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Multidegree for bifiltered D-modules

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Abstract

In commutative algebra, E. Miller and B. Sturmfels defined the notion of multidegree for multigraded modules over a multigraded polynomial ring. We apply this theory to bifiltered modules over the Weyl algebra D. The bifiltration is a combination of the standard filtration by the order of differential operators and of the so-called V-filtration along a coordinate subvariety of the ambient space defined by M. Kashiwara. The multidegree we define provides a new invariant for D-modules. We investigate its relation with the L-characteristic cycles considered by Y. Laurent. We give examples from the theory of A-hypergeometric systems $M_A(\beta)$ defined by I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky. We consider the V-filtration along the origin. When the toric projective variety defined from the matrix A is Cohen-Macaulay, we have an explicit formula for the multidegree of $M_A(\beta)$.

Introduction

We consider finite type modules over the Weyl algebra

$$D = \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle.$$

It is classical to endow D with the filtration by the order in $\partial_1, \ldots, \partial_n$, which we call the F-filration, and to endow a D-module M with a good F-filtration. For instance that leads to the notion of the characteristic variety, which is the support of $\mathrm{gr}^F(M)$, and to the characteristic cycle. M. Kashiwara introduced another kind of filtration, the V-filtration along a smooth subvariety Y of $\mathbb{C}^n$. Then one has the notion of a good (F, V)-bifiltration (c.f. [12]), and we can also consider intermediate filtrations L between F and V as developed by Y. Laurent in his theory of slopes (c.f. [11]). This leads to L-characteristic varieties (the support of $\mathrm{gr}^L(M)$) and L-characteristic cycles.

Exploring that theory with homological methods, M. Granger, T. Oaku and N. Takayama considered (F, V)-bifiltered free resolutions of finite type D-modules in [8], [15]. More precisely, dealing with local analytic D-modules, they can define minimal bifiltered free resolutions. That provides invariants attached to a bifiltered module: the ranks, also called Betti numbers, and the shifts appearing in the minimal resolution. In the category of modules over the global

Weyl algebra, (F, V)-bifiltered free resolutions still can be considered, but the minimality no longer makes sense.

Our main purpose in this paper is to introduce a new invariant, the multidegree, derived from the Betti numbers and shifts arising from any bifiltered free resolution of a (F, V)-bifiltered D-module. It will be independent of the good bifiltration, i.e. a chosen presentation of the module. We will relate this invariant to the L-characteristic cycles.

To achieve this, we use the theory of K-polynomial and multidegree, as was developed by E. Miller and B. Sturmfels in [13]. The multidegree is a generalization of the usual degree in projective geometry; it is defined for finite type multigraded modules over a polynomial ring. After reviewing this theory in Section 1, we adapt it first to F-filtered D-modules in Section 2. We obtain the notion of multidegree for a F-filtered D-module, which is independent of the good filtration. This multidegree is a monomial mT^d with $m \in \mathbb{N}$; we interpret m and d as a generic multiplicity and a generic codimension respectively.

Then we adapt the theory of multidegree to (F, V)-bifiltered D-modules in section 3. The multidegree is an element of $\mathbb{Z}[T_1, T_2]$, denoted by $\mathcal{C}_{F,V}(M; T_1, T_2)$, homogeneous in T_1, T_2. Its degree d has to be fixed because of the non-positivity of the multigrading considered: if Y is the origin in $\mathbb{C}^n$, d is the codimension of the V-homogenization module $\mathbf{R}_V(M)$. Using a proof in [12], we can show that $\mathcal{C}_{F,V}(M; T_1, T_2)$ is an invariant attached to the module, indepedently of the good bifiltration.

In section 4, we assume a strong regularity condition on the (F, V)-bifiltered module, which we call a nicely bifiltered module. We prove that in the holonomic case, this condition implies that the module has no slopes along Y. Then we show that the multidegree of such a module almost only depends on the L-characteristic cycle of the module, with L an intermediate filtration close to F or close to V. Let us note here that we have to deal with some codimensions which may alter the link between mutidegree and L-characteristic cycle: the codimension of the module $\mathbf{R}_V(M)$ may not be equal to that of $\mathrm{gr}^L(M)$.

Finally, we use the theory of hypergeometric systems to provide interesting examples in section 5. We consider the hypergeometric module $M_A(\beta)$ introduced by I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky in [5], in the case where the semigroup generated by the columns $a_1, \ldots, a_n$ of the matrix A is pointed. We take Y to be the origin in $\mathbb{C}^n$. In that case the problems about codimensions described above does not remain, and the multidegree only depends on the L-characteristic cycle if $M_A(\beta)$ is nicely bifiltered. Let $\mathrm{vol}(A)$ denotes the normalized volume of the convex hull of the set $\{0, a_1, \ldots, a_n\}$ in $\mathbb{R}^d$. Let us assume that the closure in $\mathbb{P}^n$ of the variety defined by I_A is Cohen-Macaulay. Then for generic parameters β (or for all parameters if I_A is homogeneous), niceness holds and we have:

$$\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = \mathrm{vol}(A). \sum_{j=d}^{n} \binom{n-d}{j-d} T_1^j T_2^{n-j}.$$

We give examples, computed with the computer algebra systems Singular [10] and Macaulay2 [9].

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1 Multidegree for modules over a commutative polynomial ring

1.1 Review of the theory

Let us give a review of the theory of K-polynomials and multidegrees in the commutative setting. Let $S = k[x_1, \dots, x_n]$ with k a field. A multigrading on S is given by a homomorphism of abelian groups $\deg : \mathbb{Z}^n \to \mathbb{Z}^d$ with, denoting by $e_1, \dots, e_n$ the canonical base of $\mathbb{Z}^n$, $\deg(e_i) = a_i \in \mathbb{Z}^d$. Identifying the set of monomials of S with $\mathbb{N}^n$, we have $\deg(x_1^{\alpha_1} \dots x_n^{\alpha_n}) = \sum \alpha_i a_i$, and S becomes a multigraded ring over $\mathbb{Z}^d$.

Let $M = \bigoplus_{a \in \mathbb{Z}^d} M_a$ be a multigraded S-module of finite type. For $b \in \mathbb{Z}^d$, let us denote by $S[b]$ the module S endowed with the multigrading such that for any $a \in \mathbb{Z}^d$, $S[b]_a = S_{a-b}$. A multigraded free module is a module isomorphic to $\bigoplus_{j=1}^r S[b_j]$, with $b_1, \dots, b_r \in \mathbb{Z}^d$.

Take a multigraded free resolution, i.e. a multigraded exact sequence

$$0 \to \mathcal{L}_\delta \to \cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to M \to 0,$$

with $\mathcal{L}_i$ a multigraded free module.

Definition 1.1. *For $b = (b_1, \dots, b_d) \in \mathbb{Z}^d$, the K-polynomial of $S[b]$ is defined by*

$$K(S[b]; T_1, \dots, T_d) = T_1^{b_1} \dots T_d^{b_d} \in \mathbb{Z}[T_1, \dots, T_d, T_1^{-1}, \dots, T_d^{-1}].$$

For $b_1, \dots, b_r \in \mathbb{Z}^d$, The K-polynomial of $\mathcal{L} = \bigoplus_{j=1}^r S[b_j]$ is defined by

$$K(\mathcal{L}; T_1, \dots, T_d) = \sum_j K(S[b_j]; T_1, \dots, T_d) \in \mathbb{Z}[T_1, \dots, T_d, T_1^{-1}, \dots, T_d^{-1}].$$

Then the K-polynomial of M is defined by

$$K(M; T) = \sum_i (-1)^i K(\mathcal{L}_i; T_1, \dots, T_d) \in \mathbb{Z}[T_1, \dots, T_d, T_1^{-1}, \dots, T_d^{-1}].$$

Proposition 1.1 ([13], Theorem 8.34)**.** *The definition of $K(M; T_1, \dots, T_d)$ does not depend on the multigraded free resolution.*

If we substitute $T_1, \dots, T_d$ by $1 - T_1, \dots, 1 - T_d$ in $K(M; T_1, \dots, T_d)$, we get a well-defined power series in $\mathbb{Z}[[T_1, \dots, T_d]]$. We then consider the total degree in $T_1, \dots, T_d$.

Definition 1.2. *We denote by* $\mathcal{C}(M;T_1,\dots,T_d)\in\mathbb{Z}[T_1,\dots,T_d]$ *the sum of the terms whose total degree equals* $\mathrm{codim}M$ *in* $K(M;1-T_1,\dots,1-T_d)$*. This is called the* multidegree *of* M*.*

Remind that the module M defines an algebraic cycle $\sum m_iZ_i$, where Z_i, defined by ideals $\mathfrak{p}_i$, are the irreducible components of $\mathrm{rad}(\mathrm{ann}M)$ and m_i is the multiplicity of $M_{\mathfrak{p}_i}$. It turns out that the multidegree depends only on the algebraic cycle.

Proposition 1.2 ([13], Theorem 8.53)**.** *If* $\mathfrak{p}_1,\dots,\mathfrak{p}_k$ *are the maximal dimensional associated primes of* M*, then*

$$\mathcal{C}(M;T_1,\dots,T_d)=\sum_i(\mathrm{mult}_{\mathfrak{p}_i}M)\mathcal{C}(S/\mathfrak{p}_k;T_1,\dots,T_d).$$

S is said to be positively multigraded if moreover for any $b\in\mathbb{Z}^d$, we have $\dim_kS_b<\infty$. In that case we can consider the Hilbert series

$$H(M;T_1,\dots,T_d)=\sum_{b\in\mathbb{Z}^d}(\dim_kM_b)T_1^{b_1}\dots T_d^{b_d}\in\mathbb{Z}[[T_1,\dots,T_d]].$$

If $b=(b_1,\dots,b_d)\in\mathbb{Z}^d$, let us denote by T^b the product $T_1^{b_1}\dots T_d^{b_d}$.

Proposition 1.3. *Let* S *be positively multigraded. Then*

1. $$H(M;T_1,\dots,T_d)=\frac{K(M;T_1,\dots,T_d)}{\Pi(1-T^{a_i})}$$

2. *If* $M\neq 0$*, then* $\mathcal{C}(M;T_1,\dots,T_d)\neq 0$*, moreover* $\mathcal{C}(M;T_1,\dots,T_d)$ *is the sum of the non-zero terms of least total degree in* $K(M;1-T_1,\dots,1-T_d)$*.*

The assertion 1 is [13], Theorem 8.20, and the assertion 2 follows from [13], Claim 8.54 and Exercise 8.10.

1.2 Genericity

Let $S=k[\lambda_1,\dots,\lambda_p][x_1,\dots,x_n]$ be multigraded by $\deg x_i=a_i\in\mathbb{Z}^d$ and $\deg\lambda_i=0$. We consider $\lambda_1,\dots,\lambda_p$ as parameters and study the behaviour of the K-polynomial under the specialization.

Let $\mathbb{K}=\mathrm{Frac}(k[\lambda_1,\dots,\lambda_p])$. Let $M=S^r/N$ be a multigraded finite type S-module. For $c\in k^p$, let

$$M^c=\frac{S}{\langle\lambda_1-c_1,\dots,\lambda_p-c_p\rangle}\otimes M,$$

considered as a multigraded $k[x_1,\dots,x_n]$-module. We are going to state that if c is generic, then $K(\mathbb{K}\otimes M;T)=K(M^c;T)$. More precisely, we shall describe the exceptional values of c in terms of Gröbner bases.

Let $<$ be a well-ordering on $\mathbb{N}^n\times\{1,\dots,r\}$, such that for any $\alpha,\beta,\delta\in\mathbb{N}^n$ and $i,i'\in\{1,\dots,r\}$, we have

$$(\alpha,i)<(\beta,i')\Rightarrow(\alpha+\delta,i)<(\beta+\delta,i'),$$

and let $<'$ be the well-ordering on $\mathbb{N}^p \times \mathbb{N}^n \times \{1, \dots, r\}$ defined by

$$(\alpha, \beta, i) <' (\alpha', \beta', i') \text{ iff } \begin{cases} (\beta, i) < (\beta', i') \\ \text{or } ((\beta, i) = (\beta', i') \text{ and } \alpha <_{\text{lex}} \alpha'). \end{cases}$$

Let $P_1, \dots, P_s$ be a Gröbner base of N. For $1 \leq i \leq s$, $q_i(\lambda) \in k[\lambda]$ denotes the leading coefficient, with respect to $<$, of the image of P_i in $\mathbb{K} \otimes S$. For $P \in k[x]^r$ or $P \in \mathbb{K}[x]^r$, we denote by $\mathrm{Exp}_< P \in \mathbb{N}^n \times \{1, \dots, r\}$ the leading exponent of P with respect to $<$.

Proposition 1.4 ([14], Propositions 6 and 7)**.** *1. $P_1, \dots, P_s$ is a Gröbner base of $\mathbb{K} \otimes N$.*

2. Let $c \in k^n$ such that $c \notin \bigcup_i (q_i = 0)$. Then $P_1(c), \dots, P_s(c)$ is a Gröbner base of N^c and $\mathrm{Exp}_< \mathbb{K} \otimes N = \mathrm{Exp}_< N^c$.

Proposition 1.5. *Let $c \in k^n$ such that $c \notin \bigcup_i (q_i = 0)$. Then $K(\mathbb{K} \otimes M; T) = K(M^c; T)$. Consequently $\mathcal{C}(\mathbb{K} \otimes M; T) = \mathcal{C}(M^c; T)$.*

This follows from Proposition 1.4 and from [13], Theorem 8.36 which asserts that the K-polynomial remains the same when taking the initial module with respect to any well-ordering.

2 Multidegree for F-filtered D-modules

Let $D = \mathbb{C}[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle$ be the Weyl algebra. A vector $(\mathbf{u}, \mathbf{v}) \in \mathbb{Z}^n \times \mathbb{Z}^n$ is called an *admissible weight vector* for D if for all i, $u_i + v_i \geq 0$. For $P = \sum a_{\alpha,\beta}(x) x^\alpha \partial^\beta \in D$, we define

$$\mathrm{ord}^{(\mathbf{u},\mathbf{v})}(P) = \max_{(\alpha,\beta) | a_{\alpha,\beta} \neq 0} (\sum u_i \alpha_i + \sum v_i \beta_i).$$

We then define an increasing filtration by $F_d^{(\mathbf{u},\mathbf{v})}(D) = \{P \in D, \mathrm{ord}^F(P) \leq d\}$ with $d \in \mathbb{Z}$.

In this section we consider only the weight vector $(\mathbf{0}, \mathbf{1})$; we will simply denote the associated filtration by $(F_d(D))_{d \in \mathbb{N}}$, called the F-filtration. We have $\mathrm{gr}^F(D) \simeq \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$.

Let M be a D-module. An F-filtration of M is an exhausting increasing filtration $(F_d(M))_{d \in \mathbb{N}}$ compatible with the F-filtration of D. For $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{Z}^r$, let us denote by $D^r[\mathbf{n}]$ the module D^r endowed with the F-filtration such that $F_d(D^r[\mathbf{n}]) = \bigoplus_{i=1}^r F_{d-n_i}(D)$. If N is a submodule of D^r, we endow $D^r[\mathbf{n}]/N$ with the quotient filtration, i.e.

$$F_d\left(\frac{D^r[\mathbf{n}]}{N}\right) = \frac{F_d(D^r[\mathbf{n}]) + N}{N}.$$

We say that a filtration $F_d(M)$ is good if M is isomorphic as an F-filtered D-module to a module of the type $D^r[\mathbf{n}]/N$.

Let us take a filtered free resolution

$$0 \to D^{r_\delta}[\mathbf{n}^{(\delta)}] \to \cdots \to D^{r_1}[\mathbf{n}^{(1)}] \to D^{r_0}[\mathbf{n}^{(0)}] \to M \to 0.$$

Its existence can be proved in the same way as [8], Theorem 3.4, forgetting the minimality.

Definition 2.1. *The K-polynomial of $D^r[\mathbf{n}]$ is defined by*

$$K_F(D^r[\mathbf{n}];T) = \sum_i T^{\mathbf{n}_i} \in \mathbb{Z}[T].$$

The K-polynomial of M is defined by

$$K_F(M;T) = \sum_i (-1)^i K_F(D^{r_i}[\mathbf{n}^{(i)}];T) \in \mathbb{Z}[T].$$

Proposition 2.1. *The definition of $K_F(M;T)$ does not depend on the filtered free resolution.*

Proof. Let $R = \mathrm{gr}^F(D)$, and for $\mathbf{n} = (n_1, \dots, n_r)$, $R^r[\mathbf{n}] = \oplus_{i=1}^r R[n_i]$. By grading the filtered free resolution we get a graded free resolution over the commutative ring R:

$$0 \to R^{r_\delta}[\mathbf{n}^{(\delta)}] \to \cdots \to R^{r_1}[\mathbf{n}^{(1)}] \to R^{r_0}[\mathbf{n}^{(0)}] \to \mathrm{gr}^F(M) \to 0.$$

The K-polynomial is unchanged. Then apply Proposition 1.1. □

Definition 2.2. *We denote by $\mathcal{C}_F(M;T)$ the term of least degree in T in $K_F(M;1-T)$. This is the multidegree of M with respect to F.*

Proposition 2.2. *$\mathcal{C}_F(M;T)$ does not depend on the good filtration.*

Proof. Again we argue by grading. We have $\mathcal{C}_F(M;T) = \mathcal{C}(\mathrm{gr}^F(M);T)$. Let $\mathbb{K} = \mathrm{Frac}(\mathbb{C}[x])$. We have $\mathcal{C}(\mathrm{gr}^F(M);T) = \mathcal{C}(\mathbb{K} \otimes \mathrm{gr}^F(M);T)$. The graded ring $\mathbb{K} \otimes \mathrm{gr}^F(D)$ is a positively graded ring. Hence the K-polynomial is equal to the numerator of the Hilbert series, by Proposition 1.3. The multidegree is of the form mT^d with $d = \mathrm{codim}\, \mathbb{K} \otimes \mathrm{gr}^F(M)$ (unless it is 0), and m is the multiplicity of $\mathbb{K} \otimes \mathrm{gr}^F(M)$ along the maximal ideal $\xi_1, \dots, \xi_n$. We can show that this data is independent of the good filtration in the same way as [7], Remark 12 and Proposition 25. □

Let us give some interpretation. We have $\mathcal{C}_F(M;T) = mT^d$. For $x_0 \in \mathbb{C}^n$, the graded $\mathbb{C}[\xi]$-module $(\mathrm{gr}^F(M))^{x_0}$ is defined as in the section 1.2.

Proposition 2.3. *1. m and d are equal respectively to the multiplicity and the codimension of the graded $\mathbb{C}[\xi]$-module $\mathrm{gr}^F(M)^{x_0}$ for x_0 generic. Let us denote by $\pi : T^*\mathbb{C}^n \to \mathbb{C}^n$ the canonical projection. d is equal to the codimension of the variety $\mathrm{char}\, M \cap \pi^{-1}(x_0)$ for x_0 generic.*

2. If moreover M is holonomic, then $m = \mathrm{rank}\, M = \dim_{\mathbb{K}} \mathbb{K} \otimes \mathrm{gr}^F(M)$.

Proof. 1. This is Proposition 1.5.

2. In the holonomic case, $\mathbb{K} \otimes \mathrm{gr}^F(M)$ is finite dimensional over $\mathbb{K}$, and we have

$$\dim_{\mathbb{K}} \mathbb{K} \otimes \mathrm{gr}^F(M) = H(\mathbb{K} \otimes \mathrm{gr}^F(M);T)_{|T=1}.$$

The result follows, by using Proposition 1.3.

□

3 Multidegree for (F, V)-bifiltered D-modules

Now set $D = \mathbb{C}[x_1, \dots, x_n, t_1, \dots, t_p]\langle \partial_{x_1}, \dots, \partial_{x_n}, \partial_{t_1}, \dots, \partial_{t_p}\rangle$. We still endow it with the F-filtration. We introduce the V-filtration along $t_1 = \cdots = t_p = 0$. This is the filtration defined by assigning the weight vector $(\mathbf{0}, -\mathbf{1}, \mathbf{0}, \mathbf{1})$ to the set of variables $(x, t, \partial_x, \partial_t)$. We denote this filtration by $(V_k(D))_{k\in\mathbb{Z}}$.

Then we have the (F, V)-bifiltration on D defined by $F_{d,k}(D) = F_d(D) \cap V_k(D)$ for $d, k \in \mathbb{Z}$. For $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^r$, let us denote by $D^r[\mathbf{n}][\mathbf{m}]$ the module D^r endowed with the bifiltration such that

$$F_{d,k}(D^r[\mathbf{n}][\mathbf{m}]) = \bigoplus_{i=1}^{r} F_{d-n_i,k-m_i}(D).$$

A quotient $D^r[\mathbf{n}][\mathbf{m}]/N$ is endowed with the bifiltration $F_{d,k}(D^r[\mathbf{n}][\mathbf{m}]/N) = (F_{d,k}(D^r[\mathbf{n}][\mathbf{m}]) + N)/N$.

Let M be a D-module. A good bifiltration $(F_{d,k}(M))_{d\in\mathbb{N},k\in\mathbb{Z}}$ is an exhaustive increasing bifiltration, compatible with the bifiltration $(F_{d,k}(D))$, such that M is isomorphic as a bifiltered module to a module of the type $D^r[\mathbf{n}][\mathbf{m}]/N$.

Proposition 3.1. *M admits a bifiltered free resolution, i.e. a bifiltered exact sequence*

$$0 \to D^{r_\delta}[\mathbf{n}^{(\delta)}][\mathbf{m}^{(\delta)}] \to \cdots \to D^{r_1}[\mathbf{n}^{(1)}][\mathbf{m}^{(1)}] \to D^{r_0}[\mathbf{n}^{(0)}][\mathbf{m}^{(0)}] \to M \to 0.$$

We shall prove this proposition in a constructive way. For this purpose, let us introduce some Rees algebras. First, we have the Rees algebra with respect to the F-filtration (c.f. [3]):

$$\mathcal{R}_F(D) = \bigoplus_{d} F_d(D)\tau^d.$$

This is endowed with the V-filtration :

$$V_k(\mathcal{R}_F(D)) = \bigoplus_{k\in\mathcal{Z}} F_{d,k}(\mathcal{D})\tau^d \text{ for } d \in \mathbb{N}.$$

$\mathcal{R}_F(D)$ is isomorphic to the $\mathbb{C}$-algebra generated by x_i, t_i, $(\partial_{x_i}\tau)$, $(\partial_{t_i}\tau)$, τ, subject to the relations $[\partial_{x_i}\tau, x_i] = \tau$ and $[\partial_{t_i}\tau, t_i] = \tau$, the commutators involving other pairs of generators being zero. This is a noetherian algebra. We will replace respectively the generators x_i, t_i, $\partial_{x_i}\tau$, $\partial_{t_i}\tau$, τ by x_i, t_i, ∂_{x_i}, ∂_{t_i}, h, thus we identify $\mathcal{R}_F(D)$ with the $\mathbb{C}$-algebra, denoted $D^{(h)}$, generated by x_i, t_i, ∂_{x_i}, ∂_{t_i}, h, subject to the relations

$$[\partial_{x_i}, x_i] = h \quad \text{and} \quad [\partial_{t_i}, t_i] = h.$$

An admissible weight vector for $D^{(h)}$ is a vector $(\mathbf{u}, \mathbf{v}, l) \in \mathbb{Z}^{n+p} \times \mathbb{Z}^{n+p} \times \mathbb{Z}$ such that for any i, $u_i + v_i \geq l$. A filtration is associated with such a vector by assigning it to the set of variables $(x, t, \partial_x, \partial_t, h)$. The filtration associated with $(\mathbf{u}, \mathbf{v}, l) = (\mathbf{0}, -\mathbf{1}, \mathbf{0}, \mathbf{1}, 0)$ gives the V-filtration. The bigraded ring $\mathrm{gr}^V(D^{(h)})$ is isomorphic to $D^{(h)}$ endowed with the following multigrading :

$$\deg(x_i) = (0, 0), \quad \deg(t_i) = (0, -1), \quad \deg(h) = (1, 0),$$

$$\deg(\partial_{x_i}) = (1, 0), \quad \deg(\partial_{t_i}) = (1, 1).$$

Let us denote $F_d(M) = \bigcup_k F_{d,k}(M)$. We associate with M a $\mathcal{R}_F(D)$-module $\mathcal{R}_F(M) = \oplus_d F_d(M)\tau^d$, this is endowed with a V-filtration $V_k(\mathcal{R}_F(M)) = \oplus_d F_{d,k}(M)\tau^d$.

Conversely, there exists a dehomogenizing functor ρ_F (see [8], where this functor is denoted by ρ), from the category of V-filtered graded $D^{(h)}$-modules to the category of bifiltered D-modules. A $D^{(h)}$-module is said to be h-saturated if the action of h on this module is injective. [8], Proposition 3.6 states that the functors ρ_F and $\mathcal{R}_F$ give an equivalence of categories between the category of h-saturated $D^{(h)}$-modules with good V-filtrations and the category of D-modules with good bifiltrations, and that moreover these functors are exact.

We have also the Rees algebra of D with respect to V :

$$\mathcal{R}_V(D) = \bigoplus_{k\in\mathbb{Z}} V_k(\mathcal{D})\theta^k$$

This is endowed with the following filtration :

$$F_d(\mathcal{R}_V(D)) = \bigoplus_{k\in\mathbb{Z}} F_{d,k}(\mathcal{D})\theta^k \text{ for } d \in \mathbb{N}$$

$\mathcal{R}_V(D)$ is generated as a $\mathbb{C}$-algebra by $x_i\theta^0, \partial_{x_i}\theta^0, t_i\theta^{-1}, \partial_{t_i}\theta, \theta$. Let us denote respectively those elements by $\tilde{x_i}, \tilde{\partial_{x_i}}, \tilde{t_i}, \tilde{\partial_{t_i}}, \theta$. The following lemma is clear.

Lemma 3.1. *$\mathcal{R}_V(D)$ is isomorphic to the algebra $\mathbb{C}[\tilde{x_i}, \tilde{t_i}, \theta]\langle\tilde{\partial_{x_i}}, \tilde{\partial_{t_i}}\rangle$ subject to the relations $[\tilde{\partial_{x_i}}, \tilde{x_i}] = 1$ and $[\tilde{\partial_{t_i}}, \tilde{t_i}] = 1$ for any i.*
The F-filtration is then given by assigning the weight vector $(\mathbf{0}, \mathbf{0}, 0, \mathbf{1}, \mathbf{1})$ to the set of variables $(\tilde{x}, \tilde{t}, \theta, \tilde{\partial_x}, \tilde{\partial_t})$.

Then the bigraded ring $\mathrm{gr}^F(\mathcal{R}_V(D))$ is isomorphic to the commutative polynomial ring $\mathbb{C}[\tilde{x_i}, \tilde{t_i}, \theta, \tilde{\partial_{x_i}}, \tilde{\partial_{t_i}}]$ endowed with the following multigrading :

$$\deg(\tilde{x_i}) = (0,0), \quad \deg(\tilde{t_i}) = (0,-1), \quad \deg(\theta) = (0,1),$$
$$\deg(\tilde{\partial_{x_i}}) = (1,0), \quad \deg(\tilde{\partial_{t_i}}) = (1,1).$$

Similarly, we define the Rees module associated with M with respect to V:

$$\mathcal{R}_V(M) = \bigoplus_{k\in\mathbb{Z}} V_k(M)\theta^k$$

where $V_k(M) = \bigcup_d F_{d,k}(M)$. It admits an F-filtration

$$F_d(\mathcal{R}_V(M)) = \bigoplus_{k\in\mathbb{Z}} F_{d,k}(M)\theta^k$$

such that $\mathrm{gr}^F(\mathcal{R}_V(M))$ is isomorphic to

$$\bigoplus_{d,k} \frac{F_{d,k}(M)}{F_{d-1,k}(M)}\theta^k.$$

Conversely, as it has been stated before, there exists a dehomogenizing functor ρ_V, from the category of F-filtered graded $\mathcal{R}_V(D)$-modules to the category of bifiltered D-modules. A $\mathcal{R}_V(D)$-module is said to be θ-saturated if the action of θ on this module is injective. The functors ρ_V and $\mathcal{R}_V$ give an equivalence of categories between the category of θ-saturated $\mathcal{R}_V(D)$-modules with good F-filtrations and the category of D-modules with good bifiltrations. Moreover these functors are exact.

Proof of Proposition 3.1. $\mathcal{R}_F(M)$ is a finite type $D^{(h)}$-module isomorphic as a V-filtered graded $D^{(h)}$-module to a quotient of $(D^{(h)})^r[\mathbf{m}]$. A presentation of $\mathcal{R}_F(M)$ can be obtained by means of F-adapted Gröbner bases. By replacing D by $D^{(h)}$ in [16], section 3, we can construct a V-adapted free resolution of $\mathcal{R}_F(M)$. Dehomogenizing this resolution provides a bifiltered free resolution of M.

We can use also the V-homogenization. Using [16], section 3, we construct a presentation of $\mathcal{R}_V(M)$. We take a bigraded free resolution of $\mathrm{gr}^F\mathcal{R}_V(M)$, which can be lifted to a F-adapted resolution of $\mathcal{R}_V(M)$, as in [8], Proposition 2.7. Taking ρ_V gives a bifiltered free resolution of M. □

Definition 3.1. *The K-polynomial of $D^r[\mathbf{n}][\mathbf{m}]$ with respect to (F,V) is defined by*

$$K_{F,V}(D^r[\mathbf{n}][\mathbf{m}];T_1,T_2)=\sum_i T_1^{\mathbf{n}_i}T_2^{\mathbf{m}_i}\in\mathbb{Z}[T_1,T_2,T_2^{-1}].$$

The K-polynomial of M with respect to (F,V) is defined by

$$K_{F,V}(M;T_1,T_2)=\sum_i(-1)^iK_{F,V}(D^{r_i}[\mathbf{n}^{(i)}][\mathbf{m}^{(i)}];T_1,T_2)\in\mathbb{Z}[T_1,T_2,T_2^{-1}].$$

Proposition 3.2. *The definition of $K_{F,V}(M;T_1,T_2)$ does not depend on the bifiltered free resolution.*

Proof of Proposition 3.2. A bifiltered free resolution of M induces a bigraded free resolution of $\mathrm{gr}^F(\mathcal{R}_V(M))$. Thus $K_{F,V}(M;T_1,T_2)=K(\mathrm{gr}^F(\mathcal{R}_V(D));T_1,T_2)$ and we can apply Proposition 1.1. □

Let $\mathbb{K}=\mathrm{Frac}(\mathbb{C}[x_1,\dots,x_n])$. Instead of D, we shall work with $\mathbb{K}\otimes D$. This has no influence on the bifiltration.

Definition 3.2. *We denote by $\mathcal{C}_{F,V}(M;T_1,T_2)$ the sum of the terms whose total degree in T_1,T_2 equals codim $(K\otimes gr^F(\mathcal{R}_V(M)))$ in the expansion of $K_{F,V}(M;1-T_1,1-T_2)$. This is the multidegree of M with respect to (F,V).*

Theorem 3.1. *$\mathcal{C}_{F,V}(M;T_1,T_2)$ does not depend on the good bifiltration.*

Proof. As before we take the Rees algebra with respect to V. We get

$$\mathcal{R}_V(\mathbb{K}\otimes D)\simeq\mathbb{K}[\tilde{t_i},\theta]\langle\tilde{\partial_{x_i}},\tilde{\partial_{t_i}}\rangle$$

and

$$A:=\mathrm{gr}^F(\mathcal{R}_V(\mathbb{K}\otimes D))\simeq\mathbb{K}[\tilde{t_i},\theta,\tilde{\partial_{x_i}},\tilde{\partial_{t_i}}].$$

The ring A is bigraded as follows:

$$\deg(\tilde{t_i})=(0,-1),\quad \deg(\theta)=(0,1),\quad \deg(\tilde{\partial_{x_i}})=(1,0),\quad \deg(\tilde{\partial_{t_i}})=(1,1).$$

This is not a positive grading since $\mathbb{K}[(\tilde{t_i}\theta)]=A_{0,0}$ is infinite over $\mathbb{K}$. Let

$$\tilde{M}=\mathbb{K}\otimes\mathrm{gr}^F(\mathcal{R}_V(M)).$$

A bifiltered free resolution of M induces a bigraded free resolution of $\tilde{M}$, thus $K_{F,V}(M;T_1,T_2)=K(\tilde{M};T_1,T_2)$.

Let us endow M with another good bifiltration $(F'_{d,k}(M))_{d,k}$. We denote by M' the module M endowed with this bifiltration. In view of Proposition 1.2, it is sufficient to prove

- $\mathrm{rad}(\mathrm{ann}\tilde{M}) = \mathrm{rad}(\mathrm{ann}\tilde{M}')$
- For any prime ideal $\mathfrak{p}$ of A, $\mathrm{mult}_{\mathfrak{p}}\tilde{M} = \mathrm{mult}_{\mathfrak{p}}\tilde{M}'$.

To prove these two assertions, we argue exactly in the same way as in the proof of Proposition 1.3.2 of [12]. For the convenience of the reader, we give here the details.

We shall also use the behaviour of dimensions and multiplicities in short exact sequences.

Lemma 3.2 ([7], Proposition 24)**.** *Let*

$$0 \to E \to F \to G \to 0$$

be an exact sequence of finite type A-modules, and let $\mathfrak{p}$ *be a prime ideal of A. Then*

1. $\mathrm{dim}F_{\mathfrak{p}} = \max(\mathrm{dim}E_{\mathfrak{p}}, \mathrm{dim}G_{\mathfrak{p}})$.
2. *If* $\mathrm{dim}E_{\mathfrak{p}} = \mathrm{dim}G_{\mathfrak{p}}$*, then* $\mathrm{mult}_{\mathfrak{p}}F = \mathrm{mult}_{\mathfrak{p}}E + \mathrm{mult}_{\mathfrak{p}}G$.
   If $\mathrm{dim}E_{\mathfrak{p}} < \mathrm{dim}G_{\mathfrak{p}}$*, then* $\mathrm{mult}_{\mathfrak{p}}F = \mathrm{mult}_{\mathfrak{p}}G$.
   If $\mathrm{dim}E_{\mathfrak{p}} > \mathrm{dim}G_{\mathfrak{p}}$*, then* $\mathrm{mult}_{\mathfrak{p}}F = \mathrm{mult}_{\mathfrak{p}}E$.

We will follow the proof of [12] and indicate at each step how to prove :

Claim 1. $\mathrm{rad}(\mathrm{ann}\tilde{M}) \subset \mathrm{rad}(\mathrm{ann}\tilde{M}')$,

Claim 2. $\mathrm{mult}_{\mathfrak{p}}\tilde{M} \geq \mathrm{mult}_{\mathfrak{p}}\tilde{M}'$ if $\mathrm{dim}\tilde{M}_{\mathfrak{p}} = \mathrm{dim}\tilde{M}'_{\mathfrak{p}}$.

First, since $F_{d,k}(M)$ and $F'_{d,k}(M)$ are good bifiltrations, there exist $d_0, k_0 \in \mathbb{N}$ such that for any d, k, $F_{d,k}(M) \subset F'_{d+d_0,k+k_0}(M)$. Let us denote by M'' the module M endowed with the bifiltration $(F'_{d+d_0,k+k_0}(M))_{d,k}$. The algebraic cycle associated with $\tilde{M}'$ is equal to the algebraic cycle associated with $\tilde{M}''$. Thus we can suppose $F'_{d,k}(M) \subset F_{d,k}(M)$.

Let us introduce the Rees algebra $\mathcal{R}(D)$ with respect to the bifiltration F, V, i.e.

$$\mathcal{R}(D) = \bigoplus_{d,k} F_{d,k}(D)\tau^d\theta^k.$$

This is isomorphic to the $\mathbb{C}$-algebra generated by x_i, $t_i\theta^{-1}$, $\partial_{x_i}\tau$, $\partial_{t_i}\tau\theta$, τ and θ, subject to the relations $[\partial_{x_i}\tau, x_i] = \tau$ and $[\partial_{t_i}\tau\theta, t_i\theta^{-1}] = \tau$. This is a noetherian algebra.

We define also the Rees module $\mathcal{R}(M) = \bigoplus_{d,k} F_{d,k}(M)\tau^d\theta^k$. We have

$$\mathrm{gr}^F(\mathcal{R}_V(M)) \simeq \frac{\mathcal{R}(M)}{\tau\mathcal{R}(M)}.$$

Let us suppose moreover that there exists $r \geq 1$ such that for any d, k, $F'_{d,k}(M) \subset F_{d,k}(M) \subset F'_{d+r,k}(M)$. Let $F''_{d,k}(M) = F_{d,k}(M) \cap F'_{d+1,k}(M)$. We have

$$F'_{d,k}(M) \subset F''_{d,k}(M) \subset F'_{d+1,k}(M) \quad \text{and} \quad F_{d-r+1,k}(M) \subset F''_{d,k}(M) \subset F_{d,k}(M).$$

By induction on r we can suppose $r = 1$, i.e. $\tau\mathcal{R}(M) \subset \mathcal{R}(M') \subset \mathcal{R}(M)$. Then we have the following exact sequences of $\mathrm{gr}^F\mathcal{R}_V(D)$-modules of finite type:

$$0 \to \frac{\tau\mathcal{R}(M)}{\tau\mathcal{R}(M')} \to \frac{\mathcal{R}(M')}{\tau\mathcal{R}(M')} \to \frac{\mathcal{R}(M')}{\tau\mathcal{R}(M)} \to 0$$

$$0 \to \frac{\mathcal{R}(M')}{\tau\mathcal{R}(M)} \to \frac{\mathcal{R}(M)}{\tau\mathcal{R}(M)} \to \frac{\mathcal{R}(M)}{\mathcal{R}(M')} \to 0.$$

After tensorizing by $\mathbb{K}$, we deduce $\mathrm{rad}(\mathrm{ann}\tilde{M}) = \mathrm{rad}(\mathrm{ann}\tilde{M}')$. Then using Lemma 3.2, we get $\mathrm{mult}_{\mathfrak{p}}\tilde{M} = \mathrm{mult}_{\mathfrak{p}}\tilde{M}'$.

Let $F''_{d,k}(M) = F_{d,k}(M) \cap (\cup_i F'_{i,k}(M))$. We have :

$$\mathcal{R}(M'') = \mathcal{R}(M) \cap (\cup_{i\geq 0}\tau^{-i}\mathcal{R}(M')).$$

Let $\mathcal{L}_j = \mathcal{R}(M) \cap (\cup_{0\leq i\leq j}\tau^{-i}\mathcal{R}(M'))$. This is an ascending chain of finite type sub-modules of $\mathcal{R}(M)$. Hence it is stationary and there exists an integer $r \geq 0$ such that

$$\mathcal{R}(M'') = \mathcal{R}(M) \cap \tau^{-r}\mathcal{R}(M').$$

In particular $\mathcal{R}(M'')$ is of finite type and $F''_{d,k}(M)$ is a good bifiltration.
We have $\tau^r\mathcal{R}(M'') \subset \mathcal{R}(M') \subset \mathcal{R}(M'')$, i.e. we are in the situation of the previous paragraph. This implies $\mathrm{rad}(\mathrm{ann}\tilde{M}'') = \mathrm{rad}(\mathrm{ann}\tilde{M}')$ and $\mathrm{mult}_{\mathfrak{p}}\tilde{M}'' = \mathrm{mult}_{\mathfrak{p}}\tilde{M}'$.

On the other hand, we have a canonical injection

$$\frac{\mathcal{R}(M'')}{\tau\mathcal{R}(M'')} \to \frac{\mathcal{R}(M)}{\tau\mathcal{R}(M)}.$$

Then $\mathrm{rad}(\mathrm{ann}\tilde{M}'') \subset \mathrm{rad}(\mathrm{ann}\tilde{M})$, and Claim 1 is proved. From this canonical injection, we deduce Claim 2 by using Lemma 3.2. □

4 Nicely bifiltered D-modules

In this section we consider a bifiltered D-module satisfying the following condition:

Definition 4.1. *Let M be a D-module endowed with a good bifiltration. We say that the bifiltration is* nice *if for any d, k,*

$$\left(\bigcup_{d'} F_{d',k}(M)\right) \bigcap \left(\bigcup_{k'} F_{d,k'}(M)\right) = F_{d,k}(M). \tag{1}$$

In such a case, we say that M is nicely bifiltered.

Definition 4.2. *Let N be a bigraded $gr^V(D^{(h)})$-module. N is said to be h-*saturated *if the map $N \to N$ sending m to hm is injective.*
*Let N be a bigraded $gr^F(\mathcal{R}_V(D))$-module. N is said to be θ-*saturated *if the map $N \to N$ sending m to θm is injective.*

Lemma 4.1. *The following are equivalent :*

1. *M is nicely bifiltered,*

2. $\mathrm{gr}^V(\mathcal{R}_F(M))$ *is h-saturated,*

3. $\mathrm{gr}^F(\mathcal{R}_V(M))$ *is θ-saturated.*

Proof. By definition, 2) and 3) are equivalent to the following : $\forall d, k, F_{d+1,k}(M) \cap F_{d,k+1}(M) \subset F_{d,k}(M)$. By [2], Lemma 1.1, this is equivalent to 1). □

h-saturatedness and Gröbner bases. Let us give a criterion for h-saturatedness using Gröbner bases. Using the preceding lemma, that leads to a criterion for the niceness of a bifiltration. Let in this paragraph $D^{(h)} = \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n, h\rangle$. It is graded by setting for any i, $\deg x_i = 0$, $\deg \partial_i = 1$ and $\deg h = 1$.

Let $<''$ be a well-order on $\mathbb{N}^{2n}$, compatible with sums. Then we define a well-order $<'$ on $\mathbb{N}^{2n+1}$ by

$$(\alpha, \beta, k) <' (\alpha', \beta', k') \quad \text{iff}$$

$$\begin{cases} |\beta| + k < |\beta'| + k' \\ \text{or} \quad |\beta| + k = |\beta'| + k' \quad \text{and} \quad |\beta| < |\beta'| \\ \text{or} \quad |\beta| + k = |\beta'| + k', |\beta| = |\beta'| \text{ and } (\alpha, \beta) <'' (\alpha', \beta'). \end{cases}$$

This is a well-order on the monomials of $D^{(h)}$ adapted to the F-filtration. To deal with submodules of $(D^{(h)})^r$, we define a well-ordering $<$ on $\mathbb{N}^{2n+1} \times \{1, \ldots, r\}$ by

$$(\alpha, \beta, k, i) < (\alpha', \beta', k', i') \quad \text{iff} \quad \begin{cases} (\alpha, \beta, k) <' (\alpha', \beta', k') \\ \text{or} \quad (\alpha, \beta, k) = (\alpha', \beta', k') \quad \text{and} \quad i < i'. \end{cases}$$

Note that if $(\alpha, \beta, k, i) \geq (\alpha', \beta', k', i')$ and $|\beta| + k = |\beta'| + k'$, then $k \leq k'$. If $P \in W^r$, we denote by $\mathrm{in}(P)$ the leading monomial of P.

Definition 4.3. *Let $P_1, \ldots, P_s$ be a Gröbner base of a homogeneous submodule $N \subset (D^{(h)})^r$. Such a base is called* minimal *if*

$$\forall i, \ \mathrm{Exp} P_i \notin \bigcup_{j \neq i} \left(\mathrm{Exp} P_j + \mathbb{N}^{2n+1}\right).$$

Proposition 4.1. *The following assertions are equivalent :*

1. *$(D^{(h)})^r/N$ is h-saturated.*

2. *For any minimal homogeneous Gröbner base $P_1, \ldots, P_s$ of N, for any i, h does not divide* $\mathrm{in}\, P_i$.

3. *There exists a minimal homogeneous Gröbner base $P_1, \ldots, P_s$ of N, such that for any i, h does not divide* $\mathrm{in}\, P_i$.

Proof. Let us prove 1) $\Rightarrow$ 2) Let $P_1, \ldots, P_s$ be a minimal homogeneous Gröbner base of N. Suppose that there exists i such that h divides $\mathrm{in} P_i$. Then h divides P_i by the definition of $<$. By h-saturatedness, $P_i/h \in N$. Thus

$$\mathrm{Exp} \frac{P_i}{h} \in \bigcup_{j \neq i} \left(\mathrm{Exp} P_j + \mathbb{N}^{2n+1}\right),$$

then

$$\mathrm{Exp}P_i = h\mathrm{Exp}\frac{P_i}{h} \in \bigcup_{j\neq i}\left(\mathrm{Exp}P_j + \mathbb{N}^{2n+1}\right),$$

which contradicts the minimality.

2) $\Rightarrow$ 3) is obvious. Let us show 3) $\Rightarrow$ 1). Let $P \in (D^{(h)})^r$ homogeneous such that $hP \in N$. We shall show that $P \in N$. By division, $hP = \sum Q_iP_i$ with for any i, $Q_i \in D^{(h)}$ homogeneous, $\deg(Q_iP_i) = \deg(hP)$, and $\mathrm{ord}^F(Q_iP_i) \leq \mathrm{ord}^F(hP)$.

Let us suppose that there exists i such that h does not divide Q_i. Then $\mathrm{ord}^F Q_i = \deg Q_i$. Since h does not divide P_i, we have $\mathrm{ord}^F P_i = \deg P_i$. Then

$$\mathrm{ord}^F(Q_iP_i) = \mathrm{ord}^F(Q_i) + \mathrm{ord}^F(P_i) = \deg Q_i + \deg P_i = \deg(hP).$$

But

$$\mathrm{ord}^F(Q_iP_i) \leq \mathrm{ord}^F(hP) < \deg(hP),$$

a contradiction. Thus for any i, h divides Q_i and $P = \sum(Q_i/h)P_i \in N$. □

We shall make a link between the (F, V)-multidegree and the theory of slopes of Y. Laurent, c.f. [11]. We consider intermediate filtrations L between F and V, denoted by $pF + qV$ with $p > 0$, $q > 0$, defined by

$$L_r(D) = \sum_{dp+kq\leq r} F_{d,k}(D).$$

Similarly we endow M with the L-filtration $L_r(M) = \sum_{dp+kq\leq r} F_{d,k}(M)$, which is a good filtration since taking a bifiltered free presentation

$$D^{r_1}[\mathbf{n}^{(1)}][\mathbf{m}^{(1)}] \to D^{r_0}[\mathbf{n}^{(0)}][\mathbf{m}^{(0)}] \to M \to 0,$$

we see that $\mathrm{gr}^L(M)$ is isomorphic to a quotient of $\mathrm{gr}^L(D^{r_0}[p\mathbf{n}^{(0)} + q\mathbf{m}^{(0)}])$.

On the other hand, since $\mathrm{gr}^V(M)$ is isomorphic to a quotient of $\mathrm{gr}^V(D^{r_0}[\mathbf{m}^{(0)}])[\mathbf{n}^{(0)}]$, it is endowed with a natural F-filtration. Similarly, $\mathrm{gr}^F(M)$ is isomorphic to a quotient of $\mathrm{gr}^F(D^{r_0}[\mathbf{n}^{(0)}])[\mathbf{m}^{(0)}]$, and it is endowed with a natural V-filtration.

In [2], we considered also the bigraded module

$$\mathrm{bigr}(M) = \bigoplus_{d,k} \frac{F_{d,k}(M)}{F_{d,k-1}(M) + F_{d-1,k}(M)}$$

over the ring $\mathrm{bigr}(D) \simeq \mathrm{gr}^V(\mathrm{gr}^F(D)) \simeq \mathrm{gr}^F(\mathrm{gr}^V(D))$.

Lemma 4.2. *If M is nicely bifiltered, we have*

$$\mathrm{bigr}(M) \simeq \mathrm{gr}^V(\mathrm{gr}^F(M)) \simeq \mathrm{gr}^F(\mathrm{gr}^V(M)).$$

Proof. For the sake of simplicity, we suppose that $\mathbf{n}^{(0)} = \mathbf{m}^{(0)} = \mathbf{0}$ and consider $M = D^r/N$. We have

$$F_{d,k}(M) = \frac{F_{d,k}(D^r) + N}{N}, \quad F_d(M) = \frac{F_d(D^r) + N}{N}, \quad V_k(M) = \frac{V_k(D^r) + N}{N}.$$

The niceness assumption is equivalent to the following:

$$\forall d, k,\ (F_d(D^r) + N) \cap (V_k(D^r) + N) \subset F_{d,k}(D^r) + N. \tag{2}$$

We have $\mathrm{gr}^V(M) = \mathrm{gr}^V(D^r)/\mathrm{gr}^V(N)$ with

$$\mathrm{gr}^V(N) = \bigoplus_k \frac{V_k(D^r) \cap N + V_{k-1}(D^r)}{V_{k-1}(D^r)}.$$

We naturally define

$$\begin{aligned} F_d(\mathrm{gr}^V(N)) &= F_d(\mathrm{gr}^V(D^r)) \cap \mathrm{gr}^V(N) \\ &= \bigoplus_k \frac{F_{d,k}(D^r) + V_{k-1}(D^r)}{V_{k-1}(D^r)} \bigcap \frac{V_k(D^r) \cap N + V_{k-1}(D^r)}{V_{k-1}(D^r)}. \end{aligned}$$

Thus we have

$$\mathrm{gr}^F \mathrm{gr}^V(N) = \bigoplus_{d,k} \frac{(F_{d,k}(D^r) + V_{k-1}(D^r)) \cap (V_k(D^r) \cap N + V_{k-1}(D^r))}{(F_{d-1,k}(D^r) + V_{k-1}(D^r)) \cap (V_k(D^r) \cap N + V_{k-1}(D^r))}.$$

This is included in

$$\mathrm{gr}^F \mathrm{gr}^V(D^r) = \bigoplus_{d,k} \frac{F_{d,k}(D^r)}{F_{d-1,k}(D^r) + F_{d,k-1}(D^r)}$$

via the map

$$(F_{d,k}(D^r)+V_{k-1}(D^r))\cap(V_k(D^r)\cap N+V_{k-1}(D^r)) \to F_{d,k}(D^r)+V_{k-1}(D^r) \to F_{d,k}(D^r).$$

Hence

$$\begin{aligned} \mathrm{gr}^F \mathrm{gr}^V(M) &= \mathrm{gr}^F \mathrm{gr}^V(D)/\mathrm{gr}^F \mathrm{gr}^V(N) \\ &= \frac{F_{d,k}(D^r)}{F_{d,k}(D^r) \cap (V_k(N) + V_{k-1}(D^r)) + F_{d-1,k}(D^r) + F_{d,k-1}(D^r)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathrm{bigr}_{d,k}(M) &= \frac{F_{d,k}(M)}{F_{d-1,k}(M) + F_{d,k-1}(M)} \\ &= \frac{F_{d,k}(D^r) + N}{F_{d-1,k}(D^r) + F_{d,k-1}(D^r) + N} \\ &= \frac{F_{d,k}(D^r)}{F_{d,k}(D^r) \cap (F_{d-1,k}(D^r) + F_{d,k-1}(D^r) + N)} \\ &= \frac{F_{d,k}(D^r)}{F_{d-1,k}(D^r) + F_{d,k-1}(D^r) + N \cap F_{d,k}(D^r)}. \end{aligned}$$

We have to show

$$\begin{aligned} F_{d-1,k}(D^r) + F_{d,k-1}(D^r) + N \cap F_{d,k}(D^r) &= F_{d,k}(D^r) \cap (V_k(N) + V_{k-1}(D^r)) \\ &+ F_{d-1,k}(D^r) + F_{d,k-1}(D^r). \qquad (3) \end{aligned}$$

The inclusion $\subset$ is obvious. On the other hand,

$$\begin{aligned} F_{d,k}(D^r) \cap (V_k(N) + V_{k-1}(D^r)) &= F_{d,k}(D^r) \cap (N + V_{k-1}(D^r)) \\ &\subset (F_{d,k-1}(D^r) + N) \cap F_{d,k}(D^r) \quad (\text{using } (2)) \\ &\subset F_{d,k-1}(D^r) + N \cap F_{d,k}(D^r), \end{aligned}$$

which proves (3).

We have showed that $\mathrm{bigr}(M) \simeq \mathrm{gr}^F(\mathrm{gr}^V(M))$, and by exchanging the role of F and V we show that $\mathrm{bigr}(M) \simeq \mathrm{gr}^V(\mathrm{gr}^F(M))$.

Note also that under the niceness assumption, the module $\mathrm{bigr}(N)$ is identified with a submodule of $\mathrm{bigr}(D^r)$ such that $\mathrm{bigr}(M) \simeq \mathrm{bigr}(D^r)/\mathrm{bigr}(N)$. □

Lemma 4.3 ([17], Lemma 2.1.6)**.** *For $\epsilon > 0$ small enough,*

$$\mathrm{gr}^V(\mathrm{gr}^F(M)) \simeq \mathrm{gr}^L(M) \quad \textit{with} \quad L = F + \epsilon V,$$

and

$$\mathrm{gr}^F(\mathrm{gr}^V(M)) \simeq \mathrm{gr}^L(M) \quad \textit{with} \quad L = V + \epsilon F.$$

It is known that for any L, $\mathrm{gr}^L(M)$ defines an algebraic cycle independent of the good filtration (the proof is almost the same as for the F-filtration). The variety defined by the annihilator of $\mathrm{gr}^L(M)$ is denoted by $\mathrm{char}_L(M)$. Remember that $\mathbb{K}$ denotes the fraction field of $\mathbb{C}[x]$. The module $\mathbb{K} \otimes \mathrm{gr}^L(M)$ also defines an algebraic cycle independent of the good filtration.

Proposition 4.2. *If M is nicely bifiltered, we have*

$$K_{F,V}(M; T_1, T_2) = K(\mathrm{bigr}(M); T_1, T_2) = K(\mathrm{gr}^L M; T_1, T_2)$$

with $L = V + \epsilon F$ or $L = F + \epsilon V$ with $\epsilon > 0$ small enough. Here $\mathrm{gr}^L M$ is considered as a bigraded module.

Proof. Under this assumption, any bifiltered free resolution of M induces a bigraded free resolution of $\mathrm{bigr} M$ (see [2], Theorem 1.1, forgetting the minimality). Thus $K_{F,V}(M; T_1, T_2) = K(\mathrm{bigr} M; T_1, T_2)$. But by Lemma 4.2 and Lemma 4.3, $\mathrm{bigr} M \simeq \mathrm{gr}^L(M)$. □

Remark 4.1. The multidegree $\mathcal{C}_{F,V}(M; T_1, T_2)$ has total degree

$$d = \mathrm{codim}\, \mathbb{K} \otimes \mathrm{gr}^F(\mathcal{R}_V(M)),$$

by definition. On the other hand, since the multigrading on $\mathbb{K} \otimes \mathrm{bigr}\, D$ is positive, we know that the first non-zero terms in the expansion of $K_{F,V}(M; 1 - T_1, 1 - T_2)$ have total degree equal to

$$d' = \mathrm{codim}\, (\mathbb{K} \otimes \mathrm{bigr}\, M).$$

Thus $d \leq d'$. If $d < d'$, then $\mathcal{C}_{F,V}(M; T_1, T_2) = 0$. We will see in the next section non trivial cases in which $d = d'$.

We then have, applying Proposition 1.2 :

Theorem 4.1. *The multidegree $\mathcal{C}_{F,V}(M; T_1, T_2)$ only depends on $\mathrm{codim}\, \mathbb{K} \otimes \mathrm{gr}^F(\mathcal{R}_V(M))$ and on the algebraic cycle defined by $\mathbb{K} \otimes \mathrm{gr}^L(M)$ with $L = V + \epsilon F$ or $L = F + \epsilon V$ with $\epsilon > 0$ small enough.*

Let us recall some geometric meaning related to the L-filtration. Let $X = \mathbb{C}^{n+p}$, $Y = \{t = 0\} \subset X$ and $\Lambda = T^*_Y X$ the conormal bundle. We have $\mathrm{gr}^L(D) \simeq \mathcal{O}(T^*\Lambda)$, c.f. [11]. Let $\pi : T^*\Lambda \to Y$ be the canonical projection.

By Proposition 1.5, $\mathcal{C}_{F,V}(\mathbb{K} \otimes \mathrm{gr}^L(M); T_1, T_2) = \mathcal{C}_{F,V}(\mathrm{gr}^L(M)^y; T_1, T_2)$ for $y \in Y$ generic. This depends only on the algebraic cycle on $\pi^{-1}(y)$ defined by

$\mathrm{gr}^L(M)^y$ for y generic. d' is equal to the codimension of $\mathrm{char}_L(M) \cap \pi^{-1}(y) \subset \pi^{-1}(y)$, for y generic.

For any L, we have $\mathrm{gr}^L(D) \simeq \mathrm{gr}^F(\mathrm{gr}^V(D))$ thus $\mathrm{gr}^L(D)$ is a bigraded ring. Following the theory of Y. Laurent, we say that M has no slopes along Y if for any L, the ideal $\mathrm{rad}(\mathrm{ann}\,\mathrm{gr}^L(M))$ (defining $\mathrm{char}_L(M)$) is bihomogeneous. The following means that niceness of the bifiltration is a strong regularity condition.

Proposition 4.3. *If M is a nicely bifiltered holonomic D-module, then M has no slopes along Y.*

Proof. As before, we identify $\mathcal{R}_V(D)$ with $D[\theta]$. Let us take a bifiltered free presentation

$$D^s[\mathbf{n}][\mathbf{m}] \overset{\phi_1}{\to} D^r \overset{\phi_0}{\to} M \to 0, \tag{4}$$

with $\phi_1(e_i) = P^{(i)} = \sum_j P_j^{(i)} e_j$, and let $N = \mathrm{Im}\,\phi_1$. For the sake of simplicity, we have assumed $\mathbf{n}^{(0)} = \mathbf{m}^{(0)} = \mathbf{0}$. This induces a bigraded free resolution

$$\mathrm{gr}^F(D[\theta])^s[\mathbf{n}][\mathbf{m}] \overset{\overline{\phi}_1}{\to} \mathrm{gr}^F(D[\theta])^r \overset{\overline{\phi}_0}{\to} \mathrm{gr}^F\mathcal{R}_V M \to 0.$$

Using the lifting ([8], Proposition 2.7), we can suppose that the presentation (4) is minimal, in the sense that the elements $\overline{\phi}_1(e_i)$ form a minimal set of generators of $\mathrm{Ker}\,\overline{\phi}_0$.

Let us introduce some notations in order to determine $\overline{\phi}_1(e_i)$.
If $P = \sum a_{\nu,\mu}(x,\partial_x) t^\nu \partial_t^\mu \in V_k(D)$, we define

$$H_k^V(P) = \sum a_{\nu,\mu}(x,\partial_x) t^\nu \partial_t^\mu \theta^{k-(|\mu|-|\nu|)} \in D[\theta],$$

and $H^V(P) = H^V_{\mathrm{ord}^V(P)}(P)$, the V-homogenization of P. Similarly if $P = \sum P_j e_j \in \oplus V_{m_j}(D)$, we define $H^V_{\mathbf{m}}(P) = \sum H^V_{m_j}(P_j) e_j \in (D[\theta])^r$.
Now if $P = \sum a_\beta(x,t,\partial_t,\theta)\partial_x^\beta \in F_d(D[\theta])$, we define

$$\sigma_d^F(P) = \sum_{|\beta|=d} a_\beta(x,t,\partial_t,\theta)\partial_x^\beta \in \mathrm{gr}_d^F(D[\theta]),$$

and $\sigma^F(P) = \sigma^F_{\mathrm{ord}^F(P)} P$. Similarly if $P = \sum P_j e_j \in \oplus F_{n_j}(D[\theta])$, we define $\sigma^F_{\mathbf{n}}(P) = \sum \sigma^F_{n_j}(P_j) e_j \in \mathrm{gr}^F(D[\theta])^r$.

We have

$$\overline{\phi}_1(e_i) = \sigma^F_{\mathbf{n}}(H_{\mathbf{m}}(P)).$$

For $P = \sum_{\nu,\beta,\mu} a_{\nu,\beta,\mu}(x) t^\nu \partial_x^\beta \partial_t^\mu$, let us define the Newton polygon by

$$\mathcal{P}(P) = \bigcup_{(\nu,\beta,\mu)|a_{\nu,\beta,\mu}(x)\neq 0} (|\nu|-|\mu|, |\beta|+|\mu|) - \mathbb{N}^2 \subset \mathbb{Z}^2.$$

We say that $\mathcal{P}(P)$ is *trivial* if it is equal to a translate of $(-\mathbb{N}) \times (-\mathbb{N})$.

For $1 \leq i \leq s$, let $J(i)$ be the set of integers $1 \leq j \leq r$ such that

- $\mathrm{ord}^F P_j^{(i)} = n_i$,
- $\mathrm{ord}^V P_j^{(i)} = m_i$,
- $\mathcal{P}(P_j^{(i)})$ is trivial.

We claim that for any i, the set $J(i)$ is non-empty. Otherwise, θ would divide $\overline{\phi}_1(e_i)$. By θ-saturatedness, $\overline{\phi}_1(e_i)/\theta$ would belong to $\mathrm{gr}^F\mathcal{R}_V N$, thus the presentation (4) would not be minimal.

Then $\mathrm{bigr}N$ is generated by the elements

$$\sum_{j\in J(i)} \sigma^F\sigma^V(P_j^{(i)})e_j$$

for $1 \leq i \leq s$. Let L be an intermediate filtration. We have

$$\sigma^L(P^{(i)}) = \sum_{j\in J(i)} \sigma^L(P_j^{(i)})e_j = \sum_{j\in J(i)} \sigma^F\sigma^V(P_j^{(i)})e_j.$$

Thus for any L,

$$\mathrm{bigr}N \subset \mathrm{gr}^L N. \tag{5}$$

If $\mathcal{M}$ is a $\mathrm{gr}^L(D)$-module, we denote by $\mathrm{supp}\mathcal{M}$ the zero-set of the annihilator of $\mathcal{M}$. By [19], Theorem 1.1 and [17], Theorem 2.2.1 (valid for any L), $\mathrm{char}_L(M) = \mathrm{supp}(\mathrm{gr}^L(M))$ is pure of dimension $n+p$ for any L. Since $\mathrm{bigr}N = \mathrm{gr}^F\mathrm{gr}^V(N) = \mathrm{gr}^L(N)$ for L close to V, then $\mathrm{supp}(\mathrm{bigr}M)$ is pure of dimension $n+p$.

By (5), we have for any L, $\mathrm{char}_L M \subset \mathrm{supp}(\mathrm{bigr}M)$, thus $\mathrm{char}_L M$ is the union of some irreducible components of $\mathrm{supp}(\mathrm{bigr}M)$. The irreducible components are bihomogeneous (a bihomogeneous module admits a bihomogeneous primary decomposition), so $\mathrm{char}_L M$ is bihomogeneous. □

5 Examples from the theory of hypergeometric systems

Let $D = \mathbb{C}[x_1, \ldots, x_n]\langle\partial_1, \ldots, \partial_n\rangle$. We consider the A-hypergeometric D-module $M_A(\beta) = D/H_A(\beta)$. This is a holonomic system associated with a $d \times n$ integer matrix A and $\beta_1, \ldots, \beta_d \in \mathbb{C}$ as follows. We suppose that the abelian group generated by the columns $a_1, \ldots, a_n$ of A is equal to $\mathbb{Z}^d$. Let I_A be the ideal of $\mathbb{C}[\partial_1, \ldots, \partial_n]$ generated by the elements $\partial^u - \partial^v$ with $u, v \in \mathbb{N}^n$ such that $A.u = A.v$. The hypergeometric ideal $H_A(\beta)$ is the ideal of D generated by I_A and the elements $\sum_j a_{i,j}x_j\partial_j - \beta_i$ for $i = 1, \ldots, d$. The hypergeometric modules were introduced by I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky in [5]; their holonomicity (in the general case) was proved by A. Adolphson in [1].

We endow M with the quotient F-filtration and the quotient V-filtration with respect to $x_1 = \cdots = x_n = 0$.

Let us assume that the abelian group generated by the rows of A contains a vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{Z}^n_{>0}$. That is equivalent to the fact that the semigroup generated by the columns of A is pointed. By applying the weight

vector $W = (-\mathbf{w}, \mathbf{w})$ to (x, ∂), we get a grading on D. The hypergeometric module $M_A(\beta)$ is homogeneous w.r.t. to W.

Our first topic is to strenghten the correspondence between $\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2)$ and $\mathcal{C}(\mathrm{bigr}M_A(\beta); T_1, T_2)$, i.e. to prove that the modules $\mathrm{bigr}M_A(\beta)$ and $\mathrm{gr}^F(\mathbf{R}_V(M_A(\beta)))$ have the same codimension if $M_A(\beta)$ is nicely bifiltered.

The codimension of a finite type D-module M is by definition the codimension of $gr^F(M)$, that does not depend on the good F-filtration. In fact we can make the weight vector vary as well.

Proposition 5.1 ([17], pp. 65-66)**.** *Let $(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^{2n}$ be a weight vector such that for all i, $u_i + v_i > 0$. Endow M with a good $(\mathbf{u}, \mathbf{v})$-filtration. Then* $\mathrm{codim}(\mathrm{gr}^{(\mathbf{u},\mathbf{v})}(M)) = \mathrm{codim}M$.

We have an analogous statement for $D^{(h)}$-modules, proved in the same way. Let $(\mathbf{u}, \mathbf{v}, t) \in \mathbb{N}^{2n+1}$ such that for all i, $u_i + v_i > t$. Then $\mathrm{gr}^{(\mathbf{u},\mathbf{v},t)}(D^{(h)})$ is commutative.

Definition 5.1. *Let M be a graded $D^{(h)}$-module of finite type. Endow M with a good $(\mathbf{u}, \mathbf{v}, t)$-filtration. We define* $\mathrm{codim}M = \mathrm{codim}(\mathrm{gr}^{(\mathbf{u},\mathbf{v},t)}M)$. *This depends neither on the good filtration nor on the weight vector $(\mathbf{u}, \mathbf{v}, t)$.*

Finally, since $\mathrm{gr}^V(D^{(h)}) \simeq D^{(h)}$, we define in the same way the codimension of a $\mathrm{gr}^V(D^{(h)})$-module of finite type.

We adopt the following notation. If $P = \sum a_\beta(x)\partial_x^\beta \in F_d(D)$, we define $H_d(P) = \sum a_\beta(x)\partial_x^\beta h^{d-|\beta|} \in D^{(h)}$, and the F-homogenization $H(P) = H_{\mathrm{ord}^F(P)}(P)$. If I is an ideal of D, let $H(I)$ be the ideal of $D^{(h)}$ generated by the elements $H(P)$ such that $P \in I$. We have $\mathcal{R}_F(M) = D^{(h)}/H(I)$. Similarly we define the V-homogenization, denoted by $H^V(P) \in D[\theta]$ and $H^V(I) \subset D[\theta]$.

Proposition 5.2. *Let $M = D/I$ be a W-homogeneous nicely bifiltered D-module. Then the modules M,* $\mathrm{gr}^F(\mathbf{R}_V(M))$, $\mathrm{gr}^V(\mathbf{R}_F(M))$ *and* $\mathrm{bigr}M$ *all have the same codimension.*

Proof. First, we prove that

$$\mathrm{codim}\mathcal{R}_F(M) = \mathrm{codim}M.$$

Let $<$ be a well-order on $\mathbb{N}^{2n}$ (the monomials of D) adapted to F, i.e. for any $\alpha, \alpha', \beta, \beta', |\beta| < |\beta'| \Rightarrow (\alpha, \beta) < (\alpha', \beta')$. We derive from it a well-order $<'$ on $\mathbb{N}^{2n+1}$ (the monomials of $(D^{(h)})$) in the following way:

$$(\alpha, \beta, k) <' (\alpha', \beta', k') \quad \text{iff} \quad \begin{cases} |\beta| + k < |\beta'| + k' \\ \text{or} \quad \begin{cases} |\beta| + k = |\beta'| + k' \\ \text{and} \quad (\alpha, \beta) < (\alpha', \beta'), \end{cases} \end{cases}$$

which is adapted to the F-filtration. Let $P_1, \ldots, P_s$ be a Gröbner base of I with respect to $<$. Then $H(P_1), \ldots, H(P_s)$ is a Gröbner base of $H(I)$ with respect to $<'$ (use the Buchberger criterion). We have $\sigma^F(H(P_i)) = \sigma^F(P_i) \in \mathbb{C}[x, \xi]$, thus $\mathrm{codim}(\mathrm{gr}^F(\mathbf{R}_F(M))) = \mathrm{codim}(\mathrm{gr}^F(M))$.

Now, we prove that

$$\mathrm{codim}(\mathrm{gr}^V(\mathcal{R}_F(M))) = \mathrm{codim}M.$$

The module $\mathcal{R}_F(M)$ is bihomogeneous with respect to the weight vectors $(-\mathbf{w},\mathbf{w},0)$ and $(\mathbf{0},\mathbf{1},1)$. Let $\mu=\max(w_i-1)\in\mathbb{N}$ and

$$\Lambda=(-\mathbf{1},\mathbf{1},0)-(-\mathbf{w},\mathbf{w},0)+\mu.(\mathbf{0},\mathbf{1},1)=(\mathbf{w}-\mathbf{1},(1+\mu)\mathbf{1}-\mathbf{w},\mu.\mathbf{1})\in\mathbb{N}^{2n+1}.$$

Using the bihomogeneity, a V-adapted base of $H(N)$ is also adapted to Λ, so $\mathrm{gr}^{\Lambda}(\mathcal{R}_F(M))=\mathrm{gr}^{V}(\mathcal{R}_F(M))$. Then

$$\begin{aligned}\operatorname{codim}\mathrm{gr}^{V}(\mathcal{R}_F(M)) &= \operatorname{codim}\mathrm{gr}^{(\mathbf{0},\mathbf{1},0)}\mathrm{gr}^{V}(\mathcal{R}_F(M)) \quad \text{(by definition)}\\ &= \operatorname{codim}\mathrm{gr}^{(\mathbf{0},\mathbf{1},0)}\mathrm{gr}^{\Lambda}(\mathcal{R}_F(M))\\ &= \operatorname{codim}\mathrm{gr}^{\Lambda+\epsilon.(\mathbf{0},\mathbf{1},0)}(\mathcal{R}_F(M)) \quad \text{with } \epsilon>0,\end{aligned}$$

by [17], Lemma 2.1.6, which proves our assertion since $\Lambda+\epsilon.(\mathbf{0},\mathbf{1},0)\in\mathbb{N}^{2n+1}$.

Next, let us see that

$$\operatorname{codim}(\mathrm{gr}^{F}(\mathcal{R}_V(M)))=\operatorname{codim}M.$$

We will slightly modify the problem using the niceness assumptiom. We can endow $\mathrm{gr}^F(D)\simeq\mathbb{C}[x,\xi]$ with a filtration with respect to the weight vector $(-\mathbf{1},\mathbf{1})$, which we still call the V-filtration. The module $\mathrm{gr}^F(M)\simeq\mathrm{gr}^F(D)/\mathrm{gr}^F(I)$ is naturally endowed with the quotient V-filtration. In the same way as in the proof of Lemma 4.2, we have

$$\mathrm{gr}^F(\mathcal{R}_V(M))=\mathcal{R}_V(\mathrm{gr}^F(M)).$$

Thus we are reduced to show $\operatorname{codim}(\mathcal{R}_V(\mathrm{gr}^F(M))=\operatorname{codim}M$. As before, let $\mu=\max(w_i-1)$ and define $\Lambda=V-(-\mathbf{w},\mathbf{w})+\mu.(\mathbf{0},\mathbf{1})\in\mathbb{N}^{2n}$. We have a ring isomorphism

$$\mathcal{R}_V(\mathrm{gr}^F(D))\simeq\mathrm{gr}^F(D)[\theta]\simeq\mathcal{R}_\Lambda(\mathrm{gr}^F(D)),$$

and $\mathcal{R}_V(\mathrm{gr}^F(M))\simeq\mathcal{R}_\Lambda(\mathrm{gr}^F(M))$ above this ring isomorphism. Next,

$$\begin{aligned}\operatorname{codim}\mathcal{R}_\Lambda(\mathrm{gr}^F(M)) &= \operatorname{codim}\mathrm{gr}^{\Lambda}\mathrm{gr}^F(M)\\ &= \operatorname{codim}\mathrm{gr}^{F+\epsilon\Lambda}(M)\\ &= \operatorname{codim}M.\end{aligned}$$

Finally, we show that

$$\operatorname{codim}(\mathrm{bigr}M)=\operatorname{codim}(M).$$

We have $\mathrm{bigr}M\simeq\mathrm{gr}^V\mathrm{gr}^F(M)$, by Lemma 4.2. Taking again $\Lambda=V-(-\mathbf{w},\mathbf{w})+\mu.(\mathbf{0},\mathbf{1})$, the assertion follows from $\mathrm{gr}^V\mathrm{gr}^F(M)=\mathrm{gr}^{\Lambda}\mathrm{gr}^F(M)=\mathrm{gr}^{F+\epsilon\Lambda}(M)$. □

Remark 5.1. *If $M_A(\beta)$ is nicely bifiltered, then we have*

$$\mathcal{C}_{F,V}(M_A(\beta);T_1,T_2)_{|T_2=0}=(\operatorname{rank}M_A(\beta)).T_1^n.$$

Indeed, a bifiltered free resolution induces a F-filtered free resolution, thus $K_F(M;T_1)=K_{F,V}(M;T_1,T_2)_{|T_2=1}$, so $K_F(M;1-T_1)=K_{F,V}(M;1-T_1,1-T_2)_{|T_2=0}$, and by the Proposition above, we have $\operatorname{codim}\mathrm{gr}^F(\mathcal{R}_V(M_A(\beta)))=\operatorname{codim}M=\operatorname{codim}\mathrm{gr}^F(M_A(\beta))=n$. We conclude by using Proposition 2.3.

Let us note for $1\le i\le d$, $(Ax\xi)_i=\sum_j a_{i,j}x_j\xi_j\in\mathrm{gr}^F(D)$.

Lemma 5.1. *If* $\mathrm{gr}^F(\mathbb{C}[\partial]/I_A)$ *is Cohen-Macaulay, then* $(Ax\xi)_1, \ldots, (Ax\partial)_d$ *is a regular sequence in* $\mathrm{gr}^F(D/DI_A)$.

Proof. By [13], Proposition 7.5, $\dim(C[\partial]/I_A) = d$. Using Proposition 5.1, we get $\dim(\mathrm{gr}^F(D/DI_A)) = n + d$. But $\dim(\mathbb{C}[x,\xi]/(Ax\xi + \mathrm{gr}^F(I_A)) = n$ by [18], proof of Proposition 3.8. The results follows from the Cohen-Macaulay assumption. □

5.1 The homogeneous case

We suppose moreover that the columns of A lie in a common hyperplane, i.e. $(1, \ldots, 1)$ belongs to the $\mathbb{Q}$-row span of A. Then I_A is homogeneous for the weight vector $(1, \ldots, 1)$ and $M_A(\beta)$ is V-homogeneous.

Lemma 5.2. $M_A(\beta)$ *is nicely bifiltered.*

Indeed, $M_A(\beta)$ is V-homogeneous, thus $\mathbf{R}_F(M_A(\beta))$ is also V-homogeneous, thus $\mathrm{gr}^V \mathbf{R}_F(M_A(\beta)) \simeq \mathbf{R}_F(M_A(\beta))$ is h-saturated. Then apply Lemma 4.1.

Lemma 5.3. *Let* $R = \mathrm{bigr}D$ *and* M *be a finite type bigraded* R*-module. Let* $P \in R$ *be bihomogeneous of degree* (d,k)*. If* P *is a non zero-divisor on* M *then*

1. $K_{F,V}(M/PM; T_1, T_2) = (1 - T_1^d T_2^k) K_{F,V}(M; T_1, T_2)$ *and*
2. $\mathcal{C}_{F,V}(M/PM; T_1, T_2) = (dT_1 + kT_2)\mathcal{C}_{F,V}(M; T_1, T_2)$.

Proof. Let us prove 1). If N is a bigraded R-module, let $S_{d,k}(N)$ be the bigraded module defined by $(S_{d,k}(N))_{d',k'} = N_{d'-d,k'-k}$. In particular, $S_{d,k}(D^r[\mathbf{n}][\mathbf{m}]) = D^r[\mathbf{n} + d.\mathbf{1}][\mathbf{m} + k.\mathbf{1}]$. A bigraded free resolution

$$\cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to M \to 0$$

of M induces a bigraded free resolution

$$\cdots \to S_{d,k}(\mathcal{L}_1) \to S_{d,k}(\mathcal{L}_0) \to S_{d,k}(M) \to 0$$

of $S_{d,k}(M)$. We have a bigraded exact sequence

$$0 \to S_{d,k}(M) \xrightarrow{P.} M \to \frac{M}{PM} \to 0.$$

Then taking the cone of the morphism of resolutions $S_{d,k}(\mathcal{L}_\bullet) \xrightarrow{P.} \mathcal{L}_\bullet$ gives a resolution

$$\cdots \to S_{d,k}(\mathcal{L}_1) \oplus \mathcal{L}_2 \to S_{d,k}(\mathcal{L}_0) \oplus \mathcal{L}_1 \to \mathcal{L}_0 \to \frac{M}{PM} \to 0$$

of M/PM. Then 1) follows, and 2) follows from 1). □

Let us denote by $\mathrm{vol}(A)$ the normalized volume of the convex hull in $\mathbb{R}^d$ of the set $\{0, a_1, \ldots, a_n\}$. The normalization means that the set $[0,1] \times \cdots \times [0,1] \subset \mathbb{R}^d$ has volume $d!$.

Theorem 5.1. *If $\mathbb{C}[\partial]/I_A$ is homogeneous and Cohen-Macaulay, then for any $\beta \in \mathbb{C}^d$ we have*

$$\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = \mathrm{vol}(A). \sum_{j=d}^{n} \binom{n-d}{j-d} T_1^j T_2^{n-j},$$

Proof. By Proposition 4.2, Proposition 5.2 and Lemma 5.2, $\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2)$ is equal to the sum of the terms of least degree in $K_{F,V}(\mathrm{bigr}M_A(\beta); 1-T_1, 1-T_2)$, and by Lemma 4.2 we have

$$\mathrm{bigr}M_A(\beta) \simeq \mathrm{gr}^F \mathrm{gr}^V (M_A(\beta)) = \mathrm{gr}^F (M_A(\beta)).$$

When $\mathbb{C}[\xi]/I_A$ is Cohen-Macaulay, $(Ax\xi)_1, \ldots, (Ax\xi)_d$ form a regular sequence in $\mathbb{C}[x, \xi]/I_A$, and $\mathrm{gr}^F(H_A(\beta))$ is generated by I_A and $(Ax\xi)_1, \ldots, (Ax\xi)_d$, by Lemma 5.1 and [17], Theorem 4.3.8. Using Lemma 5.3 repeatedly, we get

$$\mathcal{C}_{F,V}(\mathrm{gr}^F(M_A(\beta)); T_1, T_2) = T_1^d . \mathcal{C}_{F,V}(\mathbb{C}[x, \xi]/I_A; T_1, T_2).$$

But $\mathcal{C}_{F,V}(\mathbb{C}[x, \xi]/I_A; T_1, T_2) = \mathcal{C}_{F,F}(\mathbb{C}[\xi]/I_A; T_1, T_2)$ since $I_A \subset \mathbb{C}[\xi]$. Let $R = \mathbb{C}[\xi]$, $P(T_1, T_2) = K_{F,F}(R/I_A; T_1, T_2)$ and $Q(T) = K_F(R/I_A; T)$. Consider a graded free resolution

$$0 \to R^{r_\delta}[\mathbf{n}^{(\delta)}] \to \cdots R^{r_0}[\mathbf{n}^{(0)}] \to R/I_A \to 0$$

of R/I_A. Then we have a bigraded free resolution

$$0 \to R^{r_\delta}[\mathbf{n}^{(\delta)}][\mathbf{n}^{(\delta)}] \to \cdots R^{r_0}[\mathbf{n}^{(0)}][\mathbf{n}^{(0)}] \to R/I_A \to 0$$

of R/I_A. We deduce that $P(T_1, T_2) = Q(T_1T_2)$. We have $Q(1-T) = b_{n-d}T^{n-d} + O(n-d+1)$, with $b_{n-d} = \deg(R/I_A) \neq 0$, and $O(n-d+1)$ denotes a polynomial of valuation greater than $n-d$. By [6], Chapter 6, Theorem 2.3, $\deg(R/I_A) = \mathrm{vol}(A)$. We have

$$\begin{aligned} P(1-T_1, 1-T_2) &= Q((1-T_1)(1-T_2)) \\ &= Q(1-(T_1+T_2-T_1T_2)) \\ &= b_{n-d}(T_1+T_2)^{n-d} + O(n-d+1) \\ &= b_{n-d}\left(\sum_{i=0}^{n-d} \binom{n-d}{i} T_1^i T_2^{n-d-i}\right) + O(n-d+1), \end{aligned}$$

from which the statement follows. □

To compute the multidegree in the following examples, we used the computer algebra systems Singular [10] and Macaulay2 [9].

Example 1. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. Then I_A is generated by $\partial_1\partial_3 - \partial_2^2$. For all β, $\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = 2T_1^3 + 2T_1^2T_2$.

Example 2. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$. Then I_A is generated by $\partial_2\partial_4 - \partial_3^2, \partial_1\partial_4 - \partial_2\partial_3, \partial_1\partial_3 - \partial_2^2$. For all β, $\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = 3T_1^4 + 6T_1^3T_2 + 3T_1^2T_2^2$.

Let us give homogeneous non-Cohen-Macaulay examples from the book [17]. Using Proposition 1.5 repeatedly, we can establish the existence of a stratification of the space of the parameters β_1, β_2 by the multidegree. In the following two examples, this stratification equals the stratification by the holonomic rank.

Example 3. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$. Then I_A is generated by $\partial_2\partial_4^2 - \partial_3, \partial_1\partial_4 - \partial_2\partial_3, \partial_1\partial_3^2 - \partial_2^2\partial_4, \partial_1^2\partial_3 - \partial_2^3$. For $(\beta_1, \beta_2) \neq (1, 2)$, we have

$$\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = 4T_1^4 + 8T_1^3T_2 + 4T_1^2T_2^2.$$

For $(\beta_1, \beta_2) = (1, 2)$, we have

$$\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = 5T_1^4 + 12T_1^3T_2 + 10T_1^2T_2^2 + 4T_1T_2^3 + T_2^4.$$

Example 4. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 7 & 9 \end{pmatrix}$. Then I_A is generated by $\partial_2\partial_4 - \partial_3^2, \partial_1^2 - \partial_2\partial_3$. Let $E = \{(2, 10), (2, 12), (3, 19)\}$. For $(\beta_1, \beta_2) \notin E$, we have

$$\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = 9T_1^5 + 27T_1^4T_2 + 27T_1^3T_2^2 + 9T_1^2T_2^3.$$

For $(\beta_1, \beta_2) \in E$, we have

$$\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = 10T_1^5 + 32T_1^4T_2 + 37T_1^3T_2^2 + 19T_1^2T_2^3 + 5T_1T_2^4 + T_2^5.$$

5.2 The inhomogeneous case

Following arguments in the book [17], we extend Theorem 5.1 in the inhomogeneous case, for generic parameters β.

Theorem 5.2. *Assume that $\mathbb{C}[\partial, h]/H(I_A)$ is Cohen-Macaulay. Then for generic β, the module $M_A(\beta)$ is nicely bifiltered and*

$$\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = \mathrm{vol}(A). \sum_{j=d}^{n} \binom{n-d}{j-d} T_1^j T_2^{n-j}.$$

Here, the assumption is that the closure of the variety defined by I_A in the projective space $\mathbb{P}^n$ is Cohen-Macaulay.

Proof. First, note that the $\mathbb{C}[\partial, h]$-module $\mathbb{C}[\partial, h]/H(I_A)$ and the $\mathrm{gr}^F(\mathbb{C}[\partial])$-module $\mathrm{gr}^F(\mathbb{C}[\partial])/\mathrm{gr}^F(I_A)$ have same codimension and same projective dimension. Thus by [4], Corollary 19.15, the Cohen-Macaulayness of the former is equivalent to that of the latter.

Also,

$$\mathcal{C}(\mathrm{gr}^F(\mathbb{C}[\partial])/\mathrm{gr}^F(I_A); T) = \mathcal{C}(\mathbb{C}[\partial, h]/H(I_A); T) = \deg(\mathbb{C}[\partial, h]/H(I_A))T^{n-d}$$

and again by [6], Chapter 6, Theorem 2.3, $\deg(\mathbb{C}[\partial, h]/H(I_A)) = \mathrm{vol}(A)$.

For generic β, by [17], Theorem 3.1.3 (with $w = (1, \ldots, 1)$), and [16], Theorem 2.5,

$$H^V(H_A(\beta)) = D[\theta]H^V(I_A) + \sum_i D[\theta]((Ax\partial)_i - \beta_i).$$

By Lemma 5.1, because of the Cohen-Macaulay assumption, $(Ax\xi)_1, \ldots, (Ax\xi)_d$ is a regular sequence in $\mathrm{gr}^F(D[\theta])/\mathrm{gr}^F(I_A) = \mathrm{gr}^F(D[\theta])/\mathrm{gr}^F(H^V(I_A))$. That implies that $H^V(I_A)$ and $((Ax\partial)_i - \beta_i)_i$ form an F-involutive base of $H^V(H_A(\beta))$ (see [17], Proposition 4.3.2). Then

$$\begin{aligned}\mathrm{gr}^F(H^V(H_A(\beta))) &= \mathrm{gr}^F(D[\theta])\mathrm{gr}^F(H^V(I_A)) + \sum_i \mathrm{gr}^F(D[\theta])(Ax\partial)_i. \\ &= \mathrm{gr}^F(D[\theta])\mathrm{gr}^F(I_A) + \sum_i \mathrm{gr}^F(D[\theta])(Ax\partial)_i.\end{aligned}$$

Thus $\mathrm{gr}^F(H^V(H_A(\beta)))$ is generated by elements independent of θ; this implies that $\mathrm{gr}^F(\mathcal{R}_V(M_A(\beta)))$ is θ-saturated (consider the graduation given by the degree in θ), which is equivalent to niceness by Lemma 4.1.

We have again $\mathrm{bigr}M_A(\beta) \simeq \mathrm{gr}^F\mathrm{gr}^V(M_A(\beta))$. With same arguments as above, we show that $\mathrm{gr}^F\mathrm{gr}^V(H_A(\beta))$ is generated by $\mathrm{gr}^F(I_A)$ and $(Ax\xi)_i$ for generic β. We conclude the computation of the multidegree as in the proof of Theorem 5.1. □

To finish, let us give examples in the inhomogeneous case.

Example 5. Let $A = \begin{pmatrix} 0 & 1 & 3 \\ 4 & 3 & 2 \end{pmatrix}$. Then I_A is generated by $\partial_1^7\partial_3^4 - \partial_2^{12}$. The ring $\mathbb{C}[\partial, h]/H(I_A)$ is Cohen-Macaulay. For any β, $M_A(\beta)$ is nicely bifiltered and $\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = 12T_1^3 + 12T_1^2T_2$.

Example 6. Let $A = \begin{pmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix}$. Then I_A is generated by $\partial_2^2\partial_4^2 - \partial_3^3, \partial_1\partial_4 - \partial_2\partial_3, \partial_1\partial_3^2 - \partial_2^3\partial_4, \partial_1^2\partial_3 - \partial_2^4$. The ring $\mathbb{C}[\partial, h]/H(I_A)$ is not Cohen-Macaulay. For β generic, $M_A(\beta)$ is nicely bifiltered and

$$\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = 6T_1^4 + 12T_1^3T_2 + 6T_1^2T_2^2.$$

We could check that the couple $\beta = (-1, 2)$ is exceptional. In that case $M_A(\beta)$ is also nicely bifiltered and we have

$$\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = 7T_1^4 + 16T_1^3T_2 + 12T_1^2T_2^2 + 4T_1T_2^3 + T_2^4.$$

Let us remark that in Examples 1–6, the formula of Theorems 5.1 and 5.2 holds for generic β, sometimes without the Cohen-Macaulay assumption.

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