m,i,j=1}^d [x_{i,m} (a^T a)_{j,n} + x_{i,n} (a^T a)_{j,m} + x_{j,m} (a^T a)_{i,n} + x_{j,n} (a^T a)_{i,m}] \partial_{i,j} \partial_{m,n} \\ &= \frac{1}{2} \sum_{n,m,i,j=1}^d x_{i,m} (a^T a)_{j,n} \partial_{i,j} \partial_{m,n} + \frac{1}{2} \sum_{n,m,i,j=1}^d x_{i,n} (a^T a)_{j,m} \partial_{i,j} \partial_{m,n} \\ & \quad + \frac{1}{2} \sum_{n,m,i,j=1}^d x_{j,m} (a^T a)_{i,n} \partial_{i,j} \partial_{m,n} + \frac{1}{2} \sum_{n,m,i,j=1}^d x_{j,n} (a^T a)_{i,m} \partial_{i,j} \partial_{m,n} \\ &= \frac{1}{2} [\text{Tr}(xD^{\mathcal{M}} a^T a (D^{\mathcal{M}})^T) + \text{Tr}(x(D^{\mathcal{M}})^T a^T a (D^{\mathcal{M}})^T) + \text{Tr}(xD^{\mathcal{M}} a^T a D^{\mathcal{M}}) + \text{Tr}(x(D^{\mathcal{M}})^T a^T a D^{\mathcal{M}})] \\ &= \frac{1}{2} [\text{Tr}(x(D^{\mathcal{M}})^T a^T a (D^{\mathcal{M}})^T) + 2\text{Tr}(xD^{\mathcal{M}} a^T a D^{\mathcal{M}}) + \text{Tr}(x(D^{\mathcal{M}})^T a^T a D^{\mathcal{M}})]. \end{aligned}$$

□

Here, we have given the infinitesimal generator on  $\mathcal{M}_d(\mathbb{R})$ , while we know that the affine process  $(X_t^x)_{t \geq 0}$  takes values in  $\mathcal{S}_d^+(\mathbb{R}) \subset \mathcal{S}_d(\mathbb{R})$ . Thus, we can also look at the infinitesimal generator of this diffusion on  $\mathcal{S}_d(\mathbb{R})$ , which is defined by:

$$x \in \mathcal{S}_d^+(\mathbb{R}), L^{\mathcal{S}} f(x) = \lim_{t \rightarrow 0^+} \frac{\mathbb{E}[f(X_t^x)] - f(x)}{t} \text{ for } f \in \mathcal{C}^2(\mathcal{S}_d(\mathbb{R}), \mathbb{R}) \text{ with bounded derivatives.}$$

For  $x \in \mathcal{S}_d(\mathbb{R})$ , we denote by  $x_{\{i,j\}} = x_{i,j} = x_{j,i}$  the value of the coordinates  $(i,j)$  and  $(j,i)$ , so that  $x = \sum_{1 \leq i \leq j \leq d} x_{\{i,j\}} (e_d^{i,j} + \mathbb{1}_{i \neq j} e_d^{j,i})$  (see notations in Appendix A). For  $f \in \mathcal{C}^2(\mathcal{S}_d(\mathbb{R}), \mathbb{R})$ , we then denote by  $\partial_{\{i,j\}} f$  its derivative with respect to the coordinates  $x_{\{i,j\}}$ . We introduce

$$\begin{aligned} \pi : \mathcal{M}_d(\mathbb{R}) &\rightarrow \mathcal{S}_d(\mathbb{R}) \\ x &\mapsto (x + x^T)/2, \end{aligned}$$

that is such that  $\pi(x) = x$  for  $x \in \mathcal{S}_d(\mathbb{R})$ . Obviously,  $f \circ \pi \in \mathcal{C}^2(\mathcal{M}_d(\mathbb{R}), \mathbb{R})$  and we have

$$L^{\mathcal{S}} f(x) = L^{\mathcal{M}} f \circ \pi(x).$$

By the chain rule, we have for  $x \in \mathcal{S}_d(\mathbb{R})$ ,  $\partial_{i,j} f \circ \pi(x) = (\mathbb{1}_{i=j} + \frac{1}{2} \mathbb{1}_{i \neq j}) \partial_{\{i,j\}} f(x)$  and get the following result.

**Corollary 4 — Infinitesimal Generator on  $\mathcal{S}_d(\mathbb{R})$ .** *The infinitesimal generator on  $\mathcal{S}_d(\mathbb{R})$  associated to  $AF F_d(x, \bar{\alpha}, B, a)$  is given by:*

$$L^{\mathcal{S}} = \text{Tr}([\bar{\alpha} + B(x)] D^{\mathcal{S}}) + 2 \text{Tr}(x D^{\mathcal{S}} a^T a D^{\mathcal{S}}), \quad (8)$$

where  $D^{\mathcal{S}}$  is defined by  $D_{i,j}^{\mathcal{S}} = (\mathbb{1}_{i=j} + \frac{1}{2} \mathbb{1}_{i \neq j}) \partial_{\{i,j\}}$ , for  $1 \leq i, j \leq d$ .

Of course, the generators  $L^{\mathcal{M}}$  and  $L^{\mathcal{S}}$  are equivalent: one can be deduced from the other. However,  $L^{\mathcal{S}}$  already embeds the fact that the process lies in  $\mathcal{S}_d(\mathbb{R})$ , which reduces the dimension from  $d^2$  to  $d(d+1)/2$  and gives in practice shorter formulas. This is why we will mostly work in the sequel with infinitesimal generators on  $\mathcal{S}_d(\mathbb{R})$ . Unless it is necessary to make the distinction with  $L^{\mathcal{M}}$ , we will simply denote  $L = L^{\mathcal{S}}$ .

## 1.2 The characteristic function of Wishart processes

As for other affine processes, the characteristic function of affine processes on positive semidefinite matrices can be obtained by solving two ODEs. In the case of Wishart processes, it is possible to solve explicitly these ODEs by solving a matrix Riccati equation (see Levin [18]). Here, we give the closed formula for the Laplace transform and a precise description of its set of convergence.

**Proposition 5 —** *Let  $X_t^x \sim WIS_d(x, \alpha, b, a; t)$ ,  $q_t = \int_0^t \exp(sb) a^T a \exp(sb^T) ds$  and  $m_t = \exp(tb)$ . We introduce the set of convergence of the Laplace transform of  $X_t^x$ ,  $\mathcal{D}_{b,a;t} = \{v \in \mathcal{S}_d(\mathbb{R}), \mathbb{E}[\exp(\text{Tr}(v X_t^x))] < \infty\}$ . This is a convex open set that is given explicitly by*

$$\mathcal{D}_{b,a;t} = \{v \in \mathcal{S}_d(\mathbb{R}), \forall s \in [0, t], I_d - 2q_s v \in \mathcal{G}_d(\mathbb{R})\}. \quad (9)$$

Besides, the Laplace transform of  $X_t^x$  is well-defined for  $v = v_R + i v_I$  with  $v_R \in \mathcal{D}_{b,a;t}$ ,  $v_I \in \mathcal{S}_d(\mathbb{R})$  and is given by:

$$\mathbb{E}[\exp(\text{Tr}(v X_t^x))] = \frac{\exp(\text{Tr}[v(I_d - 2q_t v)^{-1} m_t x m_t^T])}{\det(I_d - 2q_t v)^{\frac{\alpha}{2}}}. \quad (10)$$

The characteristic function corresponds to the case  $v_R = 0$  that clearly belongs to  $\mathcal{D}_{b,a;t}$ . The proof of this result is given in Appendix B.2. Let us remark here that for  $\tilde{X}_t^x \sim WIS_d(x, \alpha, 0, I_d^n; t)$ , the formula above becomes even simpler and we have for  $v = v_R + i v_I$  such that  $v_R \in \mathcal{D}_{b,a;t}$ ,  $v_I \in \mathcal{S}_d(\mathbb{R})$ :

$$\mathbb{E}[\exp(\text{Tr}(v \tilde{X}_t^x))] = \frac{\exp(\text{Tr}[v(I_d - 2t I_d^n v)^{-1} x])}{\det(I_d - 2t I_d^n v)^{\frac{\alpha}{2}}}. \quad (11)$$

### 1.3 Some identities in law for affine processes

This section presents simple but interesting identities in law for affine processes. We first state a preliminary lemma.

**Lemma 6** — Let  $B \in \mathcal{L}(\mathcal{S}_d(\mathbb{R}))$  and  $q \in \mathcal{G}_d(\mathbb{R})$ . We define  $B_q \in \mathcal{L}(\mathcal{S}_d(\mathbb{R}))$  by  $B_q(x) = (q^T)^{-1}B(q^T x q)q^{-1}$ . Then,

$$B \text{ satisfies (4)} \iff B_q \text{ satisfies (4)}.$$

*Proof :* It is sufficient to prove the direct implication since  $B(x) = q^T B_q((q^T)^{-1}xq^{-1})q$ . Let  $x, v \in \mathcal{S}_d^+(\mathbb{R})$  such that  $\text{Tr}(xv) = 0$ . We have

$$\text{Tr}(B_q(x)v) = \text{Tr}[B(q^T x q)q^{-1}v(q^T)^{-1}] \geq 0,$$

since  $B$  satisfies (4) and  $\text{Tr}[(q^T)^{-1}xq^{-1}qvq^T] = \text{Tr}(xv) = 0$ .  $\square$

**Proposition 7** — The following identities holds:

- $AF F_d(x, \bar{\alpha}, B, a) \stackrel{Law}{=} AF F_d(x, \bar{\alpha}, B, \sqrt{a^T a})$ .
- (linear transformation) Let  $q \in \mathcal{G}_d(\mathbb{R})$ , we have

$$q^T AF F_d(x, \bar{\alpha}, B, a)q \stackrel{Law}{=} AF F_d(q^T x q, q^T \bar{\alpha} q, B_{q^{-1}}, aq),$$

where  $B_{q^{-1}}$  is defined by  $\forall y \in \mathcal{S}_d(\mathbb{R})$ ,  $B_{q^{-1}}(y) = q^T B((q^T)^{-1}yq^{-1})q$ .

*Proof :* From Proposition 3 or Corollary 4, we know that  $(X_t^x)_{t \geq 0} \sim AF F_d(x, \bar{\alpha}, B, a)$  and  $(\tilde{X}_t^x)_{t \geq 0} \sim AF F_d(x, \bar{\alpha}, B, \sqrt{a^T a})$  have the same infinitesimal generator. Therefore, they solve the same martingale problem. Thanks to the affine structure, one can show that this martingale problem has a unique solution by looking at the characteristic function. This is made in a much general case in Cuchiero et al. [6]. Similarly, we can check easily that  $q^T AF F_d(y, \bar{\alpha}, B, a)q$  and  $AF F_d(q^T y q, q^T \bar{\alpha} q, B_{q^{-1}}, aq)$  lead to the same martingale problem. Here, we remark that  $B_{q^{-1}}$  satisfies (4) thanks to Lemma 6.  $\square$

An interesting consequence of this linear transformation is given in the following corollary. It states that any affine process can be obtained as a linear transformation of an affine process for which we have  $a = I_d^n$ , where  $I_d^n = (\mathbb{1}_{i=j \leq n})_{1 \leq i, j \leq d}$  (see notations in Appendix A). Since our main goal here is to sample paths of such processes, this says us that it is sufficient to focus on this special case.

**Corollary 8** — Let  $(X_t^x)_{t \geq 0} \sim AF F_d(x, \bar{\alpha}, B, a)$  and  $n = \text{Rk}(a)$  be the rank of  $a^T a$ . Then, there exist a diagonal matrix  $\bar{\delta}$ , and a non singular matrix  $u \in \mathcal{G}_d(\mathbb{R})$  such that  $\bar{\alpha} = u^T \bar{\delta} u$ , and  $a^T a = u^T I_d^n u$ , and we have:

$$(X_t^x)_{t \geq 0} \stackrel{Law}{=} u^T AF F_d((u^{-1})^T x u^{-1}, \bar{\delta}, B_u, I_d^n) u,$$

where  $\forall y \in \mathcal{S}_d(\mathbb{R})$ ,  $B_u(y) = (u^{-1})^T B(u^T y u)u^{-1}$ .

*Proof* : Once  $u$  is given, the identity in law comes directly from Proposition 7. We give now a constructive proof of the existence of  $u$ , which takes back the arguments given by Golub and Van Loan ([12], Theorem 8.7.1). Nonetheless, we explain it entirely since it gives a practical way to get  $u$ .

Let us consider  $\bar{\alpha} + a^T a \in \mathcal{S}_d^+(\mathbb{R})$ . From the extended Cholesky decomposition given in Lemma 33 there is a matrix  $v \in \mathcal{G}_d(\mathbb{R})$  such that  $v^T \bar{\alpha} v + v^T a^T a v = I_d^r$ , where  $r = \text{Rk}(\bar{\alpha} + a^T a)$ . Since  $v^T \bar{\alpha} v \in \mathcal{S}_d^+(\mathbb{R})$ ,  $v^T a^T a v \in \mathcal{S}_d^+(\mathbb{R})$  and  $z^T I_d^r z = 0$  for  $z \in \mathbb{R}^d$  such that  $z_1 = \dots = z_r = 0$ , there are  $s_1, s_2 \in \mathcal{S}_n^+(\mathbb{R})$  such that:

$$v^T \bar{\alpha} v = \begin{pmatrix} s_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } v^T a^T a v = \begin{pmatrix} s_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let  $o_2$  be an orthogonal matrix such that  $o_2^T s_2 o_2$  is a diagonal matrix. We assume without loss of generality that only the first  $n$  elements of this diagonal are positive:  $o_2^T s_2 o_2 = \text{diag}(\eta_1, \dots, \eta_n, 0, \dots, 0)$ . We set

$o = \begin{pmatrix} o_2 & 0 \\ 0 & I_{d-r} \end{pmatrix}$  and get  $I_d^r = o^T v^T \bar{\alpha} v o + o^T v^T a^T a v o$ , which gives that  $o^T v^T \bar{\alpha} v o$  is a diagonal matrix.

Thus, we get the desired result by taking  $u = \text{diag}(\sqrt{\eta_1}, \dots, \sqrt{\eta_n}, 1, \dots, 1) o^{-1} v^{-1}$ .  $\square$

Let us make few comments on the practical implementation to compute  $u$ . For Wishart processes,  $\bar{\alpha} = \alpha \alpha^T a$ , and  $u$  can directly be obtained by using a single extended Cholesky decomposition (Lemma 33). In the general case, the computation of  $u$  mainly requires an extended Cholesky decomposition and the diagonalization of  $s_2$ .

Up to now, we have stated identities for the law of affine processes. Thanks to the explicit characteristic function of Wishart processes, we are also able to get another interesting identity on the marginal laws.

**Proposition 9** — *Let  $t > 0$ ,  $a, b \in \mathcal{M}_d(\mathbb{R})$  and  $\alpha \geq d - 1$ . Let  $m_t = \exp(tb)$ ,  $q_t = \int_0^t \exp(sb) a^T a \exp(sb^T) ds$  and  $n = \text{Rk}(q_t)$ . Then, there is  $\theta_t \in \mathcal{G}_d(\mathbb{R})$  such that  $q_t = t \theta_t I_d^n \theta_t^T$ , and we have:*

$$WIS_d(x, \alpha, b, a; t) \underset{Law}{=} \theta_t WIS_d(\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T, \alpha, 0, I_d^n; t) \theta_t^T \quad (12)$$

This proposition plays a crucial role for the exact simulation of Wishart processes. Thanks to (12), we can sample any Wishart distribution if we are able to simulate exactly the distribution  $WIS_d(x, \alpha, 0, I_d^n; t)$  for any  $x \in \mathcal{S}_d^+(\mathbb{R})$ . In Section 2, we focus on this and give a way to sample exactly  $WIS_d(x, \alpha, 0, I_d^n; t)$ . Let us stress here that we can compute the matrix  $\theta_t$  by using the extended Cholesky decomposition of  $q_t/t$ , as it is explained in the proof below.

*Proof* : We apply Lemma 33 to  $q_t/t \in \mathcal{S}_d^+(\mathbb{R})$  and consider  $(p, c_n, k_n)$  an extended Cholesky decomposition of  $q_t/t$ . We set  $\theta_t = p^{-1} \begin{pmatrix} c_n & 0 \\ k_n & I_{d-n} \end{pmatrix}$ . Then,  $\theta_t$  is invertible and it is easy to check that  $q_t = t \theta_t I_d^n \theta_t^T$ . Now, let us observe that for  $v \in \mathcal{S}_d(\mathbb{R})$ ,

$$\begin{aligned} \det(I_d - 2i q_t v) &= \det(\theta_t (\theta_t^{-1} - 2it I_d^n \theta_t^T v)) = \det(I_d - 2it I_d^n \theta_t^T v \theta_t), \\ \text{Tr}[i v (I_d - 2i q_t v)^{-1} m_t x m_t^T] &= \text{Tr}[i (\theta_t^{-1})^T \theta_t^T v (\theta_t \theta_t^{-1} - 2it \theta_t I_d^n \theta_t^T v \theta_t \theta_t^{-1})^{-1} m_t x m_t^T] \\ &= \text{Tr}[i \theta_t^T v \theta_t (I_d - 2it I_d^n \theta_t^T v \theta_t)^{-1} \theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T]. \end{aligned}$$

Let  $X_t^x \sim WIS_d(x, \alpha, b, a; t)$  and  $\tilde{X}_t^x \sim WIS_d(x, \alpha, 0, I_d^n; t)$ . Then, from (10) and (11), we get that

$$\mathbb{E}[\exp(i \text{Tr}(v X_t^x))] = \mathbb{E}[\exp(i \text{Tr}(\theta_t^T v \theta_t \tilde{X}_t^x \theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T))] = \mathbb{E}[\exp(i \text{Tr}(v \theta_t \tilde{X}_t^x \theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T \theta_t^T))],$$

which gives the result.  $\square$

Last, we have to mention that the identity (12) extends an usual identity between CIR and Bessel squared distribution. It gives when  $d = 1$ :

$$WIS_1(x, \alpha, b, a; t) \stackrel{\text{Law}}{=} a^2 \frac{e^{2bt} - 1}{2bt} WIS_1\left(\frac{2btx}{a^2(1 - e^{-2bt})}, \alpha, 0, 1; t\right).$$

In that case, this identity can also be obtained directly from the SDE. Let  $(X_t^x)_{t \geq 0} \sim WIS_1(x, \alpha, b, a)$ . We have  $dX_t^x = (\alpha^2 + 2bX_t)dt + 2a\sqrt{X_t^x}dW_t$ . Then,  $Y_t = e^{-2bt}X_t^x/a^2$  is a time-changed Bessel squared process since  $dY_t = \alpha(e^{-2bt}dt) + 2\sqrt{Y_t}(e^{-bt}dW_t)$ . We obtain  $WIS_1(x, \alpha, b, a; t) \stackrel{\text{Law}}{=} a^2 e^{2bt} WIS_1(x/a^2, \alpha, 0, 1; \frac{1-e^{-2bt}}{2b})$ . On the other and, a linear time-change gives that  $WIS_1(x, \alpha, 0, 1; \lambda t) \stackrel{\text{Law}}{=} \lambda WIS_1(x/\lambda, \alpha, 0, 1; t)$ , which leads to the same identity as (12) by taking  $\lambda = (1 - e^{-2bt})/(2bt)$ .

## 2 Exact simulation of Wishart processes

In this section, we present a new method to simulate exactly a Wishart process. It works without any restriction on the parameters. Wishart distributions have been thoroughly studied in statistics when  $\alpha \in \mathbb{N}$  (which is then called the number of degrees of freedom). Exact simulation methods have already been proposed in that case. For instance, Odell and Feiveson [21] have given an exact simulation algorithm for central Wishart distributions based on the Bartlett's decomposition, and Gleser [11] extends it to any (non-central) Wishart distribution. Bru [4] explains also when  $\alpha \in \mathbb{N}$  how Wishart processes can be obtained as a square of Ornstein-Uhlenbeck processes on matrices.

Here, our method relies on the identity in law (12) that enables us to focus on the case  $b = 0$ ,  $a = I_d^n$ . Then, we show a remarkable splitting of the infinitesimal generator as the sum of commuting operators. These operators are associated to SDE that can be solved explicitly on  $\mathcal{S}_d^+(\mathbb{R})$ , which enables us to sample any Wishart distribution.

### 2.1 A remarkable splitting for $WIS_d(x, \alpha, 0, I_d^n)$

The following theorem explains how to split the infinitesimal generator of a  $WIS_d(x, \alpha, 0, I_d^n)$  as the sum of commutative infinitesimal generators. It will play a crucial role in the sequel for the exact simulation and to get discretization schemes.

**Theorem 10** — *Let  $L$  be the generator associated to the Wishart process  $WIS_d(x, \alpha, 0, I_d^n)$  and  $L_i$  be the generator associated to  $WIS_d(x, \alpha, 0, e_d^i)$  for  $i \in \{1, \dots, d\}$ . Then, we have*

$$L = \sum_{i=1}^n L_i \text{ and } \forall i, j \in \{1, \dots, d\}, L_i L_j = L_j L_i. \quad (13)$$

*Proof* : From (8), we easily get that  $L = \sum_{i=1}^n L_i$  since  $I_d^n = \sum_{i=1}^n e_d^i$ . The commutativity property comes from a simple but tedious calculation which is left in Appendix C.1.  $\square$

Beyond the commutativity, two other features of (13) are important to notice.

- The operators  $L_i$  and  $L_j$  are the same up to the exchange of coordinates  $i$  and  $j$ .
- The processes  $WIS_d(x, \alpha, 0, e_d^i)$  and  $WIS_d(x, \alpha, 0, I_d^n)$  are well defined on  $\mathcal{S}_d^+(\mathbb{R})$  under the same hypothesis, namely  $\alpha \geq d - 1$  and  $x \in \mathcal{S}_d^+(\mathbb{R})$ .



This second property enables us the composition that we explain now. Let us consider  $t > 0$  and  $x \in \mathcal{S}_d^+(\mathbb{R})$ . We define iteratively:

$$\begin{aligned} X_t^{1,x} &\sim WIS_d(x, \alpha, 0, e_d^1; t), \\ X_t^{2, X_t^{1,x}} &\sim WIS_d(X_t^{1,x}, \alpha, 0, e_d^2; t), \\ &\dots \\ X_t^{n, \dots, X_t^{1,x}} &\sim WIS_d(X_t^{n-1, \dots, X_t^{1,x}}, \alpha, 0, e_d^n; t). \end{aligned}$$

Thus,  $X_t^{i, \dots, X_t^{1,x}}$  is sampled according to the distribution at time  $t$  of a Wishart process starting from  $X_t^{i-1, \dots, X_t^{1,x}}$  and with parameters  $(\alpha, 0, e_d^i)$ . We have the following result.

**Proposition 11** — *Let  $X_t^{n, \dots, X_t^{1,x}}$  be defined as above. Then*

$$X_t^{n, \dots, X_t^{1,x}} \sim WIS_d(x, \alpha, 0, I_d^n; t).$$

Thanks to this proposition, we can generate a sample according to  $WIS_d(x, \alpha, 0, I_d^n; t)$  as soon as we can simulate the laws  $WIS_d(x, \alpha, 0, e_d^i)$ . These laws are the same as  $WIS_d(x, \alpha, 0, I_d^1; t)$ , up to the permutation of the first and  $i^{\text{th}}$  coordinates. In the next subsection, it is explained how to draw such random variables.

It is really easy to give a formal proof of Proposition 11. Let  $f$  be a smooth function on  $\mathcal{S}_d^+(\mathbb{R})$  and  $X_t^x \sim WIS_d(x, \alpha, 0, I_d^n; t)$ . By iterating Itô's formula, we have that  $\mathbb{E}[f(X_t^x)] = \sum_{k=0}^{\infty} t^k L^k f(x)/k!$ . Similarly, we also get by using the tower property of the conditional expectation that:

$$\mathbb{E} \left[ f(X_t^{n, \dots, X_t^{1,x}}) \right] = \mathbb{E} \left[ \mathbb{E} \left[ f(X_t^{n, \dots, X_t^{1,x}}) | X_t^{n-1, \dots, X_t^{1,x}} \right] \right] = \sum_{k_n=0}^{+\infty} \frac{t^{k_n}}{k_n!} \mathbb{E} \left[ L_n^{k_n} f(X_t^{n-1, \dots, X_t^{1,x}}) \right]. \quad (14)$$

Simply by repeating this argument, we get that

$$\mathbb{E} \left[ f(X_t^{n, \dots, X_t^{1,x}}) \right] = \sum_{k_1, \dots, k_n=0}^{+\infty} \frac{t^{\sum_{i=1}^n k_i}}{k_1! \dots k_n!} L_1^{k_1} \dots L_n^{k_n} f(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (L_1 + \dots + L_n)^k f(x) = \mathbb{E}[f(X_t^x)]. \quad (15)$$

To get the second equality, we identify a Cauchy product and use that the operators  $L_1, \dots, L_n$  commute. To make this formal proof correct, one has to check that the series are well defined and can be switched with the expectation. This check is made in the Appendix C.2 for our framework and remains valid as soon as the operator  $L_i$  and  $L$  are of affine type.

## 2.2 Exact simulation for $WIS_d(x, \alpha, 0, I_d^1; t)$

In this subsection, we focus on the simulation of  $WIS_d(x, \alpha, 0, I_d^1)$  with  $\alpha \geq d-1$  and  $x \in \mathcal{S}_d^+(\mathbb{R})$ . Thanks to Proposition 11, this enable us then to simulate samples of  $WIS_d(x, \alpha, 0, I_d^n)$ . For the sake of the clearness, we start with the case of  $d = 2$  that avoids complexities due to the matrix decompositions. We deal with the general case just after.

### 2.2.1 The case $d = 2$

We start by writing explicitly the infinitesimal generator  $L_1$  of  $WIS_2(x, \alpha, 0, I_2^1)$ . From (8), we get:

$$x \in \mathcal{S}_2^+(\mathbb{R}), \quad L_1 f(x) = \alpha \partial_{\{1,1\}} f(x) + 2x_{\{1,1\}} \partial_{\{1,1\}}^2 f(x) + 2x_{\{1,2\}} \partial_{\{1,1\}} \partial_{\{1,2\}} f(x) + \frac{x_{\{2,2\}}}{2} \partial_{\{1,2\}}^2 f(x). \quad (16)$$

We show now that this operator is in fact associated to an SDE that can be explicitly solved. We will denote by  $(Z_t^1, t \geq 0)$  and  $(Z_t^2, t \geq 0)$  two independent standard Brownian motions.

When  $x_{\{2,2\}} = 0$ , we also have  $x_{\{1,2\}} = 0$  since  $x$  is nonnegative. In that case,

$$X_0^x = x, \quad d(X_t^x)_{\{1,1\}} = \alpha dt + 2\sqrt{(X_t^x)_{\{1,1\}}}dZ_t^1, \quad d(X_t^x)_{\{1,2\}} = 0, \quad d(X_t^x)_{\{2,2\}} = 0 \quad (17)$$

has the infinitesimal generator (16), which is the one of a Cox-Ingersoll-Ross SDE (or of a squared Bessel process of dimension  $\alpha$  to be more precise). By using an algorithm that samples exactly a non central chi-square distribution (see for instance Glasserman [10]), we can then sample  $WIS_2(x, \alpha, 0, I_2^1; t)$  when  $x_{\{2,2\}} = 0$ .

When  $x_{\{2,2\}} > 0$ , it is easy to check that the SDE

$$\begin{aligned} d(X_t^x)_{\{1,1\}} &= \alpha dt + 2\sqrt{(X_t^x)_{\{1,1\}} - \frac{((X_t^x)_{\{1,2\}})^2}{(X_t^x)_{\{2,2\}}}}dZ_t^1 + 2\frac{(X_t^x)_{\{1,2\}}}{\sqrt{(X_t^x)_{\{2,2\}}}}dZ_t^2 \\ d(X_t^x)_{\{1,2\}} &= \sqrt{(X_t^x)_{\{2,2\}}}dZ_t^2 \\ d(X_t^x)_{\{2,2\}} &= 0 \end{aligned} \quad (18)$$

starting from  $X_0^x = x$  has an infinitesimal generator equal to  $L_1$ . To solve (18), we set:

$$(U_t^u)_{\{1,1\}} = (X_t^x)_{\{1,1\}} - \frac{((X_t^x)_{\{1,2\}})^2}{(X_t^x)_{\{2,2\}}}, \quad (U_t^u)_{\{1,2\}} = \frac{(X_t^x)_{\{1,2\}}}{\sqrt{x_{\{2,2\}}}}, \quad (U_t^u)_{\{2,2\}} = x_{\{2,2\}}. \quad (19)$$

Here,  $u$  stands for the initial condition, i.e.  $u = U_0^u$ . We get by using Itô calculus that

$$d(U_t^u)_{\{1,1\}} = (\alpha - 1)dt + 2\sqrt{(U_t^u)_{\{1,1\}}}dZ_t^1, \quad d(U_t^u)_{\{1,2\}} = dZ_t^2 \quad \text{and} \quad d(U_t^u)_{\{2,2\}} = 0. \quad (20)$$

Therefore,  $(U_t^u)_{\{1,2\}}$  and  $(U_t^u)_{\{1,1\}}$  can be respectively be sampled by independent Gaussian and non-central chi-square variables. Then, we can get back  $X_t^x$  by inverting (19):

$$(X_t^x)_{\{1,1\}} = (U_t^u)_{\{1,1\}} + (U_t^u)_{\{1,2\}}^2, \quad (X_t^x)_{\{1,2\}} = (U_t^u)_{\{1,2\}}\sqrt{(U_t^u)_{\{2,2\}}}, \quad (X_t^x)_{\{2,2\}} = (U_t^u)_{\{2,2\}}. \quad (21)$$

The following proposition sums up and makes more precise the above result. Its proof is postponed to the general case (Theorem 13).

**Proposition 12** — *Let  $x \in \mathcal{S}_2^+(\mathbb{R})$ . Then, the process  $(X_t^x)_{t \geq 0}$  defined by either (17) when  $x_{\{2,2\}} = 0$  or (18) when  $x_{\{2,2\}} > 0$  has its infinitesimal generator equal to  $L_1$ . Moreover, the SDE (18) has a unique strong solution which is given by (21), where  $(U_t^u, t \geq 0)$  is the solution of the SDE (20) starting from  $u_{\{1,1\}} = x_{\{1,1\}} - \frac{x_{\{1,2\}}^2}{x_{\{2,2\}}} \geq 0$ ,  $u_{\{1,2\}} = \frac{x_{\{1,2\}}}{\sqrt{x_{\{2,2\}}}}$ ,  $u_{\{2,2\}} = x_{\{2,2\}}$ .*

This result gives an interesting way to figure out the dynamic associated to the operator  $L_1$ , by using a change of variable. It is interesting to notice that the CIR process  $(U_t^u)_{\{1,1\}}$  is well defined as soon as its degree  $\alpha - 1$  is nonnegative, which coincides with the condition under which the Wishart process  $WIS_2(x, \alpha, 0, I_2^1)$  is well-defined. Last, we notice that the solution of the operator  $L_1$  involves a CIR process in the diagonal term and a Brownian motion in the non diagonal one. We will get a similar structure in the general  $d$  case.

### 2.2.2 The general case when $d \geq 2$

We present now a general way to sample exactly  $WIS_d(x, \alpha, 0, I_d^1; t)$ . We first write explicitly from (8) the infinitesimal generator of  $WIS_d(x, \alpha, 0, I_d^1)$  for  $x \in \mathcal{S}_d^+(\mathbb{R})$ :

$$L_1 f(x) = \alpha \partial_{\{1,1\}} f(x) + 2x_{\{1,1\}} \partial_{\{1,1\}}^2 f(x) + 2 \sum_{\substack{1 \leq m \leq d \\ m \neq 1}} x_{\{1,m\}} \partial_{\{1,m\}} \partial_{\{1,1\}} f(x) + \frac{1}{2} \sum_{\substack{1 \leq m, l \leq d \\ m \neq 1, l \neq 1}} x_{\{m,l\}} \partial_{\{1,m\}} \partial_{\{1,l\}} f(x). \quad (22)$$

As for  $d = 2$  we will construct an SDE that has the same infinitesimal generator  $L_1$  and that can be solved explicitly. To do so, we need however to use further matrix decomposition results. In the case  $d = 2$ , we have already noticed that we choose different SDEs whether  $x_{2,2} = 0$  or not. Here, the SDE will depend on the rank of the submatrix  $(x_{i,j})_{2 \leq i, j \leq d}$ , and we set:

$$r = \text{Rk}((x_{i,j})_{2 \leq i, j \leq d}) \in \{0, \dots, d-1\}.$$

First, we consider the case where

$$\exists c_r \in \mathcal{G}_r \text{ lower triangular, } k_r \in \mathcal{M}_{d-1-r \times r}(\mathbb{R}), \quad (x)_{2 \leq i, j \leq d} = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix} \begin{pmatrix} c_r^T & k_r^T \\ 0 & 0 \end{pmatrix} =: cc^T. \quad (23)$$

With a slight abuse of notation, we consider that this decomposition also holds when  $r = 0$  with  $c = 0$ . When  $r = d-1$ ,  $c = c_r$  is simply the usual Cholesky decomposition of  $(x_{i,j})_{2 \leq i, j \leq d}$ . As it is explained in Corollary 15, we can still get such a decomposition up to a permutation of the coordinates  $\{2, \dots, d\}$ .

**Theorem 13** — *Let us consider  $x \in \mathcal{S}_d^+(\mathbb{R})$  such that (23) holds. Let  $(Z_t^l)_{1 \leq l \leq r+1}$  be a vector of independent standard Brownian motions. Then, the following SDE (convention  $\sum_{k=1}^r (\dots) = 0$  when  $r = 0$ )*

$$\begin{aligned} d(X_t^x)_{\{1,1\}} &= \alpha dt + 2\sqrt{(X_t^x)_{\{1,1\}} - \sum_{k=1}^r (\sum_{l=1}^r (c_r^{-1})_{k,l} (X_t^x)_{\{1,l+1\}})^2} dZ_t^1 \\ &\quad + 2 \sum_{k=1}^r \sum_{l=1}^r (c_r^{-1})_{k,l} (X_t^x)_{\{1,l+1\}} dZ_t^{k+1} \\ d(X_t^x)_{\{1,i\}} &= \sum_{k=1}^r c_{i-1,k} dZ_t^{k+1}, \quad i = 2, \dots, d \\ d((X_t^x)_{\{l,k\}})_{2 \leq k, l \leq d} &= 0 \end{aligned} \quad (24)$$

has a unique strong solution  $(X_t^x)_{t \geq 0}$  starting from  $x$ . It takes values in  $\mathcal{S}_d^+(\mathbb{R})$  and has the infinitesimal generator  $L_1$ . Moreover, this solution is given explicitly by:

$X_t^x =$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix} \begin{pmatrix} (U_t^u)_{\{1,1\}} + \sum_{k=1}^r ((U_t^u)_{\{1,k+1\}})^2 & ((U_t^u)_{\{1,l+1\}})_{1 \leq l \leq r}^T & 0 \\ ((U_t^u)_{\{1,l+1\}})_{1 \leq l \leq r} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r^T & k_r^T \\ 0 & 0 & I_{d-r-1} \end{pmatrix}, \quad (25)$$

where

$$\begin{aligned} d(U_t^u)_{\{1,1\}} &= (\alpha - r)dt + 2\sqrt{(U_t^u)_{\{1,1\}}} dZ_t^1, \quad u_{\{1,1\}} = x_{\{1,1\}} - \sum_{k=1}^r (u_{\{1,k+1\}})^2 \geq 0, \\ d((U_t^u)_{\{1,l+1\}})_{1 \leq l \leq r} &= (dZ_t^{l+1})_{1 \leq l \leq r}, \quad (u_{\{1,l+1\}})_{1 \leq l \leq r} = c_r^{-1} (x_{\{1,l+1\}})_{1 \leq l \leq r}. \end{aligned} \quad (26)$$

Once again, we have made a slight abuse of notation when  $r = 0$ , and (25) should be simply read as

$X_t^x = \begin{pmatrix} (U_t^u)_{\{1,1\}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  in that case. In the statement above, it may seem weird that we use for  $u$  and

$U_t^u$  the same indexation as the one for symmetric matrices while we only use its first row (or column). The reason is that we can in fact see  $X_t^x$  as a function of  $U_t^u$  by setting:

$$(U_t^u)_{\{i,j\}} = u_{\{i,j\}} = x_{\{i,j\}} \text{ for } i, j \geq 2 \text{ and } (U_t^u)_{\{1,i\}} = u_{\{1,i\}} = 0 \text{ for } r+1 \leq i \leq d. \quad (27)$$

Thus,  $(c_r, k_r, I_{d-1})$  is an extended Cholesky decomposition of  $((U_t^u)_{i,j})_{2 \leq i, j \leq d}$  and can be seen as a function of  $U_t^u$ . We get from (25) that

$$X_t^x = h(U_t^u), \text{ with } h(u) = \sum_{r=0}^{d-1} \mathbb{1}_{r=\text{Rk}[(u_{i,j})_{2 \leq i, j \leq d}]} h_r(u) \text{ and} \quad (28)$$

$$h_r(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r(u) & 0 \\ 0 & k_r(u) & I_{d-r-1} \end{pmatrix} \begin{pmatrix} u_{\{1,1\}} + \sum_{k=1}^r (u_{\{1,k+1\}})^2 & (u_{\{1,l+1\}})_{1 \leq l \leq r}^T & 0 \\ (u_{\{1,l+1\}})_{1 \leq l \leq r} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r(u)^T & k_r(u)^T \\ 0 & 0 & I_{d-r-1} \end{pmatrix},$$

where  $(c_r(u), k_r(u), I_{d-1})$  is the extended Cholesky decomposition of  $(u_{i,j})_{2 \leq i, j \leq d}$  given by some algorithm (e.g. Golub and Van Loan [12], Algorithm 4.2.4). Equation (28) will play later an important role to analyse discretization schemes.

The proof of Theorem 13 is given in Appendix C.3. It enables us to simulate exactly the distribution  $WIS_d(x, \alpha, 0, I_d^1; t)$  simply by sampling one non-central chi-square distribution for  $(U_t^u)_{\{1,1\}}$  (see Glasserman [10]) and  $r$  other independent Gaussian random variables. Like in the  $d = 2$  case, we notice that the condition which ensures that the Cox-Ingersoll-Ross process  $((U_t^u)_{\{1,1\}}, t \geq 0)$  is well defined for any  $r \in \{0, \dots, d-1\}$ , namely  $\alpha - (d-1) \geq 0$ , is the same as the one required for the definition of  $WIS_d(x, \alpha, 0, I_d^1)$ .

**Remark 14** — From (25), we get easily by a calculation made in (47) that  $\text{Rk}(X_t^x) = \text{Rk}((x_{i,j})_{2 \leq i, j \leq d}) + \mathbb{1}_{(U_t^u)_{\{1,1\}} \neq 0}$ , and therefore,

$$\text{Rk}(X_t^x) = \text{Rk}((x_{i,j})_{2 \leq i, j \leq d}) + 1, \text{ a.s.}$$

Theorem 13 assumes that the initial value  $x \in \mathcal{S}_d^+(\mathbb{R})$  satisfies (23). Now, we explain why it is still possible up to a permutation of the coordinates to be in such a case. This relies on the extended Cholesky decomposition which is stated in Lemma 33.

**Corollary 15** — Let  $(X_t^x)_{t \geq 0} \sim WIS_d(x, \alpha, 0, I_d^1)$  and  $(c_r, k_r, p)$  be an extended Cholesky decomposition of  $(x_{i,j})_{2 \leq i, j \leq d}$  (Lemma 33). Then,  $\pi = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$  is a permutation matrix, and

$$(X_t^x)_{t \geq 0} \stackrel{\text{law}}{=} \pi^T WIS_d(\pi x \pi^T, \alpha, 0, I_d^1) \pi \text{ and } ((\pi x \pi^T)_{i,j})_{2 \leq i, j \leq d} = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix} \begin{pmatrix} c_r^T & k_r^T \\ 0 & 0 \end{pmatrix} \text{ satisfies (23).}$$

*Proof* : The result is a direct implication from Proposition 7, since  $\pi^T = \pi^{-1}$  and  $\pi I_d^1 \pi^T = I_d^1$ .  $\square$

Therefore, by a combination of Corollary 15 and Theorem 13, we get a simple way to construct explicitly a process that has the infinitesimal generator  $L_1$  for any initial condition  $x \in \mathcal{S}_d^+(\mathbb{R})$ . In particular, this enables us to sample exactly the Wishart distribution  $WIS_d(x, \alpha, 0, I_d^1; t)$ . The algorithm below sums up

the whole procedure.

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**Algorithm 1:** Exact simulation for the operator  $L_1$

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**Input:**  $x \in \mathcal{S}_d^+(\mathbb{R})$ ,  $d$ ,  $\alpha \geq d - 1$  and  $t > 0$ .

**Output:**  $X$ , sampled according to  $WIS_d(x, \alpha, 0, I_d^1; t)$

Compute the extended Cholesky decomposition  $(p, k_r, c_r)$  of  $(x_{i,j})_{2 \leq i, j \leq d}$  given by Lemma 33,  $r \in \{0, \dots, d - 1\}$  (see Golub and Van Loan [12] for an algorithm);

Set  $\pi = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ ,  $\tilde{x} = \pi x \pi^T$ ,  $(u_{\{1, l+1\}})_{1 \leq l \leq r} = (c_r)^{-1}(\tilde{x}_{\{1, l+1\}})_{1 \leq l \leq r}$  and

$u_{\{1, 1\}} = \tilde{x}_{\{1, 1\}} - \sum_{k=1}^r (u_{\{1, k+1\}})^2 \geq 0$ ;

Sample independently  $r$  normal variables  $G_2, \dots, G_{r+1} \sim \mathcal{N}(0, 1)$  and  $(U_t^u)_{\{1, 1\}}$  as a CIR process at time  $t$  starting from  $u_{\{1, 1\}}$  solving  $d(U_t^u)_{\{1, 1\}} = (\alpha - r)dt + 2\sqrt{(U_t^u)_{\{1, 1\}}}dZ_t^1$  (See Glasserman [10]).

Set  $(U_t^u)_{\{1, l+1\}} = u_{\{1, l+1\}} + \sqrt{t}G_{l+1}$ ;

**return**  $X =$

$$\pi^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix} \begin{pmatrix} (U_t^u)_{\{1, 1\}} + \sum_{k=1}^r ((U_t^u)_{\{1, k+1\}})^2 & ((U_t^u)_{\{1, l+1\}})_{1 \leq l \leq r}^T & 0 \\ ((U_t^u)_{\{1, l+1\}})_{1 \leq l \leq r} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r^T & k_r^T \\ 0 & 0 & I_{d-r-1} \end{pmatrix} \pi.$$


---

Let us discuss now the complexity of this algorithm. The number of operations required by the extended Cholesky decomposition is of order  $O(d^3)$ . From a computational point of view, the permutation is handled directly and does not require any matrix multiplication so that we can consider w.l.o.g. that  $\pi = I_d$ . Since  $c_r$  is lower triangular, the calculation of  $u_{\{1, i\}}$ ,  $i = 1, \dots, r + 1$  only requires  $O(d^2)$  operations. Also, we do not perform in practice the matrix product (25), but only compute the values of  $X_{\{1, i\}}$  for  $i = 1, \dots, d$ , which requires also  $O(d^2)$  operations. Last,  $d$  samples are at most required. To sum up, it comes out that the complexity of the whole algorithm is of order  $O(d^3)$ .

## 2.3 Exact simulation for Wishart processes

We have now shown all the mathematical results that enable us to give an exact simulation method for general Wishart processes. This is made in two steps. First, by using the remarkable splitting (Proposition 11) and the exact scheme for  $WIS_d(x, \alpha, 0, I_d^1; t)$  (Theorem 13 and Corollary 15), we get an exact simulation scheme for  $WIS_d(x, \alpha, 0, I_d^n; t)$ . As we will see, it is related but extends the Bartlett's decomposition of central Wishart distribution that dates back to 1933. Then, by using the identity in law (12), we are able to sample any Wishart distribution  $WIS_d(x, \alpha, b, a; t)$ .

### 2.3.1 Exact simulation for $WIS_d(x, \alpha, 0, I_d^n; t)$

By gathering the results obtained in the subsections 2.1 and 2.2, we get a way to sample exactly the distribution  $WIS_d(x, \alpha, 0, I_d^n; t)$ . Indeed, thanks to Theorem 13 and Corollary 15 we know how to sample exactly  $WIS_d(x, \alpha, 0, I_d^1; t)$ . By a simple permutation of the first and  $k^{\text{th}}$  coordinates, we are then also able to sample according to  $WIS_d(x, \alpha, 0, e_d^k; t)$  for  $k \in \{1, \dots, d\}$ . Thus, we get by Proposition 11 an exact

simulation method to sample  $WIS_d(x, \alpha, 0, I_d^n; t)$ . It is given explicitly in the algorithm below.

---

**Algorithm 2:** Exact simulation for  $WIS_d(x, \alpha, 0, I_d^n; t)$

---

**Input:**  $x \in \mathcal{S}_d^+(\mathbb{R})$ ,  $n \leq d$ ,  $\alpha \geq d - 1$  and  $t > 0$ .

**Output:**  $X$ , sampled according to  $WIS_d(x, \alpha, 0, I_d^n; t)$

$y = x$

**for**  $k = 1$  **to**  $n$  **do**

    Set  $p_{k,1} = p_{1,k} = p_{i,i} = 1$  for  $i \notin \{1, k\}$ , and  $p_{i,j} = 0$  otherwise (permutation of the first and  $k^{\text{th}}$  coordinates).

$y = pYp$  where  $Y$  is sampled according to  $WIS_d(pyp, \alpha, 0, I_d^1; t)$  by using Algorithm 1.

**end**

**return**  $X = y$ .

---

Since this algorithm basically runs  $n$  times Algorithm 1, it requires a complexity of order  $O(nd^3)$  and therefore at most of order  $O(d^4)$ . As we have seen, the ‘‘bottleneck’’ of Algorithm 1 is the extended Cholesky decomposition which is in  $O(d^3)$ . All the other steps in Algorithm 1 require at most  $O(d^2)$  operations. A natural question for Algorithm 2 is to wonder if we can reuse the Cholesky decomposition between the for loops instead of calculating it from scratch. For example, if it were possible to get the Cholesky decomposition of loop  $k + 1$  from the one of loop  $k$  at a cost  $O(d^2)$ , the complexity of Algorithm 2 would then drop to  $O(d^3)$ . Despite our investigations, we have not been able to do so up to now.

**Remark 16** — When  $\alpha \geq 2d - 1$ , it is possible to sample  $WIS_d(x, \alpha, 0, I_d^n; t)$  in  $O(d^3)$  by another mean. If  $X_t^1 \sim WIS_d(x, d, 0, I_d^n; t)$  and  $X_t^2 \sim WIS_d(0, \alpha - d, 0, I_d^n; t)$  are independent, we can check that  $X_t^1 + X_t^2 \sim WIS_d(x, \alpha, 0, I_d^n; t)$ . Then,  $X_t^1$  can be sampled by using Proposition 30 and  $X_t^2$  by using Corollary 18 since  $X_t^2 \stackrel{Law}{=} tWIS_d(0, \alpha - d, 0, I_d^n; 1)$  from (11).

**Remark 17** — Let  $x \in \mathcal{S}_d^{+,*}(\mathbb{R})$ . From Remark 14, we get that for each  $k \in \{1, \dots, n\}$ ,  $X_t^{k, \dots, X_t^1, x} \in \mathcal{S}_d^{+,*}(\mathbb{R})$ , a.s.. In that case, the extended Cholesky decomposition that has to be computed at each step is an usual Cholesky decomposition.

### 2.3.2 The Bartlett’s decomposition revisited

Now, we would like to illustrate our exact simulation method on the particular case  $WIS_d(0, \alpha, 0, I_d^n; 1)$ , which is known in the literature as the central Wishart distribution. In that case, we can perform explicitly the composition given by Proposition 11. We will show by an induction on  $n$  that:

$$X_1^{n, \dots, X_1^{1,0}} = \begin{pmatrix} (L_{i,j})_{1 \leq i, j \leq n} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (L_{i,j}^T)_{1 \leq i, j \leq n} & 0 \\ 0 & 0 \end{pmatrix},$$

where  $(L_{i,j})_{1 \leq j < i \leq d}$  and  $L_{i,i}$  are independent random variables such that  $L_{i,j} \sim \mathcal{N}(0, 1)$  and  $(L_{i,i})^2 \sim \chi^2(\alpha - i + 1)$ , and  $L_{i,j} = 0$  for  $i < j$ .

For  $n = 1$ , we know from Theorem 13 that  $(X_1^{1,0})_{1,1} \sim \chi^2(\alpha)$  since  $d(X_t^{1,0})_{1,1} = \alpha dt + 2\sqrt{(X_t^{1,0})_{1,1}} dZ_t^1$  with  $(X_0^{1,0})_{1,1} = 0$ , and all the other elements are equal to 0. Let us assume now that the induction hypothesis is satisfied for  $n - 1$ . Then, we can apply once again Theorem 13 (up to the permutation of the first and  $n^{\text{th}}$  coordinates). We have  $\text{Rk}(X_1^{n-1, \dots, X_1^{1,0}}) = n - 1$ , a.s., and the Cholesky decomposition is directly given by  $(L_{i,j})_{1 \leq i, j \leq n-1}$ . Then, we get from (25) that there are independent variables  $L_{n,n}^2 \sim \chi^2(\alpha - n + 1)$  and

$L_{n,i} \sim \mathcal{N}(0, 1)$  for  $i \in \{1, \dots, n-1\}$  such that  $X_1^{n, \dots, x_1^{1,0}} =$

$$\begin{pmatrix} (L_{i,j})_{1 \leq i, j \leq n-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{d-n} \end{pmatrix} \begin{pmatrix} I_{n-1} & (L_{n,i})_{1 \leq i \leq n-1} & 0 \\ (L_{n,i})_{1 \leq i \leq n-1}^T & \sum_{i=1}^n L_{n,i}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (L_{i,j})_{1 \leq i, j \leq n-1}^T & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{d-n} \end{pmatrix},$$

$$\text{Since } \begin{pmatrix} I_{n-1} & (L_{n,i})_{1 \leq i \leq n-1} & 0 \\ (L_{n,i})_{1 \leq i \leq n-1}^T & \sum_{i=1}^n L_{n,i}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{n-1} & 0 & 0 \\ (L_{n,i})_{1 \leq i \leq n-1}^T & L_{n,n} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I_{n-1} & (L_{n,i})_{1 \leq i \leq n-1} & 0 \\ 0 & L_{n,n} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we conclude by induction and get the following result which is known as the Bartlett's decomposition (see Kshirsagar [17] or Kabe [16]).

**Corollary 18** — *Bartlett's Decomposition of central Wishart distributions.* With the notations above,

$$\begin{pmatrix} (L_{i,j})_{1 \leq i, j \leq n} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (L_{i,j}^T)_{1 \leq i, j \leq n} & 0 \\ 0 & 0 \end{pmatrix} \sim WIS_d(0, \alpha, 0, I_d^n; 1).$$

Therefore, the identity given by Proposition 11 that we have obtained by a remarkable splitting of the infinitesimal generator extends in a natural way the Bartlett's decomposition to non-central Wishart distributions.

### 2.3.3 Exact simulation for $WIS_d(x, \alpha, b, a; t)$

Now, thanks to the identity in law (12), we get an exact simulation scheme for  $WIS_d(x, \alpha, b, a; t)$ . The associated algorithm is given below. To the best of our knowledge, this is the first exact simulation method for non-central Wishart distributions that works for any  $\alpha \geq d-1$ . The existing methods in the literature mainly focus on the case  $\alpha \in \mathbb{N}$  that arises naturally in statistics. Namely, Odell and Feiveson [21] and Smith and Hocking [22] have proposed an exact simulation method for central Wishart distributions based on the Bartlett's decomposition. Gleser [11] has given an exact simulation method for non-central Wishart distributions, also when the degree  $\alpha$  is an integer.

---

**Algorithm 3:** Exact simulation for  $WIS_d(x, \alpha, b, a; t)$

---

**Input:**  $x \in \mathcal{S}_d^+(\mathbb{R})$ ,  $\alpha \geq d-1$ ,  $a, b \in \mathcal{M}_d(\mathbb{R})$  and  $t > 0$ .

**Output:**  $X$ , sampled according to  $WIS_d(x, \alpha, b, a; t)$ .

Calculate  $q_t = \int_0^t \exp(sb) a^T a \exp(sb^T) ds$  and  $(p, c_n, k_n)$  an extended Cholesky decomposition of  $q_t/t$ .

Set  $\theta_t = p^{-1} \begin{pmatrix} c_n & 0 \\ k_n & I_{d-n} \end{pmatrix}$  and  $m_t = \exp(tb)$ .

**return**  $X = \theta_t Y \theta_t^T$ , where  $Y \sim WIS_d(\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T, \alpha, 0, I_d^n; t)$  is sampled by Algorithm 2.

---

## 3 High order discretization schemes for Wishart and semi definite positive affine processes

Up to now we have given an exact simulation method for Wishart processes. It relies on a remarkable splitting of the infinitesimal generator given in Theorem 10. When dealing with discretization schemes, splitting operators is a powerful technique to construct schemes for SDEs from other schemes obtained on simpler SDEs. This idea of splitting originates from the seminal work of Strang [23] in the field of ODEs. As pointed by Ninomiya and Victoir [20] or Alfonsi [1], it is rather easy to analyse the weak error of schemes obtained by splitting, simply by using the same arguments as Talay and Tubaro [24] for the Euler-Maruyama scheme. Thus, we will only focus on the analysis of the weak error, i.e. the error made on marginal distributions.

Up to our knowledge, there are very few papers in the literature that deal with discretization schemes for Wishart processes. Recently, Benabid et al. [2] have proposed a Monte-Carlo method to calculate expectations on Wishart processes which is based on a Girsanov change of probability. Gauthier and Possamaï [9] introduce a moment-matching scheme for Wishart processes. Both methods are well defined under some restrictions on the parameters, and there is no theoretical result on their accuracy. Currently, Teichmann [25] is working on dedicated schemes for general affine processes by approximating their characteristic functions.

This section is structured as follows. First, we recall basic results on the splitting technique to get discretization schemes for SDEs. We will take the same framework as Alfonsi [1] since it is somehow designed for affine processes. Then, we will explain how to get high order schemes for  $WIS_d(x, \alpha, 0, I_d^n)$  from the remarkable splitting (13). From this result, we will be able to get a second order scheme for any semi definite positive affine processes and a third order scheme for Wishart processes.

### 3.1 General results on discretization schemes for SDEs

In this paragraph, we try to present the minimal knowledge that we will use later to get schemes for affine processes. We take back the framework developed in Alfonsi ([1], Section 1) and refer to this paper for further details and proofs.

Let us start with some notations. We consider a domain  $\mathbb{D} \subset \mathbb{R}^\zeta$ ,  $\zeta \in \mathbb{N}^*$ . For a multi-index  $\gamma = (\gamma_1, \dots, \gamma_\zeta) \in \mathbb{N}^\zeta$ , we define  $\partial_\gamma = \partial_1^{\gamma_1}, \dots, \partial_\zeta^{\gamma_\zeta}$  and  $|\gamma| = \sum_{i=1}^\zeta \gamma_i$  and set:

$$\mathcal{C}_{\mathbf{pol}}^\infty(\mathbb{D}) = \{f \in \mathcal{C}^\infty(\mathbb{D}, \mathbb{R}), \forall \gamma \in \mathbb{N}^\zeta, \exists C_\gamma > 0, e_\gamma \in \mathbb{N}^*, \forall x \in \mathbb{D}, |\partial_\gamma f(x)| \leq C_\gamma(1 + \|x\|^{e_\gamma})\},$$

where  $\|\cdot\|$  is a norm on  $\mathbb{R}^\zeta$ . We will say that  $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^\zeta}$  is a *good sequence* for  $f \in \mathcal{C}_{\mathbf{pol}}^\infty(\mathbb{D})$  if one has  $|\partial_\gamma f(x)| \leq C_\gamma(1 + \|x\|^{e_\gamma})$ . Mainly (but not only), we will consider in this paper  $\mathbb{D} = \mathcal{S}_d^+(\mathbb{R})$  as a subspace of  $\mathcal{S}_d(\mathbb{R}) \simeq \mathbb{R}^{d(d+1)/2}$ . In that case, it is natural to keep the same indexation as for matrices, and we rather use the notation  $\partial_\gamma = \prod_{1 \leq i \leq j \leq d} \partial_{\{i,j\}}^{\gamma_{\{i,j\}}}$  for  $\gamma = (\gamma_{\{i,j\}})_{1 \leq i \leq j \leq d} \in \mathbb{N}^{d(d+1)/2}$ .

**Definition 19** — Let  $b : \mathbb{D} \rightarrow \mathbb{R}^\zeta$ ,  $\sigma : \mathbb{D} \rightarrow \mathcal{M}_\zeta(\mathbb{R})$ . The operator  $L$  defined for  $f \in \mathcal{C}^2(\mathbb{D}, \mathbb{R})$  by:

$$Lf(x) = \sum_{i=1}^\zeta b_i(x) \partial_i f(x) + \frac{1}{2} \sum_{i,j=1}^\zeta (\sigma \sigma^T)_{i,j}(x) \partial_i \partial_j f(x) \quad (29)$$

is said to satisfy the required assumptions on  $\mathbb{D}$  if the following conditions hold:

- $\forall i, j \in \{1, \dots, \zeta\}, b_i(x), (\sigma \sigma^T)_{i,j}(x) \in \mathcal{C}_{\mathbf{pol}}^\infty(\mathbb{D})$ ,
- for any  $x \in \mathbb{D}$ , the SDE  $X_t^x = x + \int_0^t b(X_s^x) ds + \int_0^t \sigma(X_s^x) dW_s$  has a unique weak solution defined for  $t \geq 0$ , and thus  $\mathbb{P}(\forall t \geq 0, X_t^x \in \mathbb{D}) = 1$ .

In the case of affine diffusions,  $b_i(x)$  and  $(\sigma \sigma^T)_{i,j}(x)$  are affine functions of  $x$  and the operator satisfies the required assumption on the appropriated domain. Let us stress that if  $L$  satisfies the required assumption on  $\mathbb{D}$  and  $f \in \mathcal{C}_{\mathbf{pol}}^\infty(\mathbb{D})$ , all the iterated functions  $L^k f(x)$  are well defined on  $\mathbb{D}$  and belong to  $\mathcal{C}_{\mathbf{pol}}^\infty(\mathbb{D})$  for any  $k \in \mathbb{N}$ .

Let us turn now to discretization schemes. We will consider in this paper a final time horizon  $T > 0$  and the regular time grid defined by  $t_i^N = iT/N$ ,  $i = 0, \dots, N$ . When discretizing a Markovian process, a scheme is usually described as a way to sample the process at the next time step from its current position. Thus, we will say that  $(\hat{p}_x(t)(dz), t > 0, x \in \mathbb{D})$  is a *family of transition probabilities* on  $\mathbb{D}$  if  $\hat{p}_x(t)$  is a probability law on  $\mathbb{D}$  for any time-step  $t > 0$  and any position  $x \in \mathbb{D}$ . Then, a *discretization scheme* on the regular time grid associated to this family starting from  $x_0 \in \mathbb{D}$  is simply a sequence  $(\hat{X}_{t_i^N}^N, 0 \leq i \leq N)$  of  $\mathbb{D}$ -valued random variables such that:

- $\hat{X}_{t_0^N}^N = x_0$ ,



- the law of  $\hat{X}_{t_{i+1}^N}^N$  is sampled according to  $\hat{p}_{\hat{X}_{t_i^N}^N}(T/N)(dz)$  independently from the previous samples, i.e.  $\mathbb{E}[f(\hat{X}_{t_{i+1}^N}^N) | (\hat{X}_{t_j^N}^N, 0 \leq j \leq i)] = \int_{\mathbb{D}} f(z) \hat{p}_{\hat{X}_{t_i^N}^N}(T/N)(dz)$  for any bounded measurable function  $f : \mathbb{D} \rightarrow \mathbb{R}$ .

Up to the initial condition, the discretization scheme  $(\hat{X}_{t_i^N}^N, 0 \leq i \leq N)$  is entirely characterized by  $(\hat{p}_x(t)(dz), t > 0, x \in \mathbb{D})$ . With a slight abuse of language, we will then say that  $(\hat{p}_x(t)(dz), t > 0, x \in \mathbb{D})$  is a *scheme* on  $\mathbb{D}$ . We will denote by  $\hat{X}_t^x$  a random variable on  $\mathbb{D}$  which is sampled according to  $\hat{p}_x(t)$ .

**Definition 20** — *Let  $L$  be an operator that satisfies the required assumption on  $\mathbb{D}$ . A scheme  $(\hat{p}_x(t)(dz), t > 0, x \in \mathbb{D})$  is a potential weak  $\nu$ th-order scheme for the operator  $L$  if for any function  $f \in \mathcal{C}_{\mathbf{pol}}^\infty(\mathbb{D})$  with a good sequence  $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^\zeta}$ , there exist positive constants  $C$ ,  $E$ , and  $\eta$  depending only on  $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^\zeta}$  such that*

$$\forall t \in (0, \eta), \left| \mathbb{E}[f(\hat{X}_t^x)] - \left[ f(x) + \sum_{k=1}^{\nu} \frac{1}{k!} t^k L^k f(x) \right] \right| \leq Ct^{\nu+1}(1 + \|x\|^E). \quad (30)$$

When the coefficients of the SDE have a sublinear growth, we can check that the exact scheme (i.e.  $\hat{X}_t^x = X_t^x$ , where  $(X_t^x, t \geq 0)$  solves the SDE associated to  $L$ ) is a potential  $\nu$ th-order scheme for any order  $\nu \in \mathbb{N}$  ([1], Proposition 1.12). In that case, (30) is then clearly equivalent to:

$$\exists C, E, \eta > 0, \forall t \in (0, \eta), \left| \mathbb{E}[f(\hat{X}_t^x)] - \mathbb{E}[f(X_t^x)] \right| \leq Ct^{\nu+1}(1 + \|x\|^E).$$

In practice, having a potential weak  $\nu$ th-order scheme for the operator  $L$  is the main requirement to get a weak error of order  $\nu$ . This is precised by the following theorem that relies on the idea developed by Talay and Tubaro [24].

**Theorem 21** — *Let us consider an operator  $L$  that satisfies the required assumptions on  $\mathbb{D}$  and a discretization scheme  $(\hat{X}_{t_i^N}^N, 0 \leq i \leq N)$  with transition probabilities  $\hat{p}_x(t)(dz)$  on  $\mathbb{D}$  that starts from  $\hat{X}_{t_0^N}^N = x_0 \in \mathbb{D}$ . We assume that*

1.  $\hat{p}_x(t)(dz)$  is a potential weak  $\nu$ th-order scheme for the operator  $L$ ,
2. the scheme has bounded moments, i.e.:

$$\forall q \in \mathbb{N}^*, \exists N(q) \in \mathbb{N}, \sup_{N \geq N(q), 0 \leq i \leq N} \mathbb{E}[\|\hat{X}_{t_i^N}^N\|^q] < \infty, \quad (31)$$

3.  $f : \mathbb{D} \rightarrow \mathbb{R}$  is a function such that  $u(t, x) = \mathbb{E}[f(X_{T-t}^x)]$  is defined on  $[0, T] \times \mathbb{D}$ ,  $\mathcal{C}^\infty$ , solves  $\forall t \in [0, T], \forall x \in \mathbb{D}, \partial_t u(t, x) = -Lu(t, x)$ , and satisfies:

$$\forall l \in \mathbb{N}, \gamma \in \mathbb{N}^\zeta, \exists C_{l,\gamma}, e_{l,\gamma} > 0, \forall x \in \mathbb{D}, t \in [0, T], |\partial_t^l \partial_\gamma u(t, x)| \leq C_{l,\gamma}(1 + \|x\|^{e_{l,\gamma}}). \quad (32)$$

Then, there is  $K > 0$ ,  $N_0 \in \mathbb{N}$ , such that  $|\mathbb{E}[f(\hat{X}_{t_i^N}^N)] - \mathbb{E}[f(X_T^{x_0})]| \leq K/N^\nu$  for  $N \geq N_0$ .

Let us mention that condition (31) is slightly weakened with respect to Theorem 1.9 in [1]. However, we can check that (31) is in fact sufficient to make work the proof of this theorem given in [1].

Let us comment briefly the hypothesis of this theorem. The boundedness of the moments (31) holds as soon as the coefficients of the SDE have a sublinear growth (see Lemma 36 in Appendix D). It is instead much more technical to get the bounds (32) on the derivatives of  $u$ . In their original paper, Talay and Tubaro have considered stronger assumptions on  $b$  and  $\sigma$  (namely,  $\mathcal{C}^\infty$  with bounded derivatives) to get such bounds. In the case of Wishart processes, we are able to get (32) when  $f \in \mathcal{C}_{\mathbf{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$  by using the explicit

formula of the characteristic function (10). This is stated in Proposition 43. We however deem that this result still holds for the general affine case.

Now, we present results that explain how the property of being a potential weak  $\nu$ th-order scheme can be preserved when we use the splitting technique. To do so, we need to introduce first the composition of schemes. Let us consider two transition probabilities  $\hat{p}_x^1(t)(dz)$  and  $\hat{p}_x^2(t)(dz)$  on  $\mathbb{D}$ . Then, we define:

$$\hat{p}^2(t_2) \circ \hat{p}_x^1(t_1)(dz) := \int_{\mathbb{D}} \hat{p}_y^1(t_2)(dz) \hat{p}_x^1(t_1)(dy),$$

which is the law obtained when one uses first use scheme 1 with a time step  $t_1$  and then scheme 2 with a time step  $t_2$  with independent samples. More generally, if one has  $m$  transition probabilities  $\hat{p}_x^1, \dots, \hat{p}_x^m$  on  $\mathbb{D}$ , we define

$$\hat{p}^m(t_m) \circ \dots \circ \hat{p}_x^1(t_1)(dz) := \hat{p}^m(t_m) \circ (\hat{p}^{m-1}(t_{m-1}) \circ \dots \circ \hat{p}_x^1(t_1)(dz)).$$

**Proposition 22** — *Let  $L_1, L_2$  be two operators satisfying the required assumption on  $\mathbb{D}$ . Let  $\hat{p}_x^1$  and  $\hat{p}_x^2$  be respectively two potential weak  $\nu$ th-order schemes on  $\mathbb{D}$  for  $L_1$  and  $L_2$ .*

- *If  $L_1 L_2 = L_2 L_1$ ,  $\hat{p}^2(t) \circ \hat{p}_x^1(t)(dz)$  is a potential weak  $\nu$ th-order discretization scheme for  $L_1 + L_2$ .*
- *If  $\nu \geq 2$ ,  $\hat{p}^2(t/2) \circ \hat{p}_x^1(t) \circ \hat{p}_x^2(t/2)$  and  $\frac{1}{2}(\hat{p}^2(t) \circ \hat{p}_x^1(t) + \hat{p}_x^1(t) \circ \hat{p}_x^2(t))$  are potential weak second order schemes for  $L_1 + L_2$ .*

## 3.2 High order schemes for Wishart processes

In this paragraph, we will give a way to get weak  $\nu$ th-order schemes for any Wishart processes. The construction of these schemes is the same as the one used for the exact scheme. First we obtain a  $\nu$ th-order scheme for  $WIS_d(x, \alpha, 0, I_d^1)$ . Then, we get a  $\nu$ th-order scheme for  $WIS_d(x, \alpha, 0, I_d^r)$  by scheme composition. Last, we use the identity in law (12) to get a weak  $\nu$ th-order scheme for any Wishart processes.

### 3.2.1 A potential weak $\nu$ th-order scheme associated to $L_1$

In this paragraph, we present a potential weak  $\nu$ th-order scheme for the generator of  $WIS_d(x, \alpha, 0, I_d^1)$ . Roughly speaking, we obtain this scheme from the exact scheme given by Theorem 13 and Corollary 15 by replacing the Gaussian random variables with moment matching variables and the exact CIR distribution with a sample according to a potential weak  $\nu$ -th order scheme for the CIR.

**Theorem 23** — *Let  $x \in \mathcal{S}_d^+(\mathbb{R})$  and  $(c_r, k_r, p)$  be an extended Cholesky decomposition of  $(x_{i,j})_{2 \leq i, j \leq d}$ . We set  $\pi = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$  and  $\tilde{x} = \pi x \pi^T$ , so that  $\tilde{x} = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix} \begin{pmatrix} c_r^T & k_r^T \\ 0 & 0 \end{pmatrix}$ . Like in Theorem 13, we have*

$$u_{\{1,1\}} = \tilde{x}_{\{1,1\}} - \sum_{k=1}^r (u_{\{1,k+1\}})^2 \geq 0, \text{ where } (u_{\{1,l+1\}})_{1 \leq l \leq r} = c_r^{-1}(\tilde{x}_{\{1,l+1\}})_{1 \leq l \leq r},$$

and we set  $u_{\{1,i\}} = 0$  if  $r+2 \leq i \leq d$  and  $u_{\{i,j\}} = \tilde{x}_{\{i,j\}}$  if  $i, j \geq 2$ . Let  $(\hat{G}^i)_{1 \leq i \leq r}$  be a sequence of independent real variables with finite moments of any order such that:

$$\forall i \in \{1, \dots, r\}, \forall k \leq 2\nu + 1, \mathbb{E}[(\hat{G}^i)^k] = \mathbb{E}[G^k], \text{ where } G \sim \mathcal{N}(0, 1).$$

Let  $(\hat{U}_t^u)_{\{1,1\}}$  be sampled independently according to a potential weak  $\nu$ th-order scheme for the CIR process  $d(U_t^u)_{\{1,1\}} = (\alpha - r)dt + 2\sqrt{(U_t^u)_{\{1,1\}}}dZ_t^1$  starting from  $u_{\{1,1\}}$  and that satisfies the immersion property (Definition 37). We set:

$$(\hat{U}_t^u)_{\{1,i\}} = u_{\{1,i\}} + \sqrt{t}\hat{G}^i, \quad 2 \leq i \leq r+1, \quad (\hat{U}_t^u)_{\{1,i\}} = 0, \quad r+2 \leq i \leq d, \quad (\hat{U}_t^u)_{\{i,j\}} = u_{\{i,j\}} \text{ if } i, j \geq 2.$$

Then, the scheme defined by  $\hat{X}_t^x = \pi^T h_r(\hat{U}_t^u)\pi$  is a potential  $\nu$ th-order scheme for  $L_1$ , where the function  $h_r$  is defined by (28).

The proof of this result is technical and requires further definitions. It is left in Appendix 23. Second and third order schemes for the CIR process that satisfy the immersion property can be found in Alfonsi [1] (see Corollary 40). We can therefore get second (resp. third) order schemes for  $L_1$  by taking any variables that matches the five (resp. the seven) first moments of  $\mathcal{N}(0, 1)$ . This can be obtained by taking

$$\mathbb{P}(\hat{G}^i = \sqrt{3}) = \mathbb{P}(\hat{G}^i = -\sqrt{3}) = \frac{1}{6} \text{ and } \mathbb{P}(\hat{G}^i = 0) = \frac{2}{3} \quad (33)$$

$$\text{(resp. } \mathbb{P}\left(\hat{G}^i = \varepsilon\sqrt{3 + \sqrt{6}}\right) = \frac{\sqrt{6} - 2}{4\sqrt{6}}, \mathbb{P}\left(\hat{G}^i = \varepsilon\sqrt{3 - \sqrt{6}}\right) = \frac{1}{2} - \frac{\sqrt{6} - 2}{4\sqrt{6}}, \varepsilon \in \{-1, 1\}\text{)}. \quad (34)$$

### 3.2.2 A potential weak $\nu$ th-order scheme associated to $WIS_d(x, \alpha, 0, I_d^n)$

From Theorem 10, we know that the infinitesimal generator of  $WIS_d(x, \alpha, 0, I_d^n)$  is given by  $L = \sum_{i=1}^n L_i$ , where the operators  $L_i$  are the same as  $L_1$  up to the permutation of the first and  $i^{\text{th}}$  coordinate. Moreover, these operators are commuting. Thus, if  $\pi^{1 \leftrightarrow i}$  denotes the associated permutation matrix and  $\hat{X}_t^x$  is the potential  $\nu$ th-order scheme defined by Theorem 23,

$$\pi^{1 \leftrightarrow i} \hat{X}_t^x \pi^{1 \leftrightarrow i} \pi^{1 \leftrightarrow i} \pi^{1 \leftrightarrow i} \pi^{1 \leftrightarrow i} \text{ is a potential } \nu\text{th-order scheme for } L_i.$$

We then get a potential  $\nu$ th-order scheme for  $L$  thanks to Proposition 22.

**Corollary 24** — Let  $\hat{p}_x^i, i \in \{1, \dots, d\}$  be potential weak  $\nu$ th-order scheme for  $L_i$ . Then for any  $n \leq d$ ,

$$\hat{p}^n(t) \circ \dots \circ \hat{p}_x^1(t)(dz)$$

defines a potential weak  $\nu$ th-order scheme for  $WIS_d(x, \alpha, 0, I_d^n)$ .

**Remark 25** — Let  $\varepsilon > 0$  and  $\hat{X}_t^x$  be a potential weak  $\nu$ th-order scheme for  $WIS_d(x, \alpha, 0, I_d^n)$ . Since  $x + \varepsilon t^{\nu+1} I_d$  is a potential weak  $\nu$ th-order scheme for the operator  $L = 0$ , we easily get by scheme composition (Proposition 22) that  $\hat{X}_t^{x + \varepsilon t^{\nu+1} I_d}$  is also a potential weak  $\nu$ th-order scheme for  $WIS_d(x, \alpha, 0, I_d^n)$ . This scheme starts from an invertible initial condition, and by Remark 14, we only make in that case usual Cholesky decompositions on invertible matrices.

### 3.2.3 A weak $\nu$ th-order scheme for Wishart processes $WIS_d(x, \alpha, b, a)$

Now that we have a potential  $\nu$ th-order scheme for  $WIS_d(x, \alpha, 0, I_d^n)$ , we are in position to construct a scheme for any Wishart process  $WIS_d(x, \alpha, b, a)$  thanks to the identity (12). Unfortunately, we need to make some technical restrictions on  $a$  and  $b$  (namely,  $a \in \mathcal{G}_d(\mathbb{R})$  or  $ba^T a = a^T a b$ ) to show that we get like this a potential  $\nu$ th-order scheme. We however believe that this is rather due to our analysis of the error and that the scheme converges as well without this restriction. We mention in addition that we give in the next section a second order scheme based on Corollary 8 for which we can make our error analysis for any parameters.

**Proposition 26** — Let  $t > 0, a, b \in \mathcal{M}_d(\mathbb{R})$  and  $\alpha \geq d-1$ . Let  $m_t = \exp(tb), q_t = \int_0^t \exp(sb) a^T a \exp(sb^T) ds$  and  $n = \text{Rk}(a^T a)$ . We assume that either  $a \in \mathcal{G}_d(\mathbb{R})$  or  $b$  and  $a^T a$  commute. We define

- if  $n = d$ ,  $\theta_t$  as the (usual) Cholesky decomposition of  $q_t/t$ ,

- if  $n < d$ ,  $\theta_t = \sqrt{\frac{1}{t} \int_0^t \exp(sb) \exp(sb^T) ds}^{-1} \begin{pmatrix} c_n & 0 \\ k_n & I_{d-n} \end{pmatrix}$  where  $(c_n, k_n, p)$  is the extended Cholesky decomposition of  $a^T a$  otherwise.

In both cases,  $\theta_t \in \mathcal{G}_d(\mathbb{R})$ . Let  $\hat{Y}_t^y$  denote a potential weak  $\nu$ th-order scheme for  $WIS_d(y, \alpha, 0, I_d^n)$ . Then, the scheme defined by

$$\hat{X}_t^x = \theta_t \hat{Y}_t^{\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T} \theta_t^T, \quad (35)$$

is a potential weak  $\nu$ th-order scheme for  $WIS_d(x, \alpha, b, a)$ .

*Proof* : First, let us check that  $\theta_t \in \mathcal{G}_d(\mathbb{R})$  is well defined, such that  $q_t/t = \theta_t I_d^n \theta_t^T$  and satisfies:

$$\exists K, \eta > 0, \forall t \in (0, \eta), \max(\|\theta_t\|, \|\theta_t\|^{-1}) \leq K. \quad (36)$$

When  $n = d$ ,  $q_t/t$  is definite positive as a convex combination of definite positive matrices and the usual Cholesky decomposition is well defined. Moreover, (36) holds since  $q_t/t$  goes to  $a^T a$  which is invertible when  $t \rightarrow 0^+$ . When  $n < d$ , we have assumed in addition that  $b$  and  $a^T a$  commute. Therefore,  $q_t = a^T a (\int_0^t \exp(sb) \exp(sb^T) ds / t)$ . Since  $a^T a$  and  $(\int_0^t \exp(sb) \exp(sb^T) ds / t)$  are positive semidefinite matrices that commute, we have

$$q_t = \sqrt{\frac{1}{t} \int_0^t \exp(sb) \exp(sb^T) ds} a^T a \sqrt{\frac{1}{t} \int_0^t \exp(sb) \exp(sb^T) ds}.$$

Once again,  $\frac{1}{t} \int_0^t \exp(sb) \exp(sb^T) ds$  is definite positive as a convex combination of definite positive matrices and we get that  $\theta_t = \sqrt{\frac{1}{t} \int_0^t \exp(sb) \exp(sb^T) ds}^{-1} \begin{pmatrix} c_n & 0 \\ k_n & I_{d-n} \end{pmatrix} \in \mathcal{G}_d(\mathbb{R})$  satisfies  $q_t/t = \theta_t I_d^n \theta_t^T$  by Lemma 33. Similarly, (36) holds since  $p^{-1} \begin{pmatrix} c_n & 0 \\ k_n & I_{d-n} \end{pmatrix}$  does not depend on  $t$  and  $\sqrt{\frac{1}{t} \int_0^t \exp(sb) \exp(sb^T) ds}$  goes to  $I_d$  when  $t \rightarrow 0^+$ .

Let  $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$ . Let  $X_t^x \sim WIS_d(x, \alpha, b, a; t)$ . Since the exact scheme is a potential  $\nu$ th-order scheme, there are constants  $C, E, \eta > 0$  depending only on a good sequence of  $f$  such that

$$\forall t \in (0, \eta), |\mathbb{E}[f(X_t^x)] - \sum_{k=0}^{\nu} \frac{t^k}{k!} L^k f(x)| \leq C t^{\nu+1} (1 + \|x\|^E). \quad (37)$$

On the other hand we have from Proposition 9,

$$\mathbb{E}[f(\hat{X}_t^x)] - \mathbb{E}[f(X_t^x)] = \mathbb{E}[f(\theta_t \hat{Y}_t^{\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T} \theta_t^T)] - \mathbb{E}[f(\theta_t Y_t^{\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T} \theta_t^T)]. \quad (38)$$

Let us introduce  $f_{\theta_t}(y) := f(\theta_t y \theta_t^T) \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$ . By the chain rule, we have  $\partial_{\{i,j\}} f_{\theta_t}(y) = \text{Tr}[\theta_t (e_d^{i,j} + \mathbb{1}_{i \neq j} e_d^{j,i}) \theta_t^T \partial f(\theta_t y \theta_t^T)]$ , where  $(\partial f(x))_{k,l} = (\mathbb{1}_{k=l} + \frac{1}{2} \mathbb{1}_{k \neq l}) \partial_{\{k,l\}} f(x)$  and  $e_d^{i,j} = (\mathbb{1}_{k=i, l=j})_{1 \leq k, l \leq d}$ . From (36), we see that there is a good sequence  $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^{d(d+1)/2}}$  that can be obtained from a good sequence of  $f$  such that:

$$\forall t \in (0, \eta), \forall y \in \mathcal{S}_d^+(\mathbb{R}), |\partial_\gamma f_{\theta_t}(y)| \leq C_\gamma (1 + \|y\|^{e_\gamma}).$$

Therefore, we get that there are constants still denoted by  $C, E, \eta > 0$  such that

$$\forall t \in (0, \eta), \left| \mathbb{E}[f(\theta_t \hat{Y}_t^{\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T} \theta_t^T)] - \mathbb{E}[f(\theta_t Y_t^{\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T} \theta_t^T)] \right| \leq C t^{\nu+1} (1 + \|\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T\|^E). \quad (39)$$

From (36), we get that there is a constant  $K' > 0$  such that  $\|\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T\|^E \leq K' \|x\|^E$  for  $t \in (0, \eta)$ . Thus, we get the result by gathering (37), (38) and (39).  $\square$

Now, we want to show thanks to Theorem 21 that the scheme given by Proposition 26 gives indeed a weak error of order  $\nu$ . However, if we exclude the exact simulation of the CIR, we only have at our disposal (up to our knowledge) at most a third order scheme for the CIR (see [1]). It seems thus fair to state the result on the weak error in that case.

**Theorem 27** — *Let  $(X_t^x)_{t \geq 0} \sim WIS_d(x, \alpha, b, a)$  such that either  $a \in \mathcal{G}_d(\mathbb{R})$  or  $a^T a b = b a^T a$ , and  $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$ . Let  $(\hat{X}_{t_i^N}^x, 0 \leq i \leq N)$  be sampled with the scheme defined by Proposition 26 with the third order scheme for the CIR given in [1] and starting from  $x \in \mathcal{S}_d^+(\mathbb{R})$ . Then,*

$$\exists C, N_0 > 0, \forall N \geq N_0, |\mathbb{E}[f(\hat{X}_{t_i^N}^x)] - \mathbb{E}[f(X_t^x)]| \leq C/N^3.$$

*Proof :* The conditions of Theorem 21 are satisfied thanks to Propositions 26, 43 and Lemma 36.  $\square$

### 3.3 Second order schemes for affine diffusions on $\mathcal{S}_d^+(\mathbb{R})$

In this part, we present two different potential second order schemes for general affine processes. The first one is well-defined without any restriction on the parameters. The second one is faster but requires to assume in addition that  $\bar{\alpha} - da^T a \in \mathcal{S}_d^+(\mathbb{R})$ .

Let  $(X_t^x)_{t \geq 0} \sim AFF_d(x, \bar{\alpha}, B, a)$ . Thanks to Corollary 8, there is  $u \in \mathcal{G}_d(\mathbb{R})$  and a diagonal matrix  $\bar{\delta}$  such that  $\bar{\alpha} = u^T \bar{\delta} u$ ,  $a^T a = u^T I_d^n u$  and we have:

$$(X_t^x)_{t \geq 0} \stackrel{\text{law}}{=} (u^T Y_t^{(u^{-1})^T x u^{-1}})_{t \geq 0}, \text{ where } Y_t^y \sim AFF_d(y, \bar{\delta}, B_u, I_d^n).$$

**Lemma 28** — *If  $\hat{Y}_t^y$  is a potential  $\nu$ th-order scheme for  $AFF_d(y, \bar{\delta}, B_u, I_d^n)$ , then  $u^T \hat{Y}_t^{(u^{-1})^T x u^{-1}} u$  is a potential  $\nu$ th-order scheme for  $AFF_d(x, \bar{\alpha}, B, a)$ .*

*Proof :* Let  $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$ . We then have  $x \mapsto f(u^T x u) \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$ . Since  $u$  is fixed, there are constants  $C, \eta, E$  depending only on a good sequence of  $f$  such that for  $t \in (0, \eta)$ ,  $|\mathbb{E}[f(u^T \hat{Y}_t^{(u^{-1})^T x u^{-1}} u)] - \mathbb{E}[f(X_t^x)]| = |\mathbb{E}[f(u^T \hat{Y}_t^{(u^{-1})^T x u^{-1}} u)] - \mathbb{E}[f(u^T Y_t^{(u^{-1})^T x u^{-1}} u)]| \leq C t^{\nu+1} (1 + \|(u^{-1})^T x u^{-1}\|^E) \leq C' t^{\nu+1} (1 + \|x\|^E)$ , for some constant  $C' > C$ .  $\square$

We now focus on finding a scheme for  $AFF_d(y, \bar{\alpha}, B_u, I_d^n)$ . Since  $\bar{\delta}$  is a diagonal matrix such that  $\bar{\delta} - (d-1)I_d^n \in \mathcal{S}_d^+(\mathbb{R})$ , we have

$$\delta_{\min} := \min_{1 \leq i \leq n} \bar{\delta}_{i,i} \geq d-1.$$

From Corollary 4, we can write the infinitesimal generator of  $Y_t^y$

$$L = \text{Tr}([\bar{\delta} + B(x)]D^S) + 2\text{Tr}(xD^S I_d^n D^S) = \underbrace{\text{Tr}([\bar{\delta} - \delta_{\min} I_d^n + B_u(x)]D^S)}_{L_{ODE}} + \underbrace{\delta_{\min} \text{Tr}(D^S) + 2\text{Tr}(xD^S I_d^n D^S)}_{L_{WIS_d(x, \delta_{\min}, 0, I_d^n)}} \quad (40)$$

as the sum of the infinitesimal generator of  $WIS_d(x, \delta_{\min}, 0, I_d^n)$  and of the generator of the affine ODE  $x'(t) = \bar{\delta} - \delta_{\min} I_d^n + B_u(x(t))$ . Of course, this ODE can be solved explicitly. Moreover, we know by Lemma 45 that if  $x(0) = x \in \mathcal{S}_d^+(\mathbb{R})$ , the solution of this ODE remains in  $\mathcal{S}_d^+(\mathbb{R})$  since Assumption (4) holds for  $B_u$  and  $\bar{\delta} - \delta_{\min} I_d^n \in \mathcal{S}_d^+(\mathbb{R})$ . We denote by  $p_x^{ODE}(t)$  the Dirac mass at  $x(t)$ : this is an exact scheme for  $L_{ODE}$ . Thanks to Proposition 22, we get the following result.

**Proposition 29** — Let  $\hat{p}_x^W(t)$  denote a potential second order scheme for  $WIS_d(x, \delta_{\min}, 0, I_d^n)$ . With the notations above, the following schemes

$$\frac{1}{2}p^{ODE}(t) \circ \hat{p}_x^W(t) + \frac{1}{2}\hat{p}^W(t) \circ p_x^{ODE}(t) \text{ and } p^{ODE}(t/2) \circ \hat{p}^W(t) \circ p_x^{ODE}(t/2)$$

are potential second order scheme for the affine process  $AFF_d(x, \bar{\delta}, B_u, I_d^n)$ .

By combining Lemma 28 and Proposition 29, we get a potential second order scheme for any affine process. In the numerical experiments in Section 4, we have used the composition  $p^{ODE}(t/2) \circ \hat{p}^W(t) \circ p_x^{ODE}(t/2)$  even though the other one would have work as well.

Let us stress now that different splitting from (40) are possible. For example, we could have chosen instead  $L = \text{Tr}([\bar{\delta} - \beta I_d^n + B_u(x)]D^S) + \beta \text{Tr}(D^S) + 2\text{Tr}(xD^S I_d^n D^S)$  for any  $\beta \in [d-1, \delta_{\min}]$ . When  $\bar{a} - da^T a \in \mathcal{S}_d^+(\mathbb{R})$  (or equivalently when  $\bar{\delta} - dI_d^n \in \mathcal{S}_d^+(\mathbb{R})$ ), the following splitting

$$L = \underbrace{\text{Tr}([\bar{\delta} - dI_d^n + B_u(x)]D^S)}_{\tilde{L}_{ODE}} + \underbrace{d\text{Tr}(D^S) + 2\text{Tr}(xD^S I_d^n D^S)}_{L_{WIS_d(x, d, 0, I_d^n)}} \quad (41)$$

is really interesting. Indeed it is known from Bru [4] that Wishart processes can be seen as the square of an Ornstein-Uhlenbeck process on matrices and can be simulated very efficiently. More precisely, we will use the following result.

**Proposition 30** — Let  $x \in \mathcal{S}_d^+(\mathbb{R})$  and  $c \in \mathcal{M}_d(\mathbb{R})$  be such that  $c^T c = x$ . Then, the process  $X_t^x = (c + W_t I_d^n)^T (c + W_t I_d^n)$  satisfies

$$(X_t^x)_{t \geq 0} \stackrel{\text{law}}{=} WIS_d(x, d, 0, I_d^n).$$

If  $\hat{G}$  denote a  $d$ -by- $d$  matrix with independent elements sampled according to (33),  $\hat{X}_t^x = (c + \sqrt{t}\hat{G}I_d^n)^T (c + \sqrt{t}\hat{G}I_d^n)$  is a potential second order scheme for  $WIS_d(x, d, 0, I_d^n)$ .

*Proof :* We have by using Itô calculus  $dX_t^x = (c + W_t I_d^n)^T dW_t I_d^n + I_d^n dW_t^T (c + W_t I_d^n) + dI_d^n dt$ . By using Lemma 2, The quadratic covariation of  $(X_t^x)_{i,j}$  and  $(X_t^x)_{m,n}$  is given by  $d\langle (X_t^x)_{i,j}, (X_t^x)_{m,n} \rangle = (X_t^x)_{i,m} (I_d^n)_{j,n} + (X_t^x)_{i,n} (I_d^n)_{j,m} + (X_t^x)_{j,m} (I_d^n)_{i,n} + (X_t^x)_{j,n} (I_d^n)_{i,m}$ . Therefore,  $(X_t^x)_{t \geq 0}$  solves the same martingale problem as  $WIS_d(x, d, 0, I_d^n)$ , which is known to have a unique solution from Cuchiero et al. [6].

Let us show now that  $\hat{X}_t^x$  is a potential second order scheme. We can see  $c + \sqrt{t}\hat{G}I_d^n$  as the Ninomiya-Victoir scheme with moment-matching variables (see Theorem 1.18 in [1]) associated to  $\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^n \partial_{i,j}^2$  on  $\mathcal{M}_d(\mathbb{R})$ . Let  $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$ . Then,  $x \in \mathcal{M}_d(\mathbb{R}) \mapsto f(x^T x) \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{M}_d(\mathbb{R}))$  and there are constants  $C, E, \eta > 0$  depending only on a good sequence of  $f$  such that:

$$\forall t \in (0, \eta), |\mathbb{E}[f((c + \sqrt{t}\hat{G}I_d^n)^T (c + \sqrt{t}\hat{G}I_d^n))] - \mathbb{E}[f((c + W_t I_d^n)^T (c + W_t I_d^n))]| \leq Ct^{\nu+1}(1 + \|c\|^E).$$

Let us observe now that the Frobenius norm of  $c$  is  $\sqrt{\text{Tr}(c^T c)} = \sqrt{\text{Tr}(x)} \leq \sqrt{d + \text{Tr}(x^2)} \leq \sqrt{d} + \sqrt{\text{Tr}(x^2)}$ . Therefore, for any norm, there is a constant  $K > 0$  such that  $\|c\| \leq K(1 + \|x\|)$ , which gives the result.  $\square$

**Corollary 31** — We assume that  $\bar{\delta} - dI_d^n \in \mathcal{S}_d^+(\mathbb{R})$ . Let  $\hat{p}^{W_{bis}}(t)_x$  denote the second order scheme of Proposition 30, and  $p_x^{ODE}(t)$  be the exact scheme for the ODE associated to  $\text{Tr}([\bar{\delta} - dI_d^n + B_u(x)]D^S)$ . Then,  $\frac{1}{2}p^{ODE}(t) \circ \hat{p}_x^{W_{bis}}(t) + \frac{1}{2}\hat{p}^{W_{bis}}(t) \circ p_x^{ODE}(t)$  and  $p^{ODE}(t/2) \circ \hat{p}^{W_{bis}}(t) \circ p_x^{ODE}(t/2)$  are potential second-order scheme for  $AFF_d(x, \bar{\delta}, B_u, I_d^n)$ .

The scheme given by Lemma 28 and Corollary 31 is not well-defined when  $\bar{\alpha} - da^T a \notin \mathcal{S}_d^+(\mathbb{R})$ . When it is defined, it is however more efficient than the scheme given by Proposition 29 and Lemma 28. We have already mentioned in subsection 2.3.1 that the exact scheme has a cost of  $O(d^4)$ . Similarly, the schemes given by Propositions 26 and 29 have a cost of  $O(d^4)$  operations. On the contrary, the scheme given by Proposition 30 only requires one Cholesky decomposition. Thus, the scheme given by Lemma 28 and Corollary 31 has a complexity in  $O(d^3)$ . We will illustrate this in the next section.

We conclude this section by mentioning that in the case of Wishart processes, we can show that the weak error is indeed of order 2 thanks to Theorem 21, Lemma 36 and Proposition 43.

**Theorem 32** — *Let  $(X_t^x)_{t \geq 0} \sim WIS_d(x, \alpha, b, a)$ . Let  $(\hat{X}_{t_i^N}^N, 0 \leq i \leq N)$  be the scheme starting from  $x$  which is sampled either by the scheme defined by Lemma 28 and Proposition 29 (with the second order scheme for the CIR given in [1]) or, if we have in addition  $\alpha \geq d$ , by the scheme given by Lemma 28 and Corollary 31. Then,*

$$\exists C, N_0 > 0, \forall N \geq N_0, |\mathbb{E}[f(\hat{X}_{t_i^N}^N)] - \mathbb{E}[f(X_t^x)]| \leq C/N^2.$$

## 4 Numerical results on the simulation methods

The scope of this section is to compare the different simulation methods given in this paper. We still consider a time horizon  $T$  and the regular time-grid  $t_i^N = iT/N$ , for  $i = 0, \dots, N$ . In addition, we want to compare our schemes to a standard one, and we will consider the following corrected Euler-Maruyama scheme for  $AFF_d(x, \bar{\alpha}, B, a)$ :

$$\hat{X}_{t_0^N}^N = x, \hat{X}_{t_{i+1}^N}^N = \hat{X}_{t_i^N}^N + (\bar{\alpha} + B(\hat{X}_{t_i^N}^N)) \frac{T}{N} + \sqrt{(\hat{X}_{t_i^N}^N)^+} (W_{t_{i+1}^N} - W_{t_i^N}) a + a^T (W_{t_{i+1}^N} - W_{t_i^N})^T \sqrt{(\hat{X}_{t_i^N}^N)^+}, 0 \leq i \leq N-1. \quad (42)$$

Here,  $x^+$  denotes the matrix that has the same eigenvectors as  $x$  with the same eigenvalue if it is positive and a zero eigenvalue otherwise. Namely, we set  $x^+ = \text{odiag}(\lambda_1^+, \dots, \lambda_d^+) o^T$  for  $x = \text{odiag}(\lambda_1, \dots, \lambda_d) o^T$ . Thus,  $x^+$  is by construction a positive semidefinite matrix and its square root is well defined. Without this positive part, the scheme above is not well defined for any realization of  $W$ .

First, we compare the time required by the different schemes and the exact simulation. Then, we present numerical results on the convergence of the different schemes. Last, we give an application of our scheme to the Gourieroux-Sufana model in finance.

### 4.1 Time comparison between the different algorithms

In this paragraph, we compare the time required by the different schemes given in this paper. As it has already been mentioned, the complexity of the exact scheme as well as the one of the second order scheme (given by Lemma 28 and Proposition 29) and the third order scheme (given by Proposition 26) is in  $O(d^4)$  for one time-step. To be more precise, they require  $O(d^4)$  operations that mainly corresponds to  $d$  Cholesky decompositions,  $O(d^2)$  generations of Gaussian (or moment-matching) variables and  $O(d)$  generations of noncentral chi-square distributions (or second or third order schemes for the CIR). The time saved by the second and third order schemes with respect to the exact scheme only comes from the generation of random variables. For example, the generation of the moment-matching variables (33) and (34) is 2.5 faster than the generation of  $\mathcal{N}(0, 1)$  on our computer. The gain between the second or third order schemes for the CIR given in Alfonsi [1] and the exact sampling of the CIR given by Glasserman [10] is much greater, but it depends on the parameters of the CIR. When the dimension  $d$  gets larger, the absolute gain in time between the discretization schemes and the exact scheme is of course increased. However, the relative gain instead decreases to 1, because more and more time is devoted to matrix operations and Cholesky decompositions that are the same in both cases. Let us now quickly analyse the complexity of the other schemes. The second order scheme given by Lemma 28 and Corollary 31 (called ‘‘second order bis’’ later) has a complexity

Schemes	$N = 10$			$N = 30$		
	R. value	Im. value	Time	R. value	Im. value	Time
Exact (1 step)	-0.526852	-0.227962	12			
$2^{nd}$ order bis	-0.526229	-0.228663	41	-0.526486	-0.229078	125
$2^{nd}$ order	-0.526577	-0.228923	76	-0.526574	-0.228133	229
$3^{rd}$ order	-0.527021	-0.227286	82	-0.527613	-0.228376	244
Exact ( $N$ steps)	-0.526963	-0.228303	123	-0.526891	-0.227729	369
Corrected Euler	-0.525627*	-0.233863*	225	-0.525638*	-0.231449*	687
$\alpha = 3.5, d = 3, \Delta_R = 10^{-3}, \Delta_{Im} = 10^{-3}$ , exact value R. = -0.527090 and Im. = -0.228251						
Exact (1 step)	-0.591579	-0.037651	12			
$2^{nd}$ order	-0.590444	-0.037024	77	-0.590808	-0.036487	229
$3^{rd}$ order	-0.591234	-0.034847	82	-0.590818	-0.036210	246
Exact ( $N$ steps)	-0.591169	-0.036618	174	-0.592145	-0.037411	920
Corrected Euler	-0.589735*	-0.042002*	223	-0.590079*	-0.039937*	680
$\alpha = 2.2, d = 3, \Delta_R = 0.9 \times 10^{-3}, \Delta_{Im} = 1.3 \times 10^{-3}$ , exact value R. = -0.591411 and Im. = -0.036346						
Exact (1 step)	0.062712	-0.063757	181			
$2^{nd}$ order bis	0.064237	-0.063825	921	0.064573	-0.062747	2762
$2^{nd}$ order	0.064922	-0.064103	1431	0.063534	-0.063280	4283
$3^{rd}$ order	0.064620	-0.064543	1446	0.064120	-0.063122	4343
Exact ( $N$ steps)	0.063418	-0.064636	1806	0.063469	-0.064380	5408
Corrected Euler	0.068298*	-0.058491*	2312	0.061732*	-0.056882*	7113
$\alpha = 10.5, d = 10, \Delta_R = 1.4 \times 10^{-3}, \Delta_{Im} = 1.3 \times 10^{-3}$ , exact value R. = 0.063960 and Im. = -0.063544						
Exact (1 step)	-0.036869	-0.094156	177			
$2^{nd}$ order	-0.036246	-0.094196	1430	-0.035944	-0.092770	4285
$3^{rd}$ order	-0.035408	-0.093479	1441	-0.036277	-0.093178	4327
Exact ( $N$ steps)	-0.036478	-0.092860	1866	-0.036145	-0.093003	6385
Corrected Euler	-0.028685*	-0.094281*	2321	-0.030118*	-0.088988*	7144
$\alpha = 9.2, d = 10, \Delta_R = 1.4 \times 10^{-3}, \Delta_{Im} = 1.4 \times 10^{-3}$ , exact value R. = -0.036064 and Im. = -0.093275						

Table 1:  $E[\exp(-\text{Tr}(iv\hat{X}_{t/N}^N))]$  calculated by a Monte-Carlo with  $10^6$  samples for a Wishart process with  $a = I_d$ ,  $b = 0$ ,  $x = 10I_d$ ,  $v = 0.09I_d$  and  $T = 1$ . The starred numbers are those for which the exact value is outside the 95% confidence interval, and  $\Delta_R$  (resp.  $\Delta_I$ ) gives the two standard deviations value on the real (resp. imaginary) part.

in  $O(d^3)$  operations for one Cholesky decomposition and matrix multiplications, with  $O(d^2)$  generations of Gaussian variables. The complexity of the corrected Euler scheme is of the same kind. At each time-step,  $O(d^3)$  operations are needed for matrix multiplications and for diagonalizing the matrix in order to compute the square-root of its positive part. However, diagonalizing a symmetric matrix is in practice much longer than computing a Cholesky decomposition even though both algorithms are in  $O(d^3)$ . Also, one has to sample  $O(d^2)$  Gaussian variables for the Brownian increments.

In Table 4.1, we have calculated by a Monte-Carlo method one value of the characteristic function of a Wishart process. It is also known analytically thanks to (10), and we have indicated in each case the exact value. We have considered dimensions  $d = 3$  and  $d = 10$ . We have given in each case an example where  $\alpha \geq d$  and another one where  $d - 1 \leq \alpha < d$ . We have used the different algorithms presented in this paper: “ $2^{nd}$  order bis” stands for the scheme given by Lemma 28 and Corollary 31 (with the moment-matching variables (33)), “ $2^{nd}$  order” stands for the scheme given by Lemma 28 and Proposition 29 (with (33) and the second order scheme for the CIR given by [1]), “ $3^{rd}$  order” stands for the scheme given by Proposition 26 (with (34) and the third order scheme for the CIR given by [1]), and “Corrected Euler” stands for the corrected Euler-Maruyama scheme presented in the beginning of this section. For the exact scheme, we have both considered the cases with one time-step  $T$  and  $N$  time-steps  $T/N$ . Of course, the first case is sufficient to calculate an expectation that only depends on  $X_T$ , but the second case allows to compute also pathwise



expectations. For each method, we have given the value obtained and the time needed on our computer (3000 MHz CPU).

First, let us mention that the exact value is in each case in the confidence interval except for the corrected Euler scheme. As one can expect, the exact method with one time-step is by far the quickest method to compute an expectation that only depends on the final value. We put aside this case and focus now on the generation of the whole path. We see from Table 4.1 that the second and the third order schemes require roughly the same computation time. As expected, the second order scheme bis is much faster when it is defined (i.e. when  $\alpha \geq d$ ). On the contrary, the Euler scheme is much slower than the second and third order scheme, even though it should be faster for large  $d$ . This is due to the cost of the matrix diagonalization. Let us mention that the time required by the discretization schemes is proportional to  $N$  and do not depend on the parameters when the dimension is given. On the contrary, the time needed by the exact scheme may change according to  $\alpha$  and can increase considerably when  $\alpha$  is close to  $d - 1$ . To be more precise, the exact simulation method for the CIR given by Glasserman [10] uses a rejection sampling when the degree of freedom is lower than 1, which corresponds to the case  $d - 1 \leq \alpha < d$ . The rejection rate can in fact be rather high, notably when the time-step gets smaller. For  $N = 30$ ,  $d = 3$  and  $\alpha = 2.2$ , the exact scheme is four times slower than the second order scheme and 2.5 slower than the exact scheme with  $\alpha = 3.5$ .

Let us draw a conclusion from this time comparison between the different schemes. Obviously, we recommend to use the exact scheme when calculating expectations that depend on one or few dates. Instead, when calculating pathwise expectations of affine processes by Monte-Carlo, we would recommend to use in general the second order bis scheme when  $\alpha \geq d$  and the second order (or third order for Wishart processes) when  $d - 1 \leq \alpha < d$ .

## 4.2 Numerical results on the convergence

Now, we want to illustrate the theoretical results of convergence obtained in this paper for the different schemes. To do so, we have plotted for each scheme  $\mathbb{E}[\exp(-\text{Tr}(iv\hat{X}_{t_N}^N))]$  in function of the time step  $T/N$ . This expectation is calculated by a Monte-Carlo method. As for the time comparison, we illustrate the convergence for  $d = 3$  in Figure 1 and  $d = 10$  in Figure 2. Each time, we consider a case where  $\alpha \geq d$  and a case where  $d - 1 \leq \alpha < d$ , which is in general tougher. In these figures,

- scheme 1 denotes the value obtained by the exact scheme with one time-step,
- scheme 2 stands for the second order scheme given by Lemma 28 and Proposition 29,
- scheme 3 denotes the third order scheme given by Theorem 27,
- scheme 4 is the corrected Euler scheme (42).

Here, we have not plotted the convergence of the second order (bis) scheme given by Lemma 28 and Corollary 31 because it would have given almost the same convergence as the other second order scheme.

$N$	2	4	8	10	16	30
Figure 1, right	-0.000698	0.000394	0.033193	0.111991	0.185128	0.210201
Figure 2, right	0.494752	-0.464121	0.657041	0.643042	0.637585	0.619553

Table 2: Values obtained by the Euler scheme in the numerical experiments of Figures 1 and 2.

As expected, we observe in both Figures 1 and 2 convergences that fit our theoretical results. Namely, Scheme 2 converges in  $O(1/N^2)$  and Scheme 3 converges faster in  $O(1/N^3)$ . In some cases such as Figure 2, Scheme 3 already matches the exact value from  $N = 2$ . Even though it seems to converge at a  $O(1/N)$  speed, the corrected Euler scheme is clearly not competitive with respect to the other schemes. In the tough case  $d - 1 \leq \alpha < d$ , the values obtained by the Euler scheme are in fact outside the figures, and we have put the corresponding values in Table 2.

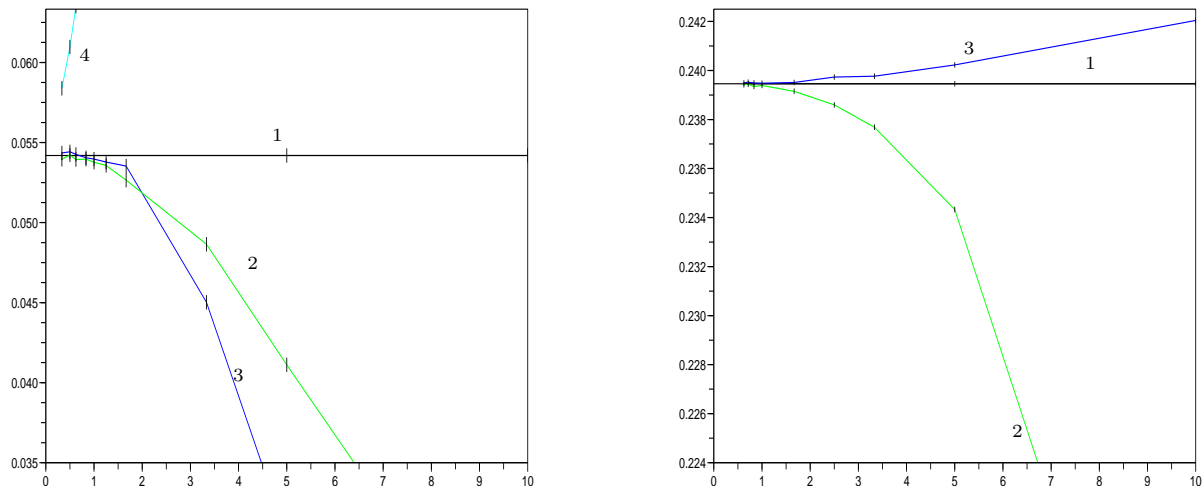


Figure 1:  $d = 3$ ,  $10^7$  Monte-Carlo samples,  $T = 10$ . The real value of  $\mathbb{E}[\exp(-\text{Tr}(iv\hat{X}_{t/N}^N))]$  in function of the time-step  $T/N$ . Left:  $v = 0.05I_d$  and Wishart parameters  $x = 0.4I_d$ ,  $\alpha = 4.5$ ,  $a = I_d$  and  $b = 0$ . Exact value: 0.054277. Right:  $v = 0.2I_d + 0.04q$  and Wishart parameters  $x = 0.4I_d + 0.2q$ ,  $\alpha = 2.22$ ,  $a = I_d$  and  $b = -0.5I_d$ . Exact value: 0.239836. Here,  $q$  is the matrix defined by:  $q_{i,j} = 1_{i \neq j}$ . The width of each point represents the 95% confidence interval.

We want to conclude this section by testing numerically the convergence of our schemes when we calculate pathwise expectations. Of course, our theoretical results only bring on the weak error, but we may hope that our schemes converge also quickly when considering more intricate expectations. In Figure 4.2, we approximate  $\mathbb{E}[\max_{0 \leq t \leq T} \text{Tr}(X_t^x)]$  with the different schemes by computing the maximum on the time-grid. The convergence seems to be roughly in  $O(1/\sqrt{N})$  for all the schemes (see Figure 4.2, left), including the exact scheme. However, the main error seems to come from the approximation of  $\max_{0 \leq t \leq T} \text{Tr}(X_t^x)$  by  $\max_{0 \leq k \leq N} \text{Tr}(X_{t_k}^x)$ . In fact, we have plotted in Figure 4.2 (right) the difference between  $\mathbb{E}[\max_{0 \leq k \leq N} \text{Tr}(\hat{X}_{t_k}^N)]$  and  $\mathbb{E}[\max_{0 \leq k \leq N} \text{Tr}(X_{t_k}^x)]$ . Then, we find convergences that are very similar to those obtained for the weak error: schemes 2 and 3 converge at a speed which is respectively compatible with  $O(1/N^2)$  and  $O(1/N^3)$ . Scheme 4 seems also to give a  $O(1/N)$  convergence. It would be hasty to draw a global conclusion from this simple example. Nonetheless, the convergence of schemes 2 and 3 is really encouraging on pathwise expectations, if we put aside the problem of approximating a function of  $(X_t^x, 0 \leq t \leq T)$  by a function of  $(X_{t_k}^x, 0 \leq k \leq N)$ .

### 4.3 An application in finance to the Gourieroux and Sufana model

In this paragraph, we want to give a possible application of  $\mathbb{E}$  of our schemes in finance. More precisely, we will consider the model introduced by Gourieroux and Sufana [13]. This is a model for  $d$  risky assets  $S_t^1, \dots, S_t^d$ . Let  $(B_t, t \geq 0)$  denote a standard Brownian motion on  $\mathbb{R}^d$  that is independent from  $(W_t, t \geq 0)$ . Then, we consider the following dynamics for the assets:

$$t \geq 0, 1 \leq l \leq d, S_t^l = S_0^l + r \int_0^t S_u^l du + \int_0^t S_u^l (\sqrt{X_u} dB_u)_l, \quad (43)$$

where  $X_t = X_0 + \int_0^t (\alpha a^T a + b X_u + X_u b^T) du + \int_0^t (\sqrt{X_u} dW_u a + a^T dW_u^T \sqrt{X_u})$  is a Wishart process. Here,  $(\sqrt{X_u} dB_u)_l$  is simply the  $l^{\text{th}}$  coordinates of the vector  $\sqrt{X_u} dB_u$ . We can easily check that the instantaneous

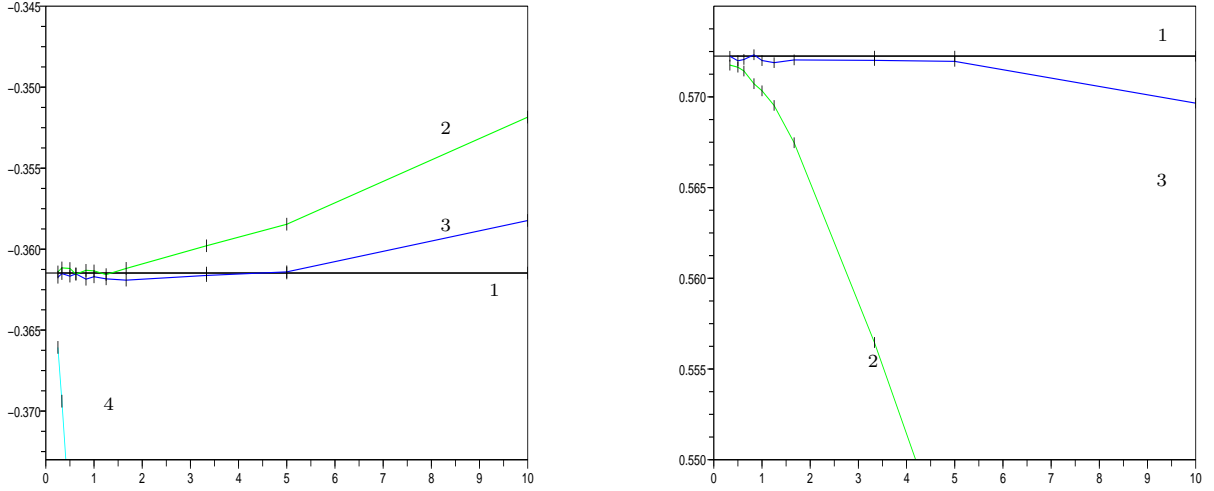


Figure 2:  $d = 10$ ,  $10^7$  Monte-Carlo samples,  $T = 10$ . Left: imaginary value of  $\mathbb{E}[\exp(-\text{Tr}(iv\hat{X}_{t/N}^N))] with  $v = 0.009I_d$  in function of the time-step  $T/N$ . Wishart parameters:  $x = 0.4I_d$ ,  $\alpha = 12.5$ ,  $b = 0$  and  $a = I_d$ . Exact value:  $-0.361586$ . Right: real value of  $\mathbb{E}[\exp(-\text{Tr}(iv\hat{X}_{t/N}^N))] with  $v = 0.009I_d$  in function of  $T/N$ . Wishart parameters:  $x = 0.4I_d$ ,  $\alpha = 9.2$ ,  $b = -0.5I_d$  and  $a = I_d$ . Exact value  $0.572241$ . The width of each point represents the 95% confidence interval.$$

quadratic covariation matrix between the log-prices of the assets is  $X_t$ . Last,  $r$  denotes the instantaneous interest rate.

To simulate both assets and the Wishart matrix, we proceed as follows. We observe that the generator of  $(S_t, X_t)$  can be written as

$$L = L^S + L^X, \text{ where } L^S = \sum_{i=1}^d r s_i \partial_{s_i} + \frac{1}{2} \sum_{i,j=1}^d s_i s_j x_{i,j} \partial_{s_i} \partial_{s_j},$$

and  $L^X$  is the generator of the Wishart process  $WIS_d(x, \alpha, b, a)$ . The operator  $L^S$  is associated to the SDE  $dS_t^l = rS_t^l + S_t^l(\sqrt{x}dB_t)_l$  that can be solved explicitly. We have indeed  $S_t^l = S_0^l \exp[(r - x_{l,l}/2)t + (\sqrt{x}B_t)_l]$ . Let us also remark that  $\sqrt{x}B_t \stackrel{Law}{=} cB_t$  if we have  $cc^T = x$ : both are centred Gaussian vectors with the same covariance matrix. In practice, it is more efficient to use  $S_t^l = S_0^l \exp[(r - x_{l,l}/2)t + (cB_t)_l]$  where  $c$  is computed with an extended Cholesky decomposition of  $x$  rather than calculating  $\sqrt{x}$ , which requires a diagonalization. Let us then denote by  $p_{(s,x)}^S(t)$  the exact scheme for  $L^S$  ( $x$  is unchanged) and  $\hat{p}_{(s,x)}^X(t)$  a second order scheme for the Wishart process  $WIS_d(x, \alpha, b, a)$  ( $s$  is unchanged). Then, we consider the following scheme

$$\frac{1}{2}(\hat{p}^X(t) \circ \hat{p}_{(s,x)}^S(t) + \hat{p}^S(t) \circ \hat{p}_{(s,x)}^X(t)),$$

i.e. we draw a Bernoulli variable of parameter  $1/2$  to decide whether we use first  $\hat{p}_{(s,x)}^X(t)$  or  $\hat{p}_{(s,x)}^S(t)$ . Under some conditions that we do not check here, this construction is known from Proposition 22 to preserve the second-order convergence. To be consistent with Subsection 4.2, this scheme is denoted by scheme 2 later in this paragraph. To compare this scheme with a more basic one, we consider the Euler-Maruyama scheme defined by (42) and

$$\hat{S}_{t_0^N}^{l,N} = S_0^l, \hat{S}_{t_{i+1}^N}^{l,N} = \hat{S}_{t_i^N}^{l,N} \left( 1 + rT/N + (\sqrt{\hat{X}_{t_i^N}^N} (B_{t_{i+1}^N}^N - B_{t_i^N}^N))_l \right), 0 \leq i \leq N-1.$$

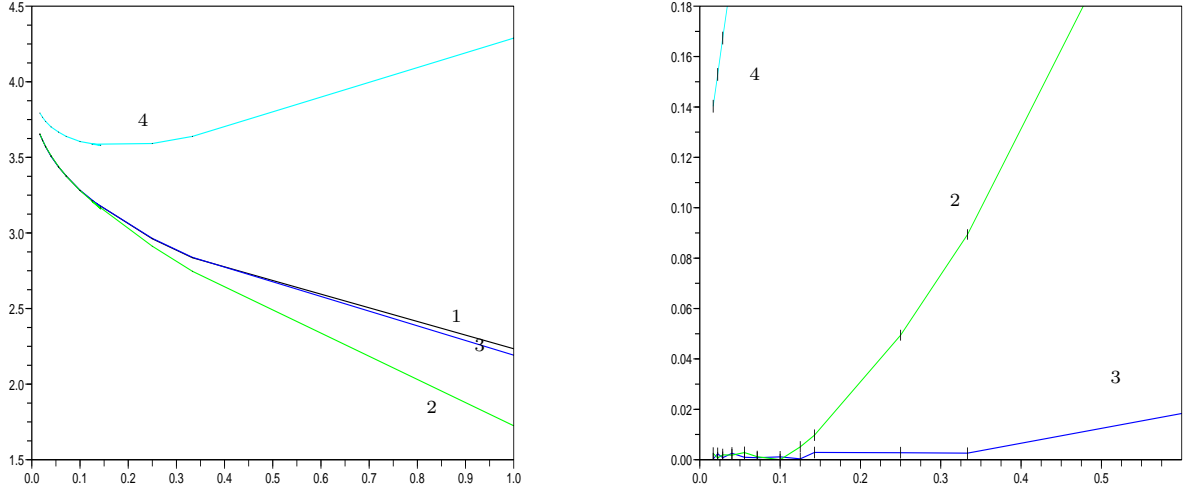


Figure 3:  $d = 3$ ,  $10^7$  Monte-Carlo samples,  $T = 1$ . Wishart parameters  $x = 0.4I_d + 0.2q$  with  $q_{i,j} = 1_{i \neq j}$ ,  $\alpha = 2.2$ ,  $b = 0$  and  $a = I_d$ . Left,  $\mathbb{E}[\max_{0 \leq k \leq N} \text{Tr}(\hat{X}_{t_k}^N)]$ , right:  $\mathbb{E}[\max_{0 \leq k \leq N} \text{Tr}(\hat{X}_{t_k}^N)] - \mathbb{E}[\max_{0 \leq k \leq N} \text{Tr}(X_{t_k}^x)]$  in function of  $T/N$ . The width of each point gives the precision up to two standard deviations.

It is denoted by scheme 4 like in Subsection 4.2.

We have plotted in Figure 4.3 the price of a put option on the maximum of two risky assets ( $d = 2$ ). The Gourieroux and Sufana model is an affine model, and the characteristic function of  $S_t$  is explicitly known (see [13]). Thus, it is possible to adapt the method proposed by Carr and Madan [5] and to calculate by numerical integration (which is possible for small dimensions) the value of this put option. We have given in Figure 4.3 the exact value obtained by this method. As one might have guessed, we observe a quadratic convergence for scheme 2 and a linear convergence for scheme 4. The benefit of using scheme 2 is clear since it already fits with the exact value from  $N = 5$  in both cases: its convergence is really satisfactory.

## 5 Conclusion and prospects

Let us draw a brief summary of this paper. Thanks to a remarkable splitting of the infinitesimal generator of Wishart processes, we have been able to sample exactly any Wishart distribution. We have also proposed a third order scheme for Wishart processes and a second order scheme for general affine diffusions. We have confirmed these rates of convergence with numerical tests and analysed the time complexity of each method. It comes out that we recommend to use the exact scheme to compute expectations that depend on one (or few) times. To calculate pathwise expectations, we instead recommend generally to use discretization schemes. More precisely, the second order scheme given by Lemma 28 and Corollary 31 has to be preferred when  $\alpha \geq d$ . Otherwise, we recommend to use the third order scheme given by Theorem 27 for Wishart processes or the second order scheme given by Lemma 28 and Proposition 29 for general affine diffusions.

Let us give now some prospects of this work. As a possible continuation of this paper, it is natural to wonder if it is possible to extend our schemes to affine diffusions on positive semidefinite matrices that include jumps (see Cuchiero et al.[6]). From a modelling point of view, we believe that Wishart processes could be used in a wide range of applications. In fact, they can be used as soon as one has to model dependence dynamics. Thus, we hope that the possibility of sampling such processes will stimulate different kinds of dependence models.

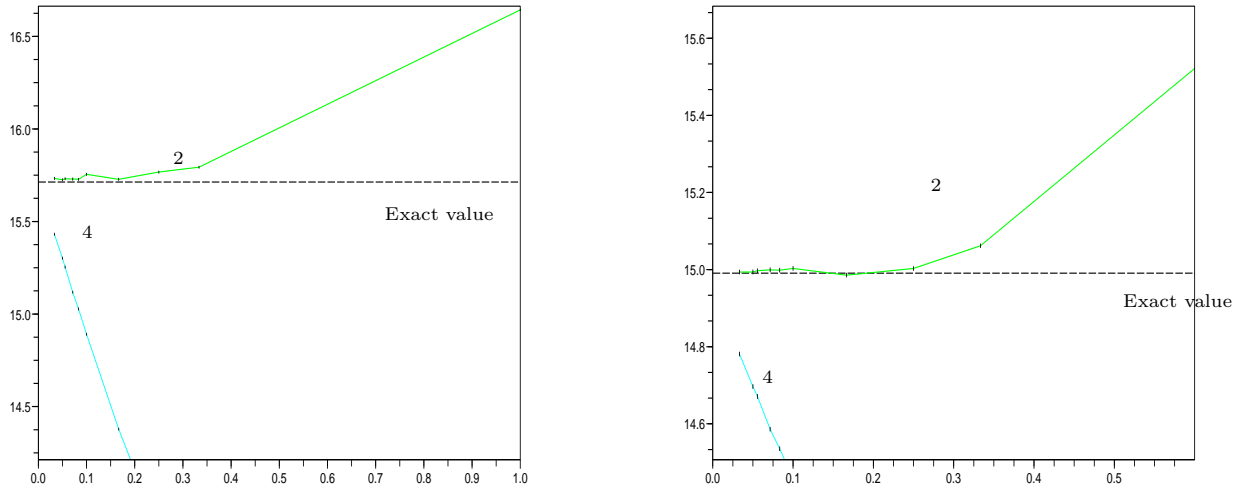


Figure 4:  $E[e^{-rT}(K - \max(\hat{S}_{t_N^N}^{1,N}, \hat{S}_{t_N^N}^{2,N}))^+]$  in function of  $T/N$ .  $d = 2$ ,  $T = 1$ ,  $K = 120$ ,  $S_0^1 = S_0^2 = 100$ , and  $r = 0.02$ . Wishart parameters:  $x = 0.04I_d + 0.02q$  with  $q_{i,j} = 1_{i \neq j}$ ,  $a = 0.2I_d$ ,  $b = 0.5I_d$  and  $\alpha = 4.5$  (left),  $\alpha = 1.05$  (right). The width of each point gives the precision up to two standard deviations ( $10^6$  Monte-Carlo samples).

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## A Notations and some results on matrices

### Notations for real matrices :

- For  $d, d' \in \mathbb{N}^*$ ,  $\mathcal{M}_d(\mathbb{R})$  denotes the real  $d$  square matrices and  $\mathcal{M}_{d \times d'}(\mathbb{R})$  the real matrices with  $d$  rows and  $d'$  columns.
- $\mathcal{S}_d(\mathbb{R})$ ,  $\mathcal{S}_d^+(\mathbb{R})$ ,  $\mathcal{S}_d^{+,*}(\mathbb{R})$ , and  $\mathcal{G}_d(\mathbb{R})$  denote respectively the set of symmetric, symmetric positive semidefinite, symmetric positive definite and non singular matrices.
- For  $x \in \mathcal{M}_d(\mathbb{R})$ ,  $x^T$ ,  $\text{adj}(x)$ ,  $\det(x)$ ,  $\text{Tr}(x)$  and  $\text{Rk}(x)$  are respectively the transpose, the adjugate, the determinant, the trace and the rank of  $x$ .
- For  $x \in \mathcal{S}_d^+(\mathbb{R})$ ,  $\sqrt{x}$  denotes the unique symmetric positive semidefinite matrix such that  $(\sqrt{x})^2 = x$
- The identity matrix is denoted by  $I_d$  and we set for  $n \leq d$ ,  $I_d^n = (\mathbb{1}_{i=j \leq n})_{1 \leq i, j \leq d}$  and  $e_d^n = (\mathbb{1}_{i=j=n})_{1 \leq i, j \leq d}$ , so that  $I_d^n = \sum_{i=1}^n e_d^i$ . We also set for  $1 \leq i, j \leq d$ ,  $e_d^{i,j} = (\mathbb{1}_{k=i, l=j})_{1 \leq k, l \leq d}$ .
- For  $x \in \mathcal{S}_d(\mathbb{R})$ , we denote by  $x_{\{i,j\}}$  the value of  $x_{i,j}$ , so that  $x = \sum_{1 \leq i \leq j \leq d} x_{\{i,j\}} (e_d^{i,j} + \mathbb{1}_{i \neq j} e_d^{j,i})$ . We use both notations in the paper: notation  $(x_{i,j})_{1 \leq i, j \leq d}$  is of course more convenient for matrix calculations while  $(x_{\{i,j\}})_{1 \leq i \leq j \leq d}$  is preferred to emphasize that we work on symmetric matrices and that we have  $x_{i,j} = x_{j,i}$ .
- For  $\lambda_1, \dots, \lambda_d \in \mathbb{R}$ ,  $\text{diag}(\lambda_1, \dots, \lambda_d)$  denotes the diagonal matrix such that  $\text{diag}(\lambda_1, \dots, \lambda_d)_{i,i} = \lambda_i$ .

Now, we present different results that are used in the paper. We first recall the extended Cholesky decomposition of positive semidefinite matrices which is used intensively and then give two other technical results.

**Lemma 33** — *Let  $q \in \mathcal{S}_d^+(\mathbb{R})$  be a matrix with rank  $r$ . Then there is a permutation matrix  $p$ , an invertible lower triangular matrix  $c_r \in \mathcal{G}_r(\mathbb{R})$  and  $k_r \in \mathcal{M}_{d-r \times r}(\mathbb{R})$  such that:*

$$pqp^T = cc^T, \quad c = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix}.$$

The triplet  $(c_r, k_r, p)$  is called an extended Cholesky decomposition of  $q$ . Besides,  $\tilde{c} = \begin{pmatrix} c_r & 0 \\ k_r & I_{d-r} \end{pmatrix} \in \mathcal{G}_d(\mathbb{R})$ , and we have:

$$q = (\tilde{c}^T p)^T I_d^r \tilde{c}^T p.$$

The proof of this result and a numerical procedure to get such a decomposition can be found in Golub and Van Loan ([12], Algorithm 4.2.4). When  $r = d$ , we can take  $p = I_d$ , and  $c_r$  is the usual Cholesky decomposition.

**Lemma 34** — *Let  $b, c \in \mathcal{S}_d(\mathbb{R})$ . If either  $b \in \mathcal{S}_d^+(\mathbb{R})$  or  $c \in \mathcal{S}_d^+(\mathbb{R})$ , then  $I_d + ibc$  is invertible. In particular, if  $b \in \mathcal{S}_d^{+,*}(\mathbb{R})$ ,  $b + ic$  is invertible.*

*Proof :* We start with the first assertion. Since  $(I_d + ibc)^T = I_d + icb$ , it is sufficient to check the case where  $c \in \mathcal{S}_d^+(\mathbb{R})$ . By a way of contradiction, let us assume that there is  $x \in \mathbb{C}^d \setminus \{0\}$  such that  $x + ibcx = 0$ . We respectively denote by  $x_R \in \mathbb{R}^d$  and  $x_I \in \mathbb{R}^d$  the real and imaginary part of  $x$ . One gets easily that  $x_R = bcx_I$  and  $x_I = -bcx_R$ . Since  $x \neq 0$ , we have necessarily  $x_R \neq 0$ ,  $cx_R \neq 0$ ,  $bcx_R \neq 0$  and  $cbcx_R \neq 0$ . Since  $c$  is nonnegative, we get by decomposing on an orthonormal basis that  $cx_R \cdot x_R > 0$  and  $cbcx_R \cdot bcx_R > 0$ .

However, we also have  $cx_R.x_R = -cx_R.bcbcx_R$ , which leads to a contradiction. The second assertion is now obvious since  $b + ic = b(I_d + ib^{-1}c)$ .  $\square$

## B Proofs of Section 1

### B.1 Proof of Lemma 2

*Proof* : From the SDE (5), we have

$$\begin{aligned} (dY_t)_{i,j} &= (C_t)_{i,j}dt + (B_t dW_t A_t + A_t^T dW_t^T B_t^T)_{i,j} \\ &= (C_t)_{i,j}dt + \sum_{k,l=1}^d ((B_t)_{i,k}(A_t)_{l,j} + (B_t)_{j,k}(A_t)_{l,i})(dW_t)_{k,l} \end{aligned}$$

Then we get the following formula for the quadratic covariation

$$\begin{aligned} d\langle (Y_t)_{i,j}, (Y_t)_{m,n} \rangle &= \sum_{k,l}^d ((B_t)_{i,k}(A_t)_{l,j} + (B_t)_{j,k}(A_t)_{l,i})((B_t)_{m,k}(A_t)_{l,n} + (B_t)_{n,k}(A_t)_{l,m})dt \\ &= \sum_{l,k}^d (B_t)_{i,k}(B_t)_{m,k}(A_t)_{l,n}(A_t)_{l,j} + \sum_{l,k}^d (B_t)_{i,k}(B_t)_{n,k}(A_t)_{l,m}(A_t)_{l,j} \\ &\quad + \sum_{l,k}^d (B_t)_{j,k}(B_t)_{m,k}(A_t)_{l,n}(A_t)_{l,i} + \sum_{l,k}^d (B_t)_{j,k}(B_t)_{n,k}(A_t)_{l,m}(A_t)_{l,i} \\ &= (B_t B_t^T)_{i,m}(A_t^T A_t)_{n,j} + (B_t B_t^T)_{i,n}(A_t^T A_t)_{m,j} \\ &\quad + (B_t B_t^T)_{j,m}(A_t^T A_t)_{n,i} + (B_t B_t^T)_{j,n}(A_t^T A_t)_{m,i}. \end{aligned} \quad (44)$$

$\square$

### B.2 Proof of Proposition 5

*Proof* : Let  $v \in \mathcal{S}_d(\mathbb{R})$  such that  $\forall s \in [0, t], I_d - 2q_s v \in \mathcal{G}_d(\mathbb{R})$ . As it is usual for affine diffusions, the Laplace transform can be formulated with ODE solutions. Namely, we will show that  $\mathbb{E}[\exp(\text{Tr}(vX_t^x))] = \exp[\phi(t, v) + \text{Tr}(\psi(t, v)x)]$ , where  $\psi$  and  $\phi$  solve following ODEs (see for example Cuchiero et al. [6]):

$$\begin{aligned} \partial_t \psi(t, v) &= \psi(t, v)b + b^T \psi(t, v) + 2\psi(t, v)a^T a \psi(t, v) \quad ; \quad \psi(0, v) = v \\ \partial_t \phi(t, v) &= \alpha \text{Tr}(\psi(t, v)) \quad ; \quad \phi(0, v) = 0. \end{aligned}$$

The function  $\psi$  solves an usual matrix Riccati ODE. As shown by Levin [18],  $\psi$  can be obtained explicitly by the mean of the following exponential matrix :

$$\begin{pmatrix} A_{1,1}(t) & A_{1,2}(t) \\ A_{2,1}(t) & A_{2,2}(t) \end{pmatrix} := \exp \left( t \begin{pmatrix} b & -2a^T a \\ 0 & -b^T \end{pmatrix} \right) = \begin{pmatrix} \exp(tb) & -2 \exp(tb) \int_0^t \exp(-sb) a^T a \exp(-sb^T) ds \\ 0 & \exp(-tb^T) \end{pmatrix}.$$

From [18], we obtain that

$$\begin{aligned} \psi(t, v) &= (\psi(0, v)A_{1,2}(t) + A_{2,2}(t))^{-1}(\psi(0, v)A_{1,1}(t) + A_{2,1}(t)) \\ &= \left( -2v \exp(tb) \int_0^t \exp(-sb) a^T a \exp(-sb^T) ds + \exp(-tb^T) \right)^{-1} v \exp(tb) \\ &= \left( -2v \int_0^t \exp(sb) a^T a \exp(sb^T) ds \exp(-tb^T) + \exp(-tb^T) \right)^{-1} v \exp(tb) \\ &= \exp(tb^T)(I_d - 2vq_t)^{-1} v \exp(tb), \end{aligned}$$

provided that  $I_d - 2q_s v$  is invertible for  $s \in [0, t]$ , which holds by assumption. Therefore we get for  $x \in \mathcal{S}_d(\mathbb{R})$ ,

$$\begin{aligned} \text{Tr}(\psi(t, v)x) &= \text{Tr}(\exp(tb^T)(I_d - 2vq_t)^{-1} v \exp(tb)x) \\ &= \text{Tr}((I_d - 2vq_t)^{-1} v \exp(tb)x \exp(tb^T)) \\ &= \text{Tr}(v(I_d - 2q_t v)^{-1} \exp(tb)x \exp(tb^T)), \end{aligned}$$



since  $v(I_d - 2q_t v)^{-1} = (I_d - 2vq_t)^{-1}v$ . As explained by Grasselli and Tebaldi ([14], section 4.2),  $\phi$  can also be calculated explicitly by the mean of the exponential matrix above, and we get:

$$\phi(t, v) = -\frac{\alpha}{2} \text{Tr}(\log[(I_d - 2vq_t) \exp(tb^T)] - t \text{Tr}(b)).$$

By using that  $\exp(\text{Tr}(\log(A))) = \det(A)$  for  $A \in \mathcal{G}_d(\mathbb{R})$ , we deduce then that

$$\begin{aligned} \exp(\phi(t, v)) &= \exp\left(\frac{\alpha}{2} t \text{Tr}(b)\right) \left(\exp(\text{Tr}(\log\{(I_d - 2vq_t) \exp(tb^T)\}))\right)^{\frac{-\alpha}{2}} \\ &= \exp\left(\frac{\alpha}{2} t \text{Tr}(b)\right) \left(\det\{(I_d - 2vq_t)\} \det\{\exp(tb^T)\}\right)^{\frac{-\alpha}{2}} \\ &= \exp\left(\frac{\alpha}{2} t \text{Tr}(b) - \frac{\alpha}{2} t \text{Tr}(b)\right) \left(\det\{(I_d - 2vq_t)\}\right)^{\frac{-\alpha}{2}} \\ &= \frac{1}{\det(I_d - 2q_t v)^{\frac{\alpha}{2}}}. \end{aligned}$$

For the last equality, we have used that  $\det(I_d - 2vq_t) = \det((I_d - 2vq_t)^T) = \det(I_d - 2q_t v)$ .

Now, it remains to show that (10) indeed holds. By Itô calculus, we get that for  $s \in (0, t)$ :

$$d \exp[\phi(t-s, v) + \text{Tr}(\psi(t-s, v) X_s^x)] = \exp[\phi(t-s, v) + \text{Tr}(\psi(t-s, v) X_s^x)] \text{Tr}[\psi(t-s, v) (\sqrt{X_s^x} dW_s a + a^T dW_s^T \sqrt{X_s^x})]. \quad (45)$$

Thus,  $\exp[\phi(t-s, v) + \text{Tr}(\psi(t-s, v) X_s^x)]$  is a positive local martingale and therefore a supermartingale, which gives that  $\mathbb{E}[\exp(\text{Tr}(v X_t^x))] \leq \exp[\phi(t, v) + \text{Tr}(\psi(t, v)x)] < \infty$ , i.e.  $\mathcal{D}_{b,a;t} \subset \tilde{\mathcal{D}}_{x,\alpha,b,a;t}$ , where

$$\mathcal{D}_{b,a;t} := \{v \in \mathcal{S}_d(\mathbb{R}), \forall s \in [0, t], I_d - 2q_s v \in \mathcal{G}_d(\mathbb{R})\} \text{ and } \tilde{\mathcal{D}}_{x,\alpha,b,a;t} := \{v \in \mathcal{S}_d(\mathbb{R}), \mathbb{E}[\exp(\text{Tr}(v X_t^x))] < \infty\}.$$

On the other hand, when  $-v \in \mathcal{S}_d^{+,*}(\mathbb{R})$ , we can check that  $\exp[\phi(t-s, v) + \text{Tr}(\psi(t-s, v) X_s^x)] \leq 1$  by observing that  $\det(I_d - 2q_t v) = \det(I_d + 2\sqrt{-v}q_t\sqrt{-v}) \geq 1$  and  $\text{Tr}(v(I_d - 2q_t v)^{-1} \exp(tb)x \exp(tb^T)) = -\text{Tr}(\sqrt{-v}(I_d + 2\sqrt{-v}q_t\sqrt{-v})^{-1} \sqrt{-v} \exp(tb)x \exp(tb^T)) \leq 0$ . In that case,  $\exp[\phi(t-s, v) + \text{Tr}(\psi(t-s, v) X_s^x)]$  is a martingale from (45), and (10) holds.

Let us observe now that  $\mathcal{D}_{b,a;t}$  is convex. In fact, we have  $\det(I_d - 2q_s v) = \det(I_d - 2\sqrt{q_s}v\sqrt{q_s})$ , and therefore  $\mathcal{D}_{b,a;t} = \{v \in \mathcal{S}_d(\mathbb{R}), \forall s \in [0, t], I_d - 2\sqrt{q_s}v\sqrt{q_s} \in \mathcal{S}_d^{+,*}(\mathbb{R})\}$  which is obviously convex. The Laplace transform  $v \mapsto \mathbb{E}[\exp(\text{Tr}(v X_t^x))]$  is an analytic function on  $\mathcal{D}_{b,a;t}$  (see for example Lemma 10.8 in [8]). The RHS of (10) is also analytic on  $\mathcal{D}_{b,a;t}$  and coincides with the Laplace transform when  $-v \in \mathcal{S}_d^{+,*}(\mathbb{R})$ . Therefore, (10) holds for  $v \in \mathcal{D}_{b,a;t}$  since  $\mathcal{D}_{b,a;t}$  is convex. Now, we can extend to complex values of  $v$ . Indeed, the RHS of (10) is well defined for  $v = v_R + iv_I$  with  $v_R \in \mathcal{D}_{b,a;t}$ , thanks to Lemma 34. Since both hand sides are analytic functions of  $v$ , (10) holds for  $v = v_R + iv_I$ .

Last, we want to show that  $\mathcal{D}_{b,a;t} = \tilde{\mathcal{D}}_{x,\alpha,b,a;t}$ . We first consider the case  $b = 0$  and assume by a way of contradiction that there is  $v \in \tilde{\mathcal{D}}_{x,\alpha,0,a;t} \setminus \mathcal{D}_{0,a;t}$  for some  $x, \alpha, a$  and  $t > 0$ . Let  $\tilde{t} = \min\{s \in [0, t], I_d - 2q_s v \notin \mathcal{G}_d(\mathbb{R})\} \in (0, t]$ . On the one hand, we have  $v \notin \mathcal{D}_{0,a;\tilde{t}}$  and  $v \in \mathcal{D}_{0,a;s}$  for  $s \in [0, \tilde{t})$ . On the other hand, we have by Jensen's inequality:

$$s \in [0, t], \exp((t-s)\text{Tr}(va^T a)) \exp(\text{Tr}(v X_s^x)) \leq \mathbb{E}[\exp(\text{Tr}(v X_t^x)) | \mathcal{F}_s],$$

which gives  $s \in [0, t] \mapsto \exp(-s\text{Tr}(va^T a)) \mathbb{E}[\exp(\text{Tr}(v X_s^x))]$  is nondecreasing and finite. Since (10) holds for  $s < \tilde{t}$ , we get that  $\mathbb{E}[\exp(\text{Tr}(v X_{\tilde{t}}^x))] = +\infty$ , which leads to a contradiction. Let us now consider the case  $b \neq 0$ . From Proposition 9 (which is a consequence of the characteristic function obtained above), we have

$$v \in \tilde{\mathcal{D}}_{x,\alpha,b,a;t} \iff \theta_t^T v \theta_t \in \mathcal{D}_{0,I_d^n;t} \iff \forall s \in [0, t], \det(I_d - 2(s/t)q_t v) \neq 0.$$

In particular,  $\tilde{\mathcal{D}}_{x,\alpha,b,a;t}$  is an open set. For  $v \in \mathcal{G}_d(\mathbb{R})$ , we have  $\det(I_d - 2(s/t)q_t v) \neq 0 \iff \det(v^{-1} - 2(s/t)q_t) \neq 0$  (resp.  $\det(I_d - 2q_s v) \neq 0 \iff \det(v^{-1} - 2q_s) \neq 0$ ). Since  $sq_t \leq s'q_t$  (resp.  $q_s \leq q_{s'}$ ) for  $s \leq s'$ , we know from Theorem 8.1.5 in [12] that the (real) eigenvalues of  $v^{-1} - 2(s/t)q_t$  (resp.  $v^{-1} - 2q_s$ ) are nonincreasing w.r.t.  $s$ . Since they are also continuous, and  $v^{-1} - 2(s/t)q_t = v^{-1} - 2q_s$  for  $s \in \{0, t\}$ , we get that  $\forall s \in [0, t], \det(v^{-1} - 2(s/t)q_t) \neq 0 \iff \forall s \in [0, t], \det(v^{-1} - 2q_s) \neq 0$ , and thus  $\tilde{\mathcal{D}}_{x,\alpha,b,a;t} \cap \mathcal{G}_d(\mathbb{R}) = \mathcal{D}_{b,a;t} \cap \mathcal{G}_d(\mathbb{R})$ . Let  $v \in \tilde{\mathcal{D}}_{x,\alpha,b,a;t}$ . Since  $\tilde{\mathcal{D}}_{x,\alpha,b,a;t}$  is an open set, there is  $\varepsilon > 0$  such that  $v \pm \varepsilon I_d \in \tilde{\mathcal{D}}_{x,\alpha,b,a;t} \cap \mathcal{G}_d(\mathbb{R})$ . Since  $\mathcal{D}_{b,a;t}$  is convex,  $v = (v + \varepsilon I_d + v - \varepsilon I_d)/2 \in \mathcal{D}_{b,a;t}$ .  $\square$

## C Proofs of Section 2

### C.1 Proof of Theorem 10

*Proof* : From (8), we get:

$$L_i = \alpha \partial_{\{i,i\}} + 2x_{\{i,i\}} \partial_{\{i,i\}}^2 + 2 \sum_{\substack{1 \leq m \leq d \\ m \neq i}} x_{\{i,m\}} \partial_{\{i,m\}} \partial_{\{i,i\}} + \frac{1}{2} \sum_{\substack{1 \leq m, l \leq d \\ m \neq i, l \neq i}} x_{\{m,l\}} \partial_{\{i,m\}} \partial_{\{i,l\}} \quad (46)$$

We want to show that  $L_i L_j = L_j L_i$  for  $i \neq j$ . Up to a permutation of the coordinates,  $L_i$  and  $L_j$  are the same operators as  $L_1$  and  $L_2$ . It is therefore sufficient to check that  $L_1 L_2 = L_2 L_1$ . By a straightforward but tedious calculation, we get

$L_1 L_2 =$

$$\begin{aligned} & \underbrace{\alpha^2 \partial_{\{1,1\}} \partial_{\{2,2\}}}_{(0)} + \underbrace{2\alpha x_{\{2,2\}} \partial_{\{1,1\}} \partial_{\{2,2\}}^2}_{(1)} + \underbrace{2\alpha \sum_{j \neq 2} x_{\{2,j\}} \partial_{\{1,1\}} \partial_{\{2,2\}} \partial_{\{2,j\}}}_{(2)} + \\ & \frac{\alpha}{2} \underbrace{(\partial_{\{1,2\}}^2)}_{(3)} + \underbrace{\sum_{j \neq 2, k \neq 2} x_{\{j,k\}} \partial_{\{1,1\}} \partial_{\{2,j\}} \partial_{\{2,k\}}}_{(4)} + \underbrace{2\alpha x_{\{1,1\}} \partial_{\{1,1\}}^2 \partial_{\{2,2\}}}_{(1)} + \underbrace{4x_{\{1,1\}} x_{\{2,2\}} \partial_{\{1,1\}}^2 \partial_{\{2,2\}}^2}_{(5)} + \\ & \underbrace{4 \sum_{j \neq 2} x_{\{1,1\}} x_{\{2,j\}} \partial_{\{1,1\}}^2 \partial_{\{2,j\}} \partial_{\{2,2\}}}_{(6)} + \underbrace{x_{\{1,1\}} (2\partial_{\{1,1\}} \partial_{\{1,2\}}^2)}_{(7)} + \underbrace{\sum_{j \neq 2, k \neq 2} x_{\{j,k\}} \partial_{\{1,1\}}^2 \partial_{\{2,j\}} \partial_{\{2,k\}}}_{(8)} + \\ & \underbrace{2\alpha \sum_{m \neq 1} x_{\{1,m\}} \partial_{\{1,1\}} \partial_{\{1,m\}} \partial_{\{2,2\}}}_{(2)} + \underbrace{4 \sum_{m \neq 1} x_{\{1,m\}} x_{\{2,2\}} \partial_{\{1,1\}} \partial_{\{1,m\}} \partial_{\{2,2\}}^2}_{(6)} + \\ & 4 \left( \underbrace{\sum_{m \neq 1, j \neq 2} x_{\{1,m\}} x_{\{2,j\}} \partial_{\{1,1\}} \partial_{\{1,m\}} \partial_{\{2,j\}} \partial_{\{2,2\}}}_{(9)} + \underbrace{x_{\{1,2\}} \partial_{\{1,1\}} \partial_{\{1,2\}} \partial_{\{2,2\}}}_{(10)} \right) \\ & + \underbrace{\sum_{m \neq 1, k \neq 2, j \neq 2} x_{\{1,m\}} x_{\{j,k\}} \partial_{\{1,1\}} \partial_{\{1,m\}} \partial_{\{2,j\}} \partial_{\{2,k\}}}_{(11)} + \underbrace{\sum_{m \neq 1, m \neq 2} x_{\{1,m\}} \partial_{\{1,2\}}^2 \partial_{\{1,m\}}}_{(12)} \\ & + 2 \underbrace{\sum_{m \neq 1, m \neq 2} x_{\{1,m\}} \partial_{\{1,1\}} \partial_{\{1,2\}} \partial_{\{2,m\}}}_{(13)} + \underbrace{x_{\{1,2\}} \partial_{\{1,2\}}^3 + 2x_{\{1,2\}} \partial_{\{1,1\}} \partial_{\{1,2\}} \partial_{\{2,2\}}}_{(14)} \\ & \frac{\alpha}{2} \underbrace{\sum_{m \neq 1, l \neq 1} x_{\{m,l\}} \partial_{\{1,m\}} \partial_{\{1,l\}} \partial_{\{2,2\}}}_{(4)} + \underbrace{\sum_{m \neq 1, l \neq 1} x_{\{2,2\}} x_{\{m,l\}} \partial_{\{1,m\}} \partial_{\{1,l\}} \partial_{\{2,2\}}^2}_{(8)} + \\ & \underbrace{\sum_{m \neq 1, l \neq 1, j \neq 2} x_{\{2,j\}} x_{\{m,l\}} \partial_{\{1,m\}} \partial_{\{1,l\}} \partial_{\{2,2\}} \partial_{\{2,j\}}}_{(11)} + 2 \underbrace{\sum_{m \neq 1, m \neq 2} x_{\{2,m\}} \partial_{\{1,2\}} \partial_{\{1,m\}} \partial_{\{2,2\}}}_{(13)} + \underbrace{2x_{\{2,2\}} \partial_{\{1,2\}}^2 \partial_{\{2,2\}}}_{(7)} + \\ & \frac{1}{4} \underbrace{\sum_{\substack{m \neq 1, l \neq 1 \\ j \neq 2, k \neq 2}} x_{\{m,l\}} x_{\{j,k\}} \partial_{\{1,m\}} \partial_{\{1,l\}} \partial_{\{2,k\}} \partial_{\{2,j\}}}_{(15)} + \underbrace{\sum_{\substack{m \neq 1, l \neq 1 \\ m \neq 2, l \neq 2}} x_{\{m,l\}} \partial_{\{1,l\}} \partial_{\{1,2\}} \partial_{\{2,m\}}}_{(16)} + \underbrace{\sum_{m \neq 1, m \neq 2} x_{\{2,m\}} \partial_{\{1,2\}}^2 \partial_{\{2,m\}}}_{(12)}. \end{aligned}$$

Obviously, we have the same formula for  $L_2L_1$  simply by exchanging the index 1 and 2. It is then sufficient to check that the above formula remains unchanged when we exchange the two index. Each term above is marked with a number. If this number is in the form  $(\bar{\tau})$ , then the associated term is symmetric. Otherwise, there exist in the formula its corresponding symmetric term which is marked with the same number.  $\square$

## C.2 Proof of Proposition 11

*Proof :* Let  $X_t^x \sim WIS_d(x, \alpha, 0, I_d^n; t)$ . We will check that for any polynomial function  $f$  of the matrix elements, we have  $\mathbb{E}[f(X_t^x)] = \mathbb{E}[f(X_t^{n, \dots, x_t^{1,x}})]$ . Let us consider a polynomial function  $f$  of degree  $m$ :

$$x \in \mathcal{S}_d(\mathbb{R}), f(x) = \sum_{\gamma \in \mathbb{N}^{d(d+1)/2}, |\gamma| \leq m} a_\gamma \bar{x}^\gamma,$$

where  $|\gamma| = \sum_{1 \leq i \leq j \leq d} |\gamma_{\{i,j\}}|$  and  $\bar{x}^\gamma = \prod_{1 \leq i \leq j \leq d} x_{\{i,j\}}^{\gamma_{\{i,j\}}}$ . Since the operator are affine, it is easy to check that  $Lf(x)$  and  $L_i f(x)$  are also polynomial functions of degree  $m$ . We set:

$$\|f\|_{\mathbb{P}} = \sum_{\gamma \in \mathbb{N}^{d(d+1)/2}, |\gamma| \leq m} |a_\gamma| \text{ and } |L| = \max_{\gamma \in \mathbb{N}^{d(d+1)/2}, |\gamma| \leq m} \|L\bar{x}^\gamma\|_{\mathbb{P}},$$

so that  $\|L^k f\|_{\mathbb{P}} \leq |L|^k \|f\|_{\mathbb{P}}$  for any  $k \in \mathbb{N}$ . Therefore, the series  $\sum_{k=0}^{\infty} t^k L^k f(x)/k!$  converges absolutely. By using  $l+1$  times Itô's formula, we get:

$$\mathbb{E}[f(X_t^x)] = \sum_{k=0}^l \frac{t^k}{k!} L^k f(x) + \int_0^t \frac{(t-s)^l}{l!} \mathbb{E}[L^{l+1} f(X_s^x)] ds.$$

Wishart processes have bounded moments since the drift and diffusion coefficients have a sublinear growth. Thus,  $C = \max_{\gamma \in \mathbb{N}^{d(d+1)/2}, |\gamma| \leq m} \sup_{s \in [0,t]} \mathbb{E}[|\bar{X}_s^\gamma|] < \infty$  and we obtain that  $|\int_0^t \frac{(t-s)^l}{l!} \mathbb{E}[L^{l+1} f(X_s^x)] ds| \leq C \|f\|_{\mathbb{P}} (t|L|)^{l+1}/(l+1)! \xrightarrow{l \rightarrow +\infty} 0$ . Thus, we have  $\mathbb{E}[f(X_t^x)] = \sum_{k=0}^{\infty} t^k L^k f(x)/k!$  and similarly we get that

$$\mathbb{E} \left[ f(X_t^{n, \dots, x_t^{1,x}}) | X_t^{n-1, \dots, x_t^{1,x}} \right] = \sum_{k_n=0}^{+\infty} \frac{t^{k_n}}{k_n!} L_n^{k_n} f(X_t^{n-1, \dots, x_t^{1,x}}).$$

Now, we remark that  $\tilde{C} = \max_{\gamma \in \mathbb{N}^{d(d+1)/2}, |\gamma| \leq m} \sup_{s \in [0,t]} \max(\mathbb{E}[|\bar{X}_t^{1,x^\gamma}|], \dots, \mathbb{E}[|\bar{X}_t^{n, \dots, x_t^{1,x} \gamma}|]) < \infty$  by using once again that Wishart processes have bounded moments. Since  $\mathbb{E}[|L_n^{k_n} f(X_t^{n-1, \dots, x_t^{1,x}})|] \leq \tilde{C} \|f\|_{\mathbb{P}} |L_n|^{k_n}$ , we can switch the expectation with the series and get (14). Then, since  $L_n^{k_n} f(x)$  are polynomial function of degree  $m$ , we can iterate this argument and finally get (15), which gives the result.  $\square$

## C.3 Proof of Theorem 13

The proof is divided into two parts. First, we prove that the SDE (24) has a unique strong solution which is given by (25) and is well defined on  $\mathcal{S}_d^+(\mathbb{R})$ . Second, we show that its infinitesimal generator is equal to the operator  $L_1$  defined in (16).

**First step.** Let us assume that  $(X_t^x)_{t \geq 0}$  is a solution to (24). We use the matrix decomposition of  $(x_{i,j})_{2 \leq i, j \leq d}$  given by (23) and set:

$$\begin{aligned} (U_t)_{\{1, l+1\}} &= \sum_{i=1}^r (c_r^{-1})_{l,i} (X_t^x)_{\{1, i+1\}}, \quad l \in \{l, \dots, r\}, \\ (U_t)_{\{1,1\}} &= (X_t^x)_{\{1,1\}} - \sum_{l=1}^r \left( \sum_{i=1}^r (c_r^{-1})_{l,i} (X_t^x)_{\{1, i+1\}} \right)^2 = (X_t^x)_{\{1,1\}} - \sum_{l=1}^r ((U_t)_{\{1, l+1\}})^2. \end{aligned}$$

We get by using Lemma 35 that:

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix} \begin{pmatrix} (U_t)_{\{1,1\}} + \sum_{k=1}^r ((U_t)_{\{1, k+1\}})^2 & ((U_t)_{\{1, l+1\}})_{1 \leq l \leq r}^T & 0 \\ ((U_t)_{\{1, l+1\}})_{1 \leq l \leq r} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r^T & k_r^T \\ 0 & 0 & I_{d-r-1} \end{pmatrix} \\ &= \begin{pmatrix} (U_t)_{\{1,1\}} + \sum_{k=1}^r ((U_t)_{\{1, k+1\}})^2 & ((U_t)_{\{1, l+1\}})_{1 \leq l \leq r}^T c_r^T & ((U_t)_{\{1, l+1\}})_{1 \leq l \leq r}^T k_r^T \\ c_r ((U_t)_{\{1, l+1\}})_{1 \leq l \leq r} & c_r c_r^T & c_r k_r^T \\ k_r ((U_t)_{\{1, l+1\}})_{1 \leq l \leq r} & k_r c_r^T & 0 \end{pmatrix} = X_t^x. \end{aligned}$$

Since  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix}$  is invertible,  $X_t^x \in \mathcal{S}_d^+(\mathbb{R})$  if, and only if:

$$\begin{aligned} \forall z \in \mathbb{R}^d, \quad z^T &\begin{pmatrix} (U_t)_{\{1,1\}} + \sum_{i=1}^r ((U_t)_{\{1, i+1\}})^2 & ((U_t)_{\{1, l\}})_{2 \leq l \leq r+1} & 0 \\ ((U_t)_{\{1, l\}})_{2 \leq l \leq r+1} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} z \\ &= z_1^2 (U_t)_{\{1,1\}} + \sum_{i=1}^r (z_{i+1} + (U_t)_{\{1, i+1\}} z_1)^2 \geq 0, \iff (U_t)_{\{1,1\}} \geq 0. \end{aligned} \quad (47)$$

In particular, we get that  $(U_0)_{\{1,1\}} = u_{\{1,1\}} \geq 0$  since  $x \in \mathcal{S}_d^+(\mathbb{R})$ . Now, by Itô calculus, we get from (24) that  $d(U_t)_{\{1, l+1\}} = \sum_{i=1}^r \sum_{k=1}^r (c_r^{-1})_{l,i} (c_r)_{i,k} dZ_t^{k+1} = dZ_t^{l+1}$  and

$$\begin{aligned} d(U_t)_{\{1,1\}} &= (\alpha - r)dt + 2\sqrt{(U_t)_{\{1,1\}}} dW_t^1 + 2 \sum_{l=1}^r \sum_{k=1}^r (c_r^{-1})_{l,k} (X_t)_{\{1, k+1\}} dW_t^{l+1} \\ &\quad - \sum_{l=1}^r 2((U_t)_{\{1, l+1\}}) dW_t^{l+1} \\ &= (\alpha - r)dt + 2\sqrt{(U_t)_{\{1,1\}}} dW_t^1. \end{aligned}$$

Thus, the solution  $(X_t^x)_{t \geq 0}$  is necessarily the one given by (25) (pathwise uniqueness holds for  $((U_t^u)_{\{1, l\}})_{1 \leq l \leq r+1}$ , and especially for the CIR diffusion  $(U_t^u)_{\{1,1\}}$  since  $\alpha \geq d - 1 \geq r$ ). Reciprocally, it is easy to check by Itô calculus that (25) solves (24).

**Second step.** Now, we want to show that  $L_1$  is the infinitesimal operator associated to the process  $(X_t^x)_{t \geq 0}$ . It is sufficient to compare the drift and the quadratic covariation of the process  $X_t^x$  with  $L_1$ . Since the drift part of  $(X_t^x)_{t \geq 0}$  clearly corresponds to the first order of  $L_1$ , we study directly the quadratic part. From (24), we have for  $i, j \in \{2, \dots, d\}^2$ :

$$\begin{aligned} d\langle (X_t^x)_{\{1,1\}}, (X_t^x)_{\{1,1\}} \rangle &= 4((X_t^x)_{\{1,1\}} - \sum_{k=1}^r [\sum_{l=1}^r (c_r^{-1})_{k,l} (X_t^x)_{\{1, l+1\}}])^2 + \sum_{k=1}^r [\sum_{l=1}^r (c_r^{-1})_{k,l} (X_t^x)_{\{1, l+1\}}]^2 \\ &= 4(X_t^x)_{\{1,1\}} dt, \\ d\langle (X_t^x)_{\{1,i\}}, (X_t^x)_{\{1,j\}} \rangle &= \sum_{k=1}^r (c_r)_{i-1,k} (c_r)_{j-1,k} dt = (cc^T)_{i-1, j-1} dt = (X_t^x)_{\{i,j\}} dt, \\ d\langle (X_t^x)_{\{1,1\}}, (X_t^x)_{\{1,i\}} \rangle &= 2 \sum_{k=1}^r \sum_{l=1}^r (c_r)_{i-1,k} (c_r^{-1})_{k,l} (X_t^x)_{\{1, l+1\}} dt = 2(X_t^x)_{\{1,i\}} dt, \text{ if } i \leq r+1, \\ d\langle (X_t^x)_{\{1,1\}}, (X_t^x)_{\{1,i\}} \rangle &= 2 \sum_{k=1}^r \sum_{l=1}^r (k_r)_{i-1-r,k} (c_r^{-1})_{k,l} (X_t^x)_{\{1, l+1\}} dt \\ &= 2 \sum_{l=1}^r (k_r c_r^{-1})_{i-1-r, l} (X_t^x)_{\{1, l+1\}} dt = 2(X_t^x)_{\{1,i\}} dt \text{ if } i > r+1, \text{ by Lemma 35.} \end{aligned}$$

Thus, we deduce that  $L_1$  is the infinitesimal generator of  $(X_t^x)_{t \geq 0}$ .

**Lemma 35** — Let  $y \in \mathcal{S}_d^+(\mathbb{R})$ . We set  $r = \text{Rk}((y_{i,j})_{2 \leq i,j \leq d})$ ,  $y_1^r = (y_{1,i+1})_{1 \leq i \leq r}$  and  $y_1^{r,d} = (y_{1,i+1})_{r+1 \leq i \leq d}$ . We assume that there are an invertible matrix  $c_r$  and a matrix  $k_r$  defined on  $\mathcal{M}_{d-r-1 \times r}(\mathbb{R})$ , such that

$$(y_{i,j})_{2 \leq i,j \leq d} = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix} \begin{pmatrix} c_r^T & k_r^T \\ 0 & 0 \end{pmatrix}.$$

Then, we have  $y_1^{r,d} = k_r c_r^{-1} y_1^r$ .

*Proof :* We set  $p = \left( \begin{array}{c|cc} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{array} \right)$  and have  $p^{-1} = \left( \begin{array}{c|cc} 1 & 0 & 0 \\ 0 & c_r^{-1} & 0 \\ 0 & -k_r c_r^{-1} & I_{d-r-1} \end{array} \right)$ . Since the matrix

$$p^{-1} y (p^{-1})^T = \left( \begin{array}{c|cc} y_{1,1} & (c_r^{-1} y_1^r)^T & (y_1^{r,d} - k_r c_r^{-1} y_1^r)^T \\ \hline c_r^{-1} y_1^r & I_r & 0 \\ y_1^{r,d} - k_r c_r^{-1} y_1^r & 0 & 0 \end{array} \right)$$

is positive semidefinite, we necessarily have  $y_1^{r,d} - k_r c_r^{-1} y_1^r = 0$ .  $\square$

## D Proofs of Section 3

**Lemma 36** — Let us consider an operator  $L$  on  $\mathbb{D}$  that satisfies the required assumption and such that  $b$  and  $\sigma$  have a sublinear growth, i.e.  $\exists K > 0, \|b(x)\|_2 + \|\sigma(x)\|_2 \leq K(1 + \|x\|_2)$ . Let us consider  $\hat{X}_t^x$  a potential  $\nu$ th-order scheme for  $L$ , with  $\nu \geq 1$ . Then, condition (31) holds.

*Proof :* We have  $\mathbb{D} \subset \mathbb{R}^\zeta$ , and we set for  $p \in \mathbb{N}^*$ ,  $f_p(x) = \left( \sum_{i=1}^\zeta x_i^2 \right)^p = (\|x\|_2)^{2p}$ ,  $x \in \mathbb{D}$ . Clearly,  $f_p \in \mathcal{C}_{\text{pol}}^\infty(\mathbb{D})$ , and since  $\hat{X}_t^x$  is a potential  $\nu$ th-order scheme for  $L$ , we have

$$\lim_{t \rightarrow 0^+} [\mathbb{E}[f_p(\hat{X}_t^x)] - f_p(x)]/t = Lf_p(x). \quad (48)$$

Since we have  $\partial_i f_p(x) = 2px_i f_{p-1}(x)$  and  $\partial_i \partial_j f_p(x) = 4p(p-1)x_i x_j f_{p-2}(x) + \mathbf{1}_{i=j}(2pf_{p-1}(x))$ , it is easy to check that  $|Lf_p(x)| \leq \sum_{i=1}^\zeta 2pK(1 + \|x\|_2)\|x\|_2^{2p-1} + \sum_{i,j=1}^\zeta 4p^2 K^2(1 + \|x\|_2)^1 \|x\|_2^{2p-2} < C_p(1 + \|x\|_2)^{2p} = C_p(1 + f_p(x))$ , for some constant  $C_p > 0$ . From (48), we get that:

$$\exists \eta > 0, \forall t \in (0, \eta), \mathbb{E}[f_p(\hat{X}_t^x)] \leq f_p(x) + tC_p(1 + f_p(x)).$$

We consider now the scheme  $(\hat{X}_{t_i^N}^N, i = 0, \dots, N)$  with  $N > T/\eta$  that starts from  $x_0 \in \mathbb{D}$ . We have  $\mathbb{E}[f_p(\hat{X}_{t_{i+1}^N}^N)] \leq \mathbb{E}[f_p(\hat{X}_{t_i^N}^N)] + C_p(T/N)(1 + \mathbb{E}[f_p(\hat{X}_{t_i^N}^N)])$ . Let  $u_0 = f_p(x_0)$  and  $u_{i+1} = u_i(1 + C_p T/N) + C_p T/N$ . We have  $\mathbb{E}[f_p(\hat{X}_{t_{i+1}^N}^N)] \leq u_i$  and  $u_i = (1 + C_p T/N)^i (f_p(x_0) + 1) - 1 \leq e^{C_p T} (f_p(x_0) + 1)$ . Therefore we have

$$\sup_{N > T/\eta} \sup_{0 \leq i \leq N} \mathbb{E}[\|\hat{X}_{t_i^N}^N\|_2^{2p}] \leq e^{C_p T} (\|x_0\|_2^{2p} + 1) < \infty,$$

which gives (31).  $\square$

## D.1 Proof of Theorem 23

Theorem 23 defines the scheme as  $\pi^T h_r(\hat{U}_t^u)\pi$ . We can prove (see later) that  $\hat{U}_t^u$  is a potential  $\nu$ th-order scheme for some operator, and it would be then easy to analyze the error if  $h_r$  were in  $\mathcal{C}_{\mathbf{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$ . Unfortunately,  $h_r$  is only smooth w.r.t. to the coordinates  $u_{\{1,1\}}, \dots, u_{\{1,d\}}$  and is not a priori in  $\mathcal{C}_{\mathbf{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$ . In fact, this is sufficient to show that  $\pi^T h_r(\hat{U}_t^u)\pi$  is a potential  $\nu$ th-order scheme because these coordinates are the only one that are modified by the scheme. This requires however some further analysis which is made below.

Let  $\mathbb{D} \subset \mathbb{R}^\zeta$  be a domain. We introduce for any domain  $\tilde{\mathbb{D}} \subset \mathbb{R}^\xi$ ,  $\xi \in \mathbb{N}^*$  the set

$$\mathcal{C}_{\mathbf{pol}}^\infty|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}}) = \{f \in \mathcal{C}^\infty(\mathbb{D} \times \tilde{\mathbb{D}}, \mathbb{R}), \forall \gamma \in \mathbb{N}^\zeta, \exists C_\gamma > 0, e_\gamma \in \mathbb{N}^*, \forall (x, \tilde{x}) \in \mathbb{D} \times \tilde{\mathbb{D}}, |\partial_\gamma f(x, \tilde{x})| \leq C_\gamma(1 + \|(x, \tilde{x})\|^{e_\gamma})\},$$

where  $\|\cdot\|$  is a norm on  $\mathbb{R}^{\zeta+\xi}$  and  $\partial_\gamma = \partial_1^{\gamma_1}, \dots, \partial_\zeta^{\gamma_\zeta}$  denotes the derivatives w.r.t. the coordinates of  $\mathbb{D}$ . For  $f \in \mathcal{C}_{\mathbf{pol}}^\infty|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$ , we will say that  $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^\zeta}$  is a *good sequence* for  $f$  if it is such that  $|\partial_\gamma f(x, \tilde{x})| \leq C_\gamma(1 + \|(x, \tilde{x})\|^{e_\gamma})$  for any  $(x, \tilde{x}) \in \mathbb{D} \times \tilde{\mathbb{D}}$ .

Let us now consider an operator  $L$  that satisfies the required assumption on  $\mathbb{D}$ . It is easy to check that all the iterated functions  $L^k f$  are well defined and belong to  $\mathcal{C}_{\mathbf{pol}}^\infty|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$  as soon as  $f \in \mathcal{C}_{\mathbf{pol}}^\infty|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$ . Let us fix  $\tilde{x} \in \tilde{\mathbb{D}}$  and consider  $\hat{X}_t^x$  sampled according to a potential weak  $\nu$ th-order scheme for  $L$ . Since  $x \mapsto f(x, \tilde{x})$  belongs to  $\mathcal{C}_{\mathbf{pol}}^\infty(\mathbb{D})$ , we know by (30) that there are constants  $C_{\tilde{x}}, E_{\tilde{x}}, \eta_{\tilde{x}}$  such that

$$\forall t \in (0, \eta_{\tilde{x}}), \left| \mathbb{E}[f(\hat{X}_t^x)] - \sum_{k=0}^{\nu} \frac{1}{k!} t^k L^k f(x, \tilde{x}) \right| \leq C_{\tilde{x}} t^{\nu+1} (1 + \|x\|^{E_{\tilde{x}}}).$$

In practice, one would like instead to get some bounds where the dependence with respect to  $\tilde{x}$  is more tractable. This is why we introduce the following definition.

**Definition 37** — *Let  $L$  be an operator that satisfies the required assumption on  $\mathbb{D}$ . We will say that a potential weak  $\nu$ th-order scheme for  $L$  satisfies the immersion property if for any  $\tilde{\mathbb{D}} \subset \mathbb{R}^\xi$ ,  $\xi \in \mathbb{N}^*$  and any function  $f \in \mathcal{C}_{\mathbf{pol}}^\infty|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$  with a good sequence  $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^\zeta}$ , there exist positive constants  $C$ ,  $E$ , and  $\eta$  depending only on  $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^\zeta}$  such that*

$$\forall t \in (0, \eta), \left| \mathbb{E}[f(\hat{X}_t^x, \tilde{x})] - \sum_{k=0}^{\nu} \frac{1}{k!} t^k L^k f(x, \tilde{x}) \right| \leq C t^{\nu+1} (1 + \|(x, \tilde{x})\|^E).$$

In practice, most of the usual schemes satisfy this property. In fact, to prove that a scheme is a potential  $\nu$ th-order scheme, it is common to use a Taylor expansion that gives generally at the same time the immersion property. We illustrate this for the exact scheme below.

**Proposition 38** — *Let  $\mathbb{D} \subset \mathbb{R}^\zeta$ ,  $b : \mathbb{D} \rightarrow \mathbb{R}^\zeta$  and  $\sigma : \mathbb{D} \rightarrow \mathcal{M}_\zeta(\mathbb{R})$  such that  $\|b(x)\| + \|\sigma(x)\| \leq C(1 + \|x\|)$  for some  $C > 0$ , and assume that  $Lf(x) = \sum_{i=1}^{\zeta} b_i(x) \partial_i f(x) + \frac{1}{2} \sum_{i,j=1}^{\zeta} (\sigma \sigma^T(x))_{i,j} \partial_i \partial_j f(x)$  satisfies the required assumption on  $\mathbb{D}$ . Then, for any  $\nu \in \mathbb{N}$ , the exact scheme is a potential weak  $\nu$ th-order scheme for  $L$  and it satisfies the immersion property.*

*Proof :* Let  $f \in \mathcal{C}_{\mathbf{pol}}^\infty|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$ . We know from the sublinear growth condition that we have bounds on the moments of  $X_t^x$ :  $\forall q \in \mathbb{N}^*, \exists C_q > 0, \forall t \in [0, 1], \mathbb{E}[\|(X_t^x, \tilde{x})\|^q] \leq C_q(1 + \|(x, \tilde{x})\|^q)$ . By iterating Itô's Formula, we get then easily for  $t \in [0, 1]$ ,

$$\mathbb{E}[f(X_t^x, \tilde{x})] = \sum_{k=0}^{\nu} \frac{t^k}{k!} L^k f(x, \tilde{x}) + \int_0^t \frac{(t-s)^\nu}{\nu!} \mathbb{E}[L^{\nu+1} f(X_s^x, \tilde{x})] ds.$$

Since  $L^{\nu+1}f \in \mathcal{C}_{\text{pol}}^\infty|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$ , there are constants  $C > 0$  and  $q \in \mathbb{N}^*$  depending only on a good sequence of  $f$  such that  $|L^{\nu+1}f(x, \tilde{x})| \leq C(1 + \|(x, \tilde{x})\|^q)$ . Thus, we deduce that  $|\mathbb{E}[f(X_t^x, \tilde{x})] - \sum_{k=0}^{\nu} \frac{t^k}{k!} L^k f(x, \tilde{x})| \leq \frac{t^{\nu+1}}{(\nu+1)!} C(1 + C_q(1 + \|(x, \tilde{x})\|^q))$ .  $\square$

Besides, the immersion property is easily preserved by scheme composition.

**Proposition 39** — *Let  $L_1$  and  $L_2$  be two operators that satisfy the required assumptions on  $\mathbb{D}$ , and assume that  $\hat{p}_x^1(t)(dz)$  and  $\hat{p}_x^2(t)(dz)$  are respectively potential weak  $\nu$ th-order discretization schemes on  $\mathbb{D}$  for these operators that satisfy the immersion property. Let  $\lambda_1, \lambda_2 > 0$  and  $\hat{X}_{\lambda_2 t, \lambda_1 t}^{2 \circ 1, x} \sim \hat{p}^2(\lambda_2 t) \circ \hat{p}_x^1(\lambda_1 t)(dz)$ . Let  $f \in \mathcal{C}_{\text{pol}}^\infty|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$ . Then, there are constants  $C, E, \nu$  that only depend on a good sequence of  $f$  such that*

$$\forall t \in (0, \eta), \left| \mathbb{E}[f(\hat{X}_{\lambda_2 t, \lambda_1 t}^{2 \circ 1, x}, \tilde{x})] - \sum_{l_1+l_2 \leq \nu} \frac{\lambda_1^{l_1} \lambda_2^{l_2}}{l_1! l_2!} t^{l_1+l_2} L_1^{l_1} L_2^{l_2} f(x, \tilde{x}) \right| \leq C t^{\nu+1} (1 + \|(x, \tilde{x})\|^E).$$

Therefore,

- If  $L_1 L_2 = L_2 L_1$ ,  $\hat{p}^2(t) \circ \hat{p}_x^1(t)(dz)$  is a potential weak  $\nu$ th-order discretization scheme for  $L_1 + L_2$  satisfying the immersion property.
- If  $\nu \geq 2$ ,  $\hat{p}^2(t/2) \circ \hat{p}_x^1(t) \circ \hat{p}_x^2(t/2)$  and  $\frac{1}{2}(\hat{p}^2(t) \circ \hat{p}_x^1(t) + \hat{p}_x^1(t) \circ \hat{p}^2(t))$  are potential weak second order schemes for  $L_1 + L_2$  satisfying the immersion property.

This proposition is a straightforward extension of Proposition 1.15, Corollary 1.16 and Theorem 1.17 of Alfonsi [1], and we do not repeat the proof here. Thanks to this result, we can prove the immersion property of the schemes that are obtained by splitting. The Ninomiya-Victoir scheme [20] which is obtained by a composition of exact schemes naturally satisfies this property. By looking at the proof of Theorem 1.18 in [1], it still satisfies this property if we replace the Gaussian samples by moment matching variables. Also, we can check that the second and third order schemes for the CIR process presented in Alfonsi [1] satisfy the immersion property.

**Corollary 40** — *The Ninomiya-Victoir scheme and the second and third order scheme for the Cox-Ingersoll-Ross process given in [1] satisfy the immersion property.*

**Corollary 41** — *Let  $L_1$  (resp.  $L_2$ ) be an operator that satisfies the required assumptions on  $\mathbb{D}_1$  (resp.  $\mathbb{D}_2$ ). Let  $\hat{X}_t^{1, x_1}$  and  $\hat{X}_t^{2, x_2}$  be potential weak  $\nu$ th-order schemes for  $L_1$  and  $L_2$  sampled independently. Then,  $(\hat{X}_t^{1, x_1}, \hat{X}_t^{2, x_2})$  is a potential weak  $\nu$ th-order schemes on  $\mathbb{D}_1 \times \mathbb{D}_2$  that satisfies the immersion property.*

*Proof* : From the immersion property, it is easy to check that  $(\hat{X}_t^{1, x_1}, x_2)$  (resp.  $(x_1, \hat{X}_t^{2, x_2})$ ) is a potential  $\nu$ th order scheme for  $L_1$  (resp.  $L_2$ ) on  $\mathbb{D}_1 \times \mathbb{D}_2$  that satisfies the immersion property. The composition of these schemes is simply  $(\hat{X}_t^{1, x_1}, \hat{X}_t^{2, x_2})$ . Since  $L_1$  and  $L_2$  operate on different domains, we have  $L_1 L_2 = L_2 L_1$ , which gives the result.  $\square$

**Proof of Theorem 23.**

Let  $x \in \mathcal{S}_d^+(\mathbb{R})$  and  $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$  and  $r = \text{Rk}((x_{i,j})_{2 \leq i, j \leq d})$ . Since the operator  $L_1$  satisfies the required

assumption, we know that the exact scheme is a potential  $\nu$ th-order scheme (Proposition 38), and there are constants  $C, E, \eta > 0$  depending on a good sequence of  $f$  such that

$$\forall t \in (0, \eta), |\mathbb{E}[f(X_t^x)] - \sum_{k=0}^{\nu} \frac{t^k}{k!} L_1^k f(x)| \leq C(1 + \|x\|^E). \quad (49)$$

On the other hand, we know from Theorem 13, equation (28) and Corollary 15 that we have

$$X_t^x = \pi^T h_r(U_t^u) \pi,$$

where  $(U_t^u)_{\{1,l\}}$  solves the SDEs (26) starting from the initial condition  $u_{1,l}$  for  $1 \leq l \leq r+1$ , and  $(U_t^u)_{\{i,j\}} = u_{\{i,j\}}$  for the other coordinates. We have also  $\hat{X}_t^x = \pi^T h_r(\hat{U}_t^u) \pi$  by construction, and it is natural to focus on the function  $u \mapsto f(\pi^T h_r(u) \pi)$ .

Let us consider the set

$$\{x \in \mathcal{S}_d(\mathbb{R}), \text{ s.t. } (x_{i,j})_{2 \leq i,j \leq d} \in \mathcal{S}_{d-1}^+(\mathbb{R}), x_{1,1} \geq 0\}.$$

It is isomorphic to  $(\mathbb{R}_+ \times \mathbb{R}^{d-1}) \times \mathcal{S}_{d-1}^+(\mathbb{R})$  by the map  $x \mapsto ((x_{1,1}, \dots, x_{1,d}), (x_{i,j})_{2 \leq i,j \leq d})$ . We have to notice now that the function  $h_r$  defined by (28) is such that  $h_r \in \mathcal{C}_{\text{pol}}^\infty|_{\mathbb{R}_+ \times \mathbb{R}^{d-1}}(\mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathcal{S}_{d-1}^+(\mathbb{R}))$ . It is indeed a polynomial function with respect to  $u_{1,1}, \dots, u_{1,d}$ . Then, it is easy to check that  $u \mapsto f(\pi^T h_r(u) \pi) \in \mathcal{C}_{\text{pol}}^\infty|_{\mathbb{R}_+ \times \mathbb{R}^{d-1}}(\mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathcal{S}_{d-1}^+(\mathbb{R}))$  since  $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$ . Moreover, by the chain rule, we can get a good sequence for this function that only depend on a good sequence of  $f$  since  $\pi$  and  $h_r$  are fixed.

By assumption,  $(\hat{U}_t^u)_{\{1,1\}}$  is a potential  $\nu$ th-order scheme for the operator  $(\alpha - r)\partial_{\{1,1\}} + 2u_{\{1,1\}}\partial_{\{1,1\}}^2$  and satisfy the immersion property. The schemes  $(\hat{U}_t^u)_{\{1,i\}}$  ( $2 \leq i \leq r+1$ ) can be seen as a Ninomiya-Victoir scheme with moment matching variables. They are therefore potential  $\nu$ th-order scheme for the operator  $\frac{1}{2}\partial_{\{1,i\}}^2$  ([1], Theorem 1.18) and satisfy the immersion property from Corollary 40. Therefore, from Corollary 41,  $((\hat{U}_t^u)_{\{1,1\}}, \dots, (\hat{U}_t^u)_{\{1,d\}})$  is a potential  $\nu$ th order scheme for  $(\alpha - r)\partial_{\{1,1\}} + 2u_{\{1,1\}}\partial_{\{1,1\}}^2 + \frac{1}{2}\sum_{i=1}^r \partial_{\{1,i\}}^2$  satisfying the immersion property. Thus, there are constants that we still denote by  $C, E, \eta > 0$  depending on a good sequence of  $f$  such that:

$$\forall t \in (0, \eta), |\mathbb{E}[f(\pi^T h_r(U_t^u) \pi)] - \mathbb{E}[f(\pi^T h_r(\hat{U}_t^u) \pi)]| \leq Ct^{\nu+1}(1 + \|u\|^E). \quad (50)$$

Now, one has to notice that  $\|u\| \leq C'(1 + \|x\|)$  for some constant  $C' > 0$  since we have  $u_{\{1,1\}} + \sum_{k=1}^r (u_{\{1,k+1\}})^2 = \tilde{x}_{\{1,1\}}$  and  $\tilde{x}$  and  $x$  have the same Frobenius norm. We get then the result by gathering (49) and (50).  $\square$

**Remark 42** — We can check by looking at the proof above that the scheme obtained for  $L_1$  also satisfies the immersion property. Since we do not need it for the construction of the further schemes for  $WIS_d(x, \alpha, b, a)$  or  $AFf_d(x, \bar{\alpha}, B, a)$ , we will no longer mention it.

## D.2 Proof of the third condition of Theorem 21 for Wishart processes

The scope of this part is to show the following result.

**Proposition 43** — Let  $(X_t^x)_{t \geq 0} \sim WIS_d(x, \alpha, b, a)$ ,  $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$ ,  $x \in \mathcal{S}_d^+(\mathbb{R})$  and  $T > 0$ . Then,  $\tilde{u}(t, x) = \mathbb{E}[f(X_t^x)]$  is  $C^\infty$  on  $[0, T] \times \mathcal{S}_d^+(\mathbb{R})$ , solves  $\partial_t \tilde{u}(t, x) = L\tilde{u}(t, x)$  and its derivatives satisfy

$$\forall l \in \mathbb{N}, \forall n \in \mathbb{N}^{\frac{d(d+1)}{2}} \exists C_{l,n}, e_{l,n} > 0, \forall x \in \mathcal{S}_d^+(\mathbb{R}), \forall t \in [0, T], \left| \partial_t^l \prod_{1 \leq i \leq j \leq d} \partial_{\{i,j\}}^{n_{\{i,j\}}} \tilde{u}(t, x) \right| \leq C_{l,n}(1 + \|x\|^{e_{l,n}}). \quad (51)$$



For technical reasons on the boundary of  $\mathcal{S}_d^+(\mathbb{R})$ , it is convenient here to work with  $f \in \mathcal{C}_{\mathbf{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$  rather than  $f \in \mathcal{C}_{\mathbf{pol}}^\infty(\mathcal{S}_d^+(\mathbb{R}))$ . The proof of Proposition 43 is based on a remarkable identity on the derivatives of the Laplace transform which is stated below.

**Lemma 44** — Let  $(X_t^x)_{t \geq 0} \underset{Law}{\sim} WIS_d(x, \alpha, b, a)$  and  $v = v_R + iv_I$  such that  $v_R \in \mathcal{D}_{b,a;t}$  and  $v_I \in \mathcal{S}_d(\mathbb{R})$ . We denote by  $\phi(t, \alpha, x, v)$  the Laplace transform of  $X_t^x$  given by (10), the other parameters  $a, b$  being fixed. Then, the derivative w.r.t  $x_{\{k,l\}}$  satisfies the following equality

$$\partial_{\{k,l\}} \phi(t, \alpha, x, v) = \phi(t, \alpha + 2, x, v) p_t^{\{k,l\}}(v),$$

where  $p_t^{\{k,l\}}$  is a polynomial function of the matrix elements of degree  $d$  defined by :

$$p_t^{\{k,l\}}(v) = \text{Tr} \left[ v \text{adj}(I_d - 2q_t v) m_t (e_d^{k,l} + \mathbb{1}_{k \neq l} e_d^{l,k}) m_t^T \right] =: \sum_{\gamma \in \mathbb{N}^{\frac{d(d+1)}{2}}, |\gamma| \leq d} a_t^{\gamma, \{k,l\}} \bar{v}^\gamma, \text{ where } \bar{v}^\gamma = \prod_{\{i,j\}} v_{\{i,j\}}^{\gamma_{\{i,j\}}}.$$

Moreover, its coefficients are bounded uniformly in time:

$$\exists K_t > 0, \forall s \in [0, t], \max_{\gamma \in \mathbb{N}^{\frac{d(d+1)}{2}}, |\gamma| \leq d} \left( |a_s^{\gamma, \{k,l\}}| \right) \leq K_t.$$

*Proof :* We get from (10),

$$\begin{aligned} \partial_{\{k,l\}} \phi(t, \alpha, x, v) &= \frac{\text{Tr} \left[ v \text{adj}(I_d - 2q_t v) m_t (e_d^{k,l} + \mathbb{1}_{k \neq l} e_d^{l,k}) m_t^T \right]}{\det(I_d - 2q_t v)} \times \frac{\exp(\text{Tr} [v(I_d - 2q_t v)^{-1} m_t x m_t^T])}{\det(I_d - 2q_t v)^{\frac{\alpha}{2}}} \\ &= \phi(t, \alpha + 2, x, v) \text{Tr} \left[ v \text{adj}(I_d - 2q_t v) m_t (e_d^{k,l} + \mathbb{1}_{k \neq l} e_d^{l,k}) m_t^T \right]. \end{aligned}$$

Since  $s \mapsto \|m_s\|$  and  $s \mapsto \|q_s\|$  are continuous functions on  $[0, t]$ , we obtain the bounds on the polynomial coefficients.  $\square$

*Proof of Proposition 43:* Let  $f \in \mathcal{C}_{\mathbf{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$ . First, let us observe that (51) is obvious when  $l = |n| = 0$ . Since we have  $\forall l \in \mathbb{N}, L^l f \in \mathcal{C}_{\mathbf{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$ , and  $\partial_t^l \tilde{u}(t, x) = \mathbb{E}(L^l f(X_t^x))$ , it is sufficient to prove (51) only for the derivatives w.r.t.  $x$ .

We first focus on the case  $|n| = 1$  and want to show that  $\partial_{\{k,l\}} \tilde{u}(t, x)$  satisfies (51). The sketch of this proof is to write  $f$  as the inverse Fourier transform of its Fourier transform and then use Lemma 44. Unfortunately,  $f$  has not a priori the required integrability to do that, and we have to introduce an auxiliary function  $f_\rho$ .

**Definition of the new function  $f_\rho$ .** Since  $\mathcal{D}_{b,a;T}$  given by (9) is an open set and  $0 \in \mathcal{D}_{b,a;T}$ , there is  $\rho > 0$  such that  $\rho I_d \in \mathcal{D}_{b,a;T}$ . Let  $\mu : \mathbb{R} \rightarrow \mathbb{R}$  be the function such that  $\mu(x) = 0$  if  $x \leq -1$  or  $x \geq 0$ ,  $\mu(x) = \exp(\frac{1}{x(x+1)})$  if  $-1 < x < 0$ . We have  $\mu \in \mathcal{C}^\infty(\mathbb{R})$ . We consider then the cutoff function  $\zeta : \mathbb{R} \rightarrow \mathbb{R} \in \mathcal{C}^\infty(\mathbb{R})$  defined as  $\forall x \in \mathbb{R}, \zeta(x) = \frac{\int_{-\infty}^x \mu(y) dy}{\int_{\mathbb{R}} \mu(y) dy}$ . It is nondecreasing, such that  $0 \leq \zeta(x) \leq 1$ ,  $\zeta(x) = 0$  if  $x \leq -1$  and  $\zeta(x) = 1$  if  $x \geq 0$ . Besides, we have  $\zeta \in \mathcal{C}_{\mathbf{pol}}^\infty(\mathbb{R})$  since all its derivatives have a compact support. Now, we define a  $\vartheta \in \mathcal{C}_{\mathbf{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$  as

$$\begin{aligned} \vartheta : \mathcal{S}_d(\mathbb{R}) &\rightarrow \mathbb{R} \\ x &\mapsto \prod_{i=1}^d \zeta(x_{\{i,i\}}) \prod_{i \neq j} \zeta(x_{\{j,j\}} x_{\{i,i\}} - x_{\{i,j\}}^2). \end{aligned}$$

It is important to notice that  $0 \leq \vartheta \leq 1$ ,  $\vartheta(x) = 1$  if  $x \in \mathcal{S}_d(\mathbb{R})$  and  $\vartheta(x) = 0$  if there is  $i \in \{1, \dots, d\}$  such that  $x_{\{i,i\}} < -1$  or  $i < j \in \{1, \dots, d\}$  such that  $x_{\{i,j\}}^2 > 1 + x_{\{i,i\}}x_{\{j,j\}}$ . Let  $\gamma \in \mathbb{N}^{d(d-1)/2}$ . Since  $f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$ , there are constants  $K, E > 0$  and  $K', E' > 0$  such that,  $\forall x \in \mathcal{S}_d(\mathbb{R})$

$$\begin{aligned} |\partial^\gamma(\vartheta f)(x)| &\leq K(1 + \|x\|^E) \prod_{i=1}^d \left( \mathbb{1}_{\{|x_{\{i,i\}}| > -1\}} \right) \prod_{1 \leq i < j \leq d} \left( \mathbb{1}_{\{x_{\{i,j\}}^2 \leq 1 + x_{\{i,i\}}x_{\{j,j\}}\}} \right) \\ &\leq K'(1 + \|(x_{\{i,i\}})_{1 \leq i \leq d}\|^{E'}) \prod_{i=1}^d \left( \mathbb{1}_{\{|x_{\{i,i\}}| > -1\}} \right) \prod_{1 \leq i < j \leq d} \left( \mathbb{1}_{\{x_{\{i,j\}}^2 \leq 1 + x_{\{i,i\}}x_{\{j,j\}}\}} \right). \end{aligned}$$

Here, the upper bound only involves the diagonal coefficients. We define

$$x \in \mathcal{S}_d(\mathbb{R}), \quad f_\rho(x) := \vartheta(x)f(x) \exp(-\text{Tr}(\rho x)),$$

and obtain from the last inequality that  $f_\rho$  belongs to the Schwartz space of rapidly decreasing functions since  $\rho > 0$ . Thus, its Fourier transform also belongs to the Schwartz space and we have

$$f_\rho(x) = \frac{1}{(2\pi)^{\frac{d(d+1)}{2}}} \int_{\mathbb{R}^{\frac{d(d+1)}{2}}} \exp(-\text{Tr}(ivx)) \mathcal{F}(f_\rho)(v) dv, \quad \text{where } \mathcal{F}(f_\rho)(v) = \int_{\mathbb{R}^{\frac{d(d+1)}{2}}} \exp(\text{Tr}(ivx)) f_\rho(x) dx,$$

and in particular  $f_\rho, \mathcal{F}(f_\rho) \in L^1(\mathcal{S}_d(\mathbb{R})) \cap L^\infty(\mathcal{S}_d(\mathbb{R}))$ .

**A new representation of  $\tilde{u}(t, x)$ .** We have  $f(x) = \exp(\rho \text{Tr}(x)) f_\rho(x)$  for  $x \in \mathcal{S}_d^+(\mathbb{R})$ , and therefore

$$\begin{aligned} \tilde{u}(t, x) &= \mathbb{E}[\exp(\text{Tr}(\rho X_t^x)) f_\rho(X_t^x)] \\ &= \frac{1}{(2\pi)^{\frac{d(d+1)}{2}}} \mathbb{E} \left[ \int_{\mathbb{R}^{\frac{d(d+1)}{2}}} \exp(\text{Tr}((-iv + \rho I_d) X_t^x)) \mathcal{F}(f_\rho)(v) dv \right] \\ &= \frac{1}{(2\pi)^{\frac{d(d+1)}{2}}} \int_{\mathbb{R}^{\frac{d(d+1)}{2}}} \mathbb{E}[\exp(\text{Tr}((-iv + \rho I_d) X_t^x))] \mathcal{F}(f_\rho)(v) dv. \end{aligned}$$

The last equality holds since  $\int_{\mathbb{R}^{\frac{d(d+1)}{2}}} |\mathbb{E}[\exp(\text{Tr}((-iv + \rho I_d) X_t^x))]| \times |\mathcal{F}(f_\rho)(v)| dv \leq \phi(t, \alpha, x, \rho I_d) \|\mathcal{F}(f_\rho)\|_1 < \infty$ . Here, we have used that  $\rho I_d \in \mathcal{D}_{b,a;T}$  to get  $\phi(t, \alpha, x, \rho I_d) < \infty$ .

**Derivation with respect to  $x_{\{k,l\}}$ ,  $k, l \in \{1, \dots, d\}$ .** From Lemma 44, we have by Lebesgue's theorem

$$\partial_{\{k,l\}} \tilde{u}(t, x) = \frac{1}{(2\pi)^{\frac{d(d+1)}{2}}} \int_{\mathbb{R}^{\frac{d(d+1)}{2}}} \phi(t, \alpha + 2, x, -iv + \rho I_d) p_t^{\{k,l\}}(\rho I_d - iv) \mathcal{F}(f_\rho)(v) dv \quad (52)$$

since  $|\partial_{\{k,l\}}^x \phi(t, \alpha, x, -iv + \rho I_d) \mathcal{F}(f_\rho)(v)| \leq |\phi(t, \alpha + 2, x, \rho I_d)| |p_t^{\{k,l\}}(\rho I_d - iv) \mathcal{F}(f_\rho)(v)|$  and  $p_t^{\{k,l\}}(\rho I_d - iv) \mathcal{F}(f_\rho)(v)$  is a rapidly decreasing function.

Let  $1 \leq k', l' \leq d$ . An integration by part gives  $\int_{\mathbb{R}} (\rho I_d - iv)_{\{k',l'\}} \exp(\text{Tr}[x(iv - \rho I_d)]) \vartheta(x) f(x) dx_{\{k',l'\}} = \left( \frac{\mathbb{1}_{k' \neq l'}}{2} + \mathbb{1}_{k'=l'} \right) \int_{\mathbb{R}} \exp(\text{Tr}[x(iv - \rho I_d)]) \partial_{\{k',l'\}} (\vartheta(x) f(x)) dx_{\{k',l'\}}$ , and thus

$$(\rho I_d - iv)_{\{k',l'\}} \mathcal{F}(\exp[-\rho \text{Tr}(x)] \vartheta(x) f(x))(v) = \left( \frac{\mathbb{1}_{k' \neq l'}}{2} + \mathbb{1}_{k'=l'} \right) \mathcal{F}(\exp[-\rho \text{Tr}(x)] \partial_{\{k',l'\}} [\vartheta(x) f(x)])(v).$$

We set  $\varphi(\gamma) = \prod_{1 \leq k' \leq l' \leq d} \left( \frac{\mathbb{1}_{k' \neq l'}}{2} + \mathbb{1}_{k'=l'} \right)^{\gamma_{\{k',l'\}}}$  for  $\gamma \in \mathbb{N}^{d(d+1)/2}$  and get by iterating the argument that:

$$\prod_{1 \leq k' \leq l' \leq d} (\rho I_d - iv)_{\{k',l'\}}^{\gamma_{\{k',l'\}}} \mathcal{F}(f_\rho)(v) = \varphi(\gamma) \mathcal{F}(\exp[-\rho \text{Tr}(x)] \partial_\gamma (\vartheta \times f)(x))(v). \quad (53)$$

Since  $p_t^{\{k,l\}}(\rho I_d - iv) = \sum_{\gamma \in \mathbb{N}^{\frac{d(d+1)}{2}}, |\gamma| \leq d} a_t^{\gamma, \{k,l\}} \prod_{1 \leq k' \leq l' \leq d} (\rho I_d - iv)_{\{k',l'\}}^{\gamma_{\{k',l'\}}}$ , we get from (52) and (53):

$$\partial_{\{k,l\}} u(t, x) = \sum_{|\gamma| \leq d} a_t^{\gamma, \{k,l\}} \varphi(\gamma) \mathbb{E}(\partial_\gamma (f \times \vartheta)(Y_t^x)) = \sum_{|\gamma| \leq d} a_t^{\gamma, \{k,l\}} \varphi(\gamma) \mathbb{E}(\partial_\gamma f(Y_t^x)), \quad (54)$$

where  $(Y_t^x)_{t \geq 0} \underset{Law}{\sim} WIS_d(x, \alpha + 2, b, a)$ . Here, we have used that  $\partial_\gamma(\vartheta \times f)(y) = \partial_\gamma f(y)$  for  $y \in \mathcal{S}_d^+(\mathbb{R})$ .

From Lemma 44  $(a_t^{\gamma, \{k, l\}})_{\gamma \in \mathbb{N}^{\frac{d(d+1)}{2}}, |\gamma| \leq d}$  is bounded for  $t \in [0, T]$ , and we get (51) when  $|n| = 1$  since  $\partial_\gamma f \in \mathcal{C}_{\text{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$ . Thanks to (54), a derivative of order  $|n|$ , can be seen as a (bounded) linear combination of derivatives of order  $|n| - 1$ , and we easily get (51) by an induction on  $|n|$ .

It remains to check that we have indeed  $\partial_t \tilde{u}(t, x) = Lu(t, x)$ . Let  $t, h > 0$ . By the Markov property, we have  $\tilde{u}(t+h, x) = \mathbb{E}[\tilde{u}(t, X_h^x)]$ . From (51) and Itô's formula, we get  $[\tilde{u}(t+h, x) - u(t, x)]/h \xrightarrow{h \rightarrow 0^+} Lu(t, x)$ .  $\square$

**Lemma 45** — Let  $\alpha, x \in \mathcal{S}_d^+(\mathbb{R})$ ,  $B \in \mathcal{L}(\mathcal{S}_d^+(\mathbb{R}))$  that satisfies (4), and  $x(t)$  be the solution of the following ODE

$$x(t) = x + \int_0^t (\alpha + B(x(s))) ds. \quad (55)$$

Then, we have  $x(t) \in \mathcal{S}_d^+(\mathbb{R})$  for  $t \geq 0$ .

*Proof :* The ODE (55) is affine and has unique solution on  $\mathcal{S}_d^+(\mathbb{R})$  which is given by

$$t \geq 0, x(t) = \exp(tB)(x) + \int_0^t \exp(sB)(\alpha) ds, \quad (56)$$

where  $\forall t \in \mathbb{R}^+, \forall x \in \mathcal{S}_d(\mathbb{R})$ ,  $\exp(tB)(x) = \sum_{k=0}^\infty \frac{t^k B^k(x)}{k!}$ ,  $B^k(x) = \underbrace{B \circ \dots \circ B}_k(x)$  such that  $B^0(x) = x$ .

We assume first that  $\alpha, x \in \mathcal{S}_d^{+,*}(\mathbb{R})$  and consider  $\tau = \inf\{t \geq 0, x(t) \notin \mathcal{S}_d^+(\mathbb{R})\}$ , with the convention  $\inf \emptyset = +\infty$ . We have  $\tau > 0$ . Let us assume by a way of contradiction that  $\tau < \infty$ . Then,  $x(\tau)$  cannot be invertible and there is  $y \in \mathcal{S}_d^+(\mathbb{R})$  such that  $y \neq 0$  and  $\text{Tr}(yx(\tau)) = 0$ . From (56) and (4), we get

$$\text{Tr}(x'(\tau)y) = \text{Tr}([B(x(\tau)) + \alpha]y) > 0,$$

since  $\alpha$  is positive definite. Therefore, there is  $\epsilon \in (0, \tau)$  such that  $\text{Tr}(yx(\tau - \epsilon)) < 0$ . Let us recall now that  $z \in \mathcal{S}_d^+(\mathbb{R}) \iff \forall y \in \mathcal{S}_d^+(\mathbb{R}), \text{Tr}(yz) \geq 0$ . Thus,  $x(\tau - \epsilon) \notin \mathcal{S}_d^+(\mathbb{R})$ , which contradicts the definition of  $\tau$ .

In the general case  $\alpha, x \in \mathcal{S}_d^+(\mathbb{R})$ , we observe that the solution (56) is continuous w.r.t.  $x$  and  $\alpha$ , and thus  $\forall t \geq 0, x(t) \in \mathcal{S}_d^+(\mathbb{R})$  since  $\mathcal{S}_d^+(\mathbb{R})$  is a closed set.  $\square$